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Quasi-isometries and isoperimetric inequalities in planar domains

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Abstract. This paper studies the stability of isoperimetric inequalities under quasi-isometries between non-exceptional Riemann surfaces endowed with their Poincaré metrics. This stability was proved by Kanai in the more general setting of Riemannian manifolds under the condition of positive injectivity radius. The present work proves the stability of the linear isoperimetric inequality for planar surfaces (genus zero surfaces) without any condition on their injectivity radii. It is also shown the stability of any non-linear isoperimetric inequality.

1. Introduction.

An interesting problem in the study of geometric properties of surfaces is to consider their stability under appropriate deformations. In the 1985, in [**20**] M. Kanai proved the quasi-isometric stability (see the definition of quasi-isometry in Section 2) of several geometric properties for a large class of Riemannian manifolds.

We shall be interested not only in his results but in the ideas behind the proofs. Concretely, those relating the manifold with a particular graph (an ε-net of the manifold) in order to study the stability of the quasi-isometry. Several authors have followed Kanai in studying the stability of some other property, or in proving the equivalence of a manifold with a different associated graph (see, e.g., [**1**], [**7**], [**16**], [**19**], [**21**], [**22**], [**28**], [**29**], [**32**], [**33**], [**35**], [**37**], [**39**]).

Quasi-isometries play a central role in the theory of Gromov hyperbolic spaces for they preserve hyperbolicity of geodesic metric spaces (see, e.g., [**17**], [**18**]).

A *non-exceptional* Riemann surface S will mean a two-dimensional manifold with a complete conformal metric of constant negative curvature -1. In this case, the universal covering space of S is the unit disk $\mathbb{D}$ endowed with its Poincaré metric. The only exceptional Riemann surfaces are the sphere, the plane, the punctured plane and the tori.

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A Riemann surface S satisfies the *α-isoperimetric inequality* ($1/2 \le \alpha \le 1$) if there exists a constant $c_\alpha(S)$ such that

$$A_S(\Omega)^\alpha \le c_\alpha(S) L_S(\partial\Omega) \tag{1.1}$$

for every relatively compact domain $\Omega \subset S$. Throughout, A_S, L_S and d_S refer to Poincaré area, length and distance of S and LII refers to the 1-isoperimetric inequality also known as the *linear isoperimetric inequality.*

There are close connections between LII and some conformal invariants of Riemann surfaces, namely the bottom of the spectrum of the Laplace-Beltrami operator, the exponent of convergence, and the Hausdorff dimensions of the sets of both bounded geodesics and escaping geodesics in the surface (see [**5**], [**6**, p. 228], [**10**], [**11**], [**12**], [**13**], [**14**], [**15**], [**24**], [**25**], [**38**, p. 333]). Isoperimetric inequalities are of interest in pure and applied mathematics (see, e.g., [**9**], [**27**]).

The *injectivity radius* $\iota(p)$ *of* $p \in S$ is defined as the supremum of those $r > 0$ such that $B_S(p, r)$ is simply connected or, equivalently, as half the infimum of the lengths of the (homotopically non-trivial) loops based at p. The *injectivity radius* $\iota(S)$ *of* S is the infimum over $p \in S$ of $\iota(p)$.

This paper considers the stability of isoperimetric inequalities under quasi-isometries between non-exceptional Riemann surfaces. This stability was proved by Kanai in [**20**] under the hypothesis $\iota(S) > 0$ in the very general setting of Riemannian manifolds. Example 2.3 in the next section shows that the stability fails without the hypothesis $\iota(S) > 0$. Since this example involves non-zero genus surfaces, it is natural to wonder if the stability holds for planar surfaces.

The main result in this paper is the following.

Theorem 1.1. *Let S and S' be quasi-isometric non-exceptional genus zero Riemann surfaces. Then S' satisfies the linear isoperimetric inequality if and only if S satisfies the linear isoperimetric inequality. Furthermore, if $f : S \longrightarrow S'$ is a c-full (a, b)-quasi-isometry, and $c_1(S') < \infty$ then $c_1(S) \le C$, where C is a universal constant which just depends on a, b, c and $c_1(S')$.*

For surfaces of positive finite genus, the following result shows that the first conclusion of Theorem 1.1 holds:

Theorem 1.2. *Let S and S' be quasi-isometric non-exceptional Riemann surfaces with finite genus. Then S' satisfies the LII if and only if S satisfies the LII.*

However Example 7.2 shows that the second conclusion of Theorem 1.1 fails in this case of positive finite genus.

The idea behind the proof of Theorem 1.1 is simple: each surface is split into a thin part (with small injectivity radius) and a thick part; a slight modification of the proof of Kanai's theorem applied to the thick part, together with some new arguments to show that the thin part is "essentially" preserved under the quasi-isometry give the theorem. The difficulty is the following: two quasi-isometric surfaces have a similar shape at a large scale (if viewed from sufficiently far), but they can look very different at a small scale (by definition a quasi-isometry may not be continuous). In particular, the image of a continuous loop by a quasi-isometry need not be a continuous curve, and thus the injectivity radii can be very different in two quasi-isometric surfaces (see, e.g., Examples 2.4 and 2.5). Theorem 1.3 deals with this situation and states that a quasi-isometry between planar surfaces maps points with small injectivity radius to points with small injectivity radius (in a precise quantitative way).

THEOREM 1.3. *Let S and S' be non-exceptional genus zero Riemann surfaces and let $f : S \longrightarrow S'$ be a c-full (a,b)-quasi-isometry. For each $\varepsilon' > 0$ there exists $\varepsilon > 0$ which just depends on ε', a, b, c, such that if $\iota(z) < \varepsilon$ then $\iota(f(z)) < \varepsilon'$. Moreover, given $\varepsilon_1 > 0$, ε can be taken so that $\varepsilon < \varepsilon_1$.*

In fact, the core of this work is devoted to proving Theorem 1.3.

We show that a very different situation appears when dealing with the α-isoperimetric inequality, $1/2 \le \alpha < 1$.

THEOREM 1.4. *Let S and S' be quasi-isometric non-exceptional Riemann surfaces with $\iota(S) > 0$, and $1/2 \le \alpha < 1$. Then S' satisfies the α-isoperimetric inequality if and only if S satisfies the α-isoperimetric inequality and $\iota(S') > 0$.*

Note that here we have no hypothesis on genus.

Hence, the behavior of the α-isoperimetric inequality in Riemann surfaces under quasi-isometries is very different in the cases $\alpha = 1$ and $\alpha < 1$.

The outline of this paper is as follows. Section 2 contains some background and examples. In Section 3 the continuity of the injectivity radius in a Riemann surface is studied. Section 4 contains some technical lemmas on quasi-isometries which will be needed in Section 5 in order to control the distortion of the injectivity radius under quasi-isometries. In Section 6 the proof of Theorem 1.1 is given, and finally, Sections 7 and 8 are devoted to generalize this theorem to finite genus surfaces and to non-linear isoperimetric inequalities, respectively.

2. Background and examples.

A function between two metric spaces $f : X \longrightarrow Y$ is said to be an (a,b)-*quasi-isometric embedding* with constants $a \ge 1$, $b \ge 0$, if

$$\frac{1}{a}\, d_X(x_1,x_2) - b \le d_Y(f(x_1),f(x_2)) \le a\, d_X(x_1,x_2) + b, \qquad \text{for every } x_1,x_2 \in X.$$

Such a quasi-isometric embedding f is a *quasi-isometry* if, furthermore, there exists a constant $c \ge 0$ such that f is *c-full*, i.e., if for every $y \in Y$ there exists $x \in X$ with $d_Y(y,f(x)) \le c$.

Two metric spaces X and Y are *quasi-isometric* if there exists a quasi-isometry between them.

An (a,b)-*quasigeodesic* in X is an (a,b)-quasi-isometric embedding between an interval of $\mathbb{R}$ and X. A *geodesic* in X is a $(1,0)$-quasigeodesic.

It is easy to check that to be quasi-isometric is an equivalence relation on the set of metric spaces.

The word *geodesic* will always be used with this meaning except for the case of either simple closed geodesics (which are just local geodesics) or geodesic loops (which are just local geodesics except in their basepoints).

The surface S will be split into thin and thick parts, and some standard tools for constructing Riemann surfaces will be needed. Doubly connected domains will be crucial.

A *collar* in a non-exceptional Riemann surface S about a simple closed geodesic σ is a doubly connected domain in S "bounded" by two Jordan curves (called the boundary curves of the collar) orthogonal to the pencil of geodesics emanating from σ; such collar is equal to $\{p \in S : d_S(p,\sigma) \le d\}$, for some positive constant d. The constant d is called the *width* of the collar.

Let S be a non-exceptional Riemann surface with a cusp q (if $S \subset \mathbb{C}$, every isolated point in ∂S is a cusp). A *collar* in S about q is a doubly connected domain in S "bounded" both by q and a Jordan curve (called the boundary curve of the collar) orthogonal to the pencil of geodesics emanating from q. It is well known that the length of the boundary curve is equal to the area of the collar (see, e.g., [**4**]). A collar of area β is called a β-collar.

A *Y-piece* is a compact bordered Riemann surface which is topologically a sphere without three disks and whose border is the union of three simple closed geodesics. Given three positive numbers a, b, c, there is a unique (up to conformal mapping) Y-piece such that their boundary curves have lengths a, b, c (see, e.g., [**31**, p. 410]). They are a standard tool for constructing Riemann surfaces ([**8**, Chapter X.3] and [**6**, Chapter 1]).

A *generalized* or *degenerated Y-piece* is a bordered or non-bordered Riemann surface which is topologically a sphere without n open disks and m points, with integers $n, m \ge 0$ and $n + m = 3$, so that the n boundary curves are simple closed geodesics and the m deleted points are cusps. Observe that a generalized Y-piece is topologically the union of a Y-piece and m cylinders, with $0 \le m \le 3$.

A *funnel* is a bordered Riemann surface which is topologically a cylinder and whose border is a simple closed geodesic. Given any positive number a, there is a unique (up to conformal mapping) funnel such that its boundary curve has length a.

We collect below two well known hyperbolic trigonometric formulae (see, e.g., [**6**, p. 454]) which will be useful.

Proposition 2.1. *The following formulae hold for polygons on the unit disk* (*and then for simply connected polygons on any non-exceptional Riemann surface*).

(1) *If a, b, c are the lengths of the sides of a right-angled geodesic triangle and c is the length of the hypotenuse, then* $\cosh c = \cosh a \cosh b$.
(2) *Let us consider a geodesic quadrilateral with three right angles and let ϕ the other angle. If α, β are the lengths of the sides which meet with angle ϕ and a is the length of the opposite side to the side with length α, then* $\sinh \alpha = \sinh a \cosh \beta$.

Example 2.3 shows that the stability of LII fails for surfaces with zero injectivity radius. We need a previous result.

Lemma 2.2. *A non-exceptional Riemann surface of finite type satisfies the LII if* *and only if it has at least a funnel.*

PROOF. Let S be a non-exceptional Riemann surface of finite type.

If S has not funnels, then it has finite area and consequently it has not LII.

Assume now that S has at least a funnel. A domain $\Omega \subset S$ is said to be a geodesic domain if $\partial\Omega$ is a finite number of simple closed geodesics, and $A_S(\Omega)$ is finite. By Lemma 6.5 with $\varepsilon = 0$, S has LII if and only if it has LII for geodesic domains. Note that there are just a finite number of geodesic domains Ω in S; since S has at least a funnel, every geodesic domain Ω in S verifies $L_S(\partial\Omega) > 0$; therefore, S satisfies the LII. □

EXAMPLE 2.3. There exist two non-exceptional Riemann surfaces S, S' and a quasi-isometry $f : S \to S'$, such that $\iota(S) = \iota(S') = 0$, S does not satisfy the LII and S' satisfies the LII.

Let us consider two isometric Y-pieces Y_1, Y_2 such that ∂Y_j is the union of three simple closed geodesics with length 1 for $j = 1, 2$. Denote by X the bordered surface obtained by pasting two boundary curves of Y_1 with two boundary curves of Y_2 (X is a torus with two holes). Let us consider a sequence $\{X_m\}_{m\ge1}$ of bordered surfaces isometric to X; denote by S_0 the bordered surface obtained by pasting a boundary curve of X_m with a boundary curve of X_{m+1} for every $m \ge 1$. Consider now a generalized Y-piece Y_0 with a cusp and such that ∂Y_0 is the union of two simple closed geodesics with length 1.

S is the (non bordered) surface obtained by pasting a funnel (with boundary of length 1) to one boundary curve of Y_0 and S_0 to the other boundary curve of Y_0. S does not satisfy the LII since $\bigcup_{m=1}^n X_m$ has area $4\pi n$ and its boundary has length 2 for every $n \ge 1$.

The surface S' is obtained by pasting a funnel (with boundary of length 1) to a generalized Y-piece Y^* with two cusps and such that ∂Y^* is a simple closed geodesics with length 1. Lemma 2.2 gives that S' satisfies the LII.

Finally, let us prove the existence of a quasi-isometry $f : S \to S'$. Fix geodesic rays γ, γ' (geodesics whose domain is the interval $[0, \infty)$) in S_0 and in the 2-collar C of a fixed cusp of S', starting in ∂S_0 and ∂C, respectively. Let us consider the bijective isometry $g : \gamma \to \gamma'$. We can easily check that the map $f_0 : S_0 \to C$ defined in the following way is a quasi-isometry: if $p(z)$ is the nearest point in γ from $z \in S_0$, then $f_0(z) = g(p(z))$. Define $f : S \to S'$ as follows: $f = f_0$ on S_0, f is any isometry between the funnels of S and S', f is any isometry between the 2-collar of the cusp of S and the 2-collar of the cusp of S' which does not intersect $f(S_0)$, and $f = u_0$ otherwise, where u_0 is any fixed point in S'. Now, we can easily check that f is a quasi-isometry.

