

On the stability of locally conformal Kähler structures

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Abstract. In this article we develop a new approach to the problem of the stability of locally conformally Kähler structures (l.c.k structures) under small deformations of complex structures and deformations of flat line bundles. We show a cohomological criterion for the stability of l.c.k structures. We apply our approach to generalizations of Hopf manifolds to obtain the stability of l.c.k structures which do not have potential in general. We give an explicit description of the cohomological obstructions of the stability of l.c.k structures on Inoue surfaces with $b_2 = 0$.

1. Introduction.

Let X be a compact complex manifold with complex structure J and a Hermitian 2-form ω on X . If there is a d -closed 1-form η such that $d\omega = \eta \wedge \omega$, then ω is called a *locally conformally Kähler structure* (l.c.k structure) on X with *Lee form* η . An l.c.k structure yields a Kähler metric on the universal covering of X . Many interesting l.c.k structures on non-Kähler manifolds have been constructed. Hopf surfaces admit l.c.k structures [12], [1]. Vaisman metrics [23] are l.c.k structures with parallel Lee form, which are constructed on the quotients of the cones of Sasakian manifolds [2]. Ornea and Vebitsky [20] studied the class of l.c.k structures with potential which is the one admitting global Kähler potential on the universal covering of X . Tricerri [22] gave l.c.k structures on certain Inoue surfaces with $b_2 = 0$ [13], which do not have potential. Fujiki and Pontecorvo [6] used the Twistor theory to provide l.c.k structures on the certain complex surfaces of type VII including hyperbolic Inoue and parabolic Inoue surfaces. Brunella constructed l.c.k structures on Kato surfaces which are complex surfaces of type VII with global spherical shell [3], [4].

Kodaira and Spencer [17] showed that Kähler structures are stable under small deformations of complex structures. More precisely, if X is Kählerian, then any small deformation X_t of $X = X_0$ is also Kählerian. Contrast to Kähler structures, l.c.k structures by Tricerri on certain Inoue surfaces with $b_2 = 0$ are not stable under small deformations of complex structures [1]. On the other hand, it turns out that l.c.k structures with potential are stable [20]. This suggests the need for further research on the stability of l.c.k structures under deformations.

The purpose of this paper is to obtain the cohomological criterion for the stability of l.c.k structures. We apply the method developed on deformations of generalized Kähler geometry [7], [8], [9], [10], [11]. An l.c.k structure ω gives a flat line bundle L over X

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which we call *the corresponding flat line bundle* to ω . Then using flat line bundle L -valued differential forms $\wedge^\bullet \otimes L$, we have the L -valued de Rham cohomology groups $H^\bullet(X, L)$ of L -valued de Rham complex $(\wedge^\bullet \otimes L, d_L)$ and the L -valued Dolbeault cohomology groups $H^{p,q}(X, L)$ of the L -valued Dolbeault complex $(\wedge^{p,\bullet} \otimes L, \bar{\partial}_L)$ (see Section 1 for more detail). The $\partial\bar{\partial}$ -lemma for L -valued forms is the following:

DEFINITION 2.1. Let X be a complex manifold and L a flat line bundle over X . Then we say that (X, L) satisfies the $\partial\bar{\partial}$ -lemma at degree (p, q) if there is an L -valued form γ of type $(p-1, q-1)$ such that

$$\partial_L \bar{\partial}_L \gamma = \partial_L \alpha \in \Gamma(X, L \otimes \wedge^{p,q}),$$

for every $\bar{\partial}_L$ -closed L -valued form α of type $(p-1, q)$.

Then we have the following criterion for the stability of l.c.k structures (see Theorem 2.2.): *If L is a flat line bundle corresponding to an l.c.k structure ω_0 on X and (X, L) satisfies the $\partial\bar{\partial}$ -lemma at degree $(1, 2)$, then the stability of l.c.k structures holds, that is, every small deformation X_t of X admits an l.c.k structure ω_t , where L is still the corresponding flat line bundle to the deformed structure ω_t .*

We also consider deformations of flat line bundle $\{L_s\}$ and try to construct a family of l.c.k structures $\{\omega_s\}$ such that L_s is the corresponding line bundle to ω_s . Then an obstruction to deformations appears as a cohomology class in $H^3(X, L)$ (Theorem 2.4). We further obtain the criterion for the stability under deformations of flat line bundles (Theorem 2.3).

In Section 2, we give preliminary results of l.c.k structures and show our main theorems. In Section 3, we prove the main theorems. Deformations of l.c.k structures are constructed as formal power series. In Section 4, we show the convergence of the formal power series. In Section 5, we discuss the stability of l.c.k structures on generalizations of Hopf manifolds. They satisfy the $\partial\bar{\partial}$ -lemma at degree $(1, 2)$ for a class of flat line bundles. The L -valued Bott-Chern cohomology groups are calculated on them. When the L -valued Bott-Chern cohomology group is not trivial, there are l.c.k structures which do not have potential. Even for the class of l.c.k structures which do not have potential, our theorems can be applied to obtain deformations of the l.c.k structures. Professor Ornea and Professor Verbitsky conjectured that every l.c.k metric on a Vaisman manifold is an l.c.k with potential (c.f. Conjecture 6.3 in [21]). Our results show that their conjecture does not hold (see Proposition 5.7 and 5.8). In Section 6, we give the stability on the class of complex surfaces with effective anti-canonical line bundle. In Section 7, we discuss obstructions to deformations of l.c.k structures on Inoue surfaces with $b_2 = 0$. There are three classes of Inoue surfaces: S_M , $S_{N,p,q,r;t}^{(+)}$ and $S_{N,p,q,r}^{(-)}$. It is known that both Inoue surfaces S_M and $S_{N,p,q,r}^{(-)}$ are rigid and the Inoue surface $S_{N,p,q,r;t}^{(+)}$ admits 1-dimensional deformations of complex structures which are parameterized by complex numbers $t \in \mathbb{C}$. Belgun [1] showed that both S_M and $S_{N,p,q,r}^{(-)}$ have l.c.k structures and yet $S_{N,p,q,r;t}^{(+)}$ admits an l.c.k structure for only real numbers $t \in \mathbb{R}$. Thus the stability theorem of l.c.k structures does not hold on $S_{N,p,q,r;t}^{(+)}$. We show that the obstruction

to the stability appears as a non-trivial cohomology class (Proposition 7.2). Further together with Belgun’s result, we show that the corresponding flat line bundle to every l.c.k structure on $S_{N,p,q,r;t}^{(+)}$ must be the canonical line bundle (Proposition 7.3). We also show there are obstructions to the stability of l.c.k structures under deformations of flat line bundles on Inoue surfaces with $b_2 = 0$. It is shown that the l.c.k structures on S_M , $S_{N,p,q,r;t}^{(+)}$ and $S_{N,p,q,r}^{(-)}$ are not stable under deformations of flat line bundles (Proposition 7.3, 7.5).

2. Stability theorem of locally conformally Kähler structures.

Let $X = (M, J)$ be a compact complex manifold of dimension n , where M is the underlying differential manifold and J is an integrable complex structure on M . Let ω be a locally conformally Kähler structure (l.c.k structure) on $X = (M, J)$ with Lee form η . The Lee form η gives the cohomology class $[\eta] \in H^1(M, \mathbb{R})$. The exponential map $\exp : \mathbb{R} \rightarrow \mathbb{R}_{>0}$ induces the map $\exp_* : H^1(M, \mathbb{R}) \rightarrow H^1(M, \mathbb{R}^*)$ and then the image of the class $[-\eta]$ by \exp_* yields a real flat line bundle $L \in H^1(M, \mathbb{R}^*)$ which is called *the corresponding flat line bundle to ω* . In order to give an explicit description of L as a Čech cocycle, we take a covering $\{U_i\}_{i \in \Lambda}$ of M such that $H^p(U_{i_1} \cap \dots \cap U_{i_q}, \mathbb{R}) = 0$, for all $p > 0$ and $i_1, \dots, i_q \in \Lambda$. Then there is a function f_i such that $\eta|_{U_i} = df_i$, on each U_i and $f_i - f_j = \lambda_{ij}$ is a constant on $U_i \cap U_j$. Then $\{e^{-\lambda_{ij}}\}_{i \in \Lambda}$ is a representative of L which gives the class the first \mathbb{R}^* -valued Čech cohomology group $\check{H}^1(M, \mathbb{R}^*)$. Thus the flat line bundle L has a trivialization on U_i with locally constant transition functions $\{(e^{-\lambda_{ij}}, U_i \cap U_j)\}$. The vector bundle of L -valued k -forms is denoted by $\wedge^k \otimes L$. An L -valued k -form $\tilde{\alpha}$ is a section of $\wedge^k \otimes L$ which is also given by a set of k -forms $\{\alpha_i\}$ such that $\tilde{\alpha}_i = e^{-\lambda_{ij}} \tilde{\alpha}_j$ for all i, j . Then the exterior derivative d induces the differential operator d_L acting on L -valued forms which yields the L -valued de Rham complex:

$$\dots \xrightarrow{d_L} \wedge^k \otimes L \xrightarrow{d_L} \wedge^{k+1} \otimes L \xrightarrow{d_L} \dots$$

We denote by $H^\bullet(X, L)$ the L -valued de Rham cohomology groups. Let d_η be the differential operator $d - \eta$ which acts on differential forms. The η -twisted cohomology groups $H_\eta^\bullet(X)$ are the cohomology groups of the η -twisted complex:

$$\dots \xrightarrow{d_\eta} \wedge^k \xrightarrow{d_\eta} \wedge^{k+1} \xrightarrow{d_\eta} \dots$$

The trivialization $\{e^{-f_i}\}_{i \in \Lambda}$ is denoted by e^{-f} which is a section of L . Then by tensoring the section e^{-f} , a k -form α is regarded as an L -valued k -form $\tilde{\alpha} := e^{-f} \otimes \alpha = \{e^{-f_i} \alpha\}_{i \in \Lambda}$. This gives an isomorphism between complexes $(\wedge^\bullet, d_\eta) \cong (\wedge^\bullet \otimes L, d_L)$ since $d_\eta = e^{f_i} \circ d \circ e^{-f_i}$ on U_i . Thus we have an isomorphism $H_\eta^k(X) \cong H^k(X, L)$. We also denote by $\wedge^{p,q} \otimes L$ the vector bundle of L -valued forms of type (p, q) on a complex manifold $X = (M, J)$. Then we have the decomposition $d_L = \partial_L + \bar{\partial}_L$, where

$$\begin{aligned} \partial_L : \wedge^{p,q} \otimes L &\longrightarrow \wedge^{p+1,q} \otimes L, \\ \bar{\partial}_L : \wedge^{p,q} \otimes L &\longrightarrow \wedge^{p,q+1} \otimes L. \end{aligned}$$

Then we have the Dolbeault complex of L -valued forms

$$\dots \xrightarrow{\bar{\partial}_L} \wedge^{p,q} \otimes L \xrightarrow{\bar{\partial}_L} \wedge^{p,q+1} \otimes L \xrightarrow{\bar{\partial}_L} \dots .$$

We denote by $H^{p,q}(X, L)$ the L -valued Dolbeault cohomology groups. The Lee form η is decomposed into $\eta^{1,0}$ and $\eta^{0,1}$, where $\eta^{1,0} \in \wedge^{1,0}$ and $\eta^{0,1} \in \wedge^{0,1}$. Let ∂_η be the operator $\partial - \eta^{1,0}$ and $\bar{\partial}_\eta = \bar{\partial} - \eta^{0,1}$ the its complex conjugate. Then we have the twisted Dolbeault cohomology groups $H_\eta^{p,q}(X)$ of the twisted Dolbeault complex:

$$\dots \xrightarrow{\bar{\partial}_\eta} \wedge^{p,q} \xrightarrow{\bar{\partial}_\eta} \wedge^{p,q+1} \xrightarrow{\bar{\partial}_\eta} \dots .$$

As in before, we have an isomorphism $H_\eta^{p,q}(X) \cong H^{p,q}(X, L)$.

