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A converse theorem for double Dirichlet series and Shintani zeta functions

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Abstract. A converse theorem for double Dirichlet series is established. As an application, we show that certain zeta functions introduced by Shintani are actually Weyl group multiple Dirichlet series associated to metaplectic Eisenstein series on $GL(2)$.

1. Introduction.

The main aim of this paper is to obtain a converse theorem for double Dirichlet series and use it to show that the Shintani zeta functions [**13**] which arise in the theory of prehomogeneous vector spaces are actually linear combinations of Mellin transforms of metaplectic Eisenstein series on $GL(2)$. The converse theorem we prove will apply to a very general family of double Dirichlet series which we now define.

Definition 1.1 (Family $\mathcal{F}_N$ of double Dirichlet series). Fix a positive integer N and a weight $1/2$ multiplier system v of the congruence subgroup $\Gamma_0(4N)$. Let m^* denote the number of inequivalent singular cusps of $\Gamma_0(4N)$ in terms of v (see beginning of Section 2 for definitions of multiplier system and singular cusps). Let $a^j_{n,\ell}$ (with $\ell, n \in \mathbb{Z}$, $\ell \geq 1$, $j = 1, \dots, m^*$) be a sequence of complex numbers which are assumed to have polynomial growth in $|n|$ and ℓ as $|n|, \ell \to \infty$.

For $s, w \in \mathbb{C}$ (with sufficiently large real parts) and an integer $N \geq 1$, we define $\mathcal{F}_N$ to be a set (family) of double Dirichlet series

$$L^{\pm}_j(s, w; \chi) = \sum_{\pm n > 0} \sum_{\ell=1}^{\infty} \frac{a^j_{n,\ell}\, \tau_n(\chi)}{\ell^w\, |n|^s},$$

where j ranges over the set $\{1, \dots, m^*\}$, D ranges over the set of integers in $\{1, \dots, (4N)^2\}$ that are co-prime to N and, for each such D, χ ranges over the Dirichlet characters (mod D). Here

$$\tau_n(\chi) := \sum_{\substack{m \,(\mathrm{mod}\, D) \\ (m,D)=1}} \chi(m) e^{2\pi i m n / D}$$

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is the Gauss sum.

The converse theorem we prove will be for the family $\mathcal{F}_N$ provided every L-function in $\mathcal{F}_N$ satisfies certain "nice properties," namely, every $L_j^{\pm}(s, w; \chi) \in \mathcal{F}_N$ is holomorphic and bounded in vertical strips and satisfies certain functional equations. We call such a family $\mathcal{F}_N$ a "nice family." The precise definition is given in Section 3. The converse theorem (Theorem 3.2) states that a "nice family" $\mathcal{F}_N$ must be a family of linear combinations of Mellin transforms of metaplectic Eisenstein series. This implies, in particular, that such a "nice family" is actually a family of WMDS (Weyl group multiple Dirichlet series) studied in [**1**]. As such it satisfies additional hidden functional equations which cannot be seen by the theory of prehomogeneous vector spaces.

The method used to prove our converse theorem is a refinement of that used in [**3**] and, as a result, the statement of the theorem is significantly simplified. In particular, we solve one of the problems we pointed out in [**3**]. Specifically, it seemed impossible to eliminate from the assumptions of the converse theorem, an additional set of functional equations which were quite unnatural. The version of the converse theorem in this paper avoids the need for these functional equations and, in addition, instead of hypergeometric functions, it uses Gamma functions which are easier to handle. The key for this simplification is Bykovskii's technique [**2**] which allows for the information contained in the extra functional equations of [**3**] to be encoded into an auxiliary variable.

The simplification is even more apparent in the scalar version of the converse theorem (Theorem 5.3) corresponding to the case of $\Gamma_0(4)$. In Section 6, we use this theorem to prove that Shintani's zeta function is essentially a Mellin transform of the metaplectic Eisenstein series for $\Gamma_0(4)$ (Theorem 6.2).

Shintani's zeta functions [**13**] have been studied extensively because of their arithmetic nature and because they are important examples of zeta functions associated to prehomogeneous vector spaces. While it has long been known that Shintani's zeta functions should be closely related to the Eisenstein series studied by Siegel [**11**], there are technical difficulties in making this relation explicit by direct computation, e.g. because of the non-square-free integers. We circumvent these problems with the use of our converse theorem and establish an explicit relation with Mellin transforms of Siegel's Eisenstein series.

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2. Metaplectic Eisenstein series.

We recall the basic terminology and notation for metaplectic Eisenstein series.

Fix a positive integer N. Let $\Gamma = \Gamma_0(4N)$ denote the group of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of determinant 1 with $a, b, c, d \in \mathbb{Z}$ and $4N|c$. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, define the weight 1/2 multiplier system

$$v(\gamma) = \left(\frac{c}{d}\right)\epsilon_d^{-1},$$

with

$$\epsilon_d = \begin{cases} 1, & d \equiv 1 (\mathrm{mod}\, 4), \\ i, & d \equiv 3 (\mathrm{mod}\, 4), \end{cases}$$

where $(\frac{c}{d})$ is the usual Kronecker symbol.

Now, we fix a set $\{\mathfrak{a}_i, i = 1, \dots, m\}$ of inequivalent cusps of $\Gamma_0(4N)$ among which the first m^* are singular with respect to v (i.e. $v(\gamma_{\mathfrak{a}}) = 1$, if $\gamma_{\mathfrak{a}}$ is the generator of the stabilizer $\Gamma_{\mathfrak{a}}$ of $\mathfrak{a}$). We choose the $\mathfrak{a}$'s so that $\mathfrak{a}_1 = \infty$ and $\mathfrak{a}_{m^*} = 0$.

For each $\mathfrak{a}$ we fix a scaling matrix $\sigma_{\mathfrak{a}}$ such that $\sigma_{\mathfrak{a}}(\infty) = \mathfrak{a}$ and $\sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \Gamma_\infty$. In particular, we select $\sigma_{\mathfrak{a}_1} = I$, $\sigma_{\mathfrak{a}_{m^*}} = W_{4N}$, where I is the identity matrix and W_{4N} is the Fricke involution $\left(\begin{smallmatrix} 0 & -1/(2\sqrt{N}) \\ 2\sqrt{N} & 0 \end{smallmatrix}\right)$.

We shall also adopt the notation that we may write M in the form $M = \left(\begin{smallmatrix} a_M & b_M \\ c_M & d_M \end{smallmatrix}\right)$. Further, the arguments of complex numbers are chosen to be in $(-\pi, \pi]$. Then, for $f : \mathfrak{H} \to \mathbb{C}$ and $\gamma \in SL_2(\mathbb{R})$, we recall the slash operator: $f|\gamma$. It is defined by the formula

$$(f|\gamma)(z) = f(\gamma z)\, \frac{(c_\gamma z + d_\gamma)^{-1/2}}{|c_\gamma z + d_\gamma|^{-1/2}},$$

and satisfies the relation

$$f|\gamma|\delta = r(\gamma, \delta) \cdot f|(\gamma\delta), \qquad (\gamma, \delta \in SL_2(\mathbb{R})),$$

where

$$r(M, N) = \frac{(c_M N z + d_M)^{1/2}(c_N z + d_N)^{1/2}}{(c_{MN} z + d_{MN})^{1/2}}, \qquad (\text{for } M, N \in SL_2(\mathbb{R})).$$

To compute $r(M, N)$ we will tacitly be using Theorem 16 of [**5**].

LEMMA 2.1. *Let* $M = \left(\begin{smallmatrix} * & * \\ m_1 & m_2 \end{smallmatrix}\right)$, $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{R})$ *and* $MS = \left(\begin{smallmatrix} * & * \\ m_1' & m_2' \end{smallmatrix}\right)$. *Then* $r(M, S) = e^{(\pi i/4) w(M,S)}$, *with*

$$w(M, S) = \begin{cases} (\mathrm{sgn}(c) + \mathrm{sgn}(m_1) - \mathrm{sgn}(m_1') - \mathrm{sgn}(m_1 c m_1')), & m_1 c m_1' \neq 0, \\ (\mathrm{sgn}(c) - 1)(1 - \mathrm{sgn}(m_1)), & m_1 c \neq 0,\ m_1' = 0, \\ (\mathrm{sgn}(c) + 1)(1 - \mathrm{sgn}(m_2)), & m_1' c \neq 0,\ m_1 = 0, \\ (1 - \mathrm{sgn}(a))(1 + \mathrm{sgn}(m_1)), & m_1 m_1' \neq 0,\ c = 0, \\ (1 - \mathrm{sgn}(a))(1 - \mathrm{sgn}(m_2)), & m_1 = c = m_1' = 0. \end{cases}$$

For convenience, for every function f on $\mathfrak{H}$ we set

$$\check{f} := e^{\pi i/4} f|W_{4N}.$$

Thus, $\check{f}(iy) = f(i/(4Ny))$ and $\check{\check{f}} = f$.

For each of the cusps $\mathfrak{a}_i$ $(i = 1, \dots, m^*)$ and $w \in \mathbb{C}$ with $\mathrm{Re}(w) > 1$, we define an Eisenstein series

$$E_i(z,w) = \sum_{\gamma \in \Gamma_{\mathfrak{a}_i} \backslash \Gamma} \frac{\mathrm{Im}(\sigma_{\mathfrak{a}_i}^{-1}\gamma z)^w}{r(\sigma_{\mathfrak{a}_i}^{-1}, \gamma) v(\gamma)} \left(\frac{c_{\sigma_{\mathfrak{a}_i}^{-1}\gamma} z + d_{\sigma_{\mathfrak{a}_i}^{-1}\gamma}}{|c_{\sigma_{\mathfrak{a}_i}^{-1}\gamma} z + d_{\sigma_{\mathfrak{a}_i}^{-1}\gamma}|} \right)^{-1/2}.$$

This Eisenstein series has a meromorphic continuation to the w-plane ([**7**, Section 10]) and, for all $\delta \in \Gamma$, it satisfies

$$E_i(\cdot, w)|\delta = v(\delta) E_i(\cdot, w).$$

Next, if T denotes matrix transpose, set

$$\boldsymbol{E}(z,w) = (E_1(z,w), \dots, E_{m^*}(z,w))^T$$

and

$$\check{\boldsymbol{E}}(z,w) = (\check{E}_1(z,w), \dots, \check{E}_{m^*}(z,w))^T.$$

Each E_i is an eigenfunction of the weight $1/2$ Laplacian

$$\Delta_{1/2} = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - \frac{iy}{2} \frac{\partial}{\partial x}$$

with eigenvalue $w(w-1)$ ([**7**, (10.10)]). This implies that, if $z := x + iy$, then, for all $i, j \in \{1, \dots, m^*\}$, there are functions $a_n^{ij}(w)$, such that

$$E_i(\cdot, w)|\sigma_{\mathfrak{a}_j} = \delta_{ij} y^w + p_{ij}(w) y^{1-w} + \sum_{n \neq 0} a_n^{ij}(w) W_{\mathrm{sgn}(n)/4,\, w-(1/2)}(4\pi |n| y) e^{2\pi i n x},$$

where δ_{ij} is the Kronecker delta and $p_{ij}(w)$ the ij-th entry of the scattering matrix $\Phi(w)$. Here, $W_{\cdot,\cdot}$ is the classical Whittaker function with integral representation

$$W_{a,b}(z) = \frac{e^{-z/2} z^a}{\Gamma(1/2 - a + b)} \int_0^\infty u^{-a-1/2+b} (1 + z^{-1}u)^{a-1/2+b} e^{-u} du$$

(cf. [**14**, p. 340]).