The following examples show that the conclusion of Theorem 1.3 does not hold if S or S' are surfaces of positive genus. In particular, Example 2.5 shows that thin parts are not in correspondence.

EXAMPLE 2.4. There exist constants a, b, c, I_1, I_2 with the following property: for each n there exist non-exceptional Riemann surfaces S_n, S'_n and a c-full (a, b)-quasi-isometry $f_n : S_n \to S'_n$, such that $\iota(z) \ge n$ for every $z \in S_n$ and $I_1 \le \iota(z) \le I_2$ for every

$z \in S'_n$.

Let S_n be the annulus with the simple closed geodesic with length $2n$. Consider an 1-net N_n of S_n; by [**20**] there exists a quasi-isometry $g_n : S_n \to N_n$ with universal constants. By [**26**, Theorem 3.4] (see also [**3**, Theorem 23]), there exist a cubic graph C_n and a quasi-isometry $h_n : N_n \to C_n$ with universal constants. Let us consider a sequence $\{Y_m\}$ of Y-pieces such that ∂Y_m is the union of three simple closed geodesics with length 1 for every m (therefore, they are isometric). It suffices to consider as S'_n the surface obtained by pasting the Y-pieces $\{Y_m\}$ following the combinatorial design of C_n. Since [**35**, Theorem 3.8] or [**39**, Theorem 4.17] provide a quasi-isometry $u_n : C_n \to S'_n$ with universal constants, there exists a quasi-isometry $f_n : S_n \to S'_n$ with universal constants.

EXAMPLE 2.5. There exist constants I_1, I_2 with the following property: there exist non-exceptional Riemann surfaces S, S' and a quasi-isometry $f : S \to S'$, such that $I_1 \le \iota(z) \le I_2$ for every $z \in S$ and $\iota(S') = 0$.

Let us consider a sequence $\{X_m\}_{m\in\mathbb{Z}}$ of bordered surfaces isometric to the torus with two holes X in Example 2.3; then S is obtained by pasting a boundary curve of X_m with a boundary curve of X_{m+1} for every $m \in \mathbb{Z}$.

Consider now two isometric generalized Y-pieces Y_3, Y_4 with a cusp and such that ∂Y_j is the union of two simple closed geodesics with length 1 for $j = 3, 4$. It suffices to consider S' as the (non bordered) surface obtained by pasting the two boundary curves of Y_3 with the two boundary curves of Y_4 (S' is a torus with two cusps).

The existence of a quasi-isometry $f : S \to S'$ can be shown with a similar argument to the one used in Example 2.3.

3. Continuity of the injectivity radius.

The following result is well-known and easy to check.

LEMMA 3.1. *Let M be a Riemannian manifold and $x, y \in M$. Then* $|\iota(x) - \iota(y)| \le d_M(x, y)$.

This last result can be improved for small values of the injectivity radius.

LEMMA 3.2. *Let S be a non-exceptional Riemann surface and $z, w \in S$. Then*

$$\iota(w) \ge \operatorname{arcsinh}\left(e^{-d_S(z,w)} \min\{1, \sinh \iota(z)\}\right).$$

In particular, if $\sinh \iota(z), \sinh \iota(w) \le 1$, *then* $|\log \sinh \iota(w) - \log \sinh \iota(z)| \le d_S(z, w)$.

PROOF. Let us choose geodesic loops γ_z and γ_w with respective base points z and w such that $\iota(z) = L_S(\gamma_z)$ and $\iota(w) = L_S(\gamma_w)$.

Assume first that γ_z and γ_w are freely homotopic. It is clear that the minimum value of $\iota(w)$ is attained if γ_z and γ_w bordered a cusp and z and w belong to the same geodesic escaping to the cusp.

As usual, consider a fundamental domain for S in the upper halfplane $\mathbb{H}$ contained in $\{z \in \mathbb{H} : 0 \le \Re z \le 1\}$ and such that $\{z \in \mathbb{H} : 0 \le \Re z \le 1, \Im z \ge 1/2\}$ corresponds to the 2-collar of this cusp in S. Let us represent γ_z (respectively, γ_w) in the upper half-plane by means of a geodesic with endpoints $i\alpha$ and $i\alpha+1$ (respectively, $i\beta$ and $i\beta+1$, with $\beta>\alpha$). Note that

$$\sinh \iota(z) = \sinh \frac{d_{\mathbb{H}}(i\alpha, i\alpha+1)}{2} = \frac{1}{2\alpha},$$
$$\sinh \iota(w) = \frac{1}{2\beta}, \quad d_S(z,w) = d_{\mathbb{H}}(i\alpha, i\beta) = \log\frac{\beta}{\alpha} = \log\frac{\sinh \iota(z)}{\sinh \iota(w)}.$$

Hence, the minimum value of $\iota(w)$ is attained with $\iota(w) = \operatorname{arcsinh}\big(e^{-d_S(z,w)}\sinh \iota(z)\big)$.

Assume now that γ_z and γ_w are not freely homotopic. Let us consider a geodesic $[z,w]$ in S and the nearest point z_0 to z in $[z,w]$ with a geodesic loop γ_{z_0} freely homotopic to γ_w such that $\iota(z_0) = L_S(g_{z_0})$. It is not difficult to see that $\iota(z_0) \ge \operatorname{arcsinh} 1$ (the injectivity radius of any point in the boundary of the 2-collar of a cusp). The previous argument gives $\iota(w) \ge \operatorname{arcsinh}(e^{-d_S(z_0,w)}\sinh \iota(z_0)) \ge \operatorname{arcsinh} e^{-d_S(z,w)}$. This finishes the proof of the first statement. The second statement is a direct consequence of the first one. □

4. Technical lemmas on quasi-isometries.

A key step in the proof of our main result in this paper (Theorem 1.1) is to control the distortion of the injectivity radius under quasi-isometric transformations (see Theorems 1.3 and 5.1). Due to the complexity of the proofs of these theorems, this section is devoted to present some technical lemmas used in their proofs.

Let us consider $H>0$, a metric space X, and a subset $Y\subseteq X$. The set $V_H(Y) := \{x\in X : d(x,Y)\le H\}$ is called the *H-neighborhood* of Y in X.

A control on how collars behave under quasi-isometries will be needed, and thus a more general definition is required:

Let us consider a quasi-isometry $f: S\to S'$, a finite or infinite geodesic $\gamma\subset S$ and a connected subset γ_0 of that geodesic. Given two positive constants h and r, then the h-neighborhood of $f(\gamma)$ in S' is an *(f,γ,γ_0,h,r)-tube* T if for every point $p\in\gamma_0$, the closed ball $\overline{B_{S'}(f(p),r)}$ is contained in T.

In principle, although a tube does not need to be doubly connected, Theorems 4.5 and 4.6 will show that they are "essentially" doubly connected, and that explain the name.

We will use several times the following result known as Collar Lemma (see [**30**]).

LEMMA 4.1. *If σ is a simple closed geodesic in a non-exceptional Riemann surface S, then there exists a collar about σ of width d, for every $0<d\le w$, where* $\cosh w = \coth(L_S(\sigma)/2)$. *Hence, if $L_S(\sigma) < 2\operatorname{arccoth}(\cosh t)$, then $w>t$.*

REMARK 4.2. Along this chapter, σ will denote a simple closed geodesic in S and w the width of the collar of σ, where $\cosh w = \coth(L_S(\sigma)/2)$.

Denote by $C_{\sigma,d}$ the collar of σ of width d and by C_σ the collar of σ of width w. It is well known that if σ_1 and σ_2 are disjoint simple closed geodesics, then $C_{\sigma_1}\cap C_{\sigma_2}=\emptyset$.

For each cusp there exists a 2-collar and 2-collars of different cusps are disjoint.

Besides, the collar C_σ of the simple closed geodesic σ does not intersect the 2-collar of a cusp (see [**30**], [**36**] and [**6**, Chapter 4]). If a λ-collar of a cusp (with $0 < \lambda \leq 2$) in a Riemann surface has boundary curve α, denote this collar by $\mathscr{C}_\alpha$. Denote also by $\mathscr{C}_\alpha$ the H-neighborhood of the 2-collar of a cusp with boundary α (now α can be a union of closed curves).

The next result deals with collars of geodesics and cusps separately:

Lemma 4.3. *Assume that S is a genus zero Riemann surface. Let f be a c-full (a,b)-quasi-isometry from S to S', and $t > 0$ a constant. Then, there exist positive constants k_1, k_2 and k depending only on a, b, c, t that satisfy the following*:

1. *Let σ be a simple closed geodesic on S with $L_S(\sigma) < k_2$, γ a geodesic perpendicular to σ contained in C_σ with $L_S(\gamma) = 2w$ and $\gamma_0 := \{p \in \gamma : d_S(p,\sigma) < w - k_1\}$. Then there exists an $(f, \gamma, \gamma_0, h, h+t)$-tube $T \subset S'$ with $h := 3a + b + c$.*
2. *Let $\mathscr{C}$ be the 2-collar of a cusp in S with boundary curve σ, γ an infinite geodesic contained in $\mathscr{C}$ perpendicular to σ and $\gamma_0 := \{p \in \gamma : d_S(p,\sigma) > k\}$. Then there exists an $(f, \gamma, \gamma_0, h, h+t)$-tube $T \subset S'$ with $h := 2a + b + c$.*

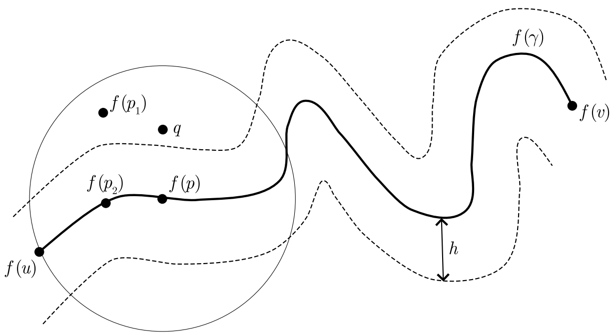
Proof. 1. Set $k_1 := 2a(3a+2b+2c+t)$ and $k_2 := 2\operatorname{arccoth}(\cosh k_1)$. Notice that $k_1 \geq 6$, since $a \geq 1$. Since S is a zero genus surface, γ is a geodesic (not just a local geodesic); therefore, $f(\gamma)$ is an (a,b)-quasigeodesic.

Seeking for a contradiction, let us assume that there exists a point $p \in \gamma_0$ such that the ball $B := \overline{B_{S'}(f(p), h+t)}$ is not contained in T. That is, there exists a point $q \in B \setminus T$ for which

$$d_{S'}(q, f(\gamma)) > h. \tag{4.2}$$

Since f is c-full, there must exist $p_1 \in S \setminus \gamma$ such that $d_{S'}(f(p_1), q) \leq c$. Let us assume that $d_S(p_1,\sigma) > w - k_1/2$. Since $p \in \gamma_0$, it means that $d_S(p,p_1) > k_1/2$. Using the fact that f is an (a,b)-quasi-isometry,

$$d_{S'}(f(p), f(p_1)) \geq \frac{1}{a}\, d_S(p,p_1) - b > \frac{k_1}{2a} - b. \tag{4.3}$$



By the triangle inequality, and using that $q \in B$,

$$d_{S'}(f(p_1), f(p)) \le d_{S'}(f(p_1), q) + d_{S'}(q, f(p)) \le 3a + b + 2c + t. \tag{4.4}$$

Combining now (4.3) and (4.4), one deduces $k_1 < 2a(3a + 2b + 2c + t)$, which contradicts the definition of k_1. Therefore, $p_1 \in C_{\sigma,w-k_1/2}$. Then, there exists a point $p_2 \in \gamma$ close enough to p_1, verifying that $d_S(p_1, p_2)$ is upper bounded by the length of one of the boundary curves of $C_{\sigma,w-k_1/2}$. Using Fermi coordinates based on σ, we can easily check that $L_S(\partial C_{\sigma,w-k_1/2})/2 \le L_S(\partial C_\sigma)/2 = L_S(\sigma)\cosh w$. Collar Lemma gives $d_S(p_1, p_2) \le L_S(\partial C_\sigma)/2 = L_S(\sigma)\cosh w = L_S(\sigma)\coth(L_S(\sigma)/2) \le 3$ since $k_1 \ge 6$ and $L_S(\sigma) < 2\,\mathrm{arccoth}(\cosh k_1)$.

On one hand, since f is an (a, b)-quasi-isometry (recall that $p_2 \in \gamma$),

$$d_{S'}(f(p_1), f(\gamma)) \le d_{S'}(f(p_1), f(p_2)) \le a d_S(p_1, p_2) + b < 3a + b. \tag{4.5}$$

On the other hand, taking into account (4.2),

$$d_{S'}(f(p_1), f(\gamma)) \ge d_{S'}(f(\gamma), q) - d_{S'}(q, f(p_1)) \ge 3a + b + c - c = 3a + b. \tag{4.6}$$

Obviously (4.6) contradicts (4.5), so such a point $q \in B \setminus T$ cannot exist and the tube T mentioned in the statement of the theorem does exist.

