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# Surface links with free abelian groups

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**Abstract.** It is known that if a classical link group is a free abelian group, then its rank is at most two. It is also known that a $k$-component 2-link group ($k > 1$) is not free abelian. In this paper, we give examples of $T^2$-links each of whose link groups is a free abelian group of rank three or four. Concerning the $T^2$-links of rank three, we determine the triple point numbers and we see that their link types are infinitely many.

## Introduction.

A *classical link* is the image of a smooth embedding of a disjoint union of circles into the Euclidean 3-space $\mathbb{R}^3$. The *link group* is the fundamental group of the link exterior. It is known [**13**, Theorem 6.3.1] that if a classical link group is a free abelian group, then its rank is at most two. A *surface link* is the image of a smooth embedding of a closed surface into the Euclidean 4-space $\mathbb{R}^4$. A *2-link* (resp. $T^2$*-link*) is a surface link whose components are homeomorphic to 2-spheres (resp. tori). It is known [**7**, Chapter 3, Corollary 2] that a $k$-component 2-link group for $k > 1$ is not a free abelian group. The aim of this paper is to give concrete examples of $T^2$-links whose link groups are free abelian.

It is known (see Remark 2.1) that a $T^2$-link called a "Hopf 2-link" [**5**] has a free abelian group of rank two. We give $T^2$-links with a free abelian group of rank three (Theorem 2.2). We also give a $T^2$-link with a free abelian group of rank four (Theorem 2.3). These $T^2$-links are "torus-covering $T^2$-links", which are $T^2$-links in the form of unbranched coverings over the standard torus.

Further we study the $T^2$-links given in Theorem 2.2 i.e. $T^2$-links each of whose link groups is a free abelian group of rank three. We determine the triple point number of each $T^2$-link (Theorem 3.1), by which we can see that their link types are infinitely many. The triple point number of each $T^2$-link is a multiple of four, and it is realized by a surface diagram in the form of a covering over the torus. For other examples of surface links (not necessarily orientable) which realize large triple point numbers, see [**6**], [**9**], [**12**], [**16**], [**17**], [**19**].

The paper is organized as follows. In Section 1, we review the definition of a torus-covering $T^2$-link, and we review a formula how to calculate its link group. In Section 2, we show Theorems 2.2 and 2.3. In Section 3, we show Theorem 3.1.

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## 1. A torus-covering $T^2$-link and its link group.

In this section, we give the definition of a torus-covering $T^2$-link $\mathcal{S}_m(a,b)$, which is determined from a pair of commuting $m$-braids $a$ and $b$ called basis braids. For the definition of a torus-covering link whose component might be of genus more than one, see [**15**]. We can compute the link group of $\mathcal{S}_m(a,b)$ by using Artin's automorphism associated with $a$ or $b$ [**15**].

### 1.1.

Let $T$ be the standard torus in $\mathbb{R}^4$, i.e. the boundary of an unknotted solid torus in $\mathbb{R}^3 \times \{0\} \subset \mathbb{R}^4$. Let $N(T)$ be a tubular neighborhood of $T$ in $\mathbb{R}^4$.

Definition 1.1. A *torus-covering* $T^2$*-link* is a surface link $F$ in $\mathbb{R}^4$ such that $F$ is embedded in $N(T)$ and $p|_F : F \to T$ is an unbranched covering map, where $p : N(T) \to T$ is the natural projection.

Let us consider a torus-covering $T^2$-link $F$. Let us fix a point $x_0$ of $T$, and take a meridian $\boldsymbol{m}$ and a longitude $\mathbf{l}$ of $T$ with the base point $x_0$. A *meridian* is an oriented simple closed curve on $T$ which bounds a 2-disk in the solid torus whose boundary is $T$ and which is not null-homologous in $T$. A *longitude* is an oriented simple closed curve on $T$ which is null-homologous in the complement of the solid torus in the three space $\mathbb{R}^3 \times \{0\}$ and which is not null-homologous in $T$. The intersections $F \cap p^{-1}(\boldsymbol{m})$ and $F \cap p^{-1}(\mathbf{l})$ are closures of classical braids. Cutting open the solid tori at the 2-disk $p^{-1}(x_0)$, we obtain a pair of classical braids. We call them *basis braids* [**15**]. The basis braids of a torus-covering $T^2$-link are commutative, and for any commutative braids $a$ and $b$, there exists a unique torus-covering $T^2$-link with basis braids $a$ and $b$ [**15**, Lemma 2.8]. For commutative $m$-braids $a$ and $b$, we denote by $\mathcal{S}_m(a,b)$ the torus-covering $T^2$-link with basis $m$-braids $a$ and $b$.