If w and $1 - w$ are not poles of any of the E_i $(i = 1, \dots, m^*)$, then, by [**7**, (10.19)],

$$\boldsymbol{E}(z, 1-w) = \Phi(1-w) \boldsymbol{E}(z,w). \tag{1}$$

3. L-functions associated to $E_i(z,w)$.

Fix a positive integer $N \geq 1$. For every positive integer D (with $(D,4N)=1$), let χ be a Dirichlet character modulo D. For every function $f : \mathfrak{H} \to \mathbb{C}$, we define its twist by

$$f(\cdot\,;\chi) = \sum_{\substack{m \,(\mathrm{mod}\, D)\\(m,D)=1}} \chi(m) f\left|\begin{pmatrix} 1 & m/D \\ 0 & 1 \end{pmatrix}\right..$$

We consider functions $f(z,w)$ of two variables $z = x+iy \in \mathfrak{H}$, $w \in \mathbb{C}$, with Fourier expansions of the form

$$f(z,w) = a(w)y^{1-w} + b(w)y^{w} + \sum_{n\neq 0} a_n(w) W_{\mathrm{sgn}(n)/4,\, w-(1/2)}(4\pi|n|y)e^{2\pi i n x}.$$

Then the twisted function $f(\cdot\,;\chi)$, in terms of z, is

$$f(z,w;\chi) = \tau_0(\chi)\big(a(w)y^{1-w} + b(w)y^{w}\big) + \sum_{n\neq 0} \tau_n(\chi) a_n(w) W_{\mathrm{sgn}(n)/4,\, w-(1/2)}(4\pi|n|y)e^{2\pi i n x},$$

where

$$\tau_n(\chi) = \sum_{\substack{m \,(\mathrm{mod}\, D)\\(m,D)=1}} \chi(m) e^{2\pi i m n/D}, \qquad (n \in \boldsymbol{Z}).$$

As shown in [**3**], we have

$$\begin{aligned} f(\cdot\,;\chi)&\left|\begin{pmatrix} 0 & -1/(2D\sqrt{N}) \\ 2D\sqrt{N} & 0 \end{pmatrix}\right. \\ &= e^{-\pi i/4}\overline{\chi(-4N)} \sum_{\substack{r \,(\mathrm{mod}\, D)\\(r,D)=1}} \overline{\chi(r)} \check{f}\left|\begin{pmatrix} D & -r \\ -4mN & t \end{pmatrix}\right| \left.\begin{pmatrix} 1 & r/D \\ 0 & 1 \end{pmatrix}\right.. \end{aligned} \tag{2}$$

For future reference we consider the Dirichlet character $\check{\chi}$ $(\mathrm{mod}\, D)$ given by

$$\check{\chi}(m) := \left(\frac{m}{D}\right)\overline{\chi(m)}.$$

Note that $\check{\chi}$ is a character since $(D,4N)=1$, D is odd and $(\frac{\cdot}{\cdot})$ is the Jacobi symbol. It satisfies $\check{\check{\chi}} = \chi$.

We are now ready to associate L-functions to metaplectic Eisenstein series.

Let $a_n^j(w)$ denote the n-th coefficient of the expansion at ∞ of $E_j(z,w)$. For $\mathrm{Re}(s)$ large enough, define

$$L_j^{\pm}(s,w) = \sum_{\pm n>0} \frac{a_n^j(w)}{|n|^s}.$$

Generally, for χ a Dirichlet character modulo D ($(D,4N)=1$), set

$$L_j^{\pm}(s,w;\chi) = \sum_{\pm n>0} \frac{\tau_n(\chi)a_n^j(w)}{|n|^s}.$$

Following [**2**], we also define the modified "completed" L-functions:

$$\Lambda_j(s,w,u;\chi) := \int_0^\infty \big(E_j((i+u)y,w;\chi) - \tau_0(\chi)\big(\delta_{j1}y^w + p_{j1}(w)y^{1-w}\big)\big)y^s\,\frac{dy}{y}.$$

We also set $\check{L}_j$ and $\check{\Lambda}_j$, for the corresponding functions associated to $\check{\boldsymbol{E}}$.

Let $u\in\mathbb{R}$ and $s,w\in\mathbb{C}$ with $\mathrm{Re}(s),\mathrm{Re}(w)$ sufficiently large. With [**6**, (13.23.4)], we have

$$\Lambda_j(s,w,u;\chi) = \boldsymbol{c}(s,w;u)\big(L_j^+(s,w;\chi),\ L_j^-(s,w;\chi)\big)^T, \tag{3}$$

where

$$\boldsymbol{c}(s,w;u) = \frac{\Gamma(w+s)\Gamma(s-w+1)}{(4\pi)^s}\cdot\Bigg(\frac{F(s+w,1+s-w,s+3/4;(1+iu)/2)}{\Gamma(s+3/4)},\\ \frac{F(s+w,1+s-w,s+5/4;(1-iu)/2)}{\Gamma(s+5/4)}\Bigg),$$

with $F(a,b,c;d)$ the Gaussian hypergeometric function.

Further, equation (2) implies that

$$E_j(\cdot,w;\chi)|W_{4ND^2} = e^{-\pi i/4}\overline{\chi(-4N)}\left(\frac{4N}{D}\right)\epsilon_D^{-1}\check{E}_j(\cdot,w;\check{\chi}), \tag{4}$$

and thus that the constant term $\check{a}_0(y,w;\chi)$ of the Fourier expansion of $E_j(\cdot,w;\chi)|W_{4ND^2}$ is

$$\check{a}_0(y,w;\chi) = \overline{\chi(-4N)}\left(\frac{4N}{D}\right)\epsilon_D^{-1}\tau_0(\check{\chi})\big(\delta_{jm^*}y^w + p_{jm^*}(w)y^{1-w}\big).$$

Evaluating at $(i-u)/(2\sqrt{N}D(u^2+1)y)$ and using

$$\left(\frac{(u+i)}{|u+i|}\right)^{1/2} = e^{\pi i/4}(1+iu)^{-1/4}(1-iu)^{1/4}, \tag{5}$$

we obtain

$$E_j\bigg(\frac{(i+u)y}{2\sqrt{N}D},w;\chi\bigg)=\overline{\chi(-4N)}\bigg(\frac{4N}{D}\bigg)\epsilon_D^{-1}\frac{(1+iu)^{1/4}}{(1-iu)^{1/4}}\check{E}_j\bigg(\frac{i-u}{2\sqrt{N}D(u^2+1)y},w,\check{\chi}\bigg).$$

Then the standard Riemann trick gives

$$\begin{aligned}
&(2\sqrt{N}D)^s\Lambda_j(s,w,u;\chi)\\
&\quad=\int_{1/\sqrt{u^2+1}}^{\infty}\bigg(E_j\bigg(\frac{(i+u)y}{2\sqrt{N}D},w;\chi\bigg)\\
&\qquad\qquad-\tau_0(\chi)\bigg(\delta_{j1}\bigg(\frac{y}{2\sqrt{N}D}\bigg)^w+p_{j1}(w)\bigg(\frac{y}{2\sqrt{N}D}\bigg)^{1-w}\bigg)\bigg)y^s\,\frac{dy}{y}\\
&\qquad+\int_0^{1/\sqrt{u^2+1}}\bigg[A\cdot\check{E}_j\bigg(\frac{i-u}{2\sqrt{N}D(u^2+1)y},w,\check{\chi}\bigg)\\
&\qquad\qquad-\tau_0(\chi)\bigg(\delta_{j1}\bigg(\frac{y}{2\sqrt{N}D}\bigg)^w+p_{j1}(w)\bigg(\frac{y}{2\sqrt{N}D}\bigg)^{1-w}\bigg)\bigg]y^s\,\frac{dy}{y}\\
&\quad=\int_{1/\sqrt{u^2+1}}^{\infty}\bigg[\bigg(E_j\bigg(\frac{(i+u)y}{2\sqrt{N}D},w;\chi\bigg)\\
&\qquad\qquad-\tau_0(\chi)\bigg(\delta_{j1}\bigg(\frac{y}{2\sqrt{N}D}\bigg)^w+p_{j1}(w)\bigg(\frac{y}{2\sqrt{N}D}\bigg)^{1-w}\bigg)\bigg)y^s\\
&\qquad\quad+A\bigg(\check{E}_j\bigg(\frac{(i-u)y}{2\sqrt{N}D},w,\check{\chi}\bigg)\\
&\qquad\quad-e^{\pi i/4}\tau_0(\check{\chi})\bigg(\delta_{jm^*}\bigg(\frac{y}{2\sqrt{N}D}\bigg)^w+p_{jm^*}(w)\bigg(\frac{y}{2\sqrt{N}D}\bigg)^{1-w}\bigg)\bigg)(y(u^2+1))^{-s}\bigg]\frac{dy}{y}\\
&\qquad+(u^2+1)^{-s}\bigg((2\sqrt{N}D)^{-w}(u^2+1)^{(s-w)/2}\bigg(A\cdot\tau_0(\check{\chi})e^{\pi i/4}\frac{\delta_{jm^*}}{s-w}-\frac{\tau_0(\chi)\delta_{j1}}{s+w}\bigg)\\
&\qquad\qquad+(2\sqrt{N}D)^{w-1}(u^2+1)^{(s+w-1)/2}\\
&\qquad\qquad\times\bigg(A\cdot\tau_0(\check{\chi})e^{\pi i/4}\frac{p_{jm^*}(w)}{w+s-1}-\frac{\tau_0(\chi)p_{j1}(w)}{s-w+1}\bigg)\bigg),
\end{aligned}\tag{6}$$

where, for convenience, we have set

$$A=\frac{(1+iu)^{1/4}}{(1-iu)^{1/4}}\overline{\chi(-4N)}\bigg(\frac{4N}{D}\bigg)\epsilon_D^{-1}.$$

By the exponential decay of $W_{\mathrm{sgn}(n)/4,w-(1/2)}(iy)$ as $y\to\infty$, the integral is convergent giving an entire function of s. This implies that $\Lambda_j(s,w;u;\chi)$ satisfies the following properties.

Property (i). The function $\Lambda_j(s,w;u;\chi)$ is meromorphic on the (s,w)-plane.

PROPERTY (ii). The function

$$(2\sqrt{N}D)^s\Lambda_j(s,w,u;\chi)$$
$$-(u^2+1)^{-s}\bigg((2\sqrt{N}D)^{-w}(u^2+1)^{(s-w)/2}\bigg(A\cdot\tau_0(\check{\chi})e^{\pi i/4}\frac{\delta_{jm^*}}{s-w}-\frac{\tau_0(\chi)\delta_{j1}}{s+w}\bigg)$$
$$+(2\sqrt{N}D)^{w-1}(u^2+1)^{(s+w-1)/2}\bigg(A\cdot\tau_0(\check{\chi})e^{\pi i/4}\frac{p_{jm^*}(w)}{w+s-1}-\frac{\tau_0(\chi)p_{j1}(w)}{s-w+1}\bigg)\bigg)$$

is EBV (entire and bounded in vertical strips).