2. The same arguments work for this situation, defining $k := a(2a + 2b + 2c + t)$. ☐

Lemma 4.4. *Let η be an (a, b)-quasi-geodesic in S and $h > 0$. Then there exists a positive constant r_0, which just depends on a, b, h, with the following property: if for some $z_0 \in \eta$ the ball $B := B_S(z_0, r)$ is simply connected and it is contained in the h-neighborhood of η, then $r \le r_0$.*

Proof. Let us define $r_0 := 2h(J + 1)$, where J is the least integer satisfying

$$J > \frac{1}{h}\left(a^2\left(\frac{a^2}{2}(3h + b) + 2b + 5h\right) + b + h\right).$$

Note that, since B is simply connected, the ball $B_1 := B_S(z_0, r/2)$ is simply connected and, besides, geodesically convex. Let I be a closed interval on the real line and $\eta : I \longrightarrow S$ a parametrization of the (a, b)-quasigeodesic. Seeking for a contradiction, let us assume that $r > 2h(J + 1)$.

Define j_1 as the least integer satisfying

$$j_1 > \frac{1}{h}\left(\frac{a^2}{2}(3h + b) + b + 2h\right).$$

There exists $\delta > 0$ such that

$$r > 2(h + \delta)(J + 1),$$

$$j_1 < \frac{1}{h+\delta}\left(\frac{a^2}{2}(3h+3\delta+b)+b+2h+2\delta\right)+2,$$

$$j_1 > \frac{1}{h+\delta}\left(\frac{a^2}{2}(3h+3\delta+b)+b+2h+2\delta\right) \tag{4.7}$$

and

$$J > \frac{1}{h+\delta}\left(a^2\left(\frac{a^2}{2}(3h+3\delta+b)+2b+5h+5\delta\right)+b+h+\delta\right).$$

Let us consider any geodesic $\gamma_1 \subset B_1$ extended in both directions from the point z_0 and a second geodesic γ_2, perpendicular to γ_1 and extended just in one direction from z_0. Let us fix points $z_1, z_2, z_3, \ldots \in \gamma_1$ in one of the directions starting at z_0 and $z_{-1}, z_{-2}, z_{-3}, \ldots \in \gamma_1$ in the opposite direction from z_0, such that $d_S(z_0, z_j) = |j|(h+\delta)$ for every j with $|j|(h+\delta) < r/2$. Analogously, choose points $w_j \in \gamma_2$ with $d_S(z_0, w_j) = j(h+\delta)$ for every $j > 0$ with $j(h+\delta) < r/2$.

Since B_1 is contained in the h-neighborhood of η, for each of these points $z_j, w_j \in B_1$ there exist points $z_j^*, w_j^* \in \eta$ verifying $d_S(z_j, z_j^*) \le h+\delta$ and $d_S(w_j, w_j^*) \le h+\delta$.

Let $t_j, s_j \in I$ be the real values such that $\eta(t_j) = z_j^*$ and $\eta(s_j) = w_j^*$ (according to this notation, $\eta(t_0) = z_0 = z_0^*$). Then $|t_j - t_k| \le a(d_S(z_j^*, z_k^*) + b) \le a(|j-k|(h+\delta) + 2h + 2\delta + b)$ and, in particular,

$$|t_j - t_{j+1}| \le a(3h+3\delta+b). \tag{4.8}$$

Note that z_J^* and z_{-J}^* are both in the ball B_1:

$$d_S(z_{\pm J}^*, z_0) \le d_S(z_{\pm J}^*, z_{\pm J}) + d_S(z_{\pm J}, z_0) \le h+\delta+J(h+\delta) = (h+\delta)(J+1) < r/2.$$

For the defined value j_1

$$\begin{aligned} |s_{j_1} - t_0| &\le a\big(d_S(w_{j_1}^*, z_0) + b\big) \le a\big(d_S(w_{j_1}^*, w_{j_1}) + d_S(w_{j_1}, z_0) + b\big) \\ &\le a(h+\delta+j_1(h+\delta)+b). \end{aligned} \tag{4.9}$$

A similar argument gives

$$|t_J - t_0| \ge \frac{1}{a}(d_S(z_0, z_J^*) - b) \ge \frac{1}{a}(J(h+\delta) - h - \delta - b). \tag{4.10}$$

Using the third inequality in (4.7) and $j_1(h+\delta) < a^2/2(3h+3\delta+b)+b+4h+4\delta$, we can easily check that

$$\frac{1}{a}(J(h+\delta) - h - \delta - b) > a(h+\delta+b+j_1(h+\delta)).$$

Therefore, comparing (4.9) and (4.10), one obtains $|t_J - t_0| > |s_{j_1} - t_0|$. Analogously, $|t_{-J} - t_0| > |s_{j_1} - t_0|$. Hence, by (4.8), there exists some $j_2 \in \mathbb{Z}$ such that

$$|s_{j_1} - t_{j_2}| \le \frac{a}{2}(3h + 3\delta + b).$$

Taking into account the above inequality,

$$\begin{aligned} d_S(w_{j_1}, z_{j_2}) &\le d_S\big(w^*_{j_1}, z^*_{j_2}\big) + 2h + 2\delta \le a|s_{j_1} - t_{j_2}| + b + 2h + 2\delta \\ &\le \frac{a^2}{2}(3h + 3\delta + b) + b + 2h + 2\delta \quad \text{and} \\ d_S(w_{j_1}, z_{j_2}) &\ge d_S(w_{j_1}, z_0) = j_1(h + \delta). \end{aligned}$$

Thus,

$$j_1(h + \delta) \le \frac{a^2}{2}(3h + 3\delta + b) + b + 2h + 2\delta,$$

which contradicts the second inequality in (4.7). Therefore, $r \le 2h(J + 1)$ as claimed. □

THEOREM 4.5. *Assume that S and S' are genus zero surfaces. Let γ be any geodesic perpendicular to σ contained in C_σ with $L_S(\gamma) = 2w$. There exist positive constants r_0, k_0, k_1, k_2, which just depend on a, b, c, so that if $\gamma_0 := \{p \in \gamma : d_S(p, \sigma) < w - k_1\}$ and $L_S(\sigma) < k_2$, then there exists an $(f, \gamma, \gamma_0, h, h + k_0)$-tube $T \subset S'$ with $h := 3a + b + c$.*

Furthermore, if u_1, u_2 are the endpoints of γ_0, and g^i is any simple geodesic loop with base point $f(u_i)$ and $L_{S'}(g^i) = 2\iota(f(u_i))$ $(i = 1, 2)$, then g^1 and g^2 bound a doubly connected set in S', and for every $z \in \gamma_0$, $\iota(f(z)) \le r_0$ and the injectivity radius in $f(z)$ is attained in the geodesic loop with base point $f(z)$ freely homotopic to a simple closed geodesic σ' in S', where σ' only depends on σ and f. Besides, $f(C_\sigma)$ is contained in the H_0-neighborhood of the collar $C_{\sigma'}$ of σ', where $H_0 := r_0 + ak_1 + b$.

Note that σ' does not depend neither on γ or γ_0.

PROOF. Let us define J as the least integer satisfying

$$J > \frac{1}{h}\left(a^2\left(\frac{a^2}{2}(3h + b) + 2b + 5h\right) + b + h\right),$$

$r_0 := 2h(J + 1)$, $k_0 := a^2(12h(J + 1) + 1/2 + 3b) + 12h(J + 1) + b + h + 3$, $k_1 := 2a(3a + 2b + 2c + k_0)$ and $k_2 := 2 \operatorname{arccoth}(\cosh(k_1 + a(4h(J + 1) + b)/2))$.

Since $L_S(\sigma) < k_2$, the width w of the collar C_σ verifies the inequality $w > k_1 + a(2r_0 + b)/2$ (see Collar Lemma), and consequently,

$$d_{S'}(f(u_2), f(u_1)) \ge \frac{1}{a} d_S(u_2, u_1) - b = \frac{2w - 2k_1}{a} - b > 4h(J + 1) = 2r_0.$$

Therefore, it is possible to choose points $x_0, x_1, \dots, x_m \in \gamma_0$, where $x_0 = u_1$, $x_m = u_2$ and

$$2r_0 < d_{S'}(f(x_j), f(x_{j-1})) \le 4r_0 \tag{4.11}$$

for $j = 1, \dots, m$.

Note that $h + k_0 > r_0$. Let us consider any r_1 with $h + k_0 > r_1 > r_0$. By Lemma 4.3 (1), taking $t = k_0$, $V_{h+k_0}(f(\gamma_0)) \subset V_h(f(\gamma))$, and thus Lemma 4.4 gives that the balls $B_{S'}(f(x_j), r_1)$ are not simply connected for $j = 0, 1, \dots, m$. Therefore, the injectivity radius $\iota(f(x_j))$ at the point $f(x_j)$ is less than r_1 for every $r_1 > r_0$, and then $\iota(f(x_j)) \le r_0$. In particular, $\iota(f(u_1)), \iota(f(u_2)) \le r_0$. Consequently, there exists a simple geodesic loop g_j with base point $f(x_j)$ and $L_{S'}(g_j) = 2\iota(f(x_j)) \le 2r_0$. In light of (4.11), $g_j \cap g_{j+1} = \emptyset$. Since S' is a genus zero surface, g_j and g_{j+1} disconnect S', and then $S' \setminus (g_j \cup g_{j+1})$ has three connected components. Consider the geodesics $\gamma_j' := [f(x_j), f(x_{j+1})] \subset S'$ for $j = 0, \dots, m-1$.

Claim. $\gamma_j' \cap g_j = \{f(x_j)\}$ and $\gamma_j' \cap g_{j+1} = \{f(x_{j+1})\}$ for $j = 0, \dots, m-1$.

Assume the claim holds. Assume that g_j is not freely homotopic to g_{j+1} for some j. It will be shown that, in that case, $B_{S'}(f(x_j), k_0) \not\subseteq V_h(f(\gamma))$, contradicting Lemma 4.3 (1). Set $\eta_j := g_j \cup \gamma_j' \cup g_{j+1} \cup (-\gamma_j')$. Note that $L_{S'}(\eta_j) \le 12r_0$. By means of a slight modification of η_j, one can construct a simple closed curve η_j' freely homotopic to η_j, with $\eta_j' \cap \{g_j \cup g_{j+1}\} = \emptyset$, $L_{S'}(\eta_j') \le 12r_0 + 1$ and $\mathcal{H}_{S'}(\eta_j, \eta_j') \le 1$ (where $\mathcal{H}_{S'}$ denotes Hausdorff distance) as follows. Without loss of generality, S' is a domain contained in $\mathbb{C}$, so take opposite orientation for g_j and g_{j+1} and let η_j be an oriented curve. If either g_j surrounds g_{j+1} or g_{j+1} surrounds g_j, choose η_j' contained in the annulus in $\mathbb{C}$ bounded by g_j and g_{j+1}. Otherwise, choose η_j' contained in the "exterior" connected component of $S' \setminus \eta_j$.

Since the curves g_j, g_{j+1} and η_j' are not trivial, g_j, g_{j+1} and η_j' disconnect S' and $S' \setminus (g_j \cup g_{j+1} \cup \eta_j')$ has four connected components, one of them bounded (with finite diameter) denoted by V; and other three unbounded (with infinite diameter). Note that $\partial V = g_j \cup g_{j+1} \cup \eta_j'$. Since there are three unbounded connected components of $S' \setminus (g_j \cup g_{j+1} \cup \eta_j')$, there must exist an unbounded connected component U with $f(u), f(v) \notin U$. Note that

$$\operatorname{diam}_{S'} \partial U \le \frac{1}{2} \max \left\{ L_{S'}(g_j), L_{S'}(g_{j+1}), L_{S'}(\eta_j') \right\} \le 6r_0 + \frac{1}{2}.$$

Since $\overline{V}$ is contained in the 1-neighborhood of η_j and $L_{S'}(\eta_j) \le 12r_0$, then

$$\operatorname{diam}_{S'} \overline{V} \le 1 + \operatorname{diam}_{S'} \eta_j + 1 \le 6r_0 + 2.$$

Assume first that $f(\gamma)$ intersects $\overline{U}$. In this case, $\operatorname{diam}_{S'}(f(\gamma) \cap \overline{U})$ is bounded above. Indeed, consider γ_0 to be oriented from u_1 to u_2 and consider points $p' := \inf\{\tau \in [u_1, u_2] : f(\tau) \in \overline{U}\}$ and $q' := \sup\{\tau \in [u_1, u_2] : f(\tau) \in \overline{U}\}$.

Given $p, q \in \gamma \cap f^{-1}(\overline{U})$, since $L_{S'}(\eta_j') \le 12r_0 + 1$, $f(\gamma)$ is a (possibly) discontinuous curve with gaps of amplitude at most b, one deduces

$$\begin{aligned} d_{S'}(f(p), f(q)) &\le a d_S(p, q) + b \le a d_S(p', q') + b \le a^2 d_{S'}(f(p'), f(q')) + a^2 b + b \\ &\le a^2(\operatorname{diam}_{S'} \partial U + 2b) + a^2 b + b \end{aligned}$$

$$\leq a^2(6r_0 + 1/2 + 3b) + b = k_0 - 6r_0 - h - 3. \tag{4.12}$$

Thus $\operatorname{diam}_{S'}(f(\gamma)\cap \overline{U}) \leq k_0 - 6r_0 - 2 - h - 1$. Consequently, if $z \in f(\gamma)\cap \overline{U}$, then $d_{S'}(f(x_j), z) \leq \operatorname{diam}_{S'} \overline{V} + \operatorname{diam}_{S'}(f(\gamma)\cap \overline{U}) \leq k_0 - h - 1$.