### 1.2.

We can compute the link group of a torus-covering $T^2$-link $\mathcal{S}_m(a,b)$ [**15**]. As preliminaries, we will give the definition of Artin's automorphism (see [**11**]). Let $c$ be an $m$-braid in a cylinder $D^2 \times [0,1]$, and let $Q_m$ be the starting point set of $c$. Let $\{h_u\}_{u\in[0,1]}$ be an isotopy of $D^2$ rel $\partial D^2$ such that $\cup_{u\in[0,1]} h_u(Q_m) \times \{u\} = c$. Let $\mathcal{A}^c : (D^2, Q_m) \to (D^2, Q_m)$ be the terminal map $h_1$, and consider the induced map $\mathcal{A}^c_* : \pi_1(D^2 - Q_m) \to \pi_1(D^2 - Q_m)$. It is known [**1**] that $\mathcal{A}^c_*$ is uniquely determined from $c$. We call $\mathcal{A}^c_*$ *Artin's automorphism* associated with $c$. Note that $\pi_1(D^2 - Q_m)$ is naturally isomorphic to the free group $F_m$ generated by the standard generators $x_1, x_2, \ldots, x_m$ of $\pi_1(D^2 - Q_m)$. By $\mathcal{A}^c_*$, the braid group $B_m$ acts on $\pi_1(D^2 - Q_m)$. It is presented by

$$\mathcal{A}^{\sigma_i}_*(x_j) = \begin{cases} x_j x_{j+1} x_j^{-1} & \text{if } j = i \\ x_{j-1} & \text{if } j = i+1 \\ x_j & \text{otherwise} \end{cases}$$

and
$$\mathcal{A}_*^{\sigma_i^{-1}}(x_j) = \begin{cases} x_{j+1} & \text{if } j = i \\ x_j^{-1}x_{j-1}x_j & \text{if } j = i+1 \\ x_j & \text{otherwise} \end{cases}$$

where $i = 1, 2, \ldots, m-1$ and $j = 1, 2, \ldots, m$.

It is known [**15**, Proposition 3.1] that the link group of $\mathcal{S}_m(a, b)$ is presented by

$$\pi_1(\mathbb{R}^4 - \mathcal{S}_m(a, b)) = \langle x_1, \ldots, x_m \mid x_j = \mathcal{A}_*^a(x_j) = \mathcal{A}_*^b(x_j), \text{ for } j = 1, 2, \ldots, m\rangle.$$

## 2. $T^2$-links whose link groups are free abelian.

In this section we show Theorems 2.2 and 2.3: There are torus-covering $T^2$-links with a free abelian group of rank three (Theorem 2.2) or four (Theorem 2.3).

REMARK 2.1. A *Hopf* 2-*link* [**5**] is a $T^2$-link which is the product of a classical Hopf link in $B^3$ with $S^1$, embedded into $\mathbb{R}^4$ via an embedding of $B^3 \times S^1$ into $\mathbb{R}^4$, where $B^3$ is a 3-ball and $S^1$ is a circle. There are two link types according to the embedding of $B^3 \times S^1$, called a standard Hopf 2-link and a twisted Hopf 2-link [**5**]. A standard (resp. twisted) Hopf 2-link is the spun $T^2$-link (resp. the turned spun $T^2$-link) of a classical Hopf link [**14**], [**2**], [**3**]. It is known [**14**], [**2**], [**3**] that the link group of the spun $T^2$-link or the turned spun $T^2$-link of a classical link $L$ is isomorphic to the classical link group of $L$. Thus we can see that a Hopf 2-link has a free abelian link group of rank two.

Let $\sigma_1, \sigma_2, \ldots, \sigma_{m-1}$ be the standard generators of $B_m$.

