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Sharp lower bound on the curvatures of ASD connections over the cylinder

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Abstract. We prove a sharp lower bound on the curvatures of non-flat ASD connections over the cylinder.

1. Introduction.

The purpose of this note is to calculate explicitly a universal lower bound on the curvatures of non-flat ASD connections over the cylinder $\mathbb{R} \times S^3$.

First we fix our conventions. Let $S^3 = \{x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\} \subset \mathbb{R}^4$ be the unit 3-sphere equipped with the Riemannian metric induced by the Euclidean metric on $\mathbb{R}^4$. Set $X := \mathbb{R} \times S^3$. We give the standard metric on $\mathbb{R}$, and X is equipped with the product metric.

Let $\mathbb{H}$ be the space of quaternions. Consider $SU(2) = \{x \in \mathbb{H} \mid |x| = 1\}$ with the Riemannian metric induced by the Euclidean metric on $\mathbb{H}$. (Hence it is isometric to S^3 above.) We naturally identify $su(2) := T_1 SU(2)$ with the imaginary part $\mathrm{Im}\mathbb{H} := \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$. Here i, j and k have length 1.

Let $E := X \times SU(2)$ be the product $SU(2)$-bundle. Let A be a connection on E, and let F_A be its curvature. F_A is an $su(2)$-valued 2-form on X. Hence for each point $p \in X$ the curvature F_A can be considered as a linear map

$$F_{A,p} : \Lambda^2(T_p X) \to su(2).$$

We denote by $|F_{A,p}|_{\mathrm{op}}$ the operator norm of this linear map. The explicit formula is as follows: Let x_1, x_2, x_3, x_4 be the normal coordinate system on X centered at p. Let $A = \sum_{i=1}^4 A_i dx_i$. Each A_i is an $su(2)$-valued function. Then $F(A)_{ij} := F_A(\partial/\partial x_i, \partial/\partial x_j) = \partial_i A_j - \partial_j A_i + [A_i, A_j]$. Since $\partial/\partial x_i \wedge \partial/\partial x_j$ $(1 \le i < j \le 4)$ form an orthonormal basis of $\Lambda^2(TX)$ at p, the norm $|F_{A,p}|_{\mathrm{op}}$ is equal to

$$\sup \left\{ \left| \sum_{1 \le i < j \le 4} a_{ij} F(A)_{ij,p} \right| \,\middle|\, a_{ij} \in \mathbb{R}, \sum_{1 \le i < j \le 4} a_{ij}^2 = 1 \right\}.$$

Let $\|F_A\|_{\mathrm{op}}$ be the supremum of $|F_{A,p}|_{\mathrm{op}}$ over $p \in X$. The main result is the following.

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THEOREM 1.1. *The minimum of $\|F_A\|_{\mathrm{op}}$ over non-flat ASD connections A on E is equal to $1/\sqrt{2}$.*

Note that we don't assume $F_A \in L^2$ in this statement. As far as I know, this kind of explicit calculations have never been done in Yang-Mills theory. (See Remark 1.3 below.) The above minimum value $1/\sqrt{2}$ is attained by the following BPST instanton ([**1**]).

EXAMPLE 1.2. We define an $SU(2)$ instanton A on $\mathbb{R}^4$ by

$$A := \mathrm{Im}\left(\frac{\bar{x}dx}{1+|x|^2}\right), \quad (x = x_1 + x_2 i + x_3 j + x_4 k).$$

By the conformal map

$$\mathbb{R} \times S^3 \to \mathbb{R}^4 \setminus \{0\}, \quad (t,\theta) \mapsto e^t \theta,$$

the connection A is transformed into an ASD connection A' on E over $\mathbb{R} \times S^3$. Then