If $f(\gamma)$ does not intersect $\overline{U}$ and $z \in \overline{U}$, then $d_{S'}(f(x_j), z) \leq \operatorname{diam}_{S'} \overline{V} \leq k_0 - h - 1$.

Therefore, in both cases, since the region U is unbounded, the ball $\overline{B_{S'}(f(x_j), k_0)}$ cannot be contained in $T = V_h(f(\gamma))$, which contradicts Lemma 4.3 (1).

Since it was shown that the closed ball $\overline{B_{S'}(f(x_j), h + k_0)}$ must be contained in T, then $\overline{B_{S'}(f(x_j), k_0)}$ must also be contained in T.

Therefore, g_j is freely homotopic to g_{j+1} for every j. Consequently, the simple geodesic loops $g_0 = g^1$ and $g_m = g^2$ with base points $f(u_1)$ and $f(u_2)$, respectively, are freely homotopic and bound a doubly connected set in S' as claimed.

By taking different sequences of points $\{x_j\}$ one can check that if $z \in \gamma_0$, and g_z is any simple geodesic loop with base point $f(z)$ and $L_{S'}(g_z) = 2\iota(f(z))$, then g_z is freely homotopic to g^1 and $\iota(f(z)) \leq r_0$.

By Theorem 4.6 below, the map f provides a bijective correspondence from the cusps of S to the cusps of S'; hence, there exists a simple closed curve σ' freely homotopic to g^1 of length l. By Collar Lemma and Proposition 2.1, the injectivity radius ι_0 at the points in $\partial C_{\sigma'}$ satisfies $\sinh \iota_0 = \sinh(l/2)\cosh w = \cosh(l/2) > 1$.

For $z \in \gamma_0$, $f(z)$ either belongs to the collar $C_{\sigma'}$, and thus to $V_{r_0}(C'_\sigma)$, or, otherwise, let us define $t := d_{S'}(f(z), \sigma') - w > 0$. Then, $\sinh r_0 \geq \sinh \iota(z) = \sinh(l/2)\cosh(w + t)$ and

$$\begin{aligned}\frac{1}{2}\, e^t < \frac{1}{2}\, e^t \sinh(l/2)\cosh w < \frac{1}{2}\, e^{w+t}\sinh(l/2) &< \cosh(w+t)\sinh(l/2)\\ &\leq \sinh r_0 < \frac{1}{2}\, e^{r_0}.\end{aligned}$$

Hence, $t < r_0$ and $f(\gamma_0) \subset V_{r_0}(C_{\sigma'})$. Given another geodesic $\tilde{\gamma}_0$ perpendicular to σ, it has been proved that $f(\tilde{\gamma}_0)$ is contained in the r_0-neighborhood of $C_{\tilde{\sigma}'}$ for some simple closed curve $\tilde{\sigma}'$ in S'; in order to check that $\tilde{\sigma}' = \sigma'$ it suffices to repeat the previous argument replacing γ_0 by any geodesic γ_1 meeting σ with an angle $\pi/2 - \varepsilon$ for small $\varepsilon > 0$. Therefore, $f(\{p \in S : d_S(p,\sigma) \leq w - k_1\})$ is contained in the r_0-neighborhood of $C_{\sigma'}$, and $f(C_\sigma)$ is contained in the $(ak_1 + b + r_0)$-neighborhood of $C_{\sigma'}$.

In order to prove the first part of the claim, assume that there exists a point ζ in $\gamma'_j \cap g_j \setminus \{f(x_j)\}$ and argue by contradiction.

Denote by g^*_j a subcurve of g_j joining $f(x_j)$ and ζ and denote by γ^*_j the subcurve of γ'_j joining $f(x_j)$ and ζ. Since γ'_j is a geodesic, $L_{S'}(\gamma^*_j) \leq L_{S'}(g^*_j)$. Choose g^*_j so that the loop $\Gamma_0 := g^*_j \cup \gamma^*_j$ with base point $f(x_j)$ is non trivial; since Γ_0 has a corner in ζ, there exists a curve Γ freely homotopic to Γ_0 (thus non trivial) with $L_{S'}(\Gamma) < L_{S'}(g^*_j) + L_{S'}(\gamma^*_j) \leq L_{S'}(g_j)$. This inequality contradicts

$$L_{S'}(g_j) = 2\iota(f(x_j)) = \inf\{L_{S'}(c) : c \text{ is a loop with base point } f(x_j)\}.$$

The proof of the second part of the claim is similar.

The claim, and hence the Theorem, hold. □

Theorem 4.6. *Assume that S' is a genus zero surface and $f : S \longrightarrow S'$ a c-full (a, b)-quasi-isometry. Let $\mathscr{C}$ be the 2-collar of a cusp in S with boundary curve σ and γ an infinite geodesic contained in $\mathscr{C}$ perpendicular to σ. There exist positive constants H, k, t, which just depends on a, b, c, so that if $\gamma_0 := \{p \in \gamma : d_S(p, \sigma) > k\}$, then there exists an $(f, \gamma, \gamma_0, h, h + t)$-tube $T \subset S'$ with $h := 2a + b + c$. Furthermore, $f(\mathscr{C})$ is contained in the H-neighborhood of the 2-collar of a cusp in S'.*

Proof. Define J as the least integer satisfying

$$J > \frac{1}{h}\left(a^2\left(\frac{a^2}{2}(3h + b) + 2b + 5h\right) + b + h\right).$$

Let us consider positive constants $t := a^2(12h(J+1)+1/2+3b)+12h(J+1)+b+h+3$, $k := a(2a + 2b + 2c + t)$ and $H := ak + b + \log\sinh(2h(J + 1))$.

The statement about the existence of the tube is given by Lemma 4.3 (2), since this lemma holds for any positive value of t, and k is defined as in the proof of Lemma 4.3.

Let us choose now two points $u_1, u_2 \in \gamma_0$ such that $d_S(u_1, u_2) > a(4h(J + 1) + b)$, which is always possible since γ_0 is infinite. Notice that

$$d_{S'}(f(u_2), f(u_1)) \geq \frac{1}{a}\, d_S(u_2, u_1) - b > 4h(J + 1).$$

Let us define the constant $C := a^2(12h(J + 1) + 1/2 + 2b) + 12h(J + 1) + ab + b + h + 3$. From this point on, the conclusion of this theorem can be obtained repeating the reasoning offered in the proof of Theorem 4.5 with C playing the role of k_0. However, since now the geodesic γ_0 is infinite, the distance between u_1 and u_2 can be arbitrarily large. Then, all the geodesic loops $\{g_j\}$ with base point $f(x_j)$, for any x_j located on γ_0 are homotopic and, besides, $L_{S'}(g_j) \leq 4h(J + 1)$. It means that $f(\mathscr{C})$ is actually contained in a neighborhood of a cusp in S', since $f(\mathscr{C})$ is not a bounded set.

Next, it will be shown that $f(\mathscr{C})$ is inside the H-neighborhood of the 2-collar of a cusp in S'. Let us choose any of the geodesic loops g_j mentioned above, and let us assume that it lies out of the 2-collar of the corresponding cusp in S'. As usual, consider a fundamental domain for S' in the upper halfplane $\mathbb{H}$ contained in $\{z \in \mathbb{H} : 0 \leq \Re z \leq 1\}$ and such that $\{z \in \mathbb{H} : 0 \leq \Re z \leq 1, \Im z \geq 1/2\}$ corresponds to the 2-collar of this cusp in S'. Let us represent g_j in the upper half-plane by means of a geodesic with endpoints $i\alpha$ and $i\alpha+1$. If $\alpha \geq 1/2$, then g_j is in the 2-collar in S'. Hence, $\alpha < 1/2$. If d is the actual length of g_j, then $\sinh(d/2) = 1/(2\alpha)$. Taking into account that $d = L_{S'}(g_j) \leq 4h(J+1)$, then $1/(2\alpha) = \sinh(d/2) \leq \sinh(2h(J + 1))$, and

$$d_{\mathbb{H}}(i\alpha, i/2) = \log\frac{1}{2\alpha} \leq \log\sinh(2h(J + 1)) =: H_1,$$

which means that x_j is in the H_1-neighborhood of the boundary curve of the 2-collar in S'. Hence, in any case, x_j is in the H_1-neighborhood of the 2-collar in S'. Therefore,

$f(\{p \in \mathscr{C} : d_S(p,\sigma) > k\})$ is contained in the H_1-neighborhood of the 2-collar in S', and $f(\mathscr{C})$ is contained in the $(H_1 + ak + b)$-neighborhood of the 2-collar of a cusp in S'. $\square$

LEMMA 4.7. *Fix two positive constants d_1 and ι_0. Let S be a non-exceptional Riemann surface, σ be a simple closed geodesic with $S\backslash\sigma$ non-connected, and x, y points in S such that $d_S(x,y) \ge d_1$ and the geodesic loops g_x, g_y with base points x, y, respectively, freely homotopic to σ, verify $L_S(g_x), L_S(g_y) \le 2\iota_0$. Let σ_x (respectively, σ_y) be the set of points in the connected component of $S \setminus \sigma$ containing x (respectively, y) which are at distance $d_S(x,\sigma)$ (respectively, $d_S(y,\sigma)$) from σ; denote by C the domain in S bounded by σ_x and σ_y, and by C_0 the set of points in C at distance greater or equal than d_2 from $\partial C = \sigma_x \cup \sigma_y$, with*

$$0 < d_2 < \operatorname{arccosh} \frac{2\cosh^2(d_1/2)}{\sqrt{4\cosh^2(d_1/2) + \sinh^2 \iota_0}}. \tag{4.13}$$

Then C_0 is non empty and $\sinh \iota(z) < 2\, e^{-d_2} \sinh \iota_0$ for every $z \in C_0$.

REMARK 4.8. An elementary computation gives that if $d_1 \ge \max\{2\iota_0, 2d_2 + \log 20\}$, then (4.13) holds.

PROOF. Define $l := L_S(\sigma)$ and x_0 (respectively, y_0) the point in σ with $d_S(x,\sigma) = d_S(x,x_0)$ (respectively, $d_S(y,\sigma) = d_S(y,y_0)$). It is clear that the maximum of the injectivity radius is attained when S is an annulus with simple closed geodesic σ, $d_S(x,y) = d_1$, $L_S(g_x)$, $L_S(g_y) = 2\iota_0$, and x, y are antipodal points with respect to σ, i.e., $d_S(x,\sigma) = d_S(y,\sigma)$ and $d_S(x_0,y_0) = l/2$. In this case, defining $u := d_S(x,\sigma) = d_S(y,\sigma)$, Proposition 2.1 gives $\cosh(d_1/2) = \cosh u \cosh(l/4)$ and $\sinh \iota_0 = \sinh(l/2)\cosh u$. Hence,

$$\sinh(l/4) = \frac{\sinh(l/2)}{2\cosh(l/4)} = \frac{\sinh \iota_0}{2\cosh(d_1/2)},$$

$$\cosh u = \frac{\cosh(d_1/2)}{\cosh(l/4)} = \frac{2\cosh^2(d_1/2)}{\sqrt{4\cosh^2(d_1/2) + \sinh^2 \iota_0}}.$$

Therefore, take $0 < d_2 < d_S(x,\sigma) = d_S(y,\sigma)$ and C_0 is not the empty set. If $u \in C_\sigma$, denote by ψ_u the geodesic loop with base point u freely homotopic to σ. For any $z \in \partial C_0$, Proposition 2.1 gives

$$\begin{aligned} \sinh(L_S(\psi_z)/2) = \sinh(l/2)\cosh(u - d_2) &\le e^{u-d_2} \sinh(l/2) \\ &< 2\, e^{-d_2} \sinh(l/2)\cosh u = 2\, e^{-d_2} \sinh \iota_0. \end{aligned}$$

Hence, for any $z \in C_0$, $\sinh(L_S(\psi_z)/2) < 2\, e^{-d_2} \sinh \iota_0$ and, consequently, $\sinh \iota(z) < 2\, e^{-d_2} \sinh \iota_0$. $\square$

LEMMA 4.9. *Let S be a non-exceptional Riemann surface and $z \in S$. If $\iota(z) <$ $\operatorname{arcsinh} 1$, then the shortest geodesic loop η with base point z is contained either in the 2-collar of a cusp or in the collar C_σ of a simple closed geodesic σ.*

Proof. Given any point p on the boundary of the 2-collar of a cusp, or on the boundary of the collar of a simple closed geodesic then $\iota(p) \geq \operatorname{arcsinh} 1$ by Collar Lemma.