For an l.c.k structure ω , we have the L -valued Kähler form $\tilde{\omega} = e^{-f} \otimes \omega$. Conversely, an L -valued Kähler form $\tilde{\omega}$ yields an l.c.k structure ω and L is the corresponding flat line bundle to ω . The inclusion $\mathbb{R}^* \rightarrow \mathcal{O}_X^*$ induces the map $H^1(X, \mathbb{R}^*) \rightarrow H^1(X, \mathcal{O}_X^*)$. Then the real flat line bundle L gives the holomorphic line bundle \mathcal{L} over X . Note that \mathcal{L} is a topologically trivial complex line bundle. Let $H^q(X, \Omega^p \otimes \mathcal{L})$ denote the cohomology group of $\Omega^p \otimes \mathcal{L}$, where Ω^p is the vector bundle of holomorphic p forms. Then we have $H^q(X, \Omega^p \otimes \mathcal{L}) \cong H^{p,q}(X, L)$ by the standard Dolbeault theorem.

DEFINITION 2.1. Let $X = (M, J)$ be a complex manifold and L a flat line bundle over X . Then we say that (X, L) satisfies the $\partial\bar{\partial}$ -lemma at degree (p, q) if there is an L -valued form γ of type $(p - 1, q - 1)$ such that

$$\partial_L \bar{\partial}_L \gamma = \partial_L \alpha \in \Gamma(X, L \otimes \wedge^{p,q})$$

for every $\bar{\partial}_L$ -closed, L -valued form α of type $(p - 1, q)$.

Note that if $H^{p-1,q}(X, L) = \{0\}$, then (X, L) satisfies the $\partial\bar{\partial}$ -lemma at (p, q) . In particular, the vanishing $H^{0,2}(X, L) = \{0\}$ yields the $\partial\bar{\partial}$ -lemma at degree $(1, 2)$.

THEOREM 2.2. Let $X = (M, J)$ be a compact, complex manifold with a locally conformally Kähler structure ω (l.c.k structure). We denote by L the corresponding flat line bundle to the l.c.k structure ω . We assume that (X, L) satisfies the $\partial\bar{\partial}$ -lemma at degree $(1, 2)$. Let $\{J_t\}$ be deformations of complex structure J which analytically depend on t and $J_0 = J$, where $|t| < \varepsilon$, for a constant $\varepsilon > 0$. Then there is a positive constant $\varepsilon' < \varepsilon$ and an analytic family of 2-forms $\{\omega_t\}$ which satisfies the followings:

- (i) $\omega_0 = \omega$,
- (ii) ω_t is an l.c.k structure on (M, J_t) for all t ,
- (iii) The line bundle L is the corresponding flat line bundle to ω_t for all t ,

where $|t| < \varepsilon'$.

Next we consider deformations of the flat line bundle L on a complex manifold $X = (M, J)$ with an l.c.k structure ω . Small deformations of the flat line bundle L

are parametrized by the first cohomology group $H^1(X, \mathbb{R})$. Let $\{L_s\}$ be deformations of flat line bundles with $L_0 = L$ which are parametrized by the real number s . Then deformations $\{L_s\}$ is given by a family of d -closed 1-forms $\{\eta_s\}$ which are representatives of $H^1(M, \mathbb{R})$. We have the following criterion for the stability of l.c.k structures under deformations of flat line bundles

THEOREM 2.3. *Let $\{L_s\}$ be deformations of flat line bundles which analytically depend on s and $L_0 = L$, where $|s| < \varepsilon$ for a constant $\varepsilon > 0$. If (X, L) satisfies the $\partial\bar{\partial}$ -lemma at degree $(1, 2)$ and $H^3(X, L) = \{0\}$, then there is a positive constant $\varepsilon' < \varepsilon$ and an analytic family of 2-forms $\{\omega_s\}$ which satisfies the followings:*

- (i) $\omega_0 = \omega$,
- (ii) ω_s is an l.c.k structure on (M, J) for all s ,
- (iii) The line bundle L_s is the corresponding flat line bundle to ω_s for all s ,

where $|s| < \varepsilon'$.

For a family of d -closed 2-forms η_s , we denote by $\dot{\eta}_0 := (d/ds)\eta_s|_{s=0}$ the infinitesimal tangent of deformations of $\{L_s\}$ at $s = 0$ which gives the class $[\dot{\eta}_0] \in H^1(X, \mathbb{R})$. Then the class $[\dot{\eta}_0 \wedge \omega] \in H^3(X, L)$ is regarded as the first obstruction to the existence of a smooth family of l.c.k structures $\{\omega_s\}$ such that L_s is the corresponding flat line bundle to ω_s .

THEOREM 2.4. *If the class $[\dot{\eta}_0 \wedge \omega] \in H^3(X, L)$ does not vanish, then X does not admit a smooth family of l.c.k forms $\{\omega_s\}$ such that L_s is the corresponding flat line bundle to ω_s for sufficiently small s .*

In Section 7, we shall show that the obstruction does not vanish for Inoue surfaces with $b_2 = 0$. Combining the above two theorems, we obtain the following theorem:

THEOREM 2.5. *Let $\{J_t\}$ be deformations of complex structure J and $\{L_s\}$ deformations of flat line bundles as in Theorems 2.2 and 2.3. If (X, L) satisfies the $\partial\bar{\partial}$ -lemma at degree $(1, 2)$ and $H^3(X, L) = \{0\}$, then there is a positive constant $\varepsilon' < \varepsilon$ and an analytic 2-parameter family of 2-forms $\{\omega_{s,t}\}$ which satisfies the followings:*

- (i) $\omega_{0,0} = \omega$,
- (ii) $\omega_{s,t}$ is an l.c.k structure on (M, J_t) for all t ,
- (iii) The line bundle L_s is the corresponding flat line bundle to $\omega_{s,t}$ for all s ,

where $|s|, |t| < \varepsilon'$.

3. Proof of main theorems.

This section is devoted to proof of theorems 2.2, 2.3, 2.4 and 2.5. We use the same notation as in the previous section. We have already shown that an l.c.k structure ω on $X = (M, J)$ with a corresponding flat line bundle L is equivalent to the L -valued Kähler form $\tilde{\omega}$. Thus Theorem 2.2 is reduced to the following theorem:

THEOREM 3.1. *Let (X, L) and $\{J_t\}$ be as in Theorem 2.2. Then there is a positive*

constant $\varepsilon' < \varepsilon$ and an analytic family of L -valued 2-forms $\{\tilde{\omega}_t\}$ which satisfies the followings:

- (i) $\tilde{\omega}_0 = \tilde{\omega}$,
- (ii) $\tilde{\omega}_t$ is an L -valued Kähler form on (M, J_t) for all t ,

where $|t| < \varepsilon'$.

From now on, we denote by ω an L -valued Kähler form on $X = (M, J)$ in stead of $\tilde{\omega}$ in this section.

The decomposition of the complex tangent bundle $T^{\mathbb{C}}M = T^{1,0} \oplus T^{0,1}$ on M gives the decomposition of the bundle of complex endmorphism $\text{End}(T^{\mathbb{C}}M) = \wedge^1 \otimes T^{\mathbb{C}}M$:

$$\text{End}(T^{\mathbb{C}}M) = E_0 \oplus E_1,$$

where $E_0 := (\wedge^{1,0} \otimes T^{0,1}) \oplus (\wedge^{0,1} \otimes T^{1,0})$ and $E_1 := (\wedge^{1,0} \otimes T^{1,0}) \oplus (\wedge^{0,1} \otimes T^{0,1})$. We denote by $E_i^{\mathbb{R}}$ the real part of E_i for $i = 0, 1$. Then we have the decomposition of the bundle of real endomorphisms $\text{End}(TM) = E_0^{\mathbb{R}} \oplus E_1^{\mathbb{R}}$. An endmorphism $a \in \text{End}(TM)$ acts on the complex structure J by the adjoint, i.e., $ad_a J := [a, J]$. The exponential e^a is regarded as an element of $\text{GL}(TM)$ for $a \in \text{End}(TM)$ which acts on the complex structure $J = J_0$ by the adjoint action, i.e., $\text{Ad}_{e^a} J := e^a \circ J \circ e^{-a}$. The linear action of $e^a \in \text{GL}(TM)$ on the L -valued Kähler form ω is denoted by $e^a \cdot \omega$. We also denote by $a \cdot \omega$ the action of $\text{End}(TM)$ on ω which is the action of the Lie algebra. We assume that there is a family of complex structures $\{J_t\}$ as in Theorem 3.1. Then there is a family of endomorphisms $\{a(t)\}$ with $a(t) \in E_0^{\mathbb{R}}$ such that

$$J_t = \text{Ad}_{e^{a(t)}} J, \tag{3.1}$$

where $a(t)$ is an analytic family (cf. Proposition 2.6 page 541 in [9]),

$$a(t) = a_1 t + a_2 \frac{t^2}{2!} + \dots$$

Since $\text{Ad}_{e^b} J = J$ for $b \in E_1^{\mathbb{R}}$, the action of $E_1^{\mathbb{R}}$ preserves the complex structure J . We assume that there is a family of endomorphisms $\{b(t)\}$ with $b(t) \in E_1^{\mathbb{R}}$ such that

$$d_L(e^{a(t)} \circ e^{b(t)} \cdot \omega) = 0 \tag{3.2}$$

By the Campbel-Hausdorff formula, there is a section $z(t) \in \text{End}(TM)$ such that

$$e^{z(t)} = e^{a(t)} \circ e^{b(t)}.$$

Then it turns out that $e^{z(t)} \cdot \omega$ is an L -valued Kähler form on (M, J_t) for each t . In fact, the diagonal action of $e^{z(t)}$ on the pair (J, ω) yields a family of pairs $(\text{Ad}_{e^{z(t)}} J, e^{z(t)} \cdot \omega)$, where $e^{z(t)} \cdot \omega$ is an L -valued Hermitian form of type $(1, 1)$ with respect to the complex structure $\text{Ad}_{e^{z(t)}} J$. Then we have

$$\text{Ad}_{e^{z(t)}} J = \text{Ad}_{e^{\alpha(t)}} \circ \text{Ad}_{e^{b(t)}} J \tag{3.3}$$

$$= \text{Ad}_{e^{\alpha(t)}} J = J_t. \tag{3.4}$$

Thus it follows from (3.2) that $(J_t, e^{z(t)} \cdot \omega)$ is an L -valued Kähler form on (M, J_t) . We shall construct a formal power series $\{b(t)\}$ with $b(t) \in E_1^{\mathbb{R}}$ which satisfies (3.2). The action of $E_1^{\mathbb{R}}$ on ω gives L -valued real forms of type $(1, 1)$ and the action of $E_0^{\mathbb{R}}$ on ω yields L -valued real forms of type $(0, 2)$ and type $(2, 0)$. Since ω is a non-degenerate 2-form at each point, it turns out that every form of type $(1, 1)$ is written as $b \cdot \omega$ for a section $b \in E_1^{\mathbb{R}}$. Thus L -valued real forms $\wedge_{\mathbb{R}}^{1,1} \otimes L$ of type $(1, 1)$ are given by $\{b \cdot \omega \mid b \in E_1^{\mathbb{R}}\}$ and we also have $(\wedge^{2,0} \oplus \wedge^{0,2})_{\mathbb{R}} \otimes L = \{a \cdot \omega \mid a \in E_0^{\mathbb{R}}\}$, where $(\wedge^{2,0} \oplus \wedge^{0,2})_{\mathbb{R}}$ is the real part of $\wedge^{2,0} \oplus \wedge^{0,2}$.