$$|F_{A',(t,\theta)}|_{\mathrm{op}} = \frac{2\sqrt{2}}{(e^t + e^{-t})^2}.$$

Hence

$$\|F_{A'}\|_{\mathrm{op}} = \frac{1}{\sqrt{2}}.$$

REMARK 1.3. The essential point of the statement of Theorem 1.1 is the explicitness of $1/\sqrt{2}$. Indeed the following general statement is easy to prove: Let Y be a closed Riemannian 3-fold, and assume that all flat $SU(2)$ connections ρ on Y satisfy the non-degeneracy condition $H^1_\rho = 0$. (See [**2**, p. 25, Definition 2.4]. S^3 satisfies this condition. More generally lens spaces $S^3/\mathbb{Z}_p$ satisfy it.) Then the infimum of $\|F_A\|_{\mathrm{op}}$ over non-flat $SU(2)$ ASD connections A on $\mathbb{R} \times Y$ is positive. The proof is just a direct application of [**2**, p. 81, Proposition 4.4]. But it is difficult to determine the value of $\inf \|F_A\|_{\mathrm{op}}$ explicitly.

Theorem 1.1 is a Yang-Mills analogy of the classical result of Lehto [**7**, Theorem 1] in complex analysis. (The formulation below is due to Eremenko [**4**, Theorem 3.2]. See also Lehto-Virtanen [**8**, Theorem 1].) Consider $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ with the length element $|dz|/|z|$. We give a metric on $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ by (naturally) identifying it with the unit 2-sphere $\{x_1^2 + x_2^2 + x_3^2 = 1\}$. For a map $f : \mathbb{C}^* \to \mathbb{C}P^1$ we denote its Lipschitz constant by $\mathrm{Lip}(f)$. Then Lehto [**7**, Theorem 1] proved that the minimum of $\mathrm{Lip}(f)$ over non-constant holomorphic maps $f : \mathbb{C}^* \to \mathbb{C}P^1$ is equal to 1. The function $f(z) = z$ attains the minimum. Eremenko [**4**, Section 3] discussed the relation between this result of Lehto and a quantitative homotopy argument of Gromov [**6**, Chapter 2, 2.12. Proposition]. Our proof of Theorem 1.1 is inspired by this idea.

2. Preliminaries: Connections over S^3.

In this section we study the method of choosing good gauges for some connections over S^3. The argument below is a careful study of [**5**, pp. 146–148]. Set $N := (1,0,0,0) \in S^3$ and $S := (-1,0,0,0) \in S^3$. Let $P := S^3 \times SU(2)$ be the product $SU(2)$-bundle over S^3. For a connection B on P we define the operator norm $\|F_B\|_{\mathrm{op}}$ in the same way as in Section 1.

Let $v_1, v_2 \in T_N S^3$ be two unit tangent vectors at N. ($|v_1| = |v_2| = 1$.) Let $\exp_N : T_N S^3 \to S^3$ be the exponential map at N. Since $|v_1| = |v_2| = 1$, we have $\exp_N(\pi v_1) = \exp_N(\pi v_2) = S$. We define a loop $l : [0, 2\pi] \to S^3$ by

$$l(t) := \begin{cases} \exp_N(t v_1) & (0 \le t \le \pi) \\ \exp_N((2\pi - t) v_2) & (\pi \le t \le 2\pi). \end{cases}$$

LEMMA 2.1. *Let B be a connection on P. Let $\mathrm{Hol}_l(B) \in SU(2)$ be the holonomy of B along the loop l. Then*

$$d(\mathrm{Hol}_l(B), 1) \le 2\pi \|F_B\|_{\mathrm{op}}.$$

Here $d(\cdot,\cdot)$ is the distance on $SU(2)$ defined by the Riemannian metric.

PROOF. This follows from the standard fact that curvature is an infinitesimal holonomy [**3**, p. 36]. (2π is half the area of the unit 2-sphere.) The explicit proof is as follows: Take a unit tangent vector $v_3 \in T_N S^3$ orthogonal to v_1 such that there is $\alpha \in [0,\pi]$ satisfying $v_2 = v_1 \cos\alpha + v_3 \sin\alpha$. Consider (the spherical polar coordinate of the totally geodesic $S^2 \subset S^3$ tangent to v_1 and v_3):

$$\Phi : [0,\alpha] \times [0,\pi] \to S^3, \quad (\theta_1, \theta_2) \mapsto \exp_N\{\theta_2(v_1 \cos\theta_1 + v_3 \sin\theta_1)\}.$$

Let Q be the pull-back of the bundle P by Φ. Since $\Phi([0,\alpha] \times \{0\}) = \{N\}$ and $\Phi([0,\alpha] \times \{\pi\}) = \{S\}$, Q admits a trivialization under which the pull-back connection $\Phi^* B$ is expressed as $\Phi^* B = B_1 d\theta_1 + B_2 d\theta_2$ with $B_1 = 0$ on $[0,\alpha] \times \{0,\pi\}$.