Therefore z must lie inside the cusp or collar. Lifting its shortest geodesic loop to the universal covering shows this will also be the case for its geodesic loop. □

Lemma 4.10. *Let S be a non-exceptional Riemann surface, σ be a simple closed geodesic in S and z a point in the collar C_σ. Then $d_S(z, \partial C_\sigma) \geq \log(1/\sinh \iota(z))$.*

Proof. If $\iota(z) \geq \operatorname{arcsinh} 1$, then $d_S(z, \partial C_\sigma) \geq 0 \geq \log(1/\sinh \iota(z))$. Assume now that $\iota(z) < \operatorname{arcsinh} 1$. By Lemma 4.9 the shortest geodesic loop η with base point z is contained in C_σ. Note that if $d := d_S(z, \sigma)$, then $d_S(z, \partial C_\sigma) = w - d$. Let $l := L_S(\sigma)$; by Collar Lemma and Proposition 2.1:

$$\sinh \iota(z) = \sinh(l/2) \cosh d = \frac{\cosh d}{\sinh w} \geq e^{d-w},$$

which implies the result. □

Lemma 4.11. *Let S be a non-exceptional genus zero Riemann surface, I and h two positive constants, σ a simple closed geodesic with $L_S(\sigma) \leq 2I$, and C_σ^h the h-neighborhood of C_σ. Denote by S_1 a connected component of $S \setminus \sigma$, and by α_1 the set of closed curves in $\partial C_\sigma^h \cap S_1$. If $p, q \in \alpha_1$, then $d_S(p, q) \leq e^h I \coth I$.*

Proof. Without loss of generality, assume that S is an annulus and σ is the simple closed geodesic in S. Define $l := L_S(\sigma)$ and $L := L_S(\alpha_1)$. Since $l/2 \leq I$ and $g(x) = x \coth x$ is an increasing function for $x > 0$,

$$\begin{aligned} d_S(p, q) \leq L/2 = (l/2) \cosh w \, \frac{\cosh(w+h)}{\cosh w} &< (l/2) \coth(l/2) \, \frac{\cosh(w+h)}{\cosh w} \\ &\leq e^h \, I \coth I. \end{aligned}$$

□

Finally, the following two lemmas are easy to check.

Lemma 4.12. *Let $g : [\alpha, \beta] \to \mathbb{R}$ be an (a, b)-quasi-isometric embedding with $g(\beta) > g(\alpha)$. If $x, y \in [\alpha, \beta]$ and $y > x + (a+1)b$, then $g(y) > g(x)$.*

Lemma 4.13. *Let $g : [0, \infty) \to [0, \infty)$ be an (a, b)-quasi-isometric embedding. If $x, y \geq 0$ and $y > x + (a+1)b$, then $g(y) > g(x)$.*

5. Stability of the injectivity radius under quasi-isometries.

Recall the notation C_σ and $\mathscr{C}_\alpha$ for collars of simple closed geodesic and cusps respectively. Also denote by $\mathscr{C}_\alpha$ the H-neighborhood of the 2-collar of a cusp with boundary α (now α can be a union of closed curves).

5.1. Proof of Theorem 1.3.

Without loss of generality assume that $0 < \varepsilon', \varepsilon \leq \operatorname{arcsinh} 1$. Fix $z \in S$ with $\iota(z) < \operatorname{arcsinh} 1$.

The proof takes advantage of the relation between $\iota(z)$ and the distance from z to the boundary of the collar of a cusp when z is in the interior of the collar of a cusp, or the distance from z to the boundary of the collar of a simple closed geodesic when z is in the collar.

Assume first that the shortest geodesic loop based on z is freely homotopic to a cusp in S. Let z belong to the interior of the 2-collar of this cusp, $\mathscr{C}_\alpha$, where α is its boundary curve. In this setting $\sinh \iota(z) = e^{-d_S(z,\alpha)} < 1$.

For every $u \in \mathscr{C}_\alpha$ define $W_\alpha(u) := d_S(u, \alpha)$. Then for any two points $u, v \in \mathscr{C}_\alpha$,

$$|W_\alpha(v) - W_\alpha(u)| \le d_S(v, u) \le |W_\alpha(v) - W_\alpha(u)| + 1, \tag{5.14}$$

By Theorem 4.6, $f(\mathscr{C}_\alpha)$ is contained in $\mathscr{C}_{\alpha'}$, the H-neighborhood of the 2-collar of a cusp in S' (now $\alpha' := \partial \mathscr{C}_{\alpha'}$ can be a union of closed curves). Let us define $W_{\alpha'}(p) := d_{S'}(p, \alpha')$ for every $p \in \mathscr{C}_{\alpha'}$. Then,

$$\begin{aligned} |W_{\alpha'}(f(v)) - W_{\alpha'}(f(u))| &\le d_{S'}(f(v), f(u)) \\ &\le |W_{\alpha'}(f(v)) - W_{\alpha'}(f(u))| + 1 + 2H, \end{aligned} \tag{5.15}$$

for any two points $u, v \in \mathscr{C}_\alpha$.

By (5.14) and (5.15),

$$\begin{aligned} |W_{\alpha'}(f(v)) - W_{\alpha'}(f(u))| &\le d_{S'}(f(v), f(u)) \le a d_S(v, u) + b \\ &\le a(|W_\alpha(v) - W_\alpha(u)| + 1) + b \\ &= a|W_\alpha(v) - W_\alpha(u)| + a + b, \\ |W_{\alpha'}(f(v)) - W_{\alpha'}(f(u))| &\ge d_{S'}(f(v), f(u)) - 1 - 2H \\ &\ge \frac{1}{a}\, d_S(v, u) - b - 1 - 2H \\ &\ge \frac{1}{a}\, |W_\alpha(v) - W_\alpha(u)| - 1 - b - 2H, \end{aligned} \tag{5.16}$$

for any two points $u, v \in \mathscr{C}_\alpha$. Therefore, (5.16) shows that there is an $(a, a+b+2H)$ quasi-isometric embedding defined from $[0, \infty)$ to $[0, \infty)$ that relates $W_\alpha(u)$ with $W_{\alpha'}(f(u))$, for every $u \in \mathscr{C}_\alpha$.

Let z_0 be the point in α so that $d_S(z, \alpha) = d_S(z, z_0)$. Then

$$W_\alpha(z) - W_\alpha(z_0) = W_\alpha(z) = \log \frac{1}{\sinh \iota(z)} > \log \frac{1}{\sinh \varepsilon}. \tag{5.17}$$

Choosing ε so that $\sinh \varepsilon < e^{-(a+1)(a+b+2H)}$,

$$W_\alpha(z) - W_\alpha(z_0) > (a + 1)(a + b + 2H).$$

By Lemma 4.13, $W_{\alpha'}(f(z)) > W_{\alpha'}(f(z_0))$, and thus (5.16) and (5.17) give

$$\begin{aligned} W_{\alpha'}(f(z)) &\geq W_{\alpha'}(f(z)) - W_{\alpha'}f((z_0)) = \big|W_{\alpha'}(f(z)) - W_{\alpha'}f((z_0))\big| \\ &\geq \frac{1}{a}\big|W_{\alpha}(z) - W_{\alpha}(z_0)\big| - 1 - b - 2H > \frac{1}{a}\log\frac{1}{\sinh\varepsilon} - 1 - b - 2H \\ &> \log\frac{1}{\sinh\varepsilon'} + H \geq H, \end{aligned}$$

since $0 < \varepsilon' \leq \operatorname{arcsinh} 1$, and if ε is taken to be

$$\sinh\varepsilon < \min\big\{(\sinh\varepsilon')^a e^{-a(3H+1+b)}, e^{-(a+1)(a+b+2H)}\big\}.$$

Since $W_{\alpha'}(f(z)) > H$, $f(z)$ is in the 2-collar of the cusp, and thus $W_{\alpha'}(f(z)) = -\log\sinh\iota(f(z)) + H$. Hence $\iota(f(z)) < \varepsilon'$.

Assume now that the shortest loop based on z is freely homotopic to a simple closed geodesic σ; then z belongs to the interior of the collar C_σ of width w. Let us consider the geodesics γ, γ_0 and the constants r_0, k_0, k_1, k_2, H_0 as in Theorem 4.5. If we require $\varepsilon \leq k_2/2$, then $l := L_S(\sigma) \leq 2\iota(z) < k_2$. By Collar Lemma and Proposition 2.1 we have that the length $2\iota_0$ of the geodesic loop freely homotopic to σ and based in any point in ∂C_σ satisfies $\sinh\iota_0 = \sinh(l/2)\cosh w = \cosh(l/2) < \cosh(k_2/2) =: k_3$; thus $\iota(u) < \operatorname{arcsinh} k_3$ for every $u \in C_\sigma$.

Denote by α_1, α_2 the simple closed curves in ∂C_σ; then $L_S(\alpha_j) = l\cosh w < 2\sinh(l/2)\cosh w = 2\cosh(l/2) < 2\cosh(k_2/2) = 2k_3$ for $j = 1, 2$. Define $W_{\alpha_j}(u) := d_S(u, \alpha_j)$ for every $u \in C_\sigma$ and $j = 1, 2$. Since S is a genus zero surface,

$$|W_{\alpha_j}(v) - W_{\alpha_j}(u)| \leq d_S(v, u) \leq |W_{\alpha_j}(v) - W_{\alpha_j}(u)| + k_3, \tag{5.18}$$

for any two points $u, v \in C_\sigma$.

By Theorem 4.5, $f(C_\sigma)$ is contained in the H_0-neighborhood of the collar $C_{\sigma'}$ of a simple closed geodesic σ' in S'.

Denote by Ψ_u the geodesic loop with base point $f(u)$ freely homotopic to σ'. Let $u \in C_\sigma$, then

$$L_{S'}(\Psi_u) \leq 2(r_0 + ak_1 + b) := 2r_0'$$

since if $d_S(u, \partial C_\sigma) \geq k_1$, Theorem 4.5 gives $L_{S'}(\Psi_u) = 2\iota(f(u)) \leq 2r_0$.

S' is also a genus zero surface, therefore $S' \setminus \sigma'$ has two connected components S_1', S_2'. Then, $f(C_\sigma)$ intersects either both of them or only one of them. In the former case, define $r_j := \sup\{d_{S'}(f(u), \sigma') \mid u \in C_\sigma, f(u) \in S_j'\}$ and $\alpha_j' := \{v \in S_j' \mid d_{S'}(v, \sigma') = r_j\}$ for $j = 1, 2$.

In the latter case, define $r_1 := \inf\{d_{S'}(f(u), \sigma') \mid u \in C_\sigma\}$, $r_2 := \sup\{d_{S'}(f(u), \sigma') \mid u \in C_\sigma\}$, and $\alpha_j' := \{v \in S_i' \mid d_{S'}(v, \sigma') = r_j\}$ for $j = 1, 2$ where i so that $S_i' \cap f(C_\sigma) \neq \emptyset$.

Let $C_{\sigma'}^0$ the domain in S' bounded by α_1' and α_2'. For any $p \in C_{\sigma'}^0$, define $W_{\alpha_j'}(p) := d_{S'}(p, \alpha_j')$, $j = 1, 2$. By Lemma 4.11, for any $u, v \in C_\sigma$,

$$\begin{aligned}\left|W_{\alpha'_j}(f(v))-W_{\alpha'_j}(f(u))\right| &\le d_{S'}(f(v),f(u))\\ &\le \left|W_{\alpha'_j}(f(v))-W_{\alpha'_j}(f(u))\right|+e^{H_0}r'_0\coth r'_0,\end{aligned}\tag{5.19}$$

By virtue of (5.18) and (5.19), for any $u,v\in C_\sigma$,

$$\begin{aligned}\left|W_{\alpha'_j}(f(v))-W_{\alpha'_j}(f(u))\right| &\le d_{S'}(f(v),f(u))\le a d_S(v,u)+b\\ &\le a(|W_{\alpha_j}(v)-W_{\alpha_j}(u)|+k_3)+b\\ &= a|W_{\alpha_j}(v)-W_{\alpha_j}(u)|+ak_3+b,\\ \left|W_{\alpha'_j}(f(v))-W_{\alpha'_j}(f(u))\right| &\ge d_{S'}(f(v),f(u))-e^{H_0}r'_0\coth r'_0\\ &\ge \frac{1}{a}\,d_S(v,u)-b-e^{H_0}r'_0\coth r'_0\\ &\ge \frac{1}{a}\,|W_{\alpha_j}(v)-W_{\alpha_j}(u)|-e^{H_0}r'_0\coth r'_0-b.\end{aligned}\tag{5.20}$$

If γ is a geodesic orthogonal to σ, and setting $k_4:=e^{H_0}r'_0\coth r'_0$, then (5.20) shows that there are two (a,ak_3+b+k_4)-quasi-isometric embeddings defined from $[0,2w]$ to $\mathbb{R}$ that relate $W_{\alpha_j}(u)$ with $W_{\alpha'_j}(f(u))$, for every $u\in\gamma$ and $j=1,2$.

Let $z\in\gamma$ and let z_j the point $z_j:=\gamma\cap\alpha_j$ for $j=1,2$. By Lemma 4.10,

$$\begin{aligned}W_{\alpha_j}(z)-W_{\alpha_j}(z_j)=W_{\alpha_j}(z)\ge d_S(z,\partial C_\sigma)\ge \log\frac{1}{\sinh\iota(z)} &> \log\frac{1}{\sinh\varepsilon}\\ &\ge (a+1)(ak_3+b+k_4),\end{aligned}$$

if ε is taken to be $\varepsilon\le \operatorname{arcsinh} e^{-(a+1)(ak_3+b+k_4)}$.