LEMMA 3.2. *We assume that (X, L) satisfies the $\partial\bar{\partial}$ -lemma at degree $(1, 2)$. If $d_L\alpha$ is a real d_L -exact form of $(\wedge^{2,1} \oplus \wedge^{1,2})_{\mathbb{R}} \otimes L$ for a real $\alpha \in \wedge^2 \otimes L$, then there is a real L -valued form β of type $(1, 1)$ such that $d_L\beta = d_L\alpha$, where β is written as $b \cdot \omega$ for $b \in E_1^{\mathbb{R}}$.*

PROOF. Let $\alpha = \alpha^{2,0} + \alpha^{1,1} + \alpha^{0,2}$ be the decomposition of α , where $\alpha^{p,q} \in \wedge^{p,q} \otimes L$ and $\alpha^{2,0} = \overline{\alpha^{0,2}}$, since α is real. Since $d_L\alpha$ consists of forms of type $(2, 1)$ and $(1, 2)$, we have $\bar{\partial}_L\alpha^{0,2} = 0$ and $\partial_L\alpha^{2,0} = 0$. Since (X, L) satisfies the $\partial\bar{\partial}$ -lemma at degree $(1, 2)$, there is an L -valued form γ of type $(0, 1)$ such that

$$\partial_L\bar{\partial}_L\gamma = \partial_L\alpha^{0,2}.$$

Then it follows that $\bar{\partial}_L\partial_L\bar{\gamma} = \bar{\partial}_L\alpha^{2,0}$. We define $\beta \in \wedge_{\mathbb{R}}^{1,1} \otimes L$ by $\beta = -\partial_L\gamma - \bar{\partial}_L\bar{\gamma} + \alpha^{1,1}$. Then β satisfies that

$$d_L\beta = d_L\alpha. \tag{□}$$

In order to obtain an estimate of β which satisfies $d_L\beta = d_L\alpha$, we use the Hodge theory of the following L -valued complex:

$$\wedge_{\mathbb{R}}^{1,1} \otimes L \xrightarrow{d_L} (\wedge^{2,1} \oplus \wedge^{1,2})_{\mathbb{R}} \otimes L \xrightarrow{d_L} (\wedge^{3,1} \oplus \wedge^{2,2} \oplus \wedge^{1,3})_{\mathbb{R}} \otimes L \xrightarrow{d_L} \dots \tag{3.5}$$

The complex (3.5) is elliptic at the terms $(\wedge^{2,1} \oplus \wedge^{1,2})_{\mathbb{R}} \otimes L$ and $(\wedge^{3,1} \oplus \wedge^{2,2} \oplus \wedge^{1,3})_{\mathbb{R}} \otimes L$. We take a Riemannian metric on M and a hermitian metric on L . Then we have the formal adjoint d_L^* of d_L and the Laplacian Δ_L and the Green operator G_L which acts on the term $(\wedge^{2,1} \oplus \wedge^{1,2})_{\mathbb{R}} \otimes L$. Then we define $\beta \in \wedge_{\mathbb{R}}^{1,1} \otimes L$ by

$$\beta = d_L^*Gd_L\alpha.$$

Then it follows from Lemma 3.2 that $d_L\alpha$ is contained in the image $d_L(\wedge_{\mathbb{R}}^{1,1} \otimes L)$. Thus by applying the Hodge theory to the elliptic complex (3.5), we obtain $d_L\beta = d_L\alpha$ and by using the elliptic estimate, we have

$$\|\beta\|_s \leq C\|\alpha\|_s,$$

where $\|\cdot\|_s$ denotes the Sobolev norm for sufficiently large $s > 0$. We shall use this estimate to show the convergence of the power series in next section.

PROOF OF THEOREM 3.1 AND THEOREM 2.2. We shall obtain a power series $b(t)$ by the induction on the degree k of t . The first term $a_1 \in E_0^{\mathbb{R}}$ is decomposed into a_1' and a_1'' , where $a_1' \in \wedge^{0,1} \otimes T^{1,0}$ and $a_1'' = \overline{a_1'}$. The component a_1' is $\bar{\partial}$ -closed which gives the Kodaira-Spencer class $[a_1'] \in H^1(X, T^{1,0})$ of deformations $\{J_t\}$. Let $d_L(e^{z(t)} \cdot \omega)_{[k]}$ be the term of $d_L(e^{z(t)} \cdot \omega)$ of degree k in t . The first term is given by

$$d_L(e^{z(t)} \cdot \omega)_{[1]} := d_L(a_1 \cdot \omega) + d_L(b_1 \cdot \omega) = 0 \tag{3.6}$$

Since $\bar{\partial}a_1' = 0$ and $a_1'' = \overline{a_1'}$, we have $d_L(a_1 \cdot \omega) \in (\wedge^{2,1} \oplus \wedge^{1,2})_{\mathbb{R}} \otimes L$. Then it follows from Lemma 3.2 that there is a $\beta \in \wedge_{\mathbb{R}}^{1,1} \otimes L$ such that $d_L(a_1 \cdot \omega) + d_L\beta = 0$. Then β is written as $b_1 \cdot \omega$ for $b_1 \in E_1^{\mathbb{R}}$. Thus we obtain that $d_L(a_1 \cdot \omega) + d_L(b_1 \cdot \omega) = 0$. Next we consider an operator $e^{-z(t)} \circ d_L \circ e^{z(t)}$ acting on L -valued differential forms. It follows that the operator $e^{-z(t)} \circ d_L \circ e^{z(t)}$ is locally written as a composition of the Lie derivative and the interior, the exterior product

$$e^{-z(t)} \circ d_L \circ e^{z(t)} = \sum_j \theta^j \wedge \mathcal{L}_{v_j} + N_j, \tag{3.7}$$

where θ^j is the exterior product by a 1-form θ^j and \mathcal{L}_{v_j} is the Lie derivative by the vector v_j and N_j is an element of $\wedge^2 \otimes TM$ which acts on differential forms by the interior and the exterior product (cf. [7, Lemma 2.4 and 2.7, page 221–223] and Definition 2.2 in [8] for more detail). We take an open covering $\{U_\alpha\}$ of M such that there is a nowhere-vanishing holomorphic n -form Ω_α on each U_α . We denote by Φ_α the pair (Ω_α, ω) consisting of Ω_α and ω . Then the diagonal action of the Lie derivative of the pair Φ_α by a vector v_j is given by

$$\mathcal{L}_{v_j}\Phi_\alpha = (\mathcal{L}_{v_j}\Omega_\alpha, \mathcal{L}_{v_j}\omega) = (a_{\alpha,j} \cdot \Omega_\alpha, a_{\alpha,j} \cdot \omega),$$

for $a_{\alpha,j} \in \text{End}(TM) = \wedge^1 \otimes TM$. Thus it follows from (3.7) that there is a section $h_\alpha \in \wedge^2 \otimes TM$ such that

$$e^{-z(t)} \circ d_L \circ e^{z(t)} \cdot \Phi_\alpha = (h_\alpha \cdot \Omega_\alpha, h_\alpha \cdot \omega),$$

where $h_\alpha = \sum_j \theta^j \wedge a_{\alpha,j} + N_j \in \wedge^2 \otimes TM$. Since $J_t = \text{Ad}_{e^{z(t)}} J$, we see that $e^{z(t)} \cdot \Omega_\alpha$ is a form of type $(n, 0)$ with respect to the complex structure J_t . Since J_t is integrable, it turns out that $d e^{z(t)} \cdot \Omega_\alpha$ is a form of type $(n, 1)$ with respect to the complex structure J_t . Thus we have that $e^{-z(t)} \circ d_L \circ e^{z(t)} \cdot \Omega_\alpha \in \wedge^{n,1}$ with respect to J . It follows that the component of h_α in $\wedge^{0,2} \otimes T^{1,0}$ must vanish. Since h_α is real, the component of h_α in $\wedge^{2,0} \otimes T^{0,1}$ vanishes also. Thus $h_\alpha \cdot \omega$ is contained in $(\wedge^{2,1} \oplus \wedge^{1,2})_{\mathbb{R}} \otimes L$ and it implies that $e^{-z(t)} \circ d_L \circ e^{z(t)} \cdot \omega = h_\alpha \cdot \omega \in (\wedge^{2,1} \oplus \wedge^{1,2})_{\mathbb{R}} \otimes L$. We assume that we

already have $b_1, \dots, b_{k-1} \in E_1^{\mathbb{R}}$ such that $d_L(e^{b(t)} \cdot \omega)_{[i]} = 0$, for all $0 \leq i \leq k - 1$, where $b(t) = \sum_i b_i(t^i/i!)$. Since $e^{z(t)} = e^{a(t)} \circ e^{b(t)}$, we have

$$\begin{aligned} d_L(e^{z(t)} \cdot \omega)_{[k]} &= \sum_{\substack{i+j=k \\ \mathbb{B}i, j \geq 0}} (e^{-z(t)})_{[j]} d_L(e^{z(t)} \cdot \omega)_{[i]} \\ &= (e^{-z(t)} \circ d_L \circ e^{z(t)} \cdot \omega)_{[k]} \end{aligned}$$

Then it follows that $d_L(e^{z(t)} \cdot \omega)_{[k]} = (h_\alpha \cdot \omega)_{[k]}$ is of type $(2, 1)$ and $(1, 2)$. The k th term $d_L(e^{z(t)} \cdot \omega)_{[k]}$ is written as the sum of the linear term and the non-linear term Ob_k

$$d_L(e^{z(t)} \cdot \omega)_{[k]} = \frac{1}{k!} d_L(b_k \cdot \omega) + \text{Ob}_k,$$

where Ob_k depends only on $a(t)$ and b_1, \dots, b_{k-1} which is d_L -exact. Thus it follows from Lemma 3.2 that there is a $b_k \in E_1^{\mathbb{R}}$ such that $d_L(e^{z(t)} \cdot \omega)_{[k]} = (1/k!)d_L(b_k \cdot \omega) + \text{Ob}_k = 0$. By our assumption of the induction, we have a solution $b(t)$ in the form of the formal power series. We shall show that the $b(t)$ is a convergent series which gives a smooth family of L -valued Kähler forms $\{\omega_t\} = \{e^{z(t)} \cdot \omega\}$ by the same method as in [7], [9], [10], [11] in next section. Thus the result follows.

As in the proof of Theorem 2.2, Theorem 2.3 is equivalent to the following:

THEOREM 3.3. *Let (X, L) , ω and $\{L_s\}$ be as in Theorem 2.3. Then there is a positive constant $\varepsilon' < \varepsilon$ and an analytic family of L_s -valued 2-forms $\{\omega_s\}$ which satisfies the followings:*

- (i) $\omega_0 = \omega$,
- (ii) ω_s is an L_s -valued Kähler on (M, J) for all s ,

where $|s| < \varepsilon'$.

PROOF OF THEOREM 3.3 AND THEOREM 2.4. Deformations of flat line bundles $\{L_s\}$ are given by a family of elements in $H^1(X, \mathbb{R})$ which are written as a family of smooth d -closed 1-forms $\{\eta_s\}$ with $\eta_0 = 0$. Then $d_L + \eta_s$ is the differential operator and a flat section of L_s is given by a section σ of L with $(d_L + \eta_s)\sigma = 0$. As before, we see that an L_s -valued Kähler form is given by an L -valued Hermitian form ω_s such that $(d_L + \eta_s)\omega_s = 0$.

Thus we shall obtain a family of section $b(s)$ with $b(s) \in E_1^{\mathbb{R}}$ such that $(d_L + \eta_s)(e^{b(s)} \cdot \omega) = 0$. The first term of the equation is given by

$$d_L(b_1 \cdot \omega) + \eta_1 \cdot \omega = 0.$$

Thus the class $[\eta_1 \cdot \omega] = [\eta_1 \wedge \omega] \in H^3(X, L)$ must vanish if there is a family of L_s -valued Kähler forms. Hence Theorem 2.4 is proved.