PROPERTY (iii). For $j=1,\dots,m^*$, we have

$$(2\sqrt{N}D)^s(1+iu)^s\Lambda_j(s,w,u;\chi)=A(2\sqrt{N}D)^{-s}(1-iu)^{-s}\check{\Lambda}_j(-s,w,-u;\check{\chi}).$$

PROPERTY (iv). Define $\boldsymbol{\Lambda}_E(s,w,u;\chi):=(\Lambda_j(s,w;u;\chi))_{j=1,\dots,m^*}^T$. Then if w and $1-w$ are not poles of $\Phi(w)$, we have the functional equation

$$\boldsymbol{\Lambda}_E(s,1-w;u;\chi)=\Phi(1-w)\boldsymbol{\Lambda}_E(s,w;u;\chi). \tag{7}$$

REMARK. The functional equations in properties (iii) and (iv) are deduced from (6) and (1) respectively.

PROPOSITION 3.1. (a) *Property* (iii) *above is equivalent to*:

PROPERTY (iii′). *For* $j=1,\dots,m^*$,

$$\left(\frac{\sqrt{N}D}{\pi}\right)^{2s}\chi(-4N)\left(\frac{4N}{D}\right)\epsilon_D\begin{pmatrix}L_j^+(s,w;\chi)\\ L_j^-(s,w;\chi)\end{pmatrix}$$
$$=\begin{pmatrix}\dfrac{\Gamma(w-s)\Gamma(1-s-w)}{\Gamma(s+(1/4))\Gamma((3/4)-s)} & \dfrac{\Gamma(w-s)\Gamma(1-s-w)}{\Gamma((1/4)+w)\Gamma((5/4)-w)}\\ \dfrac{\Gamma(w-s)\Gamma(1-s-w)}{\Gamma(w-(1/4))\Gamma((3/4)-w)} & \dfrac{\Gamma(w-s)\Gamma(1-s-w)}{\Gamma(s-(1/4))\Gamma((5/4)-s)}\end{pmatrix}\begin{pmatrix}\check{L}_j^+(-s,w;\chi)\\ \check{L}_j^-(-s,w;\chi)\end{pmatrix}.$$

(b) *Property* (iv) *above is equivalent to*:

PROPERTY (iv′). *Define* $\boldsymbol{L}_E^{\pm}(s,w;\chi):=(L_j^{\pm}(s,w;\chi))_{j=1,\dots,m^*}^T$. *Then we have the functional equations*:

$$\begin{aligned}&\boldsymbol{L}_E^+(s,1-w;\chi)=\Phi(1-w)\boldsymbol{L}_E^+(s,w;\chi)\quad \textit{and}\\ &\boldsymbol{L}_E^-(s,1-w;\chi)=\Phi(1-w)\boldsymbol{L}_E^-(s,w;\chi).\end{aligned} \tag{8}$$

PROOF OF (a). Set

$$\boldsymbol{a} = \begin{pmatrix} F\left(s+1-w, s+w, \frac{3}{4}+s; \frac{1-iu}{2}\right) \\ \frac{(1-iu)^{(1/4)-s}}{(1+iu)^{(1/4)+s}} F\left(1-s-w, w-s, \frac{5}{4}-s; \frac{1-iu}{2}\right) \end{pmatrix},$$

$$\boldsymbol{b} = \begin{pmatrix} F\left(s+1-w, s+w, \frac{5}{4}+s; \frac{1+iu}{2}\right) \\ \frac{(1-iu)^{(1/4)-s}}{(1+iu)^{(1/4)+s}} F\left(1-s-w, w-s, \frac{3}{4}-s; \frac{1+iu}{2}\right) \end{pmatrix}$$

and

$$\boldsymbol{G} = \begin{pmatrix} \frac{\Gamma((3/4)+s)\Gamma(-(1/4)-s)}{\Gamma(w-(1/4))\Gamma((3/4)-w)} & 4^s \frac{\Gamma((3/4)+s)\Gamma((1/4)+s)}{\Gamma(s+1-w)\Gamma(s+w)} \\ 4^{-s} \frac{\Gamma((5/4)-s)\Gamma(-(1/4)-s)}{\Gamma(1-s-w)\Gamma(w-s)} & \frac{\Gamma((5/4)-s)\Gamma((1/4)+s)}{\Gamma(w+(1/4))\Gamma((5/4)-w)} \end{pmatrix}.$$

With equation (3), Property (iii) can be rewritten for $j = 1, \dots, m^*$ as:

$$\left(\frac{(\sqrt{N}D/2\pi)^{2s}\Gamma(s+w)\Gamma(s-w+1)}{\overline{\chi(-4N)}(\frac{4N}{D})\epsilon_D^{-1}\Gamma(w-s)\Gamma(1-s-w)} \frac{L_j^+(s,w;\chi)}{\Gamma(s+(3/4))}, \ -\frac{\check{L}_j^-(-s,w;\check{\chi})}{\Gamma((5/4)-s)} \right) \cdot \boldsymbol{a}$$

$$= \left(-\frac{(\sqrt{N}D/2\pi)^{2s}\Gamma(s+w)\Gamma(s-w+1)}{\overline{\chi(-4N)}(\frac{4N}{D})\epsilon_D^{-1}\Gamma(w-s)\Gamma(1-s-w)} \frac{L_j^-(s,w;\chi)}{\Gamma(s+(5/4))}, \ \frac{\check{L}_j^+(-s,w;\check{\chi})}{\Gamma((3/4)-s)} \right) \cdot \boldsymbol{b}.$$

On the other hand, Kummer's relations imply that $\boldsymbol{a} = \boldsymbol{G}\boldsymbol{b}$. Since the component functions of $\boldsymbol{b}$ are linearly independent, this, an elementary computation together with the identity $|\boldsymbol{G}| = (1/4-s)/(1/4+s)$ implies the result. □

PROOF OF (b). This is a direct consequence of the linear independence of the following functions of u:

$$F\left(s+1-w, s+w, \frac{3}{4}+s; \frac{1-iu}{2}\right) \quad \text{and} \quad F\left(s+1-w, s+w, \frac{5}{4}+s; \frac{1+iu}{2}\right). \quad \square$$

4. The converse theorem.

This section is devoted to the statement and proof of our main theorem. We begin by defining a "nice family" of double Dirichlet series.

DEFINITION 4.1. Let $N \geq 1$ be an integer and $\mathcal{F}_N := \{L_j^{\pm}(s,w;\chi)\}$ a family of double Dirichlet series as in Definition 1.1. We say $\mathcal{F}_N$ is "nice" if there exists another family $\check{\mathcal{F}}_N$ (called a contragredient family) of double Dirichlet series:

$$\check{L}_j^{\pm}(s,w;\chi)=\sum_{\pm n>0}\sum_{\ell=1}^{\infty}\frac{\check{a}_{n,\ell}^{j}\,\tau_n(\chi)}{\ell^w\,|n|^s},$$

with j ranging over $\{1,\dots,m^*\}$, D over the integers in $\{1,\dots,(4N)^2\}$ that are co-prime to N and, for each such D, χ ranging over the Dirichlet characters $\pmod D$, such that the following assumptions are satisfied for all $L_j^{\pm}(s,w;\chi)\in\mathcal{F}_N$.

ASSUMPTION (a). The functions $\Lambda_j(s,w;u;\chi) := \boldsymbol{c}(s,w;u)(L_j^{+}(s,w;\chi), L_j^{-}(s,w;\chi))^T$ have meromorphic continuations to $\mathbb{C}^2$. Furthermore, there exist meromorphic functions on $\mathbb{C}$, $a_j(w)$, $b_j(w)$, $\check{a_j}(w)$, $\check{b_j}(w)$, holomorphic for $\mathrm{Re}(w)\gg 1$, such that

$$\begin{aligned}&(2\sqrt{N}D)^s\Lambda_j(s,w,u;\chi)\\&\quad-(u^2+1)^{-s}\Bigg[(2\sqrt{N}D)^{-w}(u^2+1)^{(s-w)/2}\left(A\cdot\tau_0(\check{\chi})\frac{\check{b}_j(w)}{s-w}-\frac{\tau_0(\chi)b_j(w)}{s+w}\right)\\&\qquad+(2\sqrt{N}D)^{w-1}(u^2+1)^{(s+w-1)/2}\left(A\cdot\tau_0(\check{\chi})\frac{\check{a}_j(w)}{w+s-1}-\frac{\tau_0(\chi)a_j(w)}{s-w+1}\right)\Bigg]\end{aligned}$$

are EBV for every w (with $\mathrm{Re}(w)$ large enough) and every $u\in\mathbb{R}$.

ASSUMPTION (b).

$$\begin{aligned}&\left(\frac{\sqrt{N}D}{\pi}\right)^{2s}\chi(-4N)\left(\frac{4N}{D}\right)\epsilon_D\begin{pmatrix}L_j^{+}(s,w;\chi)\\L_j^{-}(s,w;\chi)\end{pmatrix}\\&\quad=\begin{pmatrix}\dfrac{\Gamma(w-s)\Gamma(1-s-w)}{\Gamma(s+(1/4))\Gamma((3/4)-s)} & \dfrac{\Gamma(w-s)\Gamma(1-s-w)}{\Gamma((1/4)+w)\Gamma((5/4)-w)}\\[2ex]\dfrac{\Gamma(w-s)\Gamma(1-s-w)}{\Gamma(w-(1/4))\Gamma((3/4)-w)} & \dfrac{\Gamma(w-s)\Gamma(1-s-w)}{\Gamma(s-(1/4))\Gamma((5/4)-s)}\end{pmatrix}\begin{pmatrix}\check{L}_j^{+}(-s,w;\chi)\\\check{L}_j^{-}(-s,w;\chi)\end{pmatrix}.\end{aligned}$$

ASSUMPTION (c). Let $\boldsymbol{L}^{\pm}(s,w;\chi):=(L_j^{\pm}(s,w;\chi))_{j=1,\dots,m^*}^T$. We assume the functional equations

$$\begin{aligned}\boldsymbol{L}^{+}(s,1-w;\chi)&=\Phi(1-w)\boldsymbol{L}^{+}(s,w;\chi)\quad\text{and}\\\boldsymbol{L}^{-}(s,1-w;\chi)&=\Phi(1-w)\boldsymbol{L}^{-}(s,w;\chi).\end{aligned}$$

The converse theorem we will prove states that a nice family of double Dirichlet series must be the family of L-functions arising from the Mellin transforms of metaplectic Eisenstein series which were introduced in Section 2.

THEOREM 4.2 (Converse theorem for double Dirichlet series). *Fix an integer $N\geq 1$ and let $\mathcal{F}_N$ denote a "nice family" of double Dirichlet series*

$$L_j^{\pm}(s,w;\chi)=\sum_{\pm n>0}\sum_{\ell=1}^{\infty}\frac{a_{n,\ell}^j\,\tau_n(\chi)}{\ell^w\,|n|^s},$$

with j ranging over $\{1,\dots,m^*\}$, D over the integers in $\{1,\dots,(4N)^2\}$ that are co-prime to N and, for each such D, χ ranging over the Dirichlet characters $(\mathrm{mod}\ D)$.

If $\tilde{\mathcal{F}}_N$ denotes the contragredient family of $\mathcal{F}_N$, define Dirichlet series

$$a_n^j(w):=\sum_{m=1}^{\infty}\frac{a_{n,m}^j}{m^w},\qquad \check{a}_n^j(w):=\sum_{m=1}^{\infty}\frac{\check{a}_{n,m}^j}{m^w},$$

and assume that, for each fixed j,w (with $\mathrm{Re}(w)\gg 1$), $|a_n^j(w)|,|\check{a}_n^j(w)|=O(|n|^C)$ for some $C>0$, as $n\to\infty$. Also let $a_j(w)$, $b_j(w)$ be the functions associated to $\mathcal{F}_N$ by Assumption (a).