We take a smooth map $g : [0,\alpha] \times [0,\pi] \to SU(2)$ satisfying

$$g(\theta_1, 0) = 1 \quad (\forall \theta_1 \in [0,\alpha]), \quad (\partial_2 + B_2) g = 0.$$

We have $\mathrm{Hol}_l(B) = g(\alpha,\pi)^{-1} g(0,\pi)$. Then $F_{\Phi^* B}(\partial_1, \partial_2) g = [\partial_1 + B_1, \partial_2 + B_2] g = -(\partial_2 + B_2)(\partial_1 + B_1) g$. From $B_1 = 0$ on $[0,\alpha] \times \{0,\pi\}$ and Kato's inequality $|\partial_2 |(\partial_1 + B_1) g|| \le |(\partial_2 + B_2)(\partial_1 + B_1) g| = |F_{\Phi^* B}(\partial_1, \partial_2) g|$,

$$\begin{aligned} |\partial_1 g(\theta_1, \pi)| &= |(\partial_1 + B_1) g(\theta_1, \pi)| - |(\partial_1 + B_1) g(\theta_1, 0)| \\ &\le \int_{\{\theta_1\} \times [0,\pi]} |\partial_2 |(\partial_1 + B_1) g|| d\theta_2 \le \int_{\{\theta_1\} \times [0,\pi]} |F_{\Phi^* B}(\partial_1, \partial_2)| d\theta_2. \end{aligned}$$

Then

$$d(\mathrm{Hol}_l(B), 1) = d(g(0,\pi), g(\alpha,\pi)) \le \int_{[0,\alpha]\times[0,\pi]} |F_{\Phi^* B}(\partial_1, \partial_2)| d\theta_1 d\theta_2.$$

$F_{\Phi^* B}(\partial_1, \partial_2) = F_B(d\Phi(\partial/\partial\theta_1), d\Phi(\partial/\partial\theta_2))$. The vectors $d\Phi(\partial/\partial\theta_1)$ and $d\Phi(\partial/\partial\theta_2)$ are orthogonal to each other, and $|d\Phi(\partial/\partial\theta_1)| = \sin\theta_2$ and $|d\Phi(\partial/\partial\theta_2)| = 1$. Hence $|F_{\Phi^* B}(\partial_1, \partial_2)| \le \|F_B\|_{\mathrm{op}} \sin\theta_2$. From $0 \le \alpha \le \pi$,

$$d(\mathrm{Hol}_l(B), 1) \le \|F_B\|_{\mathrm{op}} \int_{[0,\alpha]\times[0,\pi]} \sin\theta_2 \, d\theta_1 d\theta_2 = 2\alpha \|F_B\|_{\mathrm{op}} \le 2\pi \|F_B\|_{\mathrm{op}}. \qquad \square$$

Let $\tau < 1/2$. Let B be a connection on P satisfying $\|F_B\|_{\mathrm{op}} \le \tau$. We construct a good connection matrix of B.

Fix $v \in T_N S^3$. By the parallel translation along the geodesic $\exp_N(tv)$ $(0 \le t \le \pi)$ we identify the fiber P_S with the fiber P_N. Let g_N and g_S be the exponential gauges (see [**5**, p. 146] or [**3**, p. 54]) centered at N and S respectively:

$$g_N : P|_{S^3\setminus\{S\}} \to (S^3 \setminus \{S\}) \times P_N, \quad g_S : P|_{S^3\setminus\{N\}} \to (S^3 \setminus \{N\}) \times P_N.$$