From our assumption in Theorem 3.3, we have $H^3(X, L) = \{0\}$. Thus it follows that $\eta_1 \wedge \omega$ is d_L -exact real form of type $(2, 1)$ and $(1, 2)$. Then applying Lemma 3.2,

we obtain $b_1 \in E_1^{\mathbb{R}}$ such that $d_L b_1 \cdot \omega + \eta_1 \cdot \omega = 0$. We shall use the induction on k . We assume that we already have b_1, \dots, b_{k-1} such that $(d_L + \eta_s)(e^{b(s)} \cdot \omega)_{[i]} = 0$ for all $0 \leq i < k$. Then we have

$$(d_L + \eta_s)(e^{b(s)} \cdot \omega)_{[k]} = (e^{-b(s)} \circ (d_L + \eta_s) \circ e^{b(s)} \cdot \omega)_{[k]}$$

Since $b(s) \in E_1^{\mathbb{R}}$, there is a h_α on each open set U_α as before such that $e^{-b(s)} \circ d_L \circ e^{b(s)} \cdot \omega = h_\alpha \cdot \omega \in (\wedge^{2,1} \oplus \wedge^{1,2})_{\mathbb{R}} \otimes L$. Further $e^{-b(s)} \circ \eta_s \circ e^{b(s)}$ is also a 1-form and then we see that $(d_L + \eta_s)(e^{b(s)} \cdot \omega)_{[k]} \in (\wedge^{2,1} \oplus \wedge^{1,2}) \otimes L$. The term $(d_L + \eta_s)(e^{b(s)} \cdot \omega)_{[k]}$ is given by

$$(d_L + \eta_s)(e^{b(s)} \cdot \omega)_{[k]} = \frac{1}{k!} d_L b_k \cdot \omega + \text{Ob}_k(\eta_s).$$

It follows from our assumption that $(d_L e^{b(s)} \cdot \omega)_{[j]} = -(\eta_s \wedge e^{b(s)} \cdot \omega)_{[j]}$ for $j = 0, 1, \dots, k-1$. Since η_s is a d -closed 1-form and its degree is greater than or equal to one, we have

$$d_L \circ (d_L + \eta_s)(e^{b(s)} \cdot \omega)_{[k]} = -(\eta_s \wedge d_L e^{b(s)} \cdot \omega)_{[k]} \tag{3.8}$$

$$= - \sum_{\substack{i+j=k \\ i \geq 1}} (\eta_s)_{[i]} (d_L e^{b(s)} \cdot \omega)_{[j]} \tag{3.9}$$

$$= \sum_{\substack{i+j=k \\ i \geq 1}} (\eta_s)_{[i]} \wedge (\eta_s \wedge e^{b(s)} \cdot \omega)_{[j]} \tag{3.10}$$

$$= (\eta_s \wedge \eta_s \wedge e^{b(s)})_{[k]} = 0. \tag{3.11}$$

Thus $\text{Ob}_k(\eta_s)$ is d_L -closed which gives a class of $H^3(X, L)$. From our assumption $H^3(X, L) = \{0\}$, $\text{Ob}_k(\eta_s)$ is a d_L -exact form of type $(2, 1)$ and $(1, 2)$. Then applying Lemma 3.2, we have $b_k \in E_1^{\mathbb{R}}$ such that $(1/k!)d_L b_k + \text{Ob}_k(\eta_s) = 0$. Thus from our assumption of the induction, we have a solution $b(s)$ in the form of the formal power series which can be shown to be a convergent series. We obtain a smooth family of L_s -valued Kähler forms $\{\omega_s\}$. □

Theorem 2.5 is also equivalent to the following:

THEOREM 3.4. *Let (X, L) , $\{J_t\}$ and $\{L_s\}$ as in Theorem 2.5. Then there is a positive constant $\varepsilon' < 1$ and an analytic 2-parameter family of L_s -valued 2 form $\{\omega_{s,t}\}$ which satisfies the followings:*

- (i) $\omega_{0,0} = \omega$,
- (ii) $\omega_{s,t}$ is an L_s -valued Kähler on (M, J_t) for all s ,

where $|s|, |t| < \varepsilon'$.

PROOF OF THEOREM 3.4. We shall use the same method as before. Deformations of complex structures $\{J_t\}$ is given by a family of endomorphisms $\{a_t\}$ with $a_t \in E_0^{\mathbb{R}}$ and deformations of flat line bundles $\{L_s\}$ is also described by a family of d -closed 1-forms

$\{\eta_s\}$. Then we shall construct a 2-parameter family of $\{b(s, t)\}$ with $b(s, t) \in E_1^{\mathbb{R}}$ such that $(d_L + \eta_s)(e^{a(t)} \circ e^{b(s, t)} \cdot \omega) = 0$. The first term of the equation in t is given by

$$d_L(a_1 \cdot \omega) + d_L(b_1 \cdot \omega) + \eta_1 \wedge \omega = 0$$

Since $d_L a_1 \cdot \omega \in \wedge^{2,1} \oplus \wedge^{1,2}$ and $[\eta_1 \wedge \omega] = 0 \in H^3(X, L)$ and $\eta_1 \wedge \omega \in (\wedge^{2,1} \oplus \wedge^{1,2})_{\mathbb{R}}$, we have a solution $b_1 \in E_1^{\mathbb{R}}$ of the first equation from Lemma 3.2. We also have a solution b_k of the k th term of the equation and the solution $b(t)$ by the same method as in the proof of Theorem 3.1. □

REMARK 3.5. If an l.c.k structure is given as an L -valued Kähler form with potential:

$$\omega = \sqrt{-1} \partial_L \bar{\partial}_L \phi,$$

for an L -valued function ϕ , then obstructions to deformations vanish. In fact, we can solve the equation (3.6) explicitly. It follows from $\bar{\partial} a'_1 = 0$ and $a''_1 = \bar{a}'_1$ that we have

$$d_L(a_1 \cdot \omega) = \sqrt{-1} d_L(a'_1 + a''_1) \cdot \partial_L \bar{\partial}_L \phi \tag{3.12}$$

$$= \sqrt{-1} d_L(\partial_L a'_1 \partial_L \phi - \bar{\partial}_L a''_1 \bar{\partial}_L \phi). \tag{3.13}$$

Thus $b_1 = \sqrt{-1}(\partial_L a'_1 \partial_L \phi - \bar{\partial}_L a''_1 \bar{\partial}_L \phi) \in \wedge_{\mathbb{R}}^{1,1} \otimes L$ gives a solution of (3.6). This is consistent with the result by Ornea and Verbitsky [20] that l.c.k structures with potential are stable under small deformations of complex structures.

REMARK 3.6. It is worth to note that our method gives a proof of the stability of Kähler structures which is different from the original proof by Kodaira-Spencer. The method by Kodaira-Spencer depends on the fact that a Kähler form is harmonic and they applied the harmonic projection to obtain the stability of Kähler structures. An l.c.k structure is, however not harmonic and we can not apply the method by Kodaira-Spencer to obtain the stability of l.c.k structures.

4. The convergence.

This section is devoted to show that both power series $b(t)$ and $z(t)$ in Proof of Theorem 3.1 of Section 3 are convergent series. We will use the almost same method as in [9] to apply the elliptic estimate of the Green operator. We develop an estimate of the obstruction Ob in Section 3 which includes the higher order term. (Note that the obstruction term in the Kodaira-Spencer theory is quadratic.) We will use the induction on the degree k of the power series in t . At first we will estimate the first terms b_1 and z_1 of power series $b(t)$ and $z(t)$ in (3.6). Next we assume that $b(t)$ and $z(t)$ satisfy the following inequalities (4.12) and (4.13) respectively. Then we will show that $b(t)$ satisfies the inequality (4.4) and then obtain the inequality (4.5).

We shall fix our notation. We denote by $\|f\|_s = \|f\|_{C^{s,\alpha}}$ the Hölder norm of a section f of a bundle with respect to a metric. Then we have an inequality, $\|fg\|_s \leq C_s \|f\|_s \|g\|_s$,

where f, g are sections and C_s is a constant. We denote by (K^\bullet, d_L) the elliptic complex (3.5) in Section 3:

$$K^1 \xrightarrow{d_L} K^2 \xrightarrow{d_L} K^3 \xrightarrow{d_L} \dots,$$

where $K^1 = \wedge_{\mathbb{R}}^{1,1} \otimes L, K^2 = (\wedge^{2,1} \oplus \wedge^{1,2})_{\mathbb{R}} \otimes L$ and $K^3 = (\wedge^{3,1} \oplus \wedge^{2,2} \oplus \wedge^{1,3})_{\mathbb{R}} \otimes L$.

We use the Schauder estimates of the elliptic operators with respect to the complex (K^\bullet, d_L) with a constant C_K . Let $P(t)$ be a formal power series in t . We denote by $(P(t))_{[k]}$ the k th coefficient of $P(t)$ and given two power series $P(t)$ and $Q(t)$, if $(P(t))_{[k]} < (Q(t))_{[k]}$ for all k , we denote it by $P(t) \ll Q(t)$. For a positive integer k , if $(P(t))_{[i]} < (Q(t))_{[i]}$ for all $i \leq k$, then we write it by $P(t) \ll_k Q(t)$. We also consider a formal power series $f(t)$ in t whose coefficients are sections of a bundle. Then we put $\|f(t)\|_s = \sum_i \|(f(t))_{[i]}\|_s t^i$. We define a convergent power series $M(t)$ by

$$M(t) = \sum_{\nu=1}^{\infty} \frac{1}{16c} \frac{(ct)^\nu}{\nu^2} = \sum_{\nu=1}^{\infty} M_\nu t^\nu,$$

where c is a constant. The series $M(t)$ is used in [15] and it turns out that the series $M(t)$ satisfies

LEMMA 4.1. $M(t)^2 \ll (1/c)M(t)$.

We put $\lambda = 1/c$. Then it follows from lemma 4.1 that $(1/l!)M(t)^l \ll (1/l!)\lambda^{l-1}M(t) = (\lambda^l/l!)(1/\lambda)M(t)$. Hence we have

LEMMA 4.2. $e^{M(t)} \ll (1/\lambda)e^\lambda M(t)$.

As in Section 3, the power series $z(t)$ is defined by the Campbel-Hausdorff formula,

$$e^{z(t)} = e^{a(t)}e^{b(t)},$$

where $z(t) = \sum_{l=0}^{\infty} (t^l/l!)z_k$ and $e^{z(t)} = \sum_{j=0}^{\infty} (1/j!)z(t)^j$. The norm of $a(t)$ in (3.1) is written as $\|a(t)\|_s = \sum_{l=1}^{\infty} (1/l!)\|a_k\|_s t^l$. Then we can assume that $\|a(t)\|_s$ satisfies

$$\|a(t)\|_s \ll K_1 M(t), \tag{4.1}$$

for a non-zero constant K_1 and λ if we take $a(t)$ sufficiently small. We will show that there exist constants K_1, K_2 and λ such that we have the following inequalities,

$$\|b(t)\|_s \ll K_2 M(t), \tag{4.2}$$

$$\|z(t)\|_s \ll M(t) \tag{4.3}$$

for sufficiently small $a(t)$. Note that K_1, K_2 and λ are determined by $a(t)$, the complex structure J and ω which do not depend on $b(t)$ and $z(t)$. The inequalities (4.2) and (4.3)

are reduced to the infinitely many inequalities on degree k

$$\|b(t)\|_s \ll_k K_2 M(t), \tag{4.4}$$

$$\|z(t)\|_s \ll_k M(t) \tag{4.5}$$

We will show both inequalities (4.4) and (4.5) by the induction on k . In this section we denote by C_i constants which do not depend on $z(t)$, $b(t)$ and k but depend on $a(t)$, J and ω . For $k = 1$, as in (3.6) of Section 3, $b_1 \cdot \omega$ satisfies the equation,

$$d_L(b_1 \cdot \omega) + d_L(a_1 \cdot \omega) = 0, \quad (b_1 \cdot \omega \in K^1)$$

Then it follows from Lemma 3.2 that $b_1 \cdot \omega$ is given by

$$b_1 \cdot \omega = -d_L^* G_L(d_L a_1 \cdot \omega), \tag{4.6}$$

where d_L^* is the adjoint operator and G_L is the Green operator of the complex (K^\bullet, d_L) . It follows from the Schauder estimate of the elliptic operators that

$$\|b_1 \cdot \omega\|_s \leq C_K \|a_1 \cdot \omega\|_s \leq C_K C_s \|a_1\|_s \|\omega\|_s \leq \frac{1}{16} C_1 K_1, \tag{4.7}$$

where $\|a_1\|_s \leq K_1 M_1 = K_1/16$ and $C_1 = C_K C_s \|\omega\|_s$. Let $\ker \omega$ be the subbundle of $E_1^{\mathbb{R}}$ which is defined by $\ker \omega = \{\gamma \in E_1^{\mathbb{R}} \mid \gamma \cdot \omega = 0\}$. We define b_1 to be a section of the orthogonal complement $(\ker \omega)^\perp$ of $\ker \omega$ in $E_1^{\mathbb{R}}$. Then we have

$$\|b_1\|_s \leq C_2 \|b_1 \omega\|. \tag{4.8}$$

Substituting (4.7) into (4.8), we have

$$\|b_1\|_s \leq \frac{1}{16} C_1 C_2 K_1 = M_1 C_1 C_2 K_1 \tag{4.9}$$

Thus if we take K_2 with $C_1 C_2 K_1 < K_2$, then we have

$$\|b_1\|_s \leq K_2 M_1, \tag{4.10}$$

Since $z_1 = a_1 + b_1$, if we take K_1 and K_2 satisfying $K_1 + K_2 < 1$, we have

$$\begin{aligned} \|z_1\|_s &\leq \|a_1\|_s + \|b_1\|_s \\ &\leq K_1 M_1 + K_2 M_1 \\ &= (K_1 + K_2) M_1 < M_1 \end{aligned} \tag{4.11}$$

It follows from (4.10), (4.11) that we have inequalities (4.4) and (4.5) for $k = 1$.