Then, for

$$\boldsymbol{f}(z,w)=(f_1(z,w),\dots,f_{m^*}(z,w))^T,$$

where

$$f_j(z,w)=a_j(w)y^{1-w}+b_j(w)y^w+\sum_{n\neq 0}a_n^j(w)W_{\mathrm{sgn}(n)/4,\,w-(1/2)}(4\pi|n|y)e^{2\pi inx},\qquad (j=1,2,\dots,m^*),$$

we have

$$\boldsymbol{f}(z,w)=A(w)\boldsymbol{E}(z,w),\tag{9}$$

where $A(w)$ is a matrix of functions and $\boldsymbol{E}(z,w)$ is the matrix of Eisenstein series given in Section 2. *If $A(w)$ is meromorphic, then, for each w and $1-w$ which are not poles of $\Phi(w)$ and $A(w)$, we have*

$$\Phi(1-w)A(w)\Phi(w)=A(1-w).\tag{10}$$

Proof. We first prove that, for every w (with $\mathrm{Re}(w)$ large enough), $f_j(\cdot,w)$ is invariant under the action $|$ of $\Gamma_0(4N)$.

For every w with $\mathrm{Re}(w)$ large enough, $j=1,\dots,m^*$, every character $\chi \bmod D$, every $u\in\mathbb{R}$ and every $y>0$ define,

$$F_j(y,w,u;\chi):=\sum_{n\neq 0}a_n^j(w)\tau_n(\chi)W_{\mathrm{sgn}(n)/4,\,w-(1/2)}(4\pi|n|y)e^{2\pi inuy},$$

$$\check{F}_j(y,w,u;\chi):=\sum_{n\neq 0}\check{a}_n^j(w)\tau_n(\chi)W_{\mathrm{sgn}(n)/4,\,w-(1/2)}(4\pi|n|y)e^{2\pi inuy}.$$

We also set

$$\check{\Lambda}_j(s,w;u;\chi) := \boldsymbol{c}(s,w;u)\big(\check{L}_j^+(s,w;\chi),\check{L}_j^-(s,w;\chi)\big)^T.$$

Since for every w ($\mathrm{Re}(w)$ large enough), $|a_n^j(w)|$, $|\check{a}_n^j(w)| = O(|n|^C)$, in the Mellin transforms of $F_j(y,w,u;\chi)$ and $\check{F}_j(y,w,u;\chi)$ we can interchange summation and integration as in (3) to get, for $\mathrm{Re}(s)$ large enough

$$\int_0^\infty F_j(y,w,u;\chi)y^s\,\frac{dy}{y} = \Lambda_j(s,w;u;\chi), \qquad \int_0^\infty \check{F}_j(y,w,u;\chi)y^s\,\frac{dy}{y} = \check{\Lambda}_j(s,w;u;\chi).$$

For each w (with $\mathrm{Re}(w)$ large enough) and for $\mathrm{Re}(s)$ large enough, the components of $\boldsymbol{c}(s,w;u)$ decay exponentially in $|s|$ as $|s|\to\infty$ and as u ranges in suitably small neighborhood of 0. ([**2**, (1.11)]). So, we can apply Mellin inversion to get

$$\begin{aligned} F_j(y,w,u;\chi) &= \frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\Lambda_j(s,w;u;\chi)y^{-s}ds,\\ \check{F}_j(y,w,u;\chi) &= \frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\check{\Lambda}_j(s,w;u;\chi)y^{-s}ds \end{aligned} \tag{11}$$

for σ_0 large enough and a line of integration to the right of the poles of Λ_j and $\check{\Lambda}_j$. By the above estimate for the components of $\boldsymbol{c}(s,w;u)$, the standard Phragmén-Lindelöf argument applies. We can, therefore, move the line of integration from σ_0 to $\sigma_1 = -\sigma_0$ to get

$$\begin{aligned} F_j(y,w;u;\chi) &= \frac{1}{2\pi i}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty}\Lambda_j(s,w;u;\chi)y^{-s}ds + \sum_{s_0\ \text{pole}}\operatorname*{Res}_{s=s_0}\Lambda_j(s,w;u;\chi)y^{-s}\\ &= \frac{1}{2\pi i}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty}\Lambda_j(s,w;u;\chi)y^{-s}ds\\ &\quad + A\tau_0(\check{\chi})\big(\check{b}_j(w)((u^2+1)4ND^2y)^{-w} + \check{a}_j(w)((u^2+1)4ND^2y)^{w-1}\big)\\ &\quad - \tau_0(\chi)\big(b_j(w)y^w + a_j(w)y^{1-w}\big). \end{aligned} \tag{12}$$

The proof of Proposition 3.1 implies that Assumption (b) in the definition of a "nice family" of double Dirichlet series is equivalent to

$$\Lambda_j(s,w;u;\chi) = A(4ND^2)^{-s}(1+u^2)^{-s}\check{\Lambda}_j(-s,w;-u;\check{\chi}). \tag{13}$$

Therefore the last integral in (12) equals

$$\begin{aligned} &\int_{\sigma_1-i\infty}^{\sigma_1+i\infty}A(4ND^2)^{-s}(1+u^2)^{-s}\check{\Lambda}_j(-s,w;-u;\check{\chi})y^{-s}ds\\ &\quad = A\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\check{\Lambda}_j(s,w;-u;\check{\chi})(4ND^2(1+u^2)y)^s ds. \end{aligned} \tag{14}$$

However, if we set

$$\begin{aligned}\check{f}_j(z,w;\check{\chi}) &:= \tau_0(\check{\chi})\big(\check{a}_j(w)y^{1-w} + \check{b}_j(w)y^w\big)\\ &\quad + \sum_{n\neq 0} \check{a}_n^j(w)\tau(\check{\chi})W_{\mathrm{sgn}(n)/4,\,w-(1/2)}(4\pi|n|y)e^{2\pi inx},\end{aligned}$$

we have

$$\begin{aligned}f_j((u+i)y,w;\chi) &= F_j(y,w;u;\chi) + \tau_0(\chi)\big(b_j(w)y^w + a_j(w)y^{1-w}\big),\\ \check{f}_j((u+i)y,w;\check{\chi}) &= \check{F}_j(y,w;u;\check{\chi}) + \tau_0(\check{\chi})\big(\check{b}_j(w)y^w + \check{a}_j(w)y^{1-w}\big).\end{aligned}$$

Therefore, (12), (14) and (11) imply that

$$\begin{aligned}f_j((u+i)y,w;\chi) &= A\check{f}_j\bigg(\frac{i-u}{4ND^2(1+u^2)y},\ w;\ \check{\chi}\bigg) = A\check{f}_j\bigg(\frac{-1}{4ND^2(u+i)y},\ w;\ \check{\chi}\bigg)\\ &= \frac{(1+iu)^{1/4}}{(1-iu)^{1/4}}\overline{\chi(-4N)}\bigg(\frac{4N}{D}\bigg)\epsilon_D^{-1}\check{f}_j\bigg(\frac{-1}{4ND^2(u+i)y},\ w;\ \check{\chi}\bigg). \qquad (15)\end{aligned}$$

Since this holds for all $y>0, u\in\mathbb{R}$, this and the elementary identity $((u+i)y/|(u+i)y|)^{1/2} = e^{\pi i/4}(1+iu)^{-1/4}(1-iu)^{1/4}$ imply that

$$f_j\bigg(\frac{-1}{4ND^2z},\ w;\ \chi\bigg) = i^{-1/2}\overline{\chi(-4N)}\bigg(\frac{4N}{D}\bigg)\epsilon_D^{-1}\check{f}_j(z,w;\check{\chi})\bigg(\frac{z}{|z|}\bigg)^{1/2}. \qquad (16)$$

Together with (2), (16) implies that

$$\begin{aligned}&\sum_{\substack{r(\mathrm{mod}\,D)\\(r,D)=1}} \overline{\chi(r)}\check{f}_j\bigg|\begin{pmatrix} D & -r\\ -4mN & t\end{pmatrix}\bigg|\begin{pmatrix}1 & r/D\\ 0 & 1\end{pmatrix}\\ &\quad = \sum_{\substack{r(\mathrm{mod}\,D)\\(r,D)=1}} \overline{\chi(r)}\bigg(\frac{4Nr}{D}\bigg)\epsilon_D^{-1}\check{f}_j\bigg|\begin{pmatrix}1 & r/D\\ 0 & 1\end{pmatrix}. \qquad (17)\end{aligned}$$

Character summation then implies that

$$\check{f}_j\bigg|\begin{pmatrix} D & -r\\ -4mN & t\end{pmatrix} = \bigg(\frac{4Nr}{D}\bigg)\epsilon_D^{-1}\check{f}_j, \qquad (18)$$

or, with Lemma 2.1,

$$f_j\bigg|\begin{pmatrix} t & m\\ 4Nr & D\end{pmatrix} = \bigg(\frac{4Nr}{D}\bigg)\epsilon_D^{-1}f_j. \qquad (19)$$

However, the matrices on the left-hand side of (19) generate Γ.

LEMMA 4.3 ([**8**]). *Let $r \in \boldsymbol{Z}_+$. For D ranging in a set of congruence classes modulo $4Nr$ ($(D, 4Nr) = 1$) choose $\left(\begin{smallmatrix} t & m \\ 4Nr & D \end{smallmatrix}\right) \in \Gamma$. Denote the set of all such matrices by S_r. Then Γ is generated by*

$$\bigcup_{r=1}^{4N} S_r \cup \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

This implies that f_i is Γ-invariant.

The rest of the proof is identical to that of Theorem 3.1 of [**3**]. (But notice that the functional equations in Assumption (c) are employed in their equivalent form analogous to (8)). □

REMARK. For $u = 0$, Assumption (a) and (13) become the equations (9) and (10) respectively, of [**3**].

5. Scalar multiple Dirichlet series.

In this section we prove a scalar converse theorem for the case of $\Gamma_0(4)$. In this case, the corresponding families of double Dirichlet series collapse to sets of two elements only and, therefore, we can formulate the result in a much simpler way than Theorem 4.2. As for the corresponding result in [**3**] we modify our notation to agree with the formalism of [**11**].

