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Degeneracy locus of critical points of the distance function on a holomorphic foliation

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Abstract. We study the geometry of transversality of holomorphic foliations of codimension one in $\mathbb{C}^n$ with spheres, from a viewpoint of dynamics of anti-holomorphic maps in the projective space. A point of non-degenerate contact of a leaf with a sphere is a hyperbolic fixed point of the corresponding dynamics. Around a point of degenerate contact, the intersection of branches of the variety of contacts is described as a bifurcation diagram of a neutral fixed point of dynamics. The Morse index for the distance function from the origin is computed as the complex dimension of an unstable manifold.

1. Introduction.

In the classical framework, given a (smooth) Morse function φ on a smooth Riemmanian manifold M, a natural dynamical system is defined on M by the flow of the gradient vector of $-\varphi$ ([**13**]). The work of R. Thom ([**15**]) suggests a similar approach to the case of foliated manifolds $(M, \mathcal{F})$, where $\mathcal{F}$ is a codimension $k \geq 1$ smooth foliation on M. R. Thom's idea is to consider a smooth function φ on M and, as r varies through $\mathbb{R}$, to study the variation of the foliated structures $([\varphi \leq r], \mathcal{F}(r))$, where $\mathcal{F}(r)$ is the foliation induced on $[\varphi \leq r] = \{x \in M,\ \varphi(x) \leq r\}$ by $\mathcal{F}$. After introducing three types of critical points (of tangential or intrinsic nature) the author discusses their effect on the pair $([\varphi \leq r], \mathcal{F}(r))$. This proposal is not however completely developed in the paper.

For a Morse function φ on a foliated manifold $(M, \mathcal{F})$ of real codimension one, we can consider yet another dynamical system, i.e., the *normal flow* on each level surface $\varphi^{-1}(r) \subset M$, $r \in \mathbb{R}$, which is a vector field obtained by projecting the normal vector of the leaf L containing $p \in \varphi^{-1}(r)$, to the tangent space $T_p(\varphi^{-1}(r)) \subset T_p(M)$. A critical point p of $\varphi|L$ will be a fixed point of the normal flow in $\varphi^{-1}(r) \subset M$, and the Morse index $m(\varphi|L; p)$ will be equal to the dimension of the unstable manifold.

An important class of foliated structures is given by holomorphic foliations in a neighborhood of the origin $0 \in \mathbb{C}^n$, where φ is the euclidian distance from the origin (see [**4**], [**5**], [**6**], [**7**], [**10**]). The case of (complex) one-dimensional foliations is addressed in [**1**], and the codimension one case in [**8**]. The conclusion is that a critical point p of $\varphi|L$ is a fixed point of the (real) gradient flow, and the Morse index at p is equal to the real dimension of the unstable manifold $W^u(p)$.

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In this article we consider a codimension one holomorphic foliation $\mathcal{F} = \mathcal{F}(\omega)$ defined by an integrable one-form $\omega = \sum_{j=1}^{n} f_j(z)dz_j$ on (an open set of) $\mathbb{C}^n$, and the distance function from the origin $\varphi(z) = \|z\|^2 = \sum_{j=1}^{n} |z_j|^2$. A level surface is a sphere $\varphi^{-1}(r^2) = S^{2n-1}(r)$, for $r > 0$. Let $\mathrm{Ker}(\omega)(p) = \{v \in T_p\mathbb{C}^n : \omega(p) \cdot v = 0\}$, $p \in \mathbb{C}^n$, and $\mathrm{Sing}(\omega) = \{p \in \mathbb{C}^n : \omega(p) = 0\}$. The distribution $\mathrm{Ker}(\omega)$ is called *transverse* to the sphere at a point p if $p \neq 0$, $p \notin \mathrm{Sing}(\omega)$ and $\mathrm{Ker}(\omega)(p) + T_p(S^{2n-1}(r)) = T_p\mathbb{R}^{2n}$, $r = \|p\|^{1/2}$, as a real subspace. The set Σ of the points $p \in \mathbb{C}^n$ where $\mathcal{F}$ is not transverse to the spheres is called *variety of contact* ([**15**]), or *polar variety* ([**10**]). We have shown in [**4**] that $\mathcal{F}$ is not transverse to the spheres at a point $z \in \mathbb{C}^n \setminus 0$ if and only if there exists $\lambda \in \mathbb{C}$ such that

$$\overline{f(z)} = \lambda z, \tag{1}$$

where $f(z) = (f_1(z), \dots, f_n(z))$, $z = (z_1, \dots, z_n)$. Thus Σ is a real analytic set given by

$$\Sigma = \{z \in \mathbb{C}^n : \ z_j\overline{f_k(z)} = z_k\overline{f_j(z)}, \forall j, k = 1, \dots, n\}.$$

The scalar $\lambda \in \mathbb{C}$ in (1) is called *conjugate-multiplier.* A point $p \in \Sigma^0 := \Sigma \setminus (\mathrm{Sing}(\omega) \cup \{0\})$ is called a *contact point.* It is called a *degenerate* contact point if it is a degenerate critical point of the distance function $\varphi|L_p$.