We assume that the following inequalities hold

$$\|b(t)\|_{k-1} \ll K_2 M(t) \tag{4.12}$$

$$\|z(t)\|_{k-1} \ll M(t). \tag{4.13}$$

Let Ob_k be the higher order term in Section 3. Then we have

LEMMA 4.3. $\text{Ob}_k = \text{Ob}_k(a_1, \dots, a_{k-1}, b_1, \dots, b_{k-1})$ satisfies the following inequality,

$$\|\text{Ob}_k\|_{s-1} \leq C(\lambda)M_k,$$

where $C(\lambda)$ satisfies $\lim_{\lambda \rightarrow 0} C(\lambda) = 0$.

PROOF. The obstruction Ob_k is given by

$$\text{Ob}_k = \sum_{l=2}^k \frac{1}{l!} (\text{ad}_{z(t)}^l d_L) \omega.$$

We also have $\|[d_L, z(t)]\omega\|_{s-1} \ll 2\|z(t)\omega\|_s$. Since $(\text{ad}_{z(t)}^l d_L) = [\text{ad}_{z(t)}^{l-1} d_L, z(t)]$, we find that

$$\|(\text{ad}_{z(t)}^l d_L)_{[k]} \omega\|_{s-1} \leq 2(2C_s)^l (\|z(t)\|_s^l \|\omega\|_s)_{[k]} \tag{4.14}$$

Hence it follows that

$$\|\text{Ob}_k\|_{s-1} = \sum_{l=2}^k \frac{1}{l!} \|(\text{ad}_{z(t)}^l d_L)_{[k]} \omega\|_{s-1} \tag{4.15}$$

$$\leq \sum_{l=2}^k \frac{1}{l!} 2(2C_s)^l (\|z(t)\|_s^l \|\omega\|_s)_{[k]} \tag{4.16}$$

Since the degree of $z(t)$ is greater than or equal to 1, it follows from our assumption (4.13) and $l \geq 2$ that we have

$$(\|z(t)\|_s^l)_{[k]} \leq (M(t))_{[k]}^l. \tag{4.17}$$

(Note that $(\|z(t)\|_s^l)_{[k]}$ consists of the term $\|z_i\|_s$, for $i < k$.) Substituting (4.17) into (4.16) and using lemma 4.2, we obtain

$$\| \text{Ob}_k \|_{s-1} \leq \sum_{l=2}^k \frac{1}{l!} 2(2C_s)^l (M(t)^l)_{[k]} \|\omega\|_s, \tag{4.18}$$

$$\leq C_3 \sum_{l=2}^k \frac{1}{l!} (2C_s)^l \lambda^{l-1} M_k \tag{4.19}$$

$$\leq C_3 \lambda^{-1} (e^{2C_s \lambda} - 1 - 2C_s \lambda) M_k \tag{4.20}$$

$$= C(\lambda) M_k.$$

where $C_3 = 2\|\omega\|_s$. Then it follows the constant $C(\lambda)$ satisfies

$$\lim_{t \rightarrow 0} C(\lambda) = 0. \tag{□}$$

LEMMA 4.4. $\|b(t)\|_s \ll_k K_2 M(t).$

PROOF. In Section 3, b_k is defined as the solution of the equation,

$$\frac{1}{k!} d_L(b_k \cdot \omega) + \frac{1}{k!} d_L(a_k \cdot \omega) + \text{Ob}_k = 0 \tag{4.21}$$

In fact $b_k \cdot \omega$ is given by

$$\frac{1}{k!} b_k \cdot \omega = -G_L d_L^*(\text{Ob}_k) - G_L d_L^* \left(\frac{1}{k!} a_k \cdot \omega \right) \tag{4.22}$$

Thus it follows from (4.8) and the Schauder estimate that

$$\left\| \frac{1}{k!} b_k \right\|_s \leq C_2 C_K \| \text{Ob}_k \|_{s-1} + C_2 C_K \left\| \frac{1}{k!} a_k \cdot \omega \right\|_s \tag{4.23}$$

Applying lemma 4.3 and (4.1) to (4.23), we have

$$\begin{aligned} \left\| \frac{1}{k!} b_k \right\|_s &\leq C_2 C_K C(\lambda) M_k + C_s C_2 C_K K_1 M_k \|\omega\|_s \\ &\leq (C_4 C(\lambda) + C_5 K_1) M_k \end{aligned} \tag{4.24}$$

where $C_4 = C_2 C_K$ and $C_5 = C_s C_2 \|\omega\|_s$. Then from (4.9) and (4.24) if we take K_2 as

$$K_2 := \max\{C_2 C_1 K_1, (C_4 C(\lambda) + C_5 K_1)\}, \tag{4.25}$$

then we have the inequality,

$$\|b(t)\|_s \ll_k K_2 M(t) \tag{4.26}$$

□

Finally we estimate z_k . It follows that

$$(z(t))_{[k]} = \frac{1}{k!} z_k = \left(e^{z(t)} - 1 - \sum_{p=2}^k \frac{1}{p!} z(t)^p \right)_{[k]}.$$

Hence we have

$$\left\| \frac{1}{k!} z_k \right\|_s \leq \| (e^{z(t)} - 1)_{[k]} \|_s + \sum_{p=2}^k \frac{1}{p!} \| (z(t)^p)_{[k]} \|_s \tag{4.27}$$

From our assumption and (4.26),

$$\| a(t) \|_s \ll K_1 M(t), \quad \| b(t) \|_s \ll_k K_2 M(t).$$

Then it follows from Lemma 4.1 and Lemma 4.2 that

$$\| e^{a(t)} - 1 \|_s \ll \frac{1}{\lambda} (e^{K_1 \lambda} - 1) M(t). \tag{4.28}$$

We also have

$$\| e^{b(t)} - 1 \|_s \ll_k \frac{1}{\lambda} (e^{K_2 \lambda} - 1) M(t) \tag{4.29}$$

Then we obtain

LEMMA 4.5. $\| z(t) \|_s \ll_k M(t).$

PROOF. It follows from Lemma 4.2 and Lemma 4.1 that we have

$$\| e^{a(t)} \|_s \ll \frac{1}{\lambda} e^{K_1 \lambda} M(t).$$

Then substituting (4.28) and (4.29) into (4.30), we have

$$\| (e^{z(t)} - 1) \|_s \ll_k \| e^{a(t)} (e^{b(t)} - 1) \|_s + \| e^{a(t)} - 1 \|_s \tag{4.30}$$

$$\ll \frac{1}{\lambda} e^{K_1 \lambda} M(t) \frac{1}{\lambda} (e^{K_2 \lambda} - 1) M(t) + \frac{1}{\lambda} (e^{K_1 \lambda} - 1) M(t) \tag{4.31}$$

Applying lemma 4.1 again, we have

$$\| (e^{z(t)} - 1) \|_s \ll_k \left(e^{K_1 \lambda} \frac{1}{\lambda} (e^{K_2 \lambda} - 1) + \frac{1}{\lambda} (e^{K_1 \lambda} - 1) \right) M(t) \tag{4.32}$$

$$\ll_k C(K_1, K_2) M(t) \tag{4.33}$$

where $C(K_1, K_2)$ is a constant which depends only on K_1 and K_2 . Since $(z(t))^p_{[k]}$ consists of terms z_i for $i < k$, it follows from our assumption of the induction that the second term of (4.27) satisfies

$$\sum_{p=2}^k \frac{1}{p!} \|(z(t)^p)_{[k]}\|_s \leq \sum_{p=2}^k \frac{1}{p!} ((C_s M(t))^p)_{[k]} \tag{4.34}$$

$$\leq \frac{1}{\lambda} (e^{C_s \lambda} - 1 - C_s \lambda) M_k \tag{4.35}$$

$$= C_1(\lambda) M_k, \tag{4.36}$$

where $\lim_{\lambda \rightarrow 0} C_1(\lambda) = 0$. Thus if we take K_1, K_2, λ which satisfy

$$C(K_1, K_2) + C_1(\lambda) \leq 1, \tag{4.37}$$

it follows from (4.27) that

$$\frac{1}{k!} \|z_k\|_s \leq (C(K_1, K_2) + C_1(\lambda)) M_k \leq M_k \tag{4.38}$$

Thus $\|z(t)\|_s \ll_k M(t)$. □

If we take $a(t)$ sufficiently small, we can take K_1, K_2 and λ with $K_1 + K_2 < 1$ which satisfy (4.25) and (4.37). Hence by the induction, it turns out that $b(t)$ and $z(t)$ in section 3 are convergent series. Note that the convergence of the power series in the proof of Theorem 3.3 and Theorem 3.4 can be shown by the same method.

5. The stability of l.c.k structures on generalizations of Hopf manifolds.

5.1. L_λ -valued Dolbeault cohomology groups.

Let B be a compact Kähler manifold with semi-positive anti-canonical line bundle $-K_B$. We denote by E a negative holomorphic Hermitian line bundle over B . Let $E \setminus \{0\}$ be the complement of the zero section of E on which \mathbb{C}^\times acts by the multiplication on each fibre. We define a compact complex manifold X to be the quotient of the complement $E \setminus \{0\}$ by the group of automorphisms generated by the multiplication of the real constant α , where $\alpha \neq 0, 1$. Thus we have a principal fibre bundle $\pi : X \rightarrow B$ with fibre elliptic curve \mathbb{C}^*/\mathbb{Z} . The complex manifold X is regarded as a generalization of the standard Hopf manifolds. In fact, the tautological line bundle over the complex projective space $\mathbb{C}P^{n-1}$ gives rise to the standard Hopf manifolds as the quotients $\mathbb{C}^n \setminus \{0\} / \mathbb{Z}$. Let h be a Hermitian metric on E such that the curvature form of the Chern connection of h is negative. We have the function $\gamma := \|\sigma\|_h^2$ on the total space of E , where σ is a section of the line bundle E . Then an l.c.k metric ω_λ on X is given by

$$\omega_\lambda = \frac{-1}{2i} \frac{\partial \bar{\partial} \gamma^\lambda}{\gamma^\lambda}, \tag{5.1}$$

for a positive real number λ . In fact, ω_λ is positive-definite since E is a negative line bundle (Note that the curvature form is negative which is given by $-i\partial\bar{\partial}\log\gamma$ for a local non-zero holomorphic section σ of E .) The Lee form of η_λ of the l.c.k form ω_λ is given by