Specifically, we set

$$j_{1/2}(\gamma, z) = v(\gamma)(cz + d)^{1/2}.$$

For every $\gamma, \delta \in \Gamma_0(4)$ and $z \in \mathfrak{H}$ we have

$$j_{1/2}(\gamma\delta, z) = j_{1/2}(\gamma, \delta z) j_{1/2}(\delta, z).$$

The group $\Gamma_0(4)$ now acts on functions f on $\mathfrak{H}$ by

$$(f|_{1/2}\gamma)(z) := f(\gamma z) j_{1/2}(\gamma, z)^{-1}, \qquad \gamma \in \Gamma_0(4).$$

Further, we will expand eigenfunctions of $\Delta_{1/2}$ in terms of the functions $y^s K_n(s, y) e^{2\pi i n x}$, where

$$K_n(s, y) = \int_{-\infty}^{\infty} \frac{e^{-2\pi i n x}}{(x^2 + y^2)^s (x + iy)^{1/2}} dx.$$

This is equivalent to the expansions in terms of $W_{\mathrm{sgn}(n)/4, w-(1/2)}(4\pi|n|y) e^{2\pi i n x}$ because of

LEMMA 5.1. *For every $n \in \mathbb{Z}$ with $n \neq 0$, $y > 0$ and $\mathrm{Re}(s)$ large enough we have*

$$K_n(s,y) = \frac{\pi^{s+(1/4)}|n|^{s-(3/4)}W_{\mathrm{sgn}(n)/4,\,s-(1/4)}(4\pi|n|y)}{e^{\pi i/4}y^{(1/4)+s}\Gamma(s+(1+\mathrm{sgn}(n))/4)}.$$

PROOF. See for instance, [**12**, pp. 84–85] and [**6**, 13.10.7]. □

The scalar converse theorem is essentially a converse theorem for a family $\mathcal{F}_1$ consisting of two double Dirichlet series

$$L^{\pm}(s,w) = \sum_{\pm n>0}\sum_{\ell=1}^{\infty}\frac{a_{n,\ell}}{\ell^w|n|^{(s-w+1)/2}}. \tag{20}$$

Note however that, in contrast to Definition 1.1, we do not index $\mathcal{F}_1$ by the (two) singular cusps of $\Gamma_0(4)$ in terms of v, or by characters. The reason we do not need to will become clear by the converse theorem we will prove. We have also normalized the exponent of $|n|$ in this way in order to be more consistent with the notation of [**11**]. We want to show that if the family $\mathcal{F}_1$ has "nice" properties, then $L^{\pm}(s,w)$ must be a linear combination of Mellin transforms of metaplectic Eisenstein series for $\Gamma_0(4)$. Accordingly, we now define the notion of a "nice" family $\mathcal{F}_1$ with root number $\epsilon = \pm 1$. We remark that the sign of the root number is independent of the sign in the L-functions $L^{\pm}(s,w)$.

DEFINITION 5.2. Let $\mathcal{F}_1$ be the family given in (20). We say $\mathcal{F}_1$ is a nice family with root number $\epsilon = \pm 1$ if the following assumptions are satisfied.

ASSUMPTION (A). The functions

$$(s+w-2)(s-w-1)L^{\pm}(s,w) \tag{21}$$

have meromorphic continuations to $s,w \in \mathbb{C}^2$ which are holomorphic if $\mathrm{Re}(w) \gg 1$. For $\mathrm{Re}(w) \gg 1$, we have the bound $(s+w-2)(s-w-1)L^{\pm}(s,w) = O(|\,\mathrm{Im}(s)|^b)$ on $\mathrm{Re}(s) = \sigma_0 \gg 1$ with $b > 0$ depending on σ_0. For $\mathrm{Re}(w) \gg 1$, we have the bound $(s+w-2)(s-w-1)L^{\pm}(s,w) = O(e^{|\,\mathrm{Im}(s)|^a})$ inside vertical strips in the s-plane.

ASSUMPTION (B). For root number $\epsilon = \pm 1$, we have the functional equation:

$$\begin{aligned}
&-\epsilon\cdot\pi^{s-(1/2)}\begin{pmatrix}L^+(1-s,w)\\ L^-(1-s,w)\end{pmatrix}\\
&\quad=\begin{pmatrix}
\dfrac{\Gamma((s+w)/2)\Gamma((1+s-w)/2)}{\Gamma((1-s)/2)\Gamma((1+s)/2)} & \dfrac{\Gamma((s+w)/2)\Gamma((1+s-w)/2)}{\Gamma(w/2)\Gamma((2-w)/2)}\\
\dfrac{\Gamma((s+w)/2)\Gamma((1+s-w)/2)}{\Gamma((w+1)/2)\Gamma((1-w)/2)} & \dfrac{\Gamma((s+w)/2)\Gamma((1+s-w)/2)}{\Gamma((s+2)/2)\Gamma(-s/2)}
\end{pmatrix}\begin{pmatrix}L^+(s,w)\\ L^-(s,w)\end{pmatrix}.
\end{aligned} \tag{22}$$

ASSUMPTION (C). Let $G(w) = \xi(2w)\Gamma(w/2)\pi^{-w/2}$ with $\xi(w) = \zeta(w)\Gamma(w/2)\cdot\pi^{-w/2}$. Then we have the functional equation:

$$G(1-w)\pi^{(1-w)/2}\begin{pmatrix}L^+(s,1-w)\\ L^-(s,1-w)\end{pmatrix}$$
$$= G(w)\pi^{w/2}\begin{pmatrix}\dfrac{\Gamma((2-w)/2)}{\Gamma((1+w)/2)} & 0\\ 0 & \dfrac{\Gamma((1-w)/2)}{\Gamma(w/2)}\end{pmatrix}\begin{pmatrix}L^+(s,w)\\ L^-(s,w)\end{pmatrix}. \tag{23}$$

THEOREM 5.3. *Let $\mathcal{F}_1$ be a nice family of double Dirichlet series*

$$L^{\pm}(s,w)=\sum_{\pm n>0}\sum_{\ell=1}^{\infty}\frac{a_{n,\ell}}{\ell^w|n|^{(s-w+1)/2}}$$

with root number $\epsilon=\pm 1$. For $w\in\mathbb{C}$ with $\mathrm{Re}(w)$ large enough, define

$$a_n(w):=\sum_{\ell=1}^{\infty}\frac{a_{n,\ell}}{\ell^w},$$

and assume that for each fixed $w\in\mathbb{C}$ ($\mathrm{Re}(w)\gg 1$) we have the bound $a_n(w)=O(|n|^C)$ for some fixed $C>0$ as $n\to\pm\infty$.

Then there exists a meromorphic function $b:\mathbb{C}\to\mathbb{C}$, holomorphic for $\mathrm{Re}(w)$ large enough, satisfying

$$b(w)\zeta(1-w)(2^{1-w}-\epsilon)=b(1-w)\zeta(w)(2^w-\epsilon) \tag{24}$$

and such that for

$$f(z,w)=b(w)y^{w/2}+\frac{b(1-w)G(1-w)}{G(w)}y^{(1-w)/2}+\sum_{n\neq 0}a_n(w)y^{w/2}K_n\Big(\frac{w}{2},y\Big)e^{2\pi inx},$$

we have

$$f(z,w)=b(w)\Big(-\epsilon\frac{e^{\pi i/4}}{\sqrt{2}}z^{-1/2}E\Big(-\frac{1}{4z},\frac{w}{2}\Big)+E\Big(z,\frac{w}{2}\Big)\Big) \tag{25}$$

for each $w\in\mathbb{C}$ for which w, $1-w$ are not poles of $b(w)$ and $E(z,w/2)$. Here

$$E(z,s)=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(4)}\frac{\mathrm{Im}(\gamma z)^s}{j_{1/2}(\gamma,z)}.$$

PROOF. We shall first introduce some auxiliary functions depending on an additional real parameter u.

For every w with $\mathrm{Re}(w)$ large enough, set

$$\Lambda(s,w;u)=\frac{\Gamma((s-w+1)/2)\Gamma((s+w)/2)}{e^{\pi i/4}2^{s-1/2}\pi^{(s-w-1)/2}}\big(G^+(s,w;u)L^+(s,w)+G^-(s,w;u)L^-(s,w)\big)$$

for each $u\in\mathbb{R}$ and each s with $\mathrm{Re}(s)$ large enough. Here

$$G^+(s,t;u):=\frac{F((s+t)/2,(s-t+1)/2,(s+1)/2;(1+iu)/2)}{\Gamma((t+1)/2)\Gamma((s+1)/2)} \tag{26}$$

and

$$G^-(s,t;u):=\frac{F((s+t)/2,(s-t+1)/2,(s+2)/2;(1-iu)/2)}{\Gamma(t/2)\Gamma((s+2)/2)}. \tag{27}$$

Further set

$$\mathcal{L}(s,w;u):=2^{s/2}(u^2+1)^{s/4}(s+w-2)(s-w+1)(s-w-1)(w+s)\Lambda(s,w;u).$$

In exactly the same way as in Proposition 3.1, we deduce that (22) is equivalent to

$$2^{s/2}(1-iu)^{s/2}\Lambda(s,w;u)=-\epsilon\,2^{(1-s)/2}(1+iu)^{(1-s)/2}\Lambda(1-s,w;-u). \tag{28}$$

Also, with (28)

$$(1-iu)^{1/4}\mathcal{L}(s,w;u)=-\epsilon\,(1+iu)^{1/4}\mathcal{L}(1-s,w;-u). \tag{29}$$

We will need two lemmas in order to state a condition implied by Assumption (A) of Definition 5.2.

Lemma 5.4. *For each fixed $u\in\mathbb{R}$, $\mathcal{L}(s,w;u)$ is meromorphic in $\mathbb{C}^2$ and holomorphic if $\mathrm{Re}(w)\gg 1$.*

Proof of Lemma 5.4. Let $w\in\mathbb{C}$ with $\mathrm{Re}(w)\gg 1$. Note that for $z\notin[1,\infty)$, the function $F(a,b,c;z)/\Gamma(c)$ is entire in a,b,c, (cf. [**6**, Section 15.2(ii)]). Therefore, with Assumption (A), the polar divisors of $\mathcal{L}(s,w;u)$ can only occur at the poles of

$$\Gamma\bigg(\frac{s-w+1}{2}\bigg)\Gamma\bigg(\frac{s+w}{2}\bigg),$$

i.e., $s=-w-2k$ or $s=w-1-2k$ $(k=0,1,\dots)$. With the functional equation (29), this implies that any polar divisors must be of the form $s=1+w+2k$ or $s=2-w+2k$. Upon substituting such values into the two-variable function $\Gamma((s-w+1)/2)\Gamma((s+w)/2)$ we deduce that only isolated points can arise as poles, which is a contradiction. □

Lemma 5.5. *Let $u\in\mathbb{R}$ and $w\in\mathbb{C}$ (with $\mathrm{Re}(w)\gg 1$) be fixed. Then for every $c>0$, and $\sigma_1<\sigma_2$,*

$$\mathcal{L}(s,w;u)=O(|\,\mathrm{Im}(s)|^{-c}) \tag{30}$$

uniformly in $\mathrm{Re}(s)$ *for all* $\sigma_1 \le \mathrm{Re}(s) \le \sigma_2$.