THEOREM 2.2. *The link group of* $\mathcal{S}_3(\sigma_1^2\sigma_2^{2n}, \Delta)$ *is a free abelian group of rank three, where* $n$ *is an integer and* $\Delta$ *is a full twist of a bundle of three parallel strings.*

PROOF. Put $S_n = \mathcal{S}_3(\sigma_1^2\sigma_2^{2n}, \Delta)$. Let us compute the link group $G_n = \pi_1(\mathbb{R}^4 - S_n)$ by applying [**15**, Proposition 3.1]. Let $x_1$, $x_2$ and $x_3$ be the generators. Then the relations concerning the basis braid $\sigma_1^2\sigma_2^{2n}$ are

$$x_1x_2 = x_2x_1, \tag{2.1}$$
$$(x_2x_3)^{|n|} = (x_3x_2)^{|n|}. \tag{2.2}$$

The other relations concerning the other basis braid $\Delta$ are

$$x_1 = (x_1x_2x_3)x_1(x_1x_2x_3)^{-1},$$
$$x_2 = (x_1x_2x_3)x_2(x_1x_2x_3)^{-1},$$
$$x_3 = (x_1x_2x_3)x_3(x_1x_2x_3)^{-1},$$

which are

$$x_1x_2x_3 = x_2x_3x_1, \tag{2.3}$$

$$x_2(x_1x_2x_3) = (x_1x_2x_3)x_2, \tag{2.4}$$

$$x_3x_1x_2 = x_1x_2x_3. \tag{2.5}$$

By (2.1), (2.3) is deformed to $x_2x_1x_3 = x_2x_3x_1$; hence

$$x_1x_3 = x_3x_1. \tag{2.6}$$

Similarly, by (2.4) and (2.1),

$$x_2x_3 = x_3x_2. \tag{2.7}$$

We can see that all the relations are generated by the three relations (2.1), (2.6) and (2.7). Thus we have

$$G_n = \langle x_1, x_2, x_3 \mid x_1x_2 = x_2x_1,\ x_2x_3 = x_3x_2,\ x_3x_1 = x_1x_3\rangle,$$

which is a free abelian group of rank three. □

Theorem 2.3. *The link group of $\mathcal{S}_4(\sigma_1^2\sigma_2^2\sigma_3^2, \Delta)$ is a free abelian group of rank four, where $\Delta$ is a full twist of a bundle of 4 parallel strings.*

Proof. Similarly to the proof of Theorem 2.2, by [**15**, Proposition 3.1], for generators $x_1$, $x_2$, $x_3$ and $x_4$, we have the following relations:

$$x_ix_{i+1} = x_{i+1}x_i, \tag{2.8}$$

where $i = 1, 2, 3$, and

$$x_i = (x_1x_2x_3x_4)x_i(x_1x_2x_3x_4)^{-1}, \tag{2.9}$$

where $i = 1, 2, 3, 4$. Using $x_1x_2 = x_2x_1$ and $x_3x_4 = x_4x_3$ of (2.8), the latter four relations (2.9) are deformed as follows:

$$x_1x_3x_4 = x_3x_4x_1, \tag{2.10}$$

$$x_2x_3x_4 = x_3x_4x_2, \tag{2.11}$$

$$x_3x_1x_2 = x_1x_2x_3, \tag{2.12}$$

$$x_4x_1x_2 = x_1x_2x_4. \tag{2.13}$$

By $x_2x_3 = x_3x_2$ of (2.8), (2.11) is deformed to $x_3x_2x_4 = x_3x_4x_2$; hence

$$x_2x_4 = x_4x_2. \tag{2.14}$$

Similarly, by $x_2x_3 = x_3x_2$ of (2.8) and (2.12),

$$x_3x_1 = x_1x_3, \tag{2.15}$$

and by (2.14) and (2.13),

$$x_4x_1 = x_1x_4. \tag{2.16}$$

We can see that all the relations are generated by the relations (2.8), (2.14), (2.15) and (2.16). Thus the link group is a free abelian group of rank four. □

## 3. The triple point numbers of the $T^2$-links with a free abelian group of rank three.

The *triple point number* of a surface link $F$ is the minimal number of triple points among all the surface diagrams of $F$. In this section we study the $T^2$-links given in Theorem 2.2 i.e. $T^2$-links each of whose link group is a free abelian group of rank three.