$$\eta_\lambda = -\lambda \frac{d\gamma}{\gamma}.$$

We denote by d_η the differential operator $d - \eta$. Then we have $d_{\eta_\lambda}\omega_\lambda = 0$. We also define operators ∂_{η_λ} and $\bar{\partial}_{\eta_\lambda}$, respectively by $\partial_{\eta_\lambda} = \partial - \eta^{1,0}$, $\bar{\partial}_{\eta_\lambda} = \bar{\partial} - \eta^{0,1}$, where we are using the decomposition $\eta = \eta^{1,0} + \eta^{0,1}$ and $\eta^{1,0}$ is a form of type $(1,0)$ and $\eta^{0,1} = \overline{\eta^{1,0}}$. Let $\rho_\lambda : \mathbb{Z} \rightarrow \mathbb{R}^\times$ be the representation of $\mathbb{Z} \cong \{\alpha^n \mid n \in \mathbb{Z}\}$ which is given by $\rho_\lambda(\alpha^n) \mapsto \alpha^{n\lambda}$, where $\lambda \in \mathbb{Z}$. Then the real flat line bundle L_λ over X is defined to be the quotient by the representation ρ_λ :

$$L_\lambda = E \setminus \{0\} \times_{\rho_\lambda} \mathbb{R}.$$

Then we see that ω_λ gives the L_λ -valued Kähler form $\tilde{\omega}_\lambda = (-1/2i)\partial\bar{\partial}\gamma^\lambda$. Let $\hat{\rho}_\lambda : \mathbb{C}^\times \rightarrow \mathbb{C}^\times$ be the representation given by $\hat{\rho}_\lambda(w) = w^\lambda$, where $w \in \mathbb{C}^\times$. Then the complex line bundle $L_{B,\lambda}$ over B is defined by

$$L_{B,\lambda} = E \setminus \{0\} \times_{\hat{\rho}_\lambda} \mathbb{C}.$$

We denote by \mathcal{L}_λ the holomorphic line bundle given by the real flat line bundle L_λ . Then we see that $\mathcal{L}_\lambda = \pi^*L_{B,\lambda}$ since $\hat{\rho}_\lambda$ is restricted to ρ_λ .

PROPOSITION 5.1. *Let λ be a positive real number. Then we have*

$$H^{0,k}(X, L_\lambda) = H^k(X, \mathcal{L}_\lambda) = \{0\}, \quad (k \geq 2)$$

PROOF. In order to obtain cohomology groups $H^k(X, \mathcal{L}_\lambda)$, we apply the Leray spectral sequence of the fibre bundle $\pi : X \rightarrow B$. The E_2 -term is given by

$$E_2^{p,q} = H^p(B, R^q\pi_*\mathcal{L}_\lambda), \quad (k = p + q).$$

Then it turns out that $R^q\pi_*\mathcal{L}_\lambda = 0$ if λ is not an integer. Thus we obtain $H^{0,k}(X, L_\lambda) = \{0\}$ if λ is not an integer. If λ is an integer, we already have that $\mathcal{L}_\lambda = \pi^*L_{B,\lambda}$. It follows from the projection formula that

$$R^q\pi_*\mathcal{L}_\lambda = R^q\pi_*\mathcal{O}_X \otimes L_{B,\lambda}.$$

(Note that \mathcal{L}_λ and $L_{B,\lambda}$ are regarded as sheaves of holomorphic sections.) Since the fibre is an elliptic curve, $R^q\pi_*\mathcal{L}_\lambda = 0$ for $q \neq 0, 1$. The form $\eta^{0,1}$ on X is $\bar{\partial}$ -closed which gives a global nowhere-vanishing section of $R^1\pi_*\mathcal{O}_X$. It implies that $R^1\pi_*\mathcal{O}_X = \mathcal{O}_B$. Thus it follows that the Leray spectral sequence degenerates at the E_2 -term. It follows that

$L_{B,-\lambda}$ is negative for $\lambda > 0$. Since $-K_B$ is semi-positive, $K_B \otimes L_{B,-\lambda}$ is negative also for $\lambda > 0$. Thus applying the Kodaira vanishing theorem, we have that $H^k(B, L_{B,\lambda}) \cong H^{n-1-k}(B, L_{B,-\lambda} \otimes K_B) = \{0\}$ and $H^{k-1}(B, L_{B,\lambda}) \cong H^{n-k}(B, L_{B,-\lambda} \otimes K_B) = \{0\}$, for $k = 2, \dots, n$. (Note that $\dim B = n - 1$.) Thus we have $H^k(X, \mathcal{L}_\lambda) = \{0\}$ for $k \geq 2$. \square

5.2. Bott-Chern cohomology groups.

A real function f on a complex manifold X is pluriharmonic if $\partial\bar{\partial}f = 0$. Let \mathcal{P} be the sheaf of real pluriharmonic functions. Then we have the short exact sequence:

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{P} \longrightarrow 0, \tag{5.2}$$

where \mathbb{R} denotes the real constant sheaf which is a subsheaf of the structure sheaf \mathcal{O}_X of X and the surjection $\mathcal{O}_X \rightarrow \mathcal{P}$ is given by taking the imaginary part of a holomorphic function. Let L be a real flat line bundle over X . The tensor product by L of the short exact sequence (5.2) induces a short exact sequence:

$$0 \longrightarrow L \longrightarrow \mathcal{L} \longrightarrow \mathcal{P} \otimes L \longrightarrow 0$$

We put $\mathcal{P}(L) := \mathcal{P} \otimes L$. Then we have the long exact sequence:

$$\dots \longrightarrow H^1(X, L) \longrightarrow H^1(X, \mathcal{L}) \longrightarrow H^1(X, \mathcal{P}(L)) \longrightarrow H^2(X, L) \longrightarrow \dots \tag{5.3}$$

Let $\wedge^{p,q}(L)$ be the sheaf of L -valued C^∞ forms of type (p, q) on X . We denote by $\wedge_{\mathbb{R}}^{p,p}(L)$ the real part of $\wedge^{p,p}(L)$. Then we have the exact sequence of sheaves:

$$0 \longrightarrow \mathcal{P}(L) \longrightarrow \wedge_{\mathbb{R}}^{0,0}(L) \xrightarrow{i\partial_L\bar{\partial}_L} \wedge_{\mathbb{R}}^{1,1}(L) \xrightarrow{d_L} (\wedge^{2,1}(L) \oplus \wedge^{1,2}(L))_{\mathbb{R}} \xrightarrow{d_L} \dots,$$

where $(\wedge^{2,1}(L) \oplus \wedge^{1,2}(L))_{\mathbb{R}}$ denotes the real part of $(\wedge^{2,1}(L) \oplus \wedge^{1,2}(L))$. Thus the cohomology group $H^1(X, \mathcal{P}(L))$ coincides with the real Bott-Chern cohomology group:

$$H^1(X, \mathcal{P}(L)) = \frac{\{\alpha \in \Gamma(X, \wedge_{\mathbb{R}}^{1,1}(L)) \mid d_L\alpha = 0\}}{\{i\partial_L\bar{\partial}_L f \mid f \in \Gamma(X, \wedge_{\mathbb{R}}^{0,0}(L))\}},$$

where $\Gamma(X, \wedge_{\mathbb{R}}^{1,1}(L))$ is the set of L -valued real C^∞ forms of type $(1, 1)$ and $\Gamma(X, \wedge_{\mathbb{R}}^{0,0}(L))$ is the set of L -valued C^∞ real functions on X^{*1} .

We recall Definition of Vaisman metric:

DEFINITION 5.2. A Vaisman metric is an l.c.k metric g whose Lee form is parallel with respect to the g .

Let $H^\bullet(Y, L)$ be the cohomology groups of the complex $(\wedge^\bullet \otimes L, d_L)$ on a compact differential manifold Y with a real flat line bundle L over Y . It was shown [18] that $H^k(Y, L)$ vanishes for all k on a locally conformally symplectic manifold Y if Y admits a

*1The author is grateful to Prof. Fujiki for this description of the Bott-Chern cohomology groups

compatible Riemannian metric on which the Lee form is parallel. Thus it is pointed out in [20] that

THEOREM 5.3 ([18]). *Let Y be a compact complex manifold with a Vaisman metric and L the corresponding flat line bundle. Then we have*

$$H^k(Y, L) = \{0\}, \quad \text{for all } k$$

Thus from the long exact sequence (5.3), we obtain

PROPOSITION 5.4. *Let Y be a compact complex manifold with a Vaisman metric. Then the L -valued Bott-Chern cohomology group is given by*

$$H^1(Y, \mathcal{P}(L)) = H^1(Y, \mathcal{L}) \equiv H^{0,1}(Y, L),$$

where $H^{0,1}(Y, L)$ denotes the L -valued Dolbeault cohomology group.

We shall apply Proposition 5.4 to the principal fibre bundle $\pi : X \rightarrow B$ in Subsection 5.1. We already show that the Leray spectral sequence degenerates at the E_2 -term and $R^1\pi_*\mathcal{O}_X$ is trivial (see Proof of Proposition 5.1). The E_2 -term of weight 1 is given by

$$E_2^{1,0} \cong H^1(B, L_{B,\lambda}), \quad E_2^{0,1} \cong H^0(B, L_{B,\lambda})$$

We already see that λ is a positive integer and $L_{B,\lambda}$ is a positive line bundle over B . Then applying the Kodaira vanishing theorem, we have $H^1(B, L_{B,\lambda}) \equiv H^{n-2}(B, K_B \otimes L_{B,-\lambda}) = \{0\}$. Thus we have $H^1(X, \mathcal{L}) \cong H^0(B, L_{B,\lambda})$. Then it follows from Proposition 5.4 that

PROPOSITION 5.5. *Let X be the complex manifold as in Subsection 5.1. Then we have*

$$H^1(X, \mathcal{P}(L_\lambda)) \equiv H^1(X, \mathcal{L}_\lambda) \cong H^0(B, L_{B,\lambda}).$$

REMARK 5.6. In our case that M is a fibre bundle $\pi : M \rightarrow B$ with fibre S^1 and η gives a generator of the first cohomology group of each fibre, we directly see that $H^k(X, L) = \{0\}$. In fact, applying the Leray spectral sequence to $p : M \rightarrow B$, we obtain that the E_2 -terms vanishes since $R^k\pi_*L = \{0\}$ for all k .

As before, the cohomology group $H^1(X, \mathcal{P}(L_\lambda))$ is written as

$$H^1(X, \mathcal{P}(L)) = \frac{\{\alpha \in \wedge_{\mathbb{R}}^{1,1} \mid d_{\eta_\lambda} \alpha = 0\}}{\{\partial_{\eta_\lambda} \bar{\partial}_{\eta_\lambda} f \mid f \in C^\infty(X, \mathbb{R})\}}.$$

We see that $\omega_\lambda + \alpha$ is an l.c.k form for a sufficiently small representative α of $H^1(X, \mathcal{P}(L))$. Thus, if the class $[\alpha] \in H^1(X, \mathcal{P}(L))$ does not vanish, the l.c.k structure $\omega_\lambda + \alpha$ does not have any potential. Then it follows from Proposition 5.5 that we have

PROPOSITION 5.7. *If $H^0(B, L_{B,\lambda})$ does not vanish, then the complex manifold X as in Subsection 5.1 admits l.c.k structures which do not have potential.*

In particular, if X is a standard Hopf manifold, $L_{B,\lambda}$ is the line bundle $\mathcal{O}(\lambda)$ over $\mathbb{C}P^{n-1}$. Then we have

$$\dim H^1(X, \mathcal{P}(L_\lambda)) = \dim H^0(\mathbb{P}^{n-1}, \mathcal{O}(\lambda)) = \binom{\lambda + n - 1}{n - 1}$$

where λ is a positive integer. Thus even a standard Hopf manifold admits a l.c.k structure which does not have potential.

Since $H^3(X, L_\lambda) = \{0\}$ and $H^2(X, \mathcal{L}_\lambda) = \{0\}$, it follows from Theorem 2.5 that we have the stability theorem of l.c.k structures on X .