PROOF OF LEMMA 5.5. Let $s_0 = \sigma_0 + i\tau_0$ with σ_0 large enough. On the vertical line $\mathrm{Re}(s) = \sigma_0$, Stirling's estimate implies that, for $\mathrm{Im}(s) \to \pm\infty$,

$$\begin{aligned}\left|\Gamma\left(\frac{s-w+1}{2}\right)\Gamma\left(\frac{s+w}{2}\right)\right| \Big/ \left|\Gamma\left(\frac{s+1}{2}\right)\right| \\ \sim \sqrt{2\pi}|\operatorname{Im}(s)|^{\alpha} e^{-(\pi/2)(|\operatorname{Im}((s-w+1)/2)|+|\operatorname{Im}((s+w)/2)|-|\operatorname{Im}((s+1)/2)|)} \\ = \sqrt{2\pi}|\operatorname{Im}(s)|^{\alpha} e^{-(\pi/4)|\operatorname{Im}(s)|}\end{aligned}$$

for an $\alpha \in \mathbb{R}$. We have (cf. e.g. [**6**, (15.8.1)]) the identity

$$F\left(\frac{s+w}{2}, \frac{s-w+1}{2}, \frac{s+1}{2}; \frac{1-iu}{2}\right) = \left(\frac{1+iu}{2}\right)^{-s/2} F\left(\frac{1-w}{2}, \frac{w}{2}, \frac{s+1}{2}; \frac{1-iu}{2}\right).$$

For a $C > 0$, $F(a,b,c;z) \sim C$ as $|c| \to \infty$, with a, b fixed, $\mathrm{Re}(z) = 1/2$ and $|\operatorname{Arg}(c)| \le \pi - \delta$ for a $\delta > 0$ ([**6**, (15.12.2)]). Also, $|((1+iu)/2)^{-s/2}| = |(1+iu)/2|^{-\operatorname{Re}(s)/2} \cdot e^{\operatorname{Arg}((1+iu)/2)\operatorname{Im}(s)/2}$. Hence the absolute value of the function multiplied to $L^+(s,w)$ in Λ is asymptotic to a constant times

$$\frac{|\operatorname{Im}(s)|^{\alpha}|(1+iu)/2|^{-\sigma_0/2}}{2^{\sigma_0-1/2}\pi^{(\sigma_0-\operatorname{Re}(w)-1)/2}} e^{-(\pi/2-\operatorname{sgn}(\operatorname{Im}(s))\operatorname{Arg}((1+iu)/2))|\operatorname{Im}(s)|/2}$$

on $\mathrm{Re}(s) = \sigma_0$ as $\mathrm{Im}(s) \to \pm\infty$. Since $|\operatorname{Arg}((1+iu)/2)| < \pi/2$, this, together with Assumption (A) of Definition 5.2, implies (30) on $\mathrm{Re}(s) = \sigma_0$ as $\mathrm{Im}(s) \to \pm\infty$ for the piece of $\mathcal{L}$ corresponding to $L^+(s,w)$. The bound for the piece corresponding to $L^-(s,w)$ is verified similarly.

To establish the corresponding bound on $\mathrm{Re}(s) = 1 - \sigma_0$ we note that, for w and u assumed fixed, (29) implies that $\mathcal{L}(1-s, w; -u) = O(\mathcal{L}(s,w;u))$ on $\mathrm{Re}(s) = 1-\sigma_0$. Equation (30) on $\mathrm{Re}(s) = \sigma_0$ we proved above implies the desired bound for that vertical line.

Finally we note that Stirling's estimate and the bound for $F(a,b,c;z)$ are uniform for s within a vertical strip. With the last part of Assumption (A) of Definition 5.2, we deduce $\mathcal{L}(s,w;u) = O(e^{|\operatorname{Im}(s)|^a})$ for some $a \in \mathbb{R}$ when $1-\sigma_0 \le \mathrm{Re}(s) \le \sigma_0$. By the Phragmén-Lindelöf principle, this completes the proof of the lemma for all intervals $[1-\sigma_0, \sigma_0]$ with σ_0 large enough and therefore, for all closed intervals. □

We are now ready to identify the function $b(w)$ mentioned in the statement of Theorem 5.3 and to state a holomorphicity and boundedness condition we will use to prove the theorem.

Let $a_1(w;u)$, $a_2(w;u)$, $a_3(w;u)$ and $a_4(w;u)$ be the residues of $2^{s/2}(u^2+1)^{s/4}\Lambda(s,w;u)$ at $2-w$, $w-1$, $w+1$ and $-w$ respectively. Then, from Lemma 5.4 we deduce that the function

$$2^{s/2}(u^2+1)^{s/4}\Lambda(s,w;u) - \frac{a_1(w;u)}{s+w-2} - \frac{a_2(w;u)}{s-w+1} - \frac{a_3(w;u)}{s-w-1} - \frac{a_4(w;u)}{s+w}$$

is holomorphic for $\mathrm{Re}(w) \gg 1$.

By the defining formula for $\mathcal{L}(s,w;u)$ we deduce that

$$a_1(w;u) = \frac{\mathcal{L}(2-w,w;u)}{2(3-2w)(1-2w)}, \tag{31}$$

$$a_2(w;u) = -\frac{\mathcal{L}(w-1,w;u)}{2(3-2w)(1-2w)}, \tag{32}$$

$$a_3(w;u) = \frac{\mathcal{L}(w+1,w;u)}{2(2w-1)(2w+1)}, \tag{33}$$

$$a_4(w;u) = -\frac{\mathcal{L}(-w,w;u)}{2(2w-1)(2w+1)}. \tag{34}$$

LEMMA 5.6. *The functions*

$$\frac{a_2(w;u)}{(1+u^2)^{(w-1)/4}} \quad \textit{and} \quad \frac{a_4(w;u)}{(1+u^2)^{-w/4}}$$

are independent of u. *As functions of* w, *they are meromorphic in* $\mathbb{C}$ *and holomorphic for* $\mathrm{Re}(w) \gg 1$.

PROOF OF LEMMA 5.6. With the defining formulas for a_2 and $\mathcal{L}(s,w;u)$ we see that u appears in

$$\frac{a_2(w;u)}{(1+u^2)^{(w-1)/4}}$$

only in the hypergeometric functions in (26) and (27). However, for our combination of arguments we obtain $F(a,0,b;z)$ for some $a,b,z \in \mathbb{C}$, which equals 1. The assertion about holomorphicity/meromorphicity in w follows from Lemma 5.4.

Similarly for $a_4(w;u)/(1+u^2)^{-w/4}$. □

This lemma implies that the following two functions are meromorphic in $\mathbb{C}$ and holomorphic if $\mathrm{Re}(w) \gg 1$:

$$a(w) := -\frac{a_2(w;u)}{2^{(1+w)/2}(1+u^2)^{(w-1)/4}}$$

and

$$b(w) := -\frac{a_4(w;u)}{2^{(2-w)/2}(1+u^2)^{-w/4}}.$$

Therefore, with the above choice of a,b, we have the following Lemma.

LEMMA 5.7. *For every $u \in \mathbb{R}$ and every w with $\mathrm{Re}(w)$ large enough,*

$$\Lambda(s,w;u) + a(w)2^{(1-s+w)/2}(u^2+1)^{(w-s-1)/4}\left(\frac{(1+iu)^{1/2}(1+u^2)^{-1/4}}{\epsilon\cdot(s+w-2)} + \frac{1}{s-w+1}\right)$$
$$+ b(w)2^{(2-s-w)/2}(u^2+1)^{-(s+w)/4}\left(\frac{(1+iu)^{1/2}(1+u^2)^{-1/4}}{\epsilon\cdot(s-w-1)} + \frac{1}{w+s}\right) \tag{35}$$

is EBV as a function of s.

PROOF OF LEMMA 5.7. With Lemma 5.6 and (29), we have

$$a_1(w;u) = -\epsilon\cdot\frac{(1+iu)^{1/4}(1-iu)^{-1/4}\mathcal{L}(w-1,w;-u)}{2(3-2w)(1-2w)}$$
$$= \epsilon\cdot\frac{(1+iu)^{1/4}}{(1-iu)^{1/4}}a_2(w;u) = -\epsilon\cdot 2^{(1+w)/2}(1+u^2)^{(w-2)/4}(1+iu)^{1/2}a(w).$$

Similarly,

$$a_3(w;u) = -\epsilon\cdot 2^{(2-w)/2}(1+u^2)^{(-w-1)/4}(1+iu)^{1/2}b(w).$$

Therefore (35) is entire.

To obtain the boundedness in a vertical strip V we observe that, since (35) is entire, it will be bounded in the rectangle $\{s\in V; |\,\mathrm{Im}(s)| \le |\,\mathrm{Im}(w)|+1\}$. For $s\in V$ with $|\,\mathrm{Im}(s)| > |\,\mathrm{Im}(w)|+1$, we have

$$|s+w-2| \ge |\,\mathrm{Im}(s)+\mathrm{Im}(w)| \ge |\,\mathrm{Im}(s)| - |\,\mathrm{Im}(w)| > 1,$$

and likewise $|s-w+1|, |s-w-1|, |s+w| > 1$. These inequalities together with Lemma 5.5 imply the boundedness in vertical strips. □

COMPLETION OF PROOF OF THEOREM 5.3. For every w with $\mathrm{Re}(w)$ large enough and every $y>0$, define

$$F(y,w;u) = \sum_{n\neq 0} a_n(w)y^{w/2}K_n\left(\frac{w}{2},y\right)e^{2\pi inuy}.$$

As in the proof of Theorem 4.2, we see that, for s with $\mathrm{Re}(s)$ large enough, we have

$$\int_0^\infty y^{s/2}F(y,w;u)\frac{dy}{y} = \Lambda(s,w;u).$$

Lemma 5.7 allows us to use the Phragmén-Lindelöf argument to see that, in the inverse Mellin transform of $\Lambda(2s,w;u)$, we can move the line of integration from σ_0 to $\sigma_1 = 1/2-\sigma_0$ to get

$$\begin{aligned}F(y,w;u) &= \frac{1}{2\pi i}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty}\Lambda(2s,w;u)y^{-s}ds + \sum_{s_0 \text{ pole}}\operatorname*{Res}_{s=s_0}\Lambda(2s,w;u)y^{-s}\\ &= \frac{1}{2\pi i}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty}\Lambda(2s,w;u)y^{-s}ds\\ &\quad - \epsilon\cdot a(w)y^{(w-2)/2}2^{w-3/2}(1+iu)^{(w-1)/2}(1-iu)^{(w-2)/2}\\ &\quad - \epsilon\cdot b(w)y^{-(w+1)/2}2^{-w-1/2}(1+iu)^{-w/2}(1-iu)^{-(w+1)/2}\\ &\quad - a(w)y^{-(w-1)/2} - b(w)y^{w/2}. \end{aligned} \tag{36}$$

Therefore, with (28), the last integral in (36) equals

$$\begin{aligned}&- 2^{1/2}\epsilon(1+iu)^{1/2}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty}(4(1+u^2))^{-s}\Lambda(1-2s,w;-u)y^{-s}ds\\ &\quad = -\epsilon\cdot(2y(1-iu))^{-1/2}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\Lambda(2s,w;-u)(4(1+u^2)y)^s ds. \end{aligned} \tag{37}$$

This together with

$$f(uy+iy,w) = F(y,w;u) + b(w)y^{w/2} + a(w)y^{(1-w)/2}$$

implies that

$$f(uy+iy,w) = -\epsilon\cdot(2y(1-iu))^{-1/2}f\left(\frac{i-u}{4(1+u^2)y},w\right). \tag{38}$$

With $u = x/y$, this gives

$$f(z,w) = -\epsilon\cdot\frac{e^{\pi i/4}}{\sqrt{2}}z^{-1/2}f\left(\frac{-1}{4z},w\right). \tag{39}$$

On the other hand, $\left(\begin{smallmatrix}0 & -1\\ 4 & 0\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & -1\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}0 & 1/4\\ -1 & 0\end{smallmatrix}\right) = \left(\begin{smallmatrix}1 & 0\\ 4 & 1\end{smallmatrix}\right)$. Then, a computation implies that, for all w with $\mathrm{Re}(w)$ large enough, $f(\cdot,w)$ is invariant under $\left(\begin{smallmatrix}1 & 0\\ 4 & 1\end{smallmatrix}\right)$, in both cases. Since $\Gamma_0(4)$ is generated by $\left(\begin{smallmatrix}1 & 0\\ 4 & 1\end{smallmatrix}\right)$ and the translations, this proves that, for all w with $\mathrm{Re}(w)$ large enough, $f(z,w)$ satisfies the weight $1/2$ transformation law for $\Gamma_0(4)$.

The growth at the cusps can be deduced as in Theorem 3.1 of [**3**] because that part does not depend on the Dirichlet series $\tilde{\Lambda}$ used in [**3**].