THEOREM 3.1. *The triple point number of $S_n = \mathcal{S}_3(\sigma_1^2\sigma_2^{2n}, \Delta)$ given in Theorem 2.2 is $4n$ for $n > 0$ and $4(1-n)$ for $n \leq 0$. Further it is realized by a surface diagram in the form of a covering over $T$, in other words, by a 3-chart on $T$ which presents $S_n$. Thus $T^2$-links with a free abelian group of rank three are infinitely many.*

Here, a 3-chart [**11**] is a finite graph with certain additional data, which we review in Section 3.1.

This section is organized as follows. In Section 3.1, we review a surface diagram and an $m$-chart on $T$ which presents a torus-covering $T^2$-link (see [**15**], [**11**]). In Section 3.2, we review the result of [**16**] which gives lower bounds of triple point numbers. In Section 3.3, we prove Theorem 3.1.

### 3.1. Surface diagrams and $m$-charts presenting torus-covering $T^2$-links.

The notion of an $m$-chart on a 2-disk was introduced by Kamada [**8**] (see also [**11**]) to present a surface braid i.e. a 2-dimensional braid in a bi-disk (see [**18**], [**11**]). An $m$-chart on a disk is obtained from the singularity set of a surface diagram of a surface braid. By a minor modification, we can define an $m$-chart on $T$ presenting a torus-covering link [**15**].

For a torus-covering $T^2$-link $F$, we consider a surface diagram in the form of a covering over the torus, as in Section 3.1.1.. Given $F$, we obtain such a surface diagram $D$, and from $D$ we obtain a graph called an $m$-chart on $T$ (without black vertices). Conversely, an $m$-chart on $T$ without black vertices presents such a surface diagram and hence a torus-covering $T^2$-link.

#### 3.1.1. Surface diagrams.

We review a surface diagram of a surface link $F$ (see [**4**]). For a projection $\pi : \mathbb{R}^4 \to \mathbb{R}^3$, the closure of the self-intersection set of $\pi(F)$ is called the singularity set. Let $\pi$ be a generic projection, i.e. the singularity set of the image $\pi(F)$ consists of double

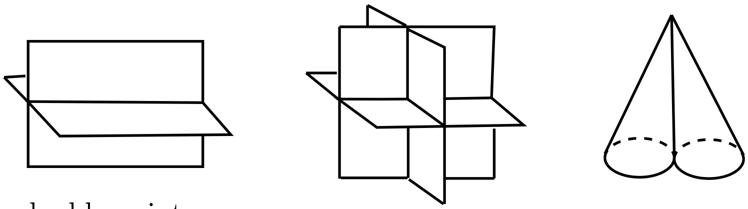


Figure 3.1. The singularity of a surface diagram.

points, isolated triple points, and isolated branch points; see Figure 3.1. The closure of the singularity set forms a union of immersed arcs and loops, which we call double point curves. Triple points (resp. branch points) form the intersection points (resp. the end points) of the double point curves. A *surface diagram* of $F$ is the image $\pi(F)$ equipped with over/under information along each double point curve with respect to the projection direction.

Throughout this paper, we consider the surface diagram of a torus-covering $T^2$-link $F$ by the projection which projects $N(T) = I \times I \times T$ to $I \times T$ for an interval $I$, where we identify $N(T)$ with $I \times I \times T$ in such a way as follows. Since $T$ is the boundary of the standard solid torus in $\mathbb{R}^3 \times \{0\}$, the normal bundle of $T$ in $\mathbb{R}^3 \times \{0\}$ is a trivial bundle. We identify it with $I \times T$. Then we identify $N(T)$ with $I \times I \times T$, where the second $I$ is an interval in the fourth axis of $\mathbb{R}^4$. Perturbing $F$ if necessary, we can assume that this projection is generic. We call this surface diagram *the surface diagram of $F$ in the form of a covering over the torus.*

### 3.1.2. From surface diagrams to $m$-charts on $T$.

Given a torus-covering $T^2$-link $F$, we obtain a graph on $T$ from the surface diagram in the form of a covering over the torus, as follows. Now we have $\mathrm{Sing}(\pi(F))$ in $I \times T$. By the definition of a torus-covering $T^2$-link, $\mathrm{Sing}(\pi(F))$ consists of double point curves and triple points, and no branch points. We can assume that the singular set of the image of $\mathrm{Sing}(\pi(F))$ by the projection to $T$ consists of a finite number of double points such that the preimages belong to double point curves of $\mathrm{Sing}(\pi(F))$. Thus the image of $\mathrm{Sing}(\pi(F))$ by the projection to $T$ forms a finite graph $\Gamma$ on $T$ such that the degree of its vertex is either 4 or 6. An edge of $\Gamma$ corresponds to a double point curve, and a vertex of degree 6 corresponds to a triple point.