Without loss of generality, label α'_1 and α'_2 so that $f(z_i)$ is closest to α'_i for $i=1,2$. Therefore, by Lemma 4.12 together with (5.20),

$$\begin{aligned}W_{\alpha'_j}(f(z))\ge \left|W_{\alpha'_j}(f(z))-W_{\alpha'_j}f((z_j))\right| &\ge \frac{1}{a}\left|W_{\alpha_j}(z)-W_{\alpha_j}(z_j)\right|-k_4-b\\ &= \frac{1}{a}\,W_{\alpha_j}(z)-k_4-b.\end{aligned}$$

By Lemma 4.10

$$d_S(z,\partial C_\sigma)\ge \log\frac{1}{\sinh\iota(z)}>\log\frac{1}{\sinh\varepsilon}.$$

If ε is taken to be so that

$$\log\frac{1}{\sinh\varepsilon}\ge a\left(1+b+k_4+\log\frac{2\sinh r'_0}{\sinh\varepsilon'}\right)=:k_1^*,$$

Lemma 4.10 gives, together with the quasi-isometric embedding,

$$W_{\alpha'_j}(f(z)) \geq \frac{1}{a} W_{\alpha_j}(z) - k_4 - b \geq \frac{1}{a} d_S(z, \partial C_\sigma) - k_4 - b > \log \frac{2\sinh r'_0}{\sinh \varepsilon'}.$$

Set $d_2 := 1 + \log(2\sinh r'_0/\sinh \varepsilon')$ and $d_1 := d_{S'}(f(z_1), f(z_2))$. In order to apply Remark 4.8, d_1 should satisfy $d_1 > \max\{2r'_0,\ 2d_2 + \log 20\}$, since $\iota(f(z_j)) \leq r'_0$. Using f is an (a,b)-quasi-isometry,

$$d_1 \geq \frac{1}{a} d_S(z_1, z_2) - b = \frac{2w}{a} - b.$$

By Collar Lemma together with $\iota(z) < \varepsilon$, the width w satisfies $\cosh w > \coth \varepsilon$. Therefore, it suffices to choose ε to be so that $\coth \varepsilon \geq \cosh((a/2)(b + \max\{2r'_0,\ 2d_2 + \log 20\}))$.

Hence, by Lemma 4.7, $\sinh \iota(f(z)) < 2\, e^{-d_2} \sinh r'_0 < \sinh \varepsilon'$ if $\iota(z) < \varepsilon$ where ε must satisfy all the above restrictions, namely:

$$0 < \varepsilon \leq \min\Bigg\{ \frac{k_2}{2},\ \operatorname{arcsinh} e^{-(a+1)(ak_3+b+k_4)},\ \operatorname{arcsinh} e^{-k_1^*},$$
$$\operatorname{arccoth} \cosh \frac{a(b + \max\{2r'_0,\ 2d_2 + \log 20\})}{2} \Bigg\}. \qquad \square$$

Note that the constant ε in Theorem 1.3 does not depend on z, f, S, S'.

Theorem 5.1. *Let S and S' be non-exceptional genus zero Riemann surfaces and $f : S \longrightarrow S'$ a c-full (a,b)-quasi-isometry. For each $\varepsilon > 0$ there exists $\varepsilon' > 0$ which just depends on ε, a, b, c, such that if $\iota(z) \geq \varepsilon$ then $\iota(f(z)) \geq \varepsilon'$.*

Proof. For each fixed $z \in S$ let us define a function $F_z : S' \longrightarrow S$ as follows: $F_z(f(z)) := z$; for each $y \in f(S) \setminus \{f(z)\}$ fix any $x \in f^{-1}(y)$ and define $F_z(y) := x$; finally, for each $y \in S' \setminus f(S)$ choose any $x \in S$ with $d_{S'}(f(x), y) \leq c$ and define $F_z(y) := x$. It is easy to check that F_z is an ab-full $(a, a(b+2c))$-quasi-isometry.

Consequently, by Theorem 1.3, for each $\varepsilon > 0$ there exists $\varepsilon' > 0$, which just depends on ε, a, b, c, such that if $\iota(p) < \varepsilon'$ then $\iota(F_z(p)) < \varepsilon$. In particular, if $\iota(f(z)) < \varepsilon'$ then $\iota(z) = \iota(F_z(f(z))) < \varepsilon$. Since ε' does not depend on z, f or F_z, then $\iota(z) < \varepsilon$ for every $z \in S$ with $\iota(f(z)) < \varepsilon'$. $\square$

6. Proof of Theorem 1.1.

This section is devoted to the proof of Theorem 1.1, which follows Kanai's approach. In Kanai's results it is essential that both $\iota(S)$ and $\iota(S')$ are positive; these conditions will be avoided due to Theorems 1.3 and 5.1 and the thick-thin decomposition of Riemann surfaces given by Margulis Lemma (see, e.g., [**2**, p. 107]). Concretely, for any $\varepsilon < \operatorname{arcsinh} 1$ any Riemann surface, S, can be partitioned into a thick part, $S_\varepsilon := \{z \in S : \iota(z) > \varepsilon\}$, and a thin part, $S \setminus S_\varepsilon$, whose components are either collars of cusps or collars of closed geodesics of length less than or equal to 2ε (see Lemma 4.9).

In order to prove Theorem 1.1, it will be shown that it suffices to consider the thick parts of S and S' for some particular choices of ε and ε', so that Kanai's insight can be brought to S_ε and $S'_{\varepsilon'}$ if we avoid the (possible) contribution to the LII given by ∂S_ε and $\partial S'_{\varepsilon'}$.

We will need the following improvement of Theorems 1.3 and 5.1.

LEMMA 6.1. *Let S and S' be non-exceptional genus zero Riemann surfaces, and $f : S \longrightarrow S'$ be a c-full (a,b)-quasi-isometry. Then, given $0 < \varepsilon$, $\varepsilon_1 < \operatorname{arcsinh} 1$, there exist $0 < \varepsilon'$, $\tilde{\varepsilon} < \varepsilon_1$, which just depend on $\varepsilon, \varepsilon_1, a, b, c$, so that*

$$f(S_\varepsilon) \subset S'_{\varepsilon'} \subset V_c(f(S_{\tilde{\varepsilon}})).$$

PROOF. Theorem 5.1 asserts that given ε there exists ε' so that the first inclusion holds. For the second one, given ε' there exists $\tilde{\varepsilon}'$ such that $S' \setminus S'_{\varepsilon'} \supset V_c(S' \setminus S'_{\tilde{\varepsilon}'})$ by Lemma 3.2. Let $z' \in S'_{\varepsilon'}$; then $V_c(z') \subset S'_{\tilde{\varepsilon}'}$ and since f is c-full, there exists $x' \in V_c(z')$ so that $x' = f(x)$ for some $x \in S_{\tilde{\varepsilon}}$ where $\tilde{\varepsilon}$ is given by $\tilde{\varepsilon}'$ in Theorem 1.3. Therefore $z' \in V_c(f(S_{\tilde{\varepsilon}}))$. Since S_t becomes larger as $t > 0$ decreases, one can obtain $0 < \varepsilon'$, $\tilde{\varepsilon} < \varepsilon_1$. $\square$

As a first goal it is going to be proved the LII intrinsic to a bordered surface, S_ε contained in S; note that S_ε is not necessarily connected. To this end, we define below the "thick" boundary of a subset of S as its intrinsic boundary in S_ε, and the "intrinsic" LII that will refered to as LII_ε.

DEFINITION 6.2. Given a non-exceptional Riemann surface S, $\varepsilon > 0$ and a domain Ω in S_ε, define

$$\partial_\varepsilon \Omega := \partial\Omega \cap S_\varepsilon = \partial\Omega \setminus \partial S_\varepsilon.$$

REMARK 6.3. If γ is a non-trivial simple closed curve, $\gamma \subset \partial_\varepsilon\Omega$, then $L_S(\gamma) > 2\varepsilon$.

DEFINITION 6.4. S_ε is said to satisfy the *ε-linear isoperimetric inequality*, LII_ε, if there exists a positive constant c, such that if Ω is a relatively compact domain in S_ε with smooth boundary, then

$$A_S(\Omega) \le c\, L_S(\partial_\varepsilon\Omega). \tag{6.21}$$

A reduction is that it suffices to prove LII_ε for intrinsic geodesic domains in S_ε. A domain $\Omega \subset S$ is said to be a *geodesic domain* if $\partial\Omega$ is a finite number of simple closed geodesics, and $A_S(\Omega)$ is finite. Note that Ω does not need to be relatively compact for it could contain a finite number of cusps. From this point of view, the boundary of a cusp will be considered as an improper geodesic of zero length. An *intrinsic geodesic domain* is a geodesic domain intrinsic to S_ε, i.e., the intersection of a geodesic domain in S with S_ε.

Let us denote by $c_1(S_\varepsilon)$ the sharp ε-linear isoperimetric constant of S_ε and by $c_{1,g}(S_\varepsilon)$ the sharp ε-linear isoperimetric constant of S_ε for intrinsic geodesic domains.

LEMMA 6.5. *Let S be a non-exceptional Riemann surface and $\varepsilon \ge 0$ so that $\varepsilon < \operatorname{arcsinh} 1$. Then,*

$$S_\varepsilon \text{ has } LII_\varepsilon \iff S_\varepsilon \text{ has } LII_\varepsilon \text{ for intrinsic geodesic domains in } S_\varepsilon.$$

In fact, $c_{1,g}(S_\varepsilon) \leq c_1(S_\varepsilon) \leq c_{1,g}(S_\varepsilon) + 2$.

Note that this lemma also holds for S, corresponding to the case $\varepsilon = 0$; Lemma 6.5 with $\varepsilon = 0$ was proved in [**15**, Lemma 1.2] and improved in [**23**, Theorem 7].

Proof. The first inequality is direct. For the second one, Collar Lemma and Bers' theorem (see [**4**]) give LII_ε with constant 2 for simply connected and doubly connected domains. It is well known that these domains satisfy LII with constant 1 (see [**15**, Lemma 1.2] and [**23**, Theorem 7]). For other domains, $\Omega \subset S_\varepsilon$, write $\partial\Omega = \bigcup_{j=1}^n g_j$, where each g_j can be assumed to be a non-trivial simple closed curve and $n \geq 3$. Consider $\tilde{\Omega}$, the intrinsic geodesic domain in S_ε bounded by $\bigcup_{j=1}^n \beta_j$ where β_j is the intrinsic geodesic in $S_\varepsilon \cup \partial S_\varepsilon$ homotopic to g_j. Then $L_S(\partial_\varepsilon\tilde{\Omega}) \leq L_S(\partial_\varepsilon\Omega)$ and $A_S(\Omega) \leq A_S(\tilde{\Omega}) + A_S(\Omega \setminus \tilde{\Omega})$ where $\Omega \setminus \tilde{\Omega}$ is a disjoint union of doubly connected domains, each component bounded by a pair β_j and g_j, or simply connected domains bounded by subsets of β_j and g_j with the same endpoints. Since $L_S(\beta_j) \leq L_S(g_j)$, applying the LII for simply and doubly connected domains to each component of $\Omega \setminus \tilde{\Omega}$ one gets $A_S(\Omega \setminus \tilde{\Omega}) \leq 2L_S(\partial_\varepsilon\Omega)$ and thus

$$A_S(\Omega) \leq A_S(\tilde{\Omega}) + 2L_S(\partial_\varepsilon\Omega) \leq c_{1,g}(S_\varepsilon)L_S(\partial_\varepsilon\Omega) + 2L_S(\partial_\varepsilon\Omega),$$

and then $c_1(S_\varepsilon) \leq c_{1,g}(S_\varepsilon) + 2$. This inequality and the first one prove the lemma. $\square$

Finally the LII in S can be deduced from the LII_ε in S_ε,

Proposition 6.6. *Let S be a non-exceptional Riemann surface. Then there exists a universal positive constant $\varepsilon_0 \leq \operatorname{arcsinh} 1$ verifying the following properties*:

1. *If S_ε has LII_ε for some $0 < \varepsilon < \varepsilon_0$, then S has LII. Moreover, $c_1(S) \leq 2c_{1,g}(S_\varepsilon) + 2$.*
2. *If S has LII, then S_ε has LII_ε for every $0 < \varepsilon < \min\{\varepsilon_0, (12c_{1,g}(S))^{-1}\}$. Moreover, $c_1(S_\varepsilon) \leq (2\pi c_{1,g}(S))/(2\pi - 1) + 2$.*

Proof. By Collar Lemma, there exists a positive constant $\varepsilon_0 \leq \operatorname{arcsinh} 1$ so that if $0 < \varepsilon < \varepsilon_0$, then $A_S(C \setminus S_\varepsilon) \leq A_S(C \cap S_\varepsilon)$ for all C collars in S, and $L_S(\eta) \leq 3\varepsilon$ for every closed curve $\eta \subseteq \partial S_\varepsilon$.

In order to prove the first item, consider any fixed geodesic domain $\Omega \subset S$. Then

$$\Omega \cap S_\varepsilon = \Omega_1 \cup \cdots \cup \Omega_m$$

with $\{\Omega_k\}$ disjoint intrinsic geodesic domains in S_ε. Since $0 < \varepsilon < \varepsilon_0$,

$$\begin{aligned} A_S(\Omega) &= A_S(\Omega \cap S_\varepsilon) + A_S(\Omega \setminus S_\varepsilon) \leq 2A_S(\Omega \cap S_\varepsilon) \\ &= 2\sum_k A_S(\Omega_k) \leq 2c_{1,g}(S_\varepsilon)\sum_k L_S(\partial_\varepsilon\Omega_k) = 2c_{1,g}(S_\varepsilon)L_S(\partial\Omega). \end{aligned}$$

Then $c_{1,g}(S) \leq 2c_{1,g}(S_\varepsilon)$ and Lemma 6.5 gives the first item.

By Lemma 6.5, the proof of the second item will follow if it is shown that when S satisfies the LII then S_ε satisfies the LII_ε for intrinsic geodesic domains. It will first be shown that, as a consequence of the LII in S, for any geodesic domain $\tilde{\Omega}$ in S, the length of the short curves of its boundary is controlled by the length of the long curves; concretely, $L_S(\partial_\varepsilon\tilde{\Omega}) \geq (2\pi - 1)L_S(\partial\tilde{\Omega} \setminus \partial_\varepsilon\tilde{\Omega})$.