Now, by [**7**, Satz 10.1 (3)], the $\Gamma_0(4)$-invariance and the moderate growth at the cusps we proved, we deduce that, for $\mathrm{Re}(w)$ large enough, $y^{1/4}f(\cdot,w-1/2)$ is the sum of a weight $1/2$ Maass cusp form g and a linear combination of the weight $1/2$ Eisenstein series at the $m^* = 2$ cusps of $\Gamma_0(4)$ that are singular in terms of v: $E_j(\cdot,w/2)$ $(j = 1,\dots,m^*)$ (in the notation of Section 2). The cusp form g must in fact vanish for $\mathrm{Re}(w)$ large enough. Otherwise, it is an eigenfunction of the Laplacian with eigen-value $(w/2)((w/2)-1)$

because it is a linear combination of $y^{1/4}f(\cdot, w-1/2)$ and $E_j(\cdot, w/2)$, $j = 1, \dots, m^*$. This is a contradiction because the discrete spectrum of $\Delta_{1/2}$ lies in $(-\infty, -3/16]$ ([**7**, Satz 5.5]), but, for $\mathrm{Re}(w)$ large enough, $(w/2)((w/2)-1)$ cannot be a real number $\leq -3/16$.

One easily sees that the singular cusps in terms of v are 0 and ∞. Therefore, for $\mathrm{Re}(w)$ large enough $y^{1/4}f(z, w-1/2)$ is a linear combination of $E_1(z, w/2)$ and $E_2(z, w/2)$. Since a computation implies that these are constant multiples of the functions $y^{1/4}E(z, w/2-1/4)$ and $y^{1/4}z^{-1/2}E(-1/(4z), w/2-1/4)$ respectively, we deduce that

$$f(z,w) = \alpha(w)z^{-1/2}E\left(-\frac{1}{4z}, \frac{w}{2}\right) + \beta(w)E\left(z, \frac{w}{2}\right) \tag{40}$$

for some functions α and β.

Upon substituting (40) into (39), and taking into account the linear independence of the functions $z^{-1/2}E(-1/(4z), w/2)$ and $E(z, w/2)$, we deduce that

$$\alpha(w) = -\epsilon \cdot \frac{e^{\pi i/4}}{\sqrt{2}}\beta(w). \tag{41}$$

However, the constant terms at infinity of $E(z, w/2)$ and $z^{-1/2}E(-1/(4z), w/2)$ are

$$y^{w/2} + \frac{2^{-2w}}{1-2^{-2w}}\frac{\xi(2w-1)}{\xi(2w)}y^{1/2-w/2} \quad \text{and} \quad \frac{e^{-\pi i/4}(1-2^{1-2w})}{2^{w-1/2}(1-2^{-2w})}\frac{\xi(2w-1)}{\xi(2w)}y^{1/2-w/2}, \tag{42}$$

respectively (cf. [**4**]). Therefore, upon comparison of the coefficients of $y^{w/2}$ on both sides of (40) we deduce that $\beta(w) = b(w)$ and, with (41), $\alpha(w) = -\epsilon \cdot (e^{\pi i/4}/\sqrt{2})b(w)$.

This implies equation (25) for $\mathrm{Re}(w)$ large enough and such that $w, 1-w$ are not poles of $a(w), b(w)$ and $E(z, w/2)$. Then $f(z, w)$ can be extended to a meromorphic function in $w \in \mathbb{C}$ by (25).

Finally, (23) implies a functional equation for $b(w)$. A computation implies that (23) is equivalent to:

$$G(w)\Lambda(s, w; u) = G(1-w)\Lambda(s, 1-w; u) \tag{43}$$

for all $u \in \mathbb{R}$. This, with (35), implies that for $\mathrm{Re}(w)$ large enough

$$\begin{aligned} &2^{(1+w)/2}(u^2+1)^{(w-1)/4}\left(\frac{(1+iu)^{1/2}(1+u^2)^{-1/4}}{s+w-2} + \frac{1}{s-w+1}\right) \\ &\quad \times (a(w)G(w) - b(1-w)G(1-w)) \\ &+ 2^{(2-w)/2}(u^2+1)^{-w/4}\left(\frac{(1+iu)^{1/2}(1+u^2)^{-1/4}}{s-w-1} + \frac{1}{w+s}\right) \\ &\quad \times (b(w)G(w) - a(1-w)G(1-w)) \end{aligned} \tag{44}$$

must be entire. Therefore, for all w with $\mathrm{Re}(w)$ large enough and such that $w, 1-w$ are not poles of $a(w)$ and of $b(w)$,

$$a(w)G(w) = b(1-w)G(1-w),$$

otherwise (44) would have a pole at $s = 2 - w$.

Thus, the constant term of $f(z, w)$ at infinity is

$$b(w)y^{w/2} + \frac{b(1-w)G(1-w)}{G(w)}y^{(1-w)/2}. \tag{45}$$

With (40), (41) and (42), we have that the coefficient of $y^{(1-w)/2}$ is also

$$b(w)\frac{\xi(2-2w)(2^{1-w}-\epsilon)}{\xi(2w)(2^w-\epsilon)}.$$

Therefore, with (45) and $\zeta(w)/\zeta(1-w) = (\Gamma((1-w)/2)\pi^{-(1-w)/2})/(\Gamma(w/2)\pi^{-w/2})$, we deduce (24). □

REMARK. We can compare Theorem 5.3 with Theorem 4.2 (for $N = D = 1$ and the trivial character) by making the change of variables $(s, w) \to (s/2 - 1/4, w/2 + 1/2)$. However, upon applying this change of variables to (iii′), one notices that some entries of the 2×2 matrix involved do not match the corresponding entries of (22).

The reason is that the normalization of the completed L-function used in Theorem 4.2 differs from that of Theorem 5.3: In (3) the denominators in $\boldsymbol{c}(s, w; u)$ contain only one Gamma function whereas in the analogous normalizer in (26) there are two. This is because of the different forms of Fourier expansion used. The first uses Whittaker W-functions but the second uses K-functions which, by Lemma 5.1, has a Gamma function in the denominator.

The effect this has on the way the transformation works is that we have different cancellations of the various Gamma functions and this accounts for the different forms of the functional equations. (But one can pass from one to the other using Lemma 5.1.)

Also, we note that in Proposition 3.1 we have a different L-function in the RHS of the equation (which we denote by $\check{\Lambda}$) whereas in Theorem 5.3 we do not. This is because in $\Gamma_0(4)$ (as in $SL_2(\boldsymbol{Z})$) we can arrange the functional equations so that we have self-contragredient L-functions (essentially by applying the equation of Proposition 3.1 to $\Lambda(s) + \check{\Lambda}(s)$).

6. Shintani's double Dirichlet series.

In [**13**], four double Dirichlet series are introduced and studied. They are defined for s_1, s_2 with $\mathrm{Re}(s_i) > 1$ by

$$\xi_i(s_1, s_2) = 2^{-1}\sum_{n,m=1}^{\infty} A(4m, (-1)^{i-1}n)m^{-s_1}n^{-s_2}$$

and

$$\xi_i^*(s_1,s_2)=\sum_{n,m=1}^{\infty}A(m,(-1)^{i-1}n)m^{-s_1}(4n)^{-s_2},$$

where $A(m,n)$ denotes the number of distinct solutions of the congruence $x^2\equiv n \bmod m$.

These series can be viewed as zeta functions associated with prehomogeneous vector spaces (cf. [**10**, Section 7.2] for a detailed discussion of this interpretation). Properties of general zeta functions associated with prehomogeneous vector spaces are proved in [**9**].

In this section, we will use Theorem 5.3 to prove that these series, appropriately normalized, are essentially Mellin transforms of linear combinations of metaplectic Eisenstein series. To this end, we first re-state Theorem 1 of [**13**] (see also [**10**, Theorem 4]) in a form which will be more convenient for our purposes.

THEOREM 6.1. (i) *For $i=1,2$, the series*

$$(s_2-1)(s_1+s_2-3/2)\xi_i(s_1,s_2)\quad\text{and}\quad(s_2-1)(s_1+s_2-3/2)\xi_i^*(s_1,s_2)$$

are absolutely convergent for $\mathrm{Re}(s_1)$, $\mathrm{Re}(s_2)>1$. *They have meromorphic continuations to $\mathbb{C}^2$ that are holomorphic in $s_1,s_2\in\mathbb{C}$ with* $\mathrm{Re}(s_1)>1$.

(ii) *The following functional equations hold*

$$\begin{aligned}&\begin{pmatrix}\xi_1(s_1,3/2-s_1-s_2)\\ \xi_2(s_1,3/2-s_1-s_2)\end{pmatrix}\\ &\quad=R(s_1,s_2)\begin{pmatrix}\sin(\pi(s_1/2+s_2)) & \sin(\pi s_1/2)\\ \cos(\pi s_1/2) & \cos(\pi(s_1/2+s_2))\end{pmatrix}\begin{pmatrix}\xi_1^*(s_1,s_2)\\ \xi_2^*(s_1,s_2)\end{pmatrix}\end{aligned}\tag{46}$$

with $R(s_1,s_2):=2^{-1}\pi^{1/2}(2/\pi)^{s_1+2s_2}\Gamma(s_2)\Gamma(s_1+s_2-1/2)$, *and*

$$\begin{aligned}&\zeta(2-2s_1)\begin{pmatrix}\xi_1^{(*)}(1-s_1,s_1+s_2-1/2)\\ \xi_2^{(*)}(1-s_1,s_1+s_2-1/2)\end{pmatrix}\\ &\quad=\frac{2}{\pi}(2\pi)^{1-2s_1}\cos\left(\frac{\pi s_1}{2}\right)\Gamma(s_1)^2\zeta(2s_1)\cdot\begin{pmatrix}\cos(\pi s_1/2) & 0\\ 0 & \sin(\pi s_1/2)\end{pmatrix}\begin{pmatrix}\xi_1^{(*)}(s_1,s_2)\\ \xi_2^{(*)}(s_1,s_2)\end{pmatrix}.\end{aligned}\tag{47}$$

Here the superscript $(*)$ indicates that the equation holds for both ξ_i and ξ_i^*.

To state our theorem we introduce some notation. For $i=1,2$ set

$$\psi_i(s_1,s_2)=\sum_{n,m=1}^{\infty}A(4m,(-1)^{i-1}n)(2m)^{-s_1}n^{-s_2}=2^{1-s_1}\xi_i(s_1,s_2),\tag{48}$$

$$\psi_i^*(s_1,s_2)=\sum_{n,m=1}^{\infty}A(m,(-1)^{i-1}n)m^{-s_1}n^{-s_2}=4^{s_2}\xi_i^*(s_1,s_2). \tag{49}$$

Further let $c_n(w)$ (resp. $c_n^*(w)$) denote the numerator of $n^{(s-w+1)/2}$ in the series expansion of

$$\psi_1\biggl(w,\frac{s-w+1}{2}\biggr)\qquad\biggl(\text{resp. }\psi_1^*\biggl(w,\frac{s-w+1}{2}\biggr)\biggr),$$

if $n>0$ and of $(-n)^{(s-w+1)/2}$ in the series expansion of

$$\psi_2\biggl(w,\frac{s-w+1}{2}\biggr)\qquad\biggl(\text{resp. }\psi_2^*\biggl(w,\frac{s-w+1}{2}\biggr)\biggr),$$

when $n<0$.