For such a graph $\Gamma$ obtained from the surface diagram, we give orientations and labels to the edges of $\Gamma$, as follows. Let us consider a path $l$ in $T$ such that $l \cap \Gamma$ is a point $P$ of an edge $e$ of $\Gamma$. Then $F \cap p^{-1}(l)$ is a classical $m$-braid with one crossing in $p^{-1}(l)$ such that $P$ corresponds to the crossing of the $m$-braid. Let $\sigma_i^{\epsilon}$ ($i \in \{1, 2, \ldots, m-1\}$, $\epsilon \in \{+1, -1\}$) be the presentation of $F \cap p^{-1}(l)$. Then label the edge $e$ by $i$, and moreover give $e$ an orientation such that the normal vector of $l$ corresponds (resp. does not correspond) to the orientation of $e$ if $\epsilon = +1$ (resp. $-1$). We call such an oriented and labeled graph an *$m$-chart of $F$* (without black vertices).

In general, we define an $m$-chart on $T$ as follows.

Definition 3.2. Let $m$ be a positive integer, and let $\Gamma$ be a finite graph on $T$. Then $\Gamma$ is called an *$m$-chart on $T$* if it satisfies the following conditions:

( i ) Every edge is oriented and labeled by an element of $\{1, 2, \ldots, m-1\}$.
( ii ) Every vertex has degree 1, 4, or 6.
(iii) The adjacent edges around each vertex are oriented and labeled as shown in Figure 3.2, where we depict a vertex of degree 1 (resp. 6) by a black vertex (resp. white vertex).

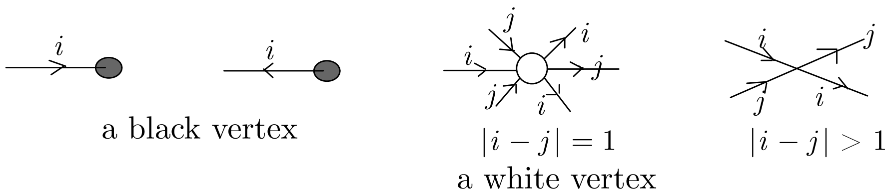


Figure 3.2. Vertices in an $m$-chart.

A black vertex presents a branch point; see [**11**]. When an $m$-chart on $T$ without black vertices is given, we can reconstruct a torus-covering $T^2$-link [**15**] (see also [**11**]).

Two $m$-charts on $T$ are *C-move equivalent* [**15**] (see also [**8**], [**10**], [**11**]) if they are related by a finite sequence of ambient isotopies of $T$ and CI, CII, CIII-moves. We show several examples of CI-moves in Figure 3.3; see [**11**] for the complete set of CI-moves and CII, CIII-moves. For two $m$-charts on $T$, their presenting torus-covering links are equivalent if the $m$-charts are C-move equivalent [**15**] (see also [**8**], [**10**], [**11**]).

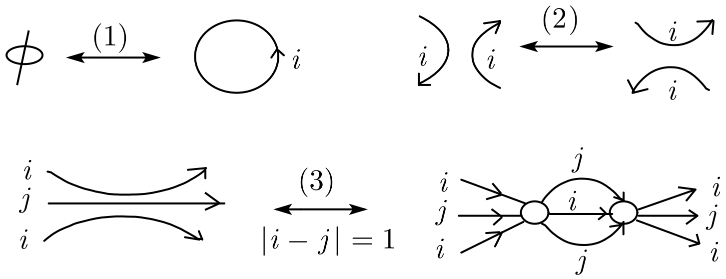


Figure 3.3. CI-moves. We give only several examples.