To this end, consider $\tilde{\Omega}$ a geodesic domain in S with $\partial\tilde{\Omega} = \bigcup_{j=1}^n \beta_j$ (each β_j is either a simple closed geodesic or a cusp and $n \geq 3$) and define $J := \{j : L_S(\beta_j) < (2c_{1,g}(S))^{-1}\}$ where $c_{1,g}(S)$ is the LII constant in S for geodesic domains. Then, if g denotes the genus of $\tilde{\Omega}$, by Gauss-Bonnet Theorem, the LLI can be written as

$$c_{1,g}(S)\Big(\sum_{j\in J} L_S(\beta_j) + \sum_{j\notin J} L_S(\beta_j)\Big) \geq 2\pi(n-2+2g),$$

and using that $(c_{1,g}(S))^{-1} \geq 2(\sharp J)^{-1}\sum_{j\in J} L_S(\beta_j)$ one gets

$$\sum_{j\notin J} L_S(\beta_j) \geq \Big(\frac{4\pi(n-2+2g)}{\sharp J} - 1\Big)\sum_{j\in J} L_S(\beta_j) \geq (2\pi - 1)\sum_{j\in J} L_S(\beta_j).$$

If $\varepsilon > 0$ is chosen so that $\varepsilon < \varepsilon_0$ and $3\varepsilon < (4c_{1,g}(S))^{-1}$, then for any $j \notin J$, β_j is in $\partial_\varepsilon\tilde{\Omega}$, and the above inequality implies $L_S(\partial_\varepsilon\tilde{\Omega}) \geq (2\pi - 1)L_S(\partial\tilde{\Omega} \setminus \partial_\varepsilon\tilde{\Omega})$ for any geodesic domain in S.

Let us show now that S_ε (with $\varepsilon < \min\{\varepsilon_0, (12c_{1,g}(S))^{-1}\}$ chosen as above) satisfies the LII_ε for intrinsic geodesic domains. If Ω is an intrinsic geodesic domain in S_ε then it can be written as $\Omega = \tilde{\Omega} \cap S_\varepsilon$, where $\tilde{\Omega}$ is a geodesic domain in S such that $\partial_\varepsilon\Omega = \partial_\varepsilon\tilde{\Omega}$ and since $\tilde{\Omega}$ satisfies the LII in S,

$$A_S(\Omega) \leq A_S(\tilde{\Omega}) \leq c_{1,g}(S)\big(L_S(\partial_\varepsilon\tilde{\Omega}) + L_S(\partial\tilde{\Omega} \setminus \partial_\varepsilon\tilde{\Omega})\big) \leq \frac{2\pi c_{1,g}(S)}{2\pi - 1} L_S(\partial_\varepsilon\Omega).$$

Then Lemma 6.5 gives the second item. □

Following Kanai's procedure, the LII will be transferred from bordered surfaces to nets and viceversa. To this end, a subset G of S is said to be δ-separated for $\delta > 0$, if $d_S(p,q) > \delta$ whenever p and q are distinct points of G. It is called *maximal* if it is maximal with respect to the order relation of inclusion.

Consider the distance d_G in G induced by the distance d_S of S. Concretely, given $p_1, p_2 \in G$, $d_G(p_1,p_2) = M$ if and only if $M \geq 0$ is the only natural number such that

$$\delta M \leq d_S(p_1,p_2) < \delta(M+1). \tag{6.22}$$

The set of neighbors of G is defined as $N(p) = \{q \in G : d_G(p,q) = 1\}$ and gives a net structure to the set G. Such net will be referred to as δ-net.

The linear isoperimetric inequality on nets is therefore defined as follows.

Definition 6.7. Let G be a net. For a subset T of G, define its *boundary* as

$\partial T := \{q \in G \setminus T : d_G(q,T) = 1\}$. It is said that G satisfies the *LII* if there exists a finite constant $c_1(G) > 0$ so that for any non-empty finite subset T of G,

$$\#T \le c_1(G)\#\partial T.$$

Let S be a Riemann surface and $0 < \varepsilon < \operatorname{arcsinh} 1$. Note that Lemma 3.2 gives that $\iota(V_\varepsilon(S_\varepsilon)) \ge c(\varepsilon)$, where $c(\varepsilon) := \operatorname{arcsinh}(e^{-\varepsilon}\sinh\varepsilon)$. The pair (G,δ) will denote a δ-net *associated* to the pair (S,ε) as follows: Set $\delta \le \iota(V_\varepsilon(S_\varepsilon))/2$, and choose a maximal δ-net G on S_ε so that

$$A_S(S_\varepsilon \cap B_S(p,\delta)) > \frac{1}{2} A_S(B_S(p,\delta)), \tag{6.23}$$

for all $p \in G$; such choice of G is possible due to Collar Lemma. Note also that G does not need to be connected.

Notice that since (G,δ) is maximal, there are no neighborhoods of points of S_ε that are not covered by balls $B_S(p,\delta)$ with $p \in G$. If this were the case one could add such point p to the net G contradicting maximality. If nevertheless there was a point q on the boundary of some balls $B_S(p,\delta)$ not covered, these balls could be slightly moved so that a neighborhood of q would not be covered and, as before, add q to the net. So, without loss of generality, $S_\varepsilon \subset \bigcup_{p\in G} B_S(p,\delta)$.

The strategy of the proof of Theorem 1.1 is as follows: Consider S and S' Riemann surfaces and $f : S \longrightarrow S'$ a quasi-isometry, (G,δ) and (G',δ') nets in (S,ε) and (S',ε'). It will be assumed that S' satisfies the *LII* that will be transferred to the net (G',δ'). Then it will be shown that (G,δ) and (G',δ') are quasi-isometric and so (G,δ) also satisfies the *LII*. Finally, this *LII* will be transferred to S. The next two results deal with transferring the *LII* between surfaces and nets: A direct application of [**20**, Lemma 4.5] is the following result:

Lemma 6.8. *Let S' be a non-exceptional Riemann surface satisfying the LII and $0 < \varepsilon' < \min\{\varepsilon_0, (12c_1(S'))^{-1}\}$, where ε_0 is the constant in Proposition* 6.6. *Let (G',δ') be a δ'-net associated to (S',ε'). Then,*

$$(G',\delta') \text{ also satisfies the } LII \text{ and } c_1(G') \le \frac{12\sinh\delta'}{\cosh(\delta'/2)-1}\, c_1(S').$$

Proof. Proposition 6.6 implies that $S'_{\varepsilon'}$ satisfies the $LII_{\varepsilon'}$ and applying Kanai's arguments in [**20**, Lemma 4.5] to $S'_{\varepsilon'}$ and G' the proof follows. □

The following lemma gives the other direction. To this end, recall Buser's local lineal isoperimetric inequality ([**6**, p. 215], [**20**, p. 411]).

Lemma 6.9. *Let (G,δ) be a δ-net associated to (S,ε). Then*

$$(G,\delta) \text{ has } LII \Longrightarrow S_\varepsilon \text{ has } LII_\varepsilon. \tag{6.24}$$

Moreover, $c_1(S_\varepsilon) \le 2\,m\,c_{1,l}(S_\varepsilon)\max\{1,\, 2\,c_1(G)(\sinh(9\delta/4)/\sinh(\delta/4))^2\} + 2$, *where*

$c_{1,l}(S_\varepsilon)$ *is the constant in the local LII and* $m =: \sup_{z\in S} \#\{p \in G : z \in B_S(p, \delta)\} < \infty$.

Proof. As in the previous lemma, it is possible to reproduce Kanai's proof in [**20**, Lemma 4.5] to get the result, mainly because it deals with a subset of S with positive injectivity radius, S_ε.

By Lemma 6.5, it suffices to consider Ω an intrinsic geodesic domain of S_ε, for which it is possible to separate ∂S_ε from $\partial_\varepsilon\Omega$ (by Collar Lemma). That is, if $p \in \partial_\varepsilon\Omega$, there exists a ball $B_S(p, 3\varepsilon)$ so that $\partial S_\varepsilon \cap B_S(p, 3\varepsilon) = \emptyset$. Following Kanai's proof, define sets $O, P_0 \subset G$

$$O := \left\{p \in G : A_S(B_S(p, \delta) \cap \Omega) > \frac{1}{2} A_S(B_S(p, \delta))\right\},$$
$$P_0 := \{p \in G \setminus O : B_S(p, \delta) \cap \Omega \neq \emptyset\},$$

so that $\Omega \subset \bigcup_{p\in O\cup P_0} B_S(p, \delta)$.

Since Ω is an intrinsic geodesic domain its boundary is a union of simple closed curves, some of them curves of ∂S_ε and the rest geodesics on S (the latter conform $\partial_\varepsilon\Omega$). Since (G, δ) is a δ-net associated to (S, ε), Collar Lemma implies that if $B_S(p, \delta)$ (for $p \in G$) intersects one curve of $\partial\Omega \setminus \partial_\varepsilon\Omega \subset \partial S_\varepsilon$ then it does not intersect any other curve of $\partial\Omega$. If this is the case, the fact that $G \subset S_\varepsilon$ and condition (6.23) imply that $p \in O$. Therefore, $B_S(p, \delta) \cap (\partial\Omega \setminus \partial_\varepsilon\Omega) = \emptyset$ for all $p \in P_0$. Since $\iota(S_\varepsilon) > \delta$ the local LII can be applied

$$\sum_{p\in P_0} A_S(B_S(p, \delta) \cap \Omega) \le c_{1,l}(S_\varepsilon) \sum_{p\in P_0} L_S(B_S(p, \delta) \cap \partial\Omega)$$
$$= c_{1,l}(S_\varepsilon) \sum_{p\in P_0} L_S(B_S(p, \delta) \cap \partial_\varepsilon\Omega) \le c_{1,l}(S_\varepsilon) m L_S(\partial_\varepsilon\Omega).$$

Now, following Kanai's estimates:

$$A_S(\Omega) \le \sum_{p\in O} A_S(B_S(p, \delta) \cap \Omega) + \sum_{p\in P_0} A_S(B_S(p, \delta) \cap \Omega) \le A(\delta)\sharp O + c_{1,l}(S_\varepsilon) m L_S(\partial_\varepsilon\Omega),$$

where $A(r) = 4\pi \sinh^2 r$ is the area of balls with radius r in $\mathbb{D}$ (the universal covering space of S). Writing $\nu := L_S(\partial_\varepsilon\Omega)/A_S(\Omega)$, then $A_S(\Omega) \le A(\delta)/(1 - c_{1,l}(S_\varepsilon)m\nu)\sharp O$. If $\nu \ge (2mc_{1,l}(S_\varepsilon))^{-1}$, then the LII_ε holds for Ω with constant $2mc_{1,l}(S_\varepsilon)$; otherwise, $\nu \le (2mc_{1,l}(S_\varepsilon))^{-1}$ and thus

$$A_S(\Omega) \le 2A(\delta)\sharp O.$$

On the other hand, points in $\partial_\varepsilon\Omega$ will be near of points in ∂O (since $\partial O \subset S_\varepsilon$). More precisely, if $p \in \partial O$ then there exists $p' \in N(p) \cap O$, and $B_S(p, \delta) \cap \sigma \neq \emptyset$ for some simple closed geodesic $\sigma \subset \partial_\varepsilon\Omega$. Note that σ separates p from p' in $B_S(p', 2\delta)$ since it is a geodesic and so $A_S(B_S(z, \delta)) \cap \Omega) = A_S(B_S(z, \delta))/2$ for $z \in \sigma$. Thus $d_S(p', \sigma) < 2\delta$

and therefore $\partial O \subset V_{2\delta}(\partial_\varepsilon \Omega)$. Let Q be a maximal δ-separated subset of $\partial_\varepsilon\Omega$; then $\bigcup_{p\in\partial O} B_S(p,\delta/2) \subset V_{5\delta/2}(\partial_\varepsilon\Omega) \subset \bigcup_{q\in Q} B_S(q,9\delta/2)$, which implies

$$\begin{aligned}
A(\delta/2)\sharp\partial O &\le \sum_{q\in Q} A_S(B_S(q,9\delta/2)) \le \frac{A(9\delta/2)}{A(\delta)} \sum_{q\in Q} A_S(B_S(q,\delta)) \\
&= \frac{2A(9\delta/2)}{A(\delta)} \sum_{q\in Q} A_S(B_S(q,\delta)\cap\Omega) \\
&\le \frac{2c_{1,l}(S_\varepsilon)A(9\delta/2)}{A(\delta)} \sum_{q\in Q} L_S(B_S(q,\delta)\cap\partial_\varepsilon\Omega) \\
&\le \frac{2mc_{1,l}(S_\varepsilon)A(9\delta/2)}{A(\delta)} L_S(\partial_\varepsilon\Omega),
\end{aligned}$$

where the local isoperimetric inequality was once again used. Combining this estimate with the previous one, and using the *LII* for G the desired result is obtained also in the case $\nu \le (2mc_{1,l}(S_\varepsilon))^{-1}$. □

As a last step, it will be constructed a quasi-isometry between the two nets (G,δ) and (G',δ') associated to (S,ε) and (S,ε') respectively with $0<\varepsilon<\operatorname{arcsinh} 1$ and $0<\varepsilon'$, $\tilde\varepsilon<\varepsilon$ given by Lemma 6.1.