(In the definition of g_S we identify P_S with P_N as in the above.) By Lemma 2.1, for $x \in S^3 \setminus \{N, S\}$,

$$d(g_N(x), g_S(x)) \le 2\pi \|F_B\|_{\mathrm{op}} \le 2\pi\tau < \pi.$$

The injectivity radius of $SU(2) = S^3$ is π (this is a crucial point of the argument). Hence there uniquely exists $u(x) \in \mathrm{ad}P_N (\cong su(2))$ satisfying

$$|u(x)| \le 2\pi \|F_B\|_{\mathrm{op}}, \quad g_S(x) = e^{u(x)} g_N(x).$$

We take and fix a cut-off function $\varphi : S^3 \to [0,1]$ such that $\varphi(x_1, x_2, x_3, x_4)$ is equal to 0 over $\{x_1 > 1/2\}$ and equal to 1 over $\{x_1 < -1/2\}$. We can define a bundle trivialization g of P all over S^3 by $g := e^{\varphi u} g_N$. Then the connection matrix $g(B)$ satisfies

$$|g(B)| \le C_\tau \|F_B\|_{\mathrm{op}}.$$

Here C_τ is a positive constant depending on τ.

3. Proof of Theorem 1.1.

In this section we denote by t the standard coordinate of $\mathbb{R}$. Let A be an ASD connection on E satisfying $\|F_A\|_{\mathrm{op}} < 1/\sqrt{2}$. We will prove that A must be flat. Set $\tau := \|F_A\|_{\mathrm{op}}/\sqrt{2} < 1/2$.

The ASD equation implies that F_A has the following form:

$$F_A = -dt \wedge (*_3 F(A|_{\{t\}\times S^3})) + F(A|_{\{t\}\times S^3}),$$

where $A|_{\{t\}\times S^3}$ is the restriction of A to $\{t\}\times S^3$ and $*_3$ is the Hodge star on $\{t\}\times S^3$. Hence

$$|F_{A,(t,\theta)}|_{\mathrm{op}} = \sqrt{2}|F(A|_{\{t\}\times S^3})_\theta|_{\mathrm{op}}.$$

Therefore

$$\|F(A|_{\{t\}\times S^3})\|_{\mathrm{op}} \le \tau < \frac{1}{2} \quad (\forall t \in \mathbb{R}).$$

Thus we can apply the construction of Section 2 to $A|_{\{t\}\times S^3}$.

Fix a bundle trivialization of E over $\mathbb{R}\times\{N\}$. (Any choice will do.) Then the construction in Section 2 gives a bundle trivialization g of E over X satisfying

$$|g(A)|_{\{t\}\times S^3}| \le C_\tau \|F(A|_{\{t\}\times S^3})\|_{\mathrm{op}} \quad (\forall t \in \mathbb{R}).$$

Set $A' := g(A)$. We consider the Chern-Simons functional

$$cs(A') := \mathrm{tr}\left(A' \wedge F_{A'} - \frac{1}{3}A'^3\right).$$

For $R > 0$

$$\begin{aligned}\int_{[-R,R]\times S^3} |F_A|^2 d\,\mathrm{vol} &= \int_{\{R\}\times S^3} cs(A') - \int_{\{-R\}\times S^3} cs(A') \quad (\text{because } A \text{ is ASD})\\ &\le \mathrm{const}_\tau \left(\|F(A|_{\{R\}\times S^3})\|_{\mathrm{op}} + \|F(A|_{\{-R\}\times S^3})\|_{\mathrm{op}}\right). \qquad (1)\end{aligned}$$

Here we have used $|A'|_{\{\pm R\}\times S^3}| \le C_\tau \|F(A|_{\{\pm R\}\times S^3})\|_{\mathrm{op}}$ and $\|F(A|_{\{\pm R\}\times S^3})\|_{\mathrm{op}} \le \tau$. Let $R \to +\infty$. Then we get

$$\int_X |F_A|^2 d\,\mathrm{vol} < +\infty.$$

This implies that the curvature F_A has an exponential decay at the ends (see [**2**, Theorem 4.2]). In particular

$$\|F(A|_{\{\pm R\}\times S^3})\|_{\mathrm{op}} \to 0 \quad (R \to +\infty).$$

By the above (1)

$$\int_X |F_A|^2 d\,\mathrm{vol} = 0.$$

This shows $F_A \equiv 0$. So A is flat.

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