### 3.2. Triple point numbers.

For a surface link $F$, we denote by $t(F)$ the triple point number of $F$. It is shown [**16**] that for a pure $m$-braid $b$ ($m \geq 3$) and an integer $n$, a lower bound of $t(\mathcal{S}_m(b, \Delta^n))$ is given by using the linking numbers of $\hat{b}$, and for a particular $b$, we can determine the triple point number. Here $\hat{b}$ denotes the closure of $b$.

For a pure 3-braid $b$, it follows from [**16**] that we can give a lower bound of $t(\mathcal{S}_3(b, \Delta))$ as follows. We define the $i$th component of $\hat{b}$ by the component constructed by the $i$th string of $\hat{b}$ ($i = 1, 2, 3$). For positive integers $i$ and $j$ with $i \neq j$, the *linking number* of the $i$th and $j$th components of a classical link $L$, denoted by $\mathrm{Lk}_{i,j}(L)$, is the total number of positive crossings minus the total number of negative crossings of a diagram of $L$ such that the under-arc (resp. over-arc) is from the $i$th (resp. $j$th) component. Put $\mu = \sum_{i<j} |\mathrm{Lk}_{i,j}(\hat{b})|$, and put $\nu = \nu_{1,2,3} + \nu_{2,3,1} + \nu_{3,1,2}$, where $\nu_{i,j,k} = \min_{i,j,k}\{|\mathrm{Lk}_{i,j}(\hat{b})|, |\mathrm{Lk}_{j,k}(\hat{b})|\}$ if $\mathrm{Lk}_{i,j}(\hat{b})\mathrm{Lk}_{j,k}(\hat{b}) > 0$ and otherwise zero. Then, by [**16**],

$$t(\mathcal{S}_3(b, \Delta)) \geq 4(\mu - \nu).$$

In particular, let $b$ be a 3-braid presented by a braid word which is an element of a monoid generated by $\sigma_1^2$ and $\sigma_2^{-2}$; note that $b$ is a pure braid. Then

$$t(\mathcal{S}_3(b, \Delta)) = 4\mu,$$

and the triple point number is realized by a surface diagram in the form of a covering over the torus [**16**].

### 3.3. Proof of Theorem 3.1.

Put $b = \sigma_1^2\sigma_2^{2n}$. We use the notations given in Section 3.2. Since $\mathrm{Lk}_{i,j}(\hat{b}) = 1$ (resp. $n$) if $\{i,j\} = \{1,2\}$ (resp. $\{2,3\}$) and otherwise zero, we can see that $\mu = 1 + |n|$.

Let us consider the case for $n \leq 0$. Since $b$ has the presentation which is an element of a monoid generated by $\sigma_1^2$ and $\sigma_2^{-2}$, $t(S_n) = 4\mu$ by [**16**]; thus $t(S_n) = 4(1-n)$ $(n \leq 0)$, and the triple point number is realized by a surface diagram in the form of a covering over the torus by [**16**].

Let us consider the case for $n > 0$. Since $\mathrm{Lk}_{i,j}(\hat{b}) = 1$ (resp. $n$) if $\{i,j\} = \{1,2\}$ (resp. $\{2,3\}$) and otherwise zero, we can see that $\nu_{i,j,k} = 1$ if $(i,j,k) = (1,2,3)$ and zero if $(i,j,k) = (2,3,1)$ or $(3,1,2)$; thus $\nu = 1$, and hence $t(S_n) \geq 4(\mu - \nu) = 4n$ by [**16**].

It remains to show that there is a surface diagram of $S_n$ $(n > 0)$ with $4n$ triple points. It suffices to draw a 3-chart $\Gamma$ on $T$ which presents $S_n$ such that $\Gamma$ has exactly $4n$ white vertices. We draw $\Gamma$ which presents $S_n$, and deform it to a 3-chart with $4n$ white vertices by C-moves, as follows. First we draw $\Gamma$ as a 3-chart which consists of $2n+2$ parts as follows, where we assume that a full twist $\Delta$ has the presentation $\Delta = (\sigma_1\sigma_2\sigma_1)^2$.

( i ) The part of $\Gamma$ with basis braids $\sigma_1$ and $\Delta$. We have two copies.
( ii ) The part of $\Gamma$ with basis braids $\sigma_2$ and $\Delta$. We have $2n$ copies.