With this notation we have

THEOREM 6.2. *There are meromorphic functions $b_1(w),b_2(w)$ on $\mathbb{C}$ which are holomorphic for* $\mathrm{Re}(w)$ *large enough such that*

$$\begin{aligned}&2\sum_{n\neq0}c_n(w)y^{w/2}K_n\biggl(\frac{w}{2},y\biggr)e^{2\pi inx}\\&\quad+\frac{G(1-w)}{G(w)}(b_1(1-w)+b_2(1-w))y^{(1-w)/2}+(b_1(w)+b_2(w))y^{w/2}\\&=(b_1(w)+b_2(w))E\biggl(z,\frac{w}{2}\biggr)+\frac{e^{\pi i/4}}{\sqrt{2}}(b_1(w)-b_2(w))z^{-1/2}E\biggl(-\frac{1}{4z},\frac{w}{2}\biggr)\end{aligned}$$

for all w that are not poles of $b_i(w),b_i(1-w)$ and $G(w),G(1-w)$. Further, $b_1(w),b_2(w)$ satisfy

$$\zeta(1-w)(2^{1-w}-(-1)^i)b_i(w)=\zeta(w)(2^w-(-1)^i)b_i(1-w),\quad i=1,2.$$

PROOF. We will apply Theorem 5.3 to

$$\boldsymbol{L}_1(s,w):=\begin{pmatrix}L_1^+(s,w)\\L_1^-(s,w)\end{pmatrix}:=\begin{pmatrix}\psi_1\biggl(w,\dfrac{s-w+1}{2}\biggr)+\psi_1^*\biggl(w,\dfrac{s-w+1}{2}\biggr)\\\psi_2\biggl(w,\dfrac{s-w+1}{2}\biggr)+\psi_2^*\biggl(w,\dfrac{s-w+1}{2}\biggr)\end{pmatrix}$$

$$\biggl(\text{resp. }\boldsymbol{L}_2(s,w):=\begin{pmatrix}L_2^+(s,w)\\L_2^-(s,w)\end{pmatrix}:=\begin{pmatrix}\psi_1\biggl(w,\dfrac{s-w+1}{2}\biggr)-\psi_1^*\biggl(w,\dfrac{s-w+1}{2}\biggr)\\\psi_2\biggl(w,\dfrac{s-w+1}{2}\biggr)-\psi_2^*\biggl(w,\dfrac{s-w+1}{2}\biggr)\end{pmatrix}\biggr).$$

Firstly, it is clear that, for fixed w with $\mathrm{Re}(w)$ large enough and for $\mathrm{Re}(s)$ large enough, $L_1^{\pm}$ (resp. $L_2^{\pm}$) form a family of double Dirichlet series $\mathcal{F}_1$ of the form shown in (20) for some $a_{n,\ell} \in \mathbb{C}$ of polynomial growth.

Further, since by Theorem 6.1 (i), $\xi_i^{(*)}(s_1, s_2)$ converge for $\mathrm{Re}(s_1)$, $\mathrm{Re}(s_2) > 1$, $L_1^{\pm}(s, w)$ (resp. $L_2^{\pm}(s, w)$) converge absolutely as series of the form (20), for fixed $w \in \mathbb{C}$ with $\mathrm{Re}(w)$ large enough and for $s \in \mathbb{C}$ with $\mathrm{Re}(s)$ large enough. This implies the required bound for the numerators

$$c_n(w) + c_n^*(w) \quad \big(\text{resp. } c_n(w) - c_n^*(w)\big),$$

of $|n|^{(s-w+1)/2}$ in the series expansion of $L_1^{\pm}(s, w)$ (resp. $L_2^{\pm}(s, w)$).

We next show that $L_1^{\pm}$ (resp. $L_2^{\pm}$) form a "nice" family of root number $\epsilon = -1$ (resp. $\epsilon = 1$). We will first verify Assumptions (B) and (C) of Definition 5.2. With (46) and the identity $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ we deduce that ψ_i and ψ_i^* satisfy

$$\begin{aligned}
&\begin{pmatrix} \psi_1(s_1, 3/2 - s_1 - s_2) \\ \psi_2(s_1, 3/2 - s_1 - s_2) \end{pmatrix} \\
&= \pi^{3/2-s_1-2s_2} \begin{pmatrix} \dfrac{\Gamma(s_2)\Gamma(s_1+s_2-1/2)}{\Gamma(s_2+s_1/2)\Gamma(1-s_1/2-s_2)} & \dfrac{\Gamma(s_2)\Gamma(s_1+s_2-1/2)}{\Gamma(s_1/2)\Gamma(1-s_1/2)} \\ \dfrac{\Gamma(s_2)\Gamma(s_1+s_2-1/2)}{\Gamma((1-s_1)/2)\Gamma((1+s_1)/2)} & \dfrac{\Gamma(s_2)\Gamma(s_1+s_2-1/2)}{\Gamma(1/2-s_1/2-s_2)\Gamma(1/2+s_1/2+s_2)} \end{pmatrix} \\
&\quad \times \begin{pmatrix} \psi_1^*(s_1, s_2) \\ \psi_2^*(s_1, s_2) \end{pmatrix}. \qquad (50)
\end{aligned}$$

From this and an inversion of the 2×2 matrix on the RHS we deduce that the same functional equation is satisfied with the ψ_i and ψ_i^* interchanged. Therefore, with $s_1 = w$ and $s_2 = (s - w + 1)/2$ we deduce that $\boldsymbol{L}_1(s, w)$ (resp. $\boldsymbol{L}_2(s, w)$) satisfies (22) thus confirming Assumption (B) of Definition 5.2.

Furthermore, multiplying both sides of (47) with $\Gamma(1 - s_1)$ and using the identity $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$, we deduce, for $i = 1, 2$,

$$\begin{aligned}
&\zeta(2-2s_1)\Gamma(1-s_1) \begin{pmatrix} \xi_1^{(*)}(1-s_1, s_1+s_2-1/2) \\ \xi_2^{(*)}(1-s_1, s_1+s_2-1/2) \end{pmatrix} \\
&\quad = 2^{2-2s_1}\pi^{-2s_1}\Gamma(s_1)\frac{\pi}{\sin(\pi s_1)}\cos\left(\frac{\pi s_1}{2}\right)\zeta(2s_1) \\
&\qquad \times \begin{pmatrix} \sin(\pi(1-s_1)/2) & 0 \\ 0 & \sin(\pi s_1/2) \end{pmatrix} \begin{pmatrix} \xi_1^{(*)}(s_1, s_2) \\ \xi_2^{(*)}(s_1, s_2) \end{pmatrix}. \qquad (51)
\end{aligned}$$

Therefore

$$(2\pi)^{s_1}\zeta(2-2s_1)\Gamma(1-s_1)\Gamma\left(\frac{1-s_1}{2}\right)\begin{pmatrix}\xi_1^{(*)}(1-s_1,s_1+s_2-1/2)\\ \xi_2^{(*)}(1-s_1,s_1+s_2-1/2)\end{pmatrix}$$
$$=(2\pi)^{1-s_1}\zeta(2s_1)\Gamma(s_1)\Gamma\left(\frac{s_1}{2}\right)\begin{pmatrix}\dfrac{\Gamma(1-s_1/2)}{\Gamma((1+s_1)/2)} & 0\\ 0 & \dfrac{\Gamma((1-s_1)/2)}{\Gamma(s_1/2)}\end{pmatrix}\begin{pmatrix}\xi_1^{(*)}(s_1,s_2)\\ \xi_2^{(*)}(s_1,s_2)\end{pmatrix}. \tag{52}$$

This implies immediately

$$\pi^{s_1}\zeta(2-2s_1)\Gamma(1-s_1)\Gamma\left(\frac{1-s_1}{2}\right)\begin{pmatrix}\psi_1(1-s_1,s_1+s_2-1/2)\\ \psi_2(1-s_1,s_1+s_2-1/2)\end{pmatrix}$$
$$=\pi^{1-s_1}\zeta(2s_1)\Gamma(s_1)\Gamma\left(\frac{s_1}{2}\right)\begin{pmatrix}\dfrac{\Gamma(1-s_1/2)}{\Gamma((1+s_1)/2)} & 0\\ 0 & \dfrac{\Gamma((1-s_1)/2)}{\Gamma(s_1/2)}\end{pmatrix}\begin{pmatrix}\psi_1(s_1,s_2)\\ \psi_2(s_1,s_2)\end{pmatrix}. \tag{53}$$

Further, applying (52) to ξ_i^* and multiplying both sides with $2^{s_1}4^{s_2-1/2}$ we deduce the functional equation (53) for ψ_i^*. The substitution $s_1=w$ and $s_2=(s-w+1)/2$ then implies (23), confirming Assumption (C) of Definition 5.2.

To verify Assumption (A) we use Theorem 6.1 (i). With the change of variables $s_1=w$ and $s_2=(s-w+1)/2$ we deduce that $(s+w-2)(s-w-1)L_i^{\pm}(s,w)$ $(i=1,2)$ are meromorphic in $\mathbb{C}^2$ and holomorphic in $\{(s,w);s\in\mathbb{C},\mathrm{Re}(w)>1\}$.

Next, for fixed w with $\mathrm{Re}(w)$ large enough, consider s with $\mathrm{Re}(s)=\sigma_0$ large enough (e.g. such that $\mathrm{Re}(s-w+1)/2>1$). Since, by Theorem 6.1 (i), $\xi_i^{(*)}(w,(s-w+1)/2)$ are absolutely convergent for such s,w, $L_i^{\pm}(s,w)$ are bounded on the vertical line $\mathrm{Re}(s)=\sigma_0$. This implies the second part of Assumption (A) of Definition 5.2.

Finally, the proof of Theorem 1 of [**13**] implies that, for $\mathrm{Re}(w)$ large enough, $\xi_i^{(*)}(s,w)$ are, for some b, of order $e^{|\operatorname{Im} s|^b}$. From this we deduce the last part of Assumption (A) of Definition 5.2.

Therefore, all conditions of Theorem 5.3 are satisfied for $\boldsymbol{L}_1$ (resp. $\boldsymbol{L}_2$). Hence, if we set

$$f_1(z,w)=\sum_{n\neq0}(c_n(w)+c_n^*(w))y^{w/2}K_n\left(\frac{w}{2},y\right)e^{2\pi inx}$$

and

$$f_2(z,w)=\sum_{n\neq0}(c_n(w)-c_n^*(w))y^{w/2}K_n\left(\frac{w}{2},y\right)e^{2\pi inx},$$

we deduce that, for some functions $b_1(w)$ (resp. $b_2(w)$) satisfying the conditions of the theorem, we have

$$f_1(z,w)+\frac{b_1(1-w)G(1-w)}{G(w)}y^{(1-w)/2}+b_1(w)y^{w/2}$$
$$=b_1(w)\left(\frac{e^{\pi i/4}}{\sqrt{2}}z^{-1/2}E\left(-\frac{1}{4z},\frac{w}{2}\right)+E\left(z,\frac{w}{2}\right)\right)$$

and

$$f_2(z,w)+\frac{b_2(1-w)G(1-w)}{G(w)}y^{(1-w)/2}+b_2(w)y^{w/2}$$
$$=b_2(w)\left(-\frac{e^{\pi i/4}}{\sqrt{2}}z^{-1/2}E\left(-\frac{1}{4z},\frac{w}{2}\right)+E\left(z,\frac{w}{2}\right)\right)$$

for each $w\in\mathbb{C}$ for which $w,1-w$ are not poles of $b_i(w)$ and $G(w)$.

Adding these two equations we deduce the theorem. □

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