PROPOSITION 5.8. *Let $\omega := \omega_\lambda + \alpha$ be the l.c.k structure on X , where ω_λ is the l.c.k structure in (5.1) of Subsection 5.1 and α is a representative of a class of the L_λ -valued Bott-Chern cohomology group $H^1(X, \mathcal{P}(L))$. Let $\{L_s\}$ be deformations of flat line bundles and $\{J_t\}$ deformations of complex structures on X as in Theorem 2.5. Then there is an analytic 2-parameter family of 2-forms $\{\omega_{s,t}\}$ such that $\omega_{s,t}$ is an l.c.k structure on (X, J_t) with the corresponding flat line bundle L_s .*

REMARK 5.9. Professor Ornea and Professor Verbitsky conjectured that every l.c.k. metric on a Vaisman manifold is an l.c.k. with potential (c.f. Conjecture 6.3 in [21]). Proposition 5.7 and Proposition 5.8 show that their conjecture does not hold.

6. Complex surfaces with effective anti-canonical line bundle.

Let ω be a locally conformally Kähler structure on a compact complex surface S and L the corresponding flat line bundle to ω . We denote by \mathcal{L} the holomorphic line bundle given by L . By the Serre duality, we have $H^2(S, \mathcal{L}) \cong H^0(S, K \otimes \mathcal{L}^{-1})$.

PROPOSITION 6.1. *We assume that the anti-canonical line bundle $-K$ is effective and $H^0(X, \mathcal{L}^{-1}) = \{0\}$. Then an l.c.k structure ω is stable under small deformations of S , that is, every small deformation of S admits a locally conformally Kähler structure.*

PROOF. It suffices to show that $H^2(S, \mathcal{L}) = \{0\}$. Since $-K$ is effective, K is an ideal sheaf and we have the injective map $H^0(S, K \otimes \mathcal{L}^{-1}) \rightarrow H^0(S, \mathcal{L}^{-1})$. Since $H^0(S, \mathcal{L}^{-1}) = \{0\}$, we have $H^0(S, K \otimes \mathcal{L}^{-1}) \cong H^2(S, \mathcal{L}) = \{0\}$. Thus the result follow from Theorem 2.3. □

7. Inoue surfaces with $b_2 = 0$.

These non-Kähler surfaces, called Inoue surfaces, were introduced by Inoue [13]. An Inoue surface S satisfies the following two conditions:

- (i) The first Betti number $b_1(S)$ is equal to one and the second Betti number $b_2(S)$ vanishes,
- (ii) S contains no curves.

There are three kinds of Inoue surfaces: S_M , $S_{N,p,q,r;t}^{(+)}$ and $S_{N,p,q,r}^{(-)}$, all of them being compact quotients of $\mathbb{H} \times \mathbb{C}$ by discrete groups of holomorphic automorphisms, where \mathbb{H} denotes the upper half plane of the complex numbers \mathbb{C} .

7.1. Inoue surfaces $S_{N,p,q,r;t}^{(+)}$

Inoue surface $S_{N,p,q,r;t}^{(+)}$ is the quotient of $\mathbb{H} \times \mathbb{C}$ by discrete group $G_{N,p,q,r;t}^{(+)}$

$$S_{N,p,q,r;t}^{(+)} = \mathbb{H} \times \mathbb{C} / G_{N,p,q,r;t}^{(+)}$$

We denote by (w, z) holomorphic coordinates of $\mathbb{H} \times \mathbb{C}$, where $w = w_1 + \sqrt{-1}w_2 \in \mathbb{H}$ and $z = z_1 + \sqrt{-1}z_2 \in \mathbb{C}$. The group $G_{N,p,q,r;t}^{(+)}$ is generated by the following analytic automorphisms:

$$\phi_0 : (w, z) \mapsto (\alpha w, z + t) \tag{7.1}$$

$$\phi_i : (w, z) \mapsto (w + a_i, z + b_i w + c_i), \quad i = 1, 2 \tag{7.2}$$

$$\phi_3 : (w, z) \mapsto \left(w, z + \frac{(b_1 a_2 - b_2 a_1)}{r} \right) \tag{7.3}$$

where a_i, b_i, c_i, r are real constants which are obtained by choosing a unimodular matrix $N = (n_{ij}) \in \text{SL}(2, \mathbb{Z})$ with two real eigenvalues $\alpha > 1$ and $1/\alpha$, and two real eigenvectors (a_1, a_2) and (b_1, b_2) corresponding to α and $1/\alpha$, respectively, and three integers p, q, r ($r \neq 0$). Two constants c_1 and c_2 are defined to be a solution of the following equation:

$$(c_1, c_2) = (c_1, c_2)N^t + (e_1, e_2) + \frac{(b_1 a_2 - b_2 a_1)}{r}(p, q),$$

where $e_i = (1/2)n_{i,1}(n_{i,1} - 1)a_1 b_1 + (1/2)n_{i,2}(n_{i,2} - 1)a_2 b_2 + n_{i,1}n_{i,2}b_1 a_1$, for $i = 1, 2$. Note that constants α, a_i, b_i, c_i, r are real. Then the action of $G_{N,p,q,r;t}^{(+)}$ on $\mathbb{H} \times \mathbb{C}$ is properly discontinuous and has no fixed points. We have a basis $\{\theta^1, \theta^2\}$ of invariant 1-forms of type $(1, 0)$ under the action of $G_{N,p,q,r;t}^{(+)}$:

$$\theta^1 = \frac{dw}{w_2}, \quad \theta^2 = \frac{z_2}{w_2} dw - dz, \tag{7.4}$$

Note that θ^1 and θ^2 are not holomorphic. We also have a basis of invariant vectors of type $(1, 0)$ under the action of $G_{N,p,q,r;t}^{(+)}$:

$$X_1 = w_2 \frac{\partial}{\partial w} + z_2 \frac{\partial}{\partial z}, \quad X_2 = -\frac{\partial}{\partial z} \tag{7.5}$$

Then we have

$$d\theta^1 = -\frac{dw_2}{w_2} \wedge \theta^1 = -\frac{\theta^1 - \bar{\theta}^1}{2i} \wedge \theta^1 = \frac{1}{2i} \bar{\theta}^1 \wedge \theta^1 \tag{7.6}$$

$$d\theta^2 = \theta^1 \wedge \frac{1}{2i}(\theta^2 - \bar{\theta}^2) \tag{7.7}$$

We define ω by $\omega = (1/i)(\theta^1 \wedge \bar{\theta}^1 + \theta^2 \wedge \bar{\theta}^2)$. Then we have

$$d\omega = \frac{dw_2}{w_2} \wedge \omega \tag{7.8}$$

Thus ω is an l.c.k structure with Lee form $\eta = dw_2/w_2$ [22]. We denote by d_η the operator $d - \eta$. Then $d_\eta \omega = 0$. We define $\tilde{\omega}$ by $\tilde{\omega} = (1/w_2)\omega$. Then we have $\phi_0^* \tilde{\omega} = \alpha^{-1} \tilde{\omega}$, $\phi_i^* \tilde{\omega} = \tilde{\omega}$ for $i = 1, 2, 3$ and $d\tilde{\omega} = 0$. Thus $\tilde{\omega}$ is a flat line bundle L -valued Kähler form. A holomorphic 2-form $\Omega := dw \wedge dz$ satisfies $\phi_0^* \Omega = \alpha \Omega$ and $\phi_i^* \Omega = \Omega$ for $i = 1, 2, 3$. Thus Ω gives a nowhere-vanishing L^{-1} -valued holomorphic 2-form. Hence we see that $K = L$. For simplicity, we denote by $S_t^{(+)}$ the Inoue surface $S_{N,p,q,r;t}^{(+)}$. Then we obtain

LEMMA 7.1. $H^{0,2}(S_t^{(+)}, L) \cong \mathbb{C}$.

PROOF. Since $K = L$, we have

$$H^{0,2}(S_t^{(+)}, L) \cong H^0(S_t^{(+)}, K \otimes L^{-1}) \cong H^0(S_t^{(+)}, \mathcal{O}) \cong \mathbb{C}. \quad \square$$

The Inoue surface $S_t^{(+)}$ admits a 1-dimensional deformations of complex structures and then the Kodaira-Spencer class is given by

$$[X_2 \otimes \bar{\theta}^1] = \left[\frac{\partial}{\partial z} \otimes \frac{d\bar{w}}{w_2} \right] \in H^1(S_t^{(+)}, \Theta)$$

The following complex is a key point of deformations of l.c.k structures which is equivalent to the complex (3.5). The obstruction to the stability of l.c.k structures arises as a cohomology class at the term $(\wedge^{2,1} \oplus \wedge^{1,2})_{\mathbb{R}}$:

$$\wedge_{\mathbb{R}}^{1,1} \xrightarrow{d_\eta} (\wedge^{2,1} \oplus \wedge^{1,2})_{\mathbb{R}} \xrightarrow{d_\eta} (\wedge^{3,1} \oplus \wedge^{2,2} \oplus \wedge^{1,3})_{\mathbb{R}} \xrightarrow{d_\eta} \dots \tag{7.9}$$

The Kodaira-Spencer class $[X_2 \otimes \bar{\theta}^1] \in H^1(S_t^{(+)}, \Theta)$ acts on ω by

$$(X_2 \otimes \bar{\theta}^1) \cdot \omega = \frac{1}{i}(\bar{\theta}^1 \wedge \bar{\theta}^2) \tag{7.10}$$

Then we have

$$d_\eta((X_2 \otimes \bar{\theta}^1) \cdot \omega) = \frac{2}{i(w_2)^2} dw_2 \wedge d\bar{w} \wedge d\bar{z} \tag{7.11}$$

$$= \theta^1 \wedge \bar{\theta}^1 \wedge \bar{\theta}^2 \tag{7.12}$$

$$= i\theta^1 \wedge \bar{\theta}^1 \wedge i(dz_1 - idz_2) \tag{7.13}$$

Then the real part and the imaginary part of $d_\eta((X_2 \otimes \bar{\theta}^1) \cdot \omega)$ give classes of the cohomology group of the complex (7.9), respectively.

PROPOSITION 7.2. *The class $[i\theta^1 \wedge \bar{\theta}^1 \wedge dz_1]$ vanishes, that is, the form $i\theta^1 \wedge \bar{\theta}^1 \wedge dz_1$ is written as $d_\eta \alpha$ for $\alpha \in \wedge_{\mathbb{R}}^{1,1}$. However the class $[i\theta^1 \wedge \bar{\theta}^1 \wedge dz_2]$ does not vanish, where we are considering the cohomology classes of the complex (7.9).*

This reflects the result by Belgun that the Inoue surface $S_{N,p,q,r;t}^{(+)}$ admits l.c.k metrics for $t \in \mathbb{R}$, however does not admit l.c.k metric for $t \in \mathbb{C} \setminus \mathbb{R}$.