We draw the part (i) as in Figure 3.4 and we denote the white vertices by $t_{i1}$ and $t_{i2}$ as in Figure 3.4 for the $i$th copy $(i = 1, 2)$. We draw the part (ii) as in Figure 3.5 and we denote the white vertices by $t_{i1}$ and $t_{i2}$ as in Figure 3.5 for the $(i-2)$th copy

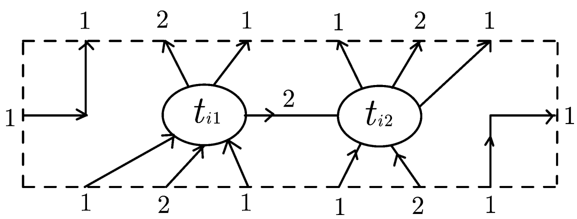


Figure 3.4. White vertices $t_{i1}$ and $t_{i2}$ $(i = 1, 2)$.

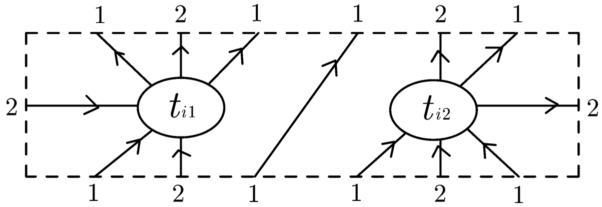


Figure 3.5. White vertices $t_{i1}$ and $t_{i2}$ ($i = 3, 4, \dots, 2n + 2$), for $n > 0$.

($i = 3, 4, \dots, 2n + 2$). There are $4n + 4$ white vertices in $\Gamma$. Let us apply a CI-move as in Figure 3.3 (3) to the pair $\{t_{21}, t_{31}\}$ of white vertices in $\Gamma$, and then to the pair $\{t_{(2n+2)2}, t_{12}\}$; then we can eliminate the four white vertices, and the resulting 3-chart has $4n$ white vertices. Hence $t(S_n) = 4n$ ($n > 0$), and the triple point number is realized by this 3-chart on $T$. □

REMARK 3.3. There is an oriented $T^2$-link as in Figure 3.6 with a free abelian group of rank three and with the triple point number zero. It is a ribbon $T^2$-link (see [**4**] for the definition of a ribbon surface link). We briefly show that the link group is free abelian, as follows. In the surface diagram, there are six broken sheets (see [**4**]), consisting of three pairs of a sheet attached with $x_i$ and a small disk $D_i$ such that each pair forms the $i$th component of the $T^2$-link ($i = 1, 2, 3$). Let us attach $y_i$ to each $D_i$. The link group has the presentation with generators $x_i$ and $y_i$ ($i = 1, 2, 3$) and the relations which are given around each double point curve (see [**4**], [**11**]). The singularity set consists of double point curves which form six circles. Around each circle in the $i$th component which does not bound $D_i$ ($i = 1, 2, 3$), there are three broken sheets such that one is an over-sheet with $x_i$ and the other two are under-sheets with the same generator $x_{i+1}$, where $x_4 = x_1$; together with the orientation, the relation is $x_i = x_{i+1} x_i x_{i+1}^{-1}$, see [**4**], [**11**]. Around each circle $\partial D_i$ ($i = 1, 2, 3$), there are three broken sheets such that one is an over-sheet with $x_{i+1}$ and the other two are under-sheets with $x_i$ and $y_i$ respectively, where $x_4 = x_1$; together with the orientation, the relation is $y_i = x_{i+1} x_i x_{i+1}^{-1}$, see [**4**], [**11**]. Thus the link group is a free abelian group of rank three.

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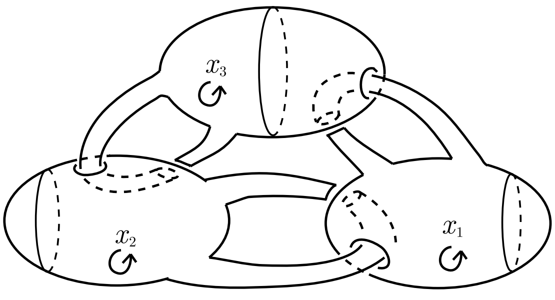


Figure 3.6. A ribbon $T^2$-link with a free abelian group of rank three.

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