PROPOSITION 6.10. *The nets (G,δ) and (G',δ') are quasi-isometric. More precisely, there is a C'-full (A,B)-quasi-isometry $g: G \longrightarrow G'$, with $A = a\max\{\delta'/\delta, \delta/\delta'\}$, $B = 5 + (a\delta/\delta') + b/\delta'$ and $C' = 2 + (a(2\delta + C(\varepsilon,\tilde\varepsilon)) + 2b + c)/\delta'$ where $C(\varepsilon,\tilde\varepsilon)$ is the maximum diameter of the connected components of $S_{\tilde\varepsilon}\setminus S_\varepsilon$ where $\tilde\varepsilon$ is given by Lemma* 6.1.

Moreover, for any $X\subset G$, $\#X \le \mu\#g(X)$ where $\mu \le 13^{a(2\delta'+b)/\delta}$.

REMARK 6.11. No connectivity is assumed for either G or G'. Note that the constant $C(\varepsilon,\tilde\varepsilon)$ does not depend on S due to Margulis Lemma.

In [**20**, Lemma 4.2] Kanai proves that the *LII* on graphs is preserved by quasi-isometries; thus an immediate consequence is:

COROLLARY 6.12. *For (G,δ) and (G',δ') as above,*

$$(G,\delta) \text{ satisfies the } LII \iff (G',\delta') \text{ satisfies the } LII.$$

Moreover, $c_1(G) \le \mu 12^{A(B+2C-1)+C-2} c_1(G')$, with μ as in Proposition 6.10.

PROOF. The function g will be defined as follows:

Given $p_1 \in G$, there exists at least one point $p_1' \in G'$ so that $p_1' \in B_{S'}(f(p_1), 2\delta')$, since $f(S_\varepsilon) \subset S'_{\varepsilon'}$ by Lemma 6.1. Define $g(p_1) := p_1'$.

Consider two points $p_1, p_2 \in G$ and suppose $d_{G'}(g(p_1), g(p_2)) = M$ for some $M \ge 0$; that is,

$$M\delta' \leq d_{S'}(g(p_1), g(p_2)) < (M+1)\delta'.$$

Transferring this property to f:

$$\begin{aligned} d_{S'}(f(p_1), f(p_2)) &\leq d_{S'}(f(p_1), g(p_1)) + d_{S'}(f(p_2), g(p_2)) + d_{S'}(g(p_1), g(p_2)) \\ &\leq 4\delta' + (M+1)\delta' = (5+M)\delta'. \end{aligned}$$

This estimate together with f being an (a,b)-quasi-isometry give:

$$\frac{1}{a} d_S(p_1, p_2) - b \leq d_{S'}(f(p_1), f(p_2)) \leq (5+M)\delta' \leq \delta'(5 + d_{G'}(g(p_1), g(p_2))),$$

that is,

$$\frac{1}{a\delta'} d_S(p_1, p_2) - \left(\frac{b}{\delta'} + 5\right) \leq d_{G'}(g(p_1), g(p_2)).$$

Use the fact that $\delta d_G(p_1, p_2) \leq d_S(p_1, p_2)$ to finally conclude

$$\frac{\delta}{a\delta'} d_G(p_1, p_2) - \left(5 + \frac{b}{\delta'}\right) \leq d_{G'}(g(p_1), g(p_2)).$$

The other direction follows from an analogous argument, obtaining in this case:

$$d_{G'}(g(p_1), g(p_2)) \leq \frac{a\delta}{\delta'} d_G(p_1, p_2) + \left(4 + \frac{b + a\delta}{\delta'}\right).$$

Finally it will be shown that

$$G' \subset \bigcup_{p \in G} B'_G(g(p), C').$$

Take $q \in G' \subset S'_{\varepsilon'}$ and $0 < \tilde{\varepsilon} < \varepsilon$ given by Lemma 6.1 such that $S'_{\varepsilon'} \subset V_c(f(S_{\tilde{\varepsilon}}))$; then $q \in V_c(f(S_{\tilde{\varepsilon}}))$. Therefore, there exist $\tilde{x} \in S_{\tilde{\varepsilon}}$ and $x \in S_\varepsilon$ so that $d_S(x, \tilde{x}) < C(\varepsilon, \tilde{\varepsilon})$ and $d_{S'}(f(\tilde{x}), q) \leq c$ (f is c-full). Since G is a maximal δ-net in S_ε there exists $p \in G$ such that $d_S(p, x) < 2\delta$. Let $r \in G'$ be given by $r := g(p)$; then $d_{S'}(r, f(p)) < 2\delta'$.

These facts together with f being an (a,b)-quasi-isometry, give:

$$d_{S'}(q, g(p)) \leq a(2\delta + C(\varepsilon, \tilde{\varepsilon})) + 2b + c + 2\delta'.$$

Since $d_{G'}(q, g(p)) \leq d_{S'}(q, g(p))/\delta'$, then $d_{G'}(q, g(p)) \leq 2 + (a(2\delta + C(\varepsilon, \tilde{\varepsilon})) + 2b + c)/\delta'$.

Finally, $\#X \leq \mu \# g(X)$ where $\mu \leq 13^{(a(2\delta'+b))/\delta}$ will be shown. It is easy to check that for Riemann surfaces, the number of points $p \in G$ contained in a ball of radius δ is at most 13 since they are δ-separated. By the way g was defined, if $p, q \in G$, with $g(p) = g(q)$, then $d_S(p, q) \leq a(2\delta' + b)$. And thus the corollary follows. □

Finally, the combination of all previous results will give the proof of Theorem 1.1.

6.1. Proof of Theorem 1.1.

Assume that S' has LII. If ε_0 is the constant in Proposition 6.6, let us fix $0 < \varepsilon < \varepsilon_0$ and let $0 < \varepsilon'$, $\tilde{\varepsilon} < \min\{\varepsilon_0, (12c_1(S'))^{-1}\}$ given by Lemma 6.1. Let (G', δ') be a net associated to (S', ε'). Since S' has LII, by Lemma 6.8, G' has LII. If (G, δ) is a net associated to (S, ε), then Proposition 6.10 gives that (G, δ) and (G', δ') are quasi-isometric, and Corollary 6.12 concludes that (G, δ) has LII. Lemma 6.9 states that S_ε has LII_ε and, since $0 < \varepsilon < \varepsilon_0$, Lemma 6.6 gives that S has LII.

Moreover, the isoperimetric constant obtained $c_1(S) < \infty$ depends just on $\varepsilon, a, b, c, c_1(S')$. In order to avoid the dependence on ε, it suffices to take $\varepsilon = \varepsilon_0/2$, since ε_0 is a universal constant. □

7. Surfaces with finite genus.

In order to obtain a similar result to Theorem 1.1 for surfaces with finite genus, the following lemma is needed.

Lemma 7.1. *Let S be a non-exceptional Riemann surface with finite genus and infinite area. Let $\sigma_1, \dots, \sigma_k$ be a set of pairwise disjoint simple closed geodesics in S such that $S \setminus \{\sigma_1 \cup \cdots \cup \sigma_k\}$ is connected and has not genus; denote by S_0 the bordered surface obtained as the completion of $S \setminus \{\sigma_1 \cup \cdots \cup \sigma_k\}$. Then the following facts hold:*

- *S and S_0 are quasi-isometric.*
- *S satisfies the LII if and only if S_0 satisfies the LII.*

Proof. Theorem 2.2 in [**34**] gives the first statement.

In order to prove the second one, assume that S_0 satisfies the LII (the other implication is direct). Seeking for a contradiction let us suppose that S does not satisfy the LII. Hence, by Lemma 6.5 there exists a sequence of geodesic domains Ω_n in S with $A_S(\Omega_n)/L_S(\partial\Omega_n) \to \infty$. Since S_0 satisfies the LII, without loss of generality, assume that there exists $1 \le j_n \le k$ with $\sigma_{j_n} \subset \partial\Omega_n$ for each n; furthermore, since $L_S(\sigma_1 \cup \cdots \cup \sigma_k)$ is a fixed number (and then bounded), one can also assume $A_S(\Omega_n) \le c$ for some constant c and every n, and $L_S(\partial\Omega_n) \to 0$ (note that $L_S(\partial\Omega_n) > 0$ since S has infinite area).

Let us consider a ball $B_S(z, r)$ in S with $\sigma_1 \cup \cdots \cup \sigma_k \subset B_S(z, r)$; let us choose now $R > r$ with $A_S(B_S(z, R) \setminus B_S(z, r)) > c$ (this is possible since S has infinite area). Let us define $u := \min\{L_S(\sigma) : \sigma$ is a simple closed geodesic with $\sigma \cap B_S(z, R) \ne \emptyset\}$. Since $A_S(B_S(z, R) \setminus B_S(z, r)) > A_S(\Omega_n)$ and $\sigma_1 \cup \cdots \cup \sigma_k \subset B_S(z, r)$, there exists a simple closed geodesic $\sigma^n \subseteq \partial\Omega_n$ with $\sigma^n \cap B_S(z, R) \ne \emptyset$ and $\sigma^n \ne \sigma_j$ for $j = 1, \dots, k$ (since $S \setminus \{\sigma_1 \cup \cdots \cup \sigma_k\}$ is connected and S has infinite area, there is no geodesic domain Ω in S with $\partial\Omega \subseteq \sigma_1 \cup \cdots \cup \sigma_k$). Hence,

$$A_S(\Omega_n) \le c\, \frac{L_S(\sigma^n)}{u} \le \frac{c}{u}\, L_S(\partial\Omega_n),$$

which contradicts $A_S(\Omega_n)/L_S(\partial\Omega_n) \to \infty$. □

7.1. Proof of Theorem 1.2.

It is not difficult to check that S has finite area if and only if S' has finite area; in this case, S and S' do not satisfy the LII. Otherwise, the theorem is a consequence of Lemma 7.1 and Theorem 1.1 (which also holds for bordered surfaces whose border is a finite union of simple closed geodesics; it suffices to take $\varepsilon, \varepsilon', \tilde{\varepsilon}$ less than half the minimum of the lengths of these simple closed geodesics). □

It is not possible to obtain a quantitative version of Theorem 1.2, as shows the following example.

Example 7.2. There exist constants a, b, c, with the following property: for each n there exist non-exceptional Riemann surfaces with finite genus S_n satisfying the LII and a c-full (a, b)-quasi-isometry $f_n : S_n \to S_1$, with $\lim_{n\to\infty} c_1(S_n) = \infty$.

Let us consider the bordered surfaces Y_1 and X in Example 2.3, and a sequence $\{X_m\}_{m\ge1}$ of bordered surfaces isometric to X; denote by R_n the bordered surface obtained from $X_1, \dots, X_n$ by pasting a boundary curve of X_m with a boundary curve of X_{m+1} for every $1 \le m \le n-1$ (R_n is a surface with genus n and two boundary curves). Consider now a generalized Y-piece Y_0 with a cusp and such that ∂Y_0 is the union of two simple closed geodesics with length 1. Denote by R_0 the bordered surface obtained by pasting two boundary curves of Y_1 with two boundary curves of Y_0 (R_0 is a torus with a cusp and a hole). S_n is the (non bordered) surface obtained by pasting a funnel (with boundary of length 1) to one boundary curve of R_n and R_0 to the other boundary curve of R_n.

Lemma 2.2 gives that S_n satisfies the LII for every n.

The domain $\bigcup_{m=1}^n X_m$ in S_n has area $4\pi n$ and its boundary has length 2 for every $n \ge 1$. This implies that $\lim_{n\to\infty} c_1(S_n) = \infty$.

Finally, let us prove the existence of c-full (a, b)-quasi-isometries $f_n : S_n \to S_1$. Fix geodesic rays γ_n in S_n starting in $\partial(R_0 \cup R_n)$. Let us consider the bijective isometry $g_n : \gamma_n \to \gamma_1$. We can easily check that the map $h_n : R_0 \cup R_n \to R_0 \cup R_1$ defined in the following way is a quasi-isometry with constants which do not depend on n: if $p_n(z)$ is the nearest point in γ_n from $z \in R_0 \cup R_n$, then $h_n(z) = g_n(p_n(z))$. Define $f_n : S_n \to S_1$ as follows: $f_n = h_n$ on the interior of $R_0 \cup R_n$, and f_n is any isometry between the funnels of S_n and S_1. Now, we can easily check that f is a quasi-isometry with constants which do not depend on n.

8. Non-linear isoperimetric inequalities.

This section deals with α-isoperimetric inequalities with $1/2 \le \alpha < 1$, which have a very different behavior from LII.

Proposition 8.1. *If a Riemann surface S satisfies $\iota(S) = 0$, then S does not satisfy the α-isoperimetric inequality for each $1/2 \le \alpha < 1$.*

Proof. Seeking for a contradiction, let us assume that S satisfies the α-isoperimetric inequality for some $1/2 \le \alpha < 1$.

If S has a cusp, let us consider the a-collars $C(a)$ of the cusp, with $0 < a \le 2$.

It is well known that $A_S(C(a)) = L_S(\partial C(a)) = a$; hence, $a^\alpha \le c_\alpha a$, which gives a contradiction if $a \to 0^+$.

If S has no cusp, then there exists a sequence of simple closed geodesics $\{\sigma_n\}$ with $\lim_{n\to\infty} L_S(\sigma_n) = 0$. Denote by C_n the collar of σ_n of width 1. It is well known that $A_S(C_n) = 2L_S(\sigma_n)\sinh 1$ and $L_S(\partial C_n) = 2L_S(\sigma_n)\cosh 1$; hence, $(2L_S(\sigma_n)\sinh 1)^\alpha \le c_\alpha 2L_S(\sigma_n)\cosh 1$, which gives a contradiction if $n \to \infty$. □

Theorem 1.4 follows from Proposition 8.1 and [**20**, Theorem 4.1].

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