PROOF. Since we have

$$d_\eta(\theta^1 \wedge \bar{\theta}^2) = d_\eta(\bar{\theta}^1 \wedge \theta^2) = \frac{1}{i}\theta^1 \wedge \bar{\theta}^1 \wedge dz_1,$$

the class $[i\theta^1 \wedge \bar{\theta}^1 \wedge dz_1]$ vanishes. We apply the Hodge theory to the complex (7.9). Let $(d_\eta)^*$ be the formal adjoint of the differential operator $\wedge_{\mathbb{R}}^{1,1} \xrightarrow{d_\eta} (\wedge^{2,1} \oplus \wedge^{1,2})_{\mathbb{R}}$ in terms of the Hermitian metric ω . Then the formal adjoint $(d_\eta)^*$ is given by

$$(d_\eta)^* = -\pi_{\wedge^{1,1}} \circ (*^{-1}d_{-\eta}*),$$

where $\pi_{\wedge^{1,1}}$ denotes the projection to forms of type (1, 1) and note that $d_{-\eta} = d + \eta$. Thus we have $(d_\eta)^*(i\theta^1 \wedge \bar{\theta}^1 \wedge dz_2) = 0$. Since the form $i\theta^1 \wedge \bar{\theta}^1 \wedge dz_2$ is harmonic, the class $[i\theta^1 \wedge \bar{\theta}^1 \wedge dz_2]$ does not vanish. □

The surface $S_t^{(+)}$ admits a 1-dimensional family of flat line bundles since $b_1(S_t^{(+)})$ is equal to 1. We already see that the corresponding flat line bundle L to the l.c.k structure ω on $S_t^{(+)}$ by Tricerri is the canonical line bundle K . It is natural to ask whether there is an l.c.k structure which admits a different corresponding flat line bundle. However we have

PROPOSITION 7.3. *Let ω' be an l.c.k structure on the Inoue surface $S := S_{N,p,q,r;t}^{(+)}$. Then the corresponding flat line bundle L' to ω' must be the canonical line bundle.*

PROOF. Since the Inoue surface S admits no curves, $H^0(S, K \otimes (L')^{-1}) = \{0\}$. If ω' gives a flat line bundle $L' \neq K$, then $H^2(S, L') \cong H^0(S, K \otimes (L')^{-1}) = \{0\}$. Then we can apply the stability theorem to obtain a family of l.c.k structures $\{\omega_t\}$, where $t \in \mathbb{C}$ is a parameter of deformations of complex structures. However it follows from Belgun's result that there is no such family of l.c.k forms. Thus L' must be K . □

7.2. Inoue surfaces S_M .

The Inoue surfaces S_M are also the quotient surfaces $S_M = \mathbb{H} \times \mathbb{C} / G_M$, where \mathbb{H} is the upper half plane of the complex numbers \mathbb{C} and G_M is a group of analytic automorphisms of $\mathbb{H} \times \mathbb{C}$. Let $M \in \text{SL}(3, \mathbb{Z})$ be a unimodular matrix with a real eigenvalue $\alpha > 1$ and two complex conjugate eigenvalues $\beta \neq \bar{\beta}$. Let (a_1, a_2, a_3) be a real eigenvector corresponding to α and (b_1, b_2, b_3) a eigenvector corresponding to β . Then G_M is the group generated

by the following automorphisms:

$$\phi_0(w, z) \mapsto (\alpha w, \beta z) \tag{7.14}$$

$$\phi_i(w, z) \mapsto (w + a_i, z + b_i), \quad \text{for } i = 1, 2, 3. \tag{7.15}$$

Then the action of G_M is properly discontinuous and has no fixed points. Thus $S_M = \mathbb{H} \times \mathbb{C}/G_M$ is a compact complex surface which is differentially a fibre bundle over the circle S^1 with the 3-torus as fibre. On S_M , an l.c.k form ω is defined by

$$\omega = -i \left(\frac{dw \wedge d\bar{w}}{(w_2)^2} + w_2 dz \wedge d\bar{z} \right).$$

A basis of invariant forms of type $(1, 0)$ is given by $\theta^1 = dw/w_2, \theta^2 = (w_2)^{1/2} dz$. Then ω is written as $\omega = -i(\theta^1 \wedge \bar{\theta}^1 + \theta^2 \wedge \bar{\theta}^2)$. By using $d\theta^1 = (1/2i)\bar{\theta}^1 \wedge \theta^1, d\theta^2 = (1/2)((\theta^1 - \bar{\theta}^1)/2i) \wedge \theta^2$, we have

$$d(\theta^1 \wedge \bar{\theta}^1) = 0, \quad d(\theta^2 \wedge \bar{\theta}^2) = \eta \wedge \theta^2 \wedge \bar{\theta}^2, \tag{7.16}$$

Thus ω is an l.c.k structure with Lee form $\eta = dw_2/w_2$.

We put $d_\eta = d - \eta$. Then we have $d_\eta \omega = 0$. As in the previous section, we have the complex by using the operator d_η :

$$\dots \xrightarrow{d_\eta} \wedge^k \xrightarrow{d_\eta} \wedge^{k+1} \xrightarrow{d_\eta} \dots$$

The cohomology group of the complex is denoted by $H_\eta^k(S_M)$. Then we already see that $H_\eta^k(S_M) \cong H^k(S_M, L)$. Then we obtain

LEMMA 7.4.

$$[\eta \wedge \omega] \neq 0 \in H_\eta^3(S_M) \cong H^3(S_M, L).$$

PROOF. We apply the Hodge theory to the complex (\wedge^\bullet, d_η) . Let d_η^* be the formal adjoint operator of d_η with respect to the Hermitian form ω . Then d_η^* is given by

$$d_\eta^* = (-1)^k *^{-1} (d_{-\eta})^* = (-1)^k *^{-1} (d + \eta)^*$$

where d_η^* acts on k -forms. Note that η changes into $-\eta$. Then we obtain

$$*(\eta \wedge \omega) = * \left(\frac{dw_2}{w_2} \wedge \omega \right) = -\frac{dw_1}{w_2} \tag{7.17}$$

$$d_{-\eta} * (\eta \wedge \omega) = -(d + \eta) \frac{dw_1}{w_2} \tag{7.18}$$

$$= \frac{dw_2}{w_2} \wedge \frac{dw_1}{w_2} - \frac{dw_2}{w_2} \wedge \frac{dw_1}{w_2} = 0 \tag{7.19}$$

Thus $\eta \wedge \omega$ is a harmonic form and the class $[\eta \wedge \omega] \in H^3_\eta(S_M)$ does not vanish. \square

Thus we apply Theorem 2.4 and obtain

PROPOSITION 7.5. *Let ω be the l.c.k structure on S_M and L the corresponding flat line bundle to ω . We denote by $\{L_s\}$ deformations of flat line bundles with $L_0 = L$ which is given by d -closed 1-forms $\{\eta_s\}$, where $[\eta] \neq 0 \in H^1(S_M)$. Then S_M does not admit a smooth family of l.c.k structures $\{\omega_s\}$ such that L_s is the corresponding flat line bundle to ω_s .*

7.3. Inoue surfaces $S_{N,p,q,r}^{(-)}$.

As in the previous section, the surfaces $S_{N,p,q,r}^{(-)}$ are defined as quotient complex manifolds $\mathbb{H} \times \mathbb{C}/G_{N,p,q,r}^{(-)}$. The group $G_{N,p,q,r}^{(-)}$ is generated by the following automorphisms:

$$\phi_0 : (w, z) \mapsto (\alpha w, -z), \tag{7.20}$$

$$\phi_i : (w, z) \mapsto (w + a_i, z + b_i w + c_i), \quad i = 1, 2 \tag{7.21}$$

$$\phi_3 : (w, z) \mapsto \left(w, z + \frac{(b_1 a_2 - b_2 a_1)}{r} \right), \tag{7.22}$$

where real constants α, a_i, b_i, c_i, r are the same as in the subsection of $S_{N,p,q,r;t}^{(+)}$. Then we have a following basis of forms of type $(1, 0)$ on $\mathbb{H} \times \mathbb{C}$:

$$\theta^1 = \frac{dw}{w_2}, \quad \theta^2 = \frac{z_2}{w_2} dw - dz$$

The forms θ^1 and θ^2 are invariant under the action of ϕ_i for $i = 1, 2$ and ϕ_3 . Yet we have $\phi_0^* \theta^1 = \theta^1$ and $\phi_0^* \theta^2 = -\theta^2$. Thus $\omega = -i(\theta \wedge \bar{\theta}^1 + \theta^2 \wedge \bar{\theta}^2)$ is an invariant form which is an l.c.k structure on $S_{N,p,q,r}^{(-)}$, that is,

$$d\omega = \frac{dw_2}{w_2} \wedge \omega.$$

The l.c.k form ω gives the corresponding flat line bundle L . Deformations of flat line bundles $\{L_s\}$ with $L_0 = L$ are given by a class $[s\eta] \in H^1(S_{N,p,q,r}^{(-)}, \mathbb{R})$. Then we have

PROPOSITION 7.6. *The Inoue surface $S_{N,p,q,r}^{(-)}$ does not admit a smooth family of l.c.k structures $\{\omega_s\}$ such that L_s is the corresponding flat line bundle to ω_s .*

PROOF. Since $\eta \wedge \omega$ is harmonic as in the proof of Lemma 7.4, it follows that the class $[\eta \wedge \omega] \in H^3_\eta(S^{(-)})$ does not vanish. Then the result follows from Theorem 2.4. \square

References

- [1] F. A. Belgun, On the metric structure of non-Kähler complex surfaces, *Math. Ann.*, **317** (2000), 1–40.
- [2] P. C. Boyer and K. Galicki, *Sasakian Geometry*, Oxford Math. Monogr., Oxford Science Publications, Oxford University Press, Oxford, 2008.
- [3] M. Brunella, Locally conformally Kähler metrics on certain non-Kählerian surfaces, *Math. Ann.*, **346** (2010), 629–639.
- [4] M. Brunella, Locally conformally Kähler metrics on Kato surfaces, *Nagoya Math. J.*, **202** (2011), 77–81.
- [5] S. Dragomir and L. Ornea, *Locally Conformal Kähler Geometry*, Progr. Math., **155**, Birkhäuser Boston, Inc., Boston, MA, 1998.
- [6] A. Fujiki and M. Pontecorvo, Anti-self-dual bihermitian structures on Inoue surfaces, *J. Differential Geom.*, **85** (2010), 15–71.
- [7] R. Goto, Moduli spaces of topological calibrations, Calabi-Yau, hyper-Kähler, G_2 and Spin(7) structures, *Internat. J. Math.*, **15** (2004), 211–257.
- [8] R. Goto, On deformations of generalized Calabi-Yau, hyperKähler, G_2 and Spin(7) structures, arXiv:math.DG/0512211.
- [9] R. Goto, Deformations of generalized complex and generalized Kähler structures, *J. Differential Geom.*, **84** (2010), 525–560.
- [10] R. Goto, Poisson structures and generalized Kähler submanifolds, *J. Math. Soc. Japan*, **61** (2009), 107–132.
- [11] R. Goto, Unobstructed K-deformations of generalized complex structures and bi-Hermitian structures, *Adv. Math.*, **231** (2012), 1041–1067.
- [12] P. Gauduchon and L. Ornea, Locally conformally Kähler metrics on Hopf surfaces, *Ann. Inst. Fourier (Grenoble)*, **48** (1998), 1107–1127.
- [13] M. Inoue, On surfaces of Class VII₀, *Invent. Math.*, **24** (1974), 269–310.
- [14] M. Ise, On the geometry of Hopf manifolds, *Osaka Math. J.*, **12** (1960), 387–402.
- [15] K. Kodaira, *Complex Manifolds and Deformation of Complex Structures*, Grundlehren Math. Wiss., **283**, Springer-Verlag, New York, 1986.
- [16] K. Kodaira and D. C. Spencer, On deformations of complex, analytic structures. I, II, *Ann. Math. (2)*, **67** (1958), 328–466.
- [17] K. Kodaira and D. C. Spencer, On deformations of complex analytic structures. III, stability theorems for complex structures, *Ann. Math. (2)*, **71** (1960), 43–76.
- [18] M. de León, B. López, J. C. Marrero and E. Padrón, On the computation of the Lichnerowicz-Jacobi cohomology, *J. Geom. Phys.*, **44** (2003), 507–522.
- [19] D. Mall, The cohomology of line bundles on Hopf manifolds, *Osaka J. Math.*, **28** (1991), 999–1015.
- [20] L. Ornea and M. Verbitsky, Locally conformal Kähler manifolds with potential, *Math. Ann.*, **348** (2010), 25–33.
- [21] L. Ornea and M. Verbitsky, Morse-Novikov cohomology of locally conformally Kähler manifolds, *J. Geom. Phys.*, **59** (2009), 295–305.
- [22] F. Tricerri, Some examples of locally conformal Kähler manifolds, *Rend. Sem. Mat. Univ. Politec. Torino*, **40** (1982), 81–92.
- [23] I. Vaisman, Generalized Hopf manifolds, *Geom. Dedicata*, **13** (1982), 231–255.

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