A problem in this situation is that we do not have a straightforward way to define a 'normal flow' on the spheres $S^{2n-1}(r)$. In fact, the anti-holomorphic vector field $\mathrm{grad}(\omega)(z) = \sum_{j=1}^{n} \overline{f_j(z)}\partial/\partial z_j$ ([**12**]) gives rise to a real vector field on the spheres $S^{2n-1}(r)$, but a contact point is not necessarily a singularity of this real flow because the conjugate-multiplier λ in (1) is not necessarily real. On the other side, if we consider a real 2-field on $S^{2n-1}(r)$ derived from $\mathrm{grad}(\omega)$, then it is not integrable in the classical sense of Frobenius.

1.1. Computing Morse indices and studying the contact variety.

Next we pass to describe our main results. First, we give two results showing how we can compute the Morse index explicitly for the first time. The dynamical motivation is as follows. Assume, for a moment, that $\omega = \sum_{j=1}^{n} f_j(z)dz_j$ is a homogeneous integrable one-form on $\mathbb{C}^n$, $n \geq 3$, having a simple (and isolated) singularity at the origin ([**3**], [**5**]), where $f_j(z)$ are homogeneous polynomials of a fixed degree d. By Malgrange's theorem [**11**], we may suppose without loss of generality that ω has a first integral g, $\omega = dg$, which is a homogeneous polynomial of degree $d + 1$ ([**4**]). The gradient field induces an anti-holomorphic discrete dynamical system on $\mathbb{CP}^{n-1}$:

$$\pi_n \circ \overline{f} \circ \pi_n^{-1} : (z_1 : \cdots : z_n) \mapsto (\overline{f_1(z)} : \cdots : \overline{f_n(z)})$$

where $\pi_n \colon \mathbb{C}^n \setminus \{0\} \to \mathbb{CP}^{n-1}$ denotes the canonical projection. By (1), it is now straightforward to see that $p \in \mathbb{C}^n \setminus 0$ is a contact point if and only if $\pi_n(p) \in \mathbb{CP}^{n-1}$ is a fixed point of $\pi_n \circ \overline{f} \circ \pi_n^{-1}$.

If ω is an inhomogeneous polynomial form or a holomorphic form in general, the gradient vector field grad(ω) does not necessarily induce a well-defined map on $\mathbb{CP}^{n-1}$, but we can still consider a local dynamics as follows. Let $p \in \Sigma^0$, and

$$H_p = \{z \in \mathbb{C}^n : \ {}^t\bar{p} \cdot (z - p) = 0\}$$

a hyperplane which is tangent at p to the sphere. Let

$$\overline{F_p} = \pi_n \circ \overline{f} \circ (\pi_n | H_p)^{-1} \tag{2}$$

be a map of a neighborhood of $\pi_n(p)$ in $\mathbb{CP}^{n-1}$ to $\mathbb{CP}^{n-1}$. We state our first result as follows:

THEOREM 1. *Let $\mathcal{F} = \mathcal{F}(\omega)$ be a holomorphic foliation defined by an integrable holomorphic one-form ω in a neighborhood of the origin $0 \in \mathbb{C}^n$. Let $p \in \Sigma^0$ be a contact point, and L_p the leaf of $\mathcal{F}$ containing p. Then p is a non-degenerate critical point of the distance function $\varphi | L_p$, if and only if $\pi_n(p) \in \mathbb{CP}^{n-1}$ is a hyperbolic fixed point of the mapping $\overline{F_p}$. If this is the case, then we have*

$$m(\varphi | L_p; p) = \dim_{\mathbb{C}} W^u(\overline{F_p}; \pi_n(p)).$$

where $m(\varphi | L_p; p)$ denotes the Morse Index of $\varphi | L_p$ at p, and $W^u(\overline{F_p}; \pi_n(p))$ is the unstable manifold of $\pi_n(p)$ in the dynamics $\overline{F_p}$.

Moreover, as for the original homogeneous case, we have the following:

COROLLARY 1. *Let $\omega = \sum_{j=1}^n f_j(z) dz_j$, where $f_j(z)$ are homogeneous polynomials of a same degree d. Let $p \in \Sigma^0$ be a contact point of the foliation $\mathcal{F}(\omega)$ with a sphere, and L_p the leaf containing p. Then p is a non-degenerate critical point of the distance function $\varphi | L_p$ if and only if $\pi_n(p)$ is a hyperbolic fixed point of $\overline{F} := \pi_n \circ \overline{f} \circ \pi_n^{-1}$. In this case, we have*

$$m(\varphi | L_p; p) = \dim_{\mathbb{C}} W^u(\overline{F}; \pi_n(p)).$$

Note that $W^u(\overline{F_p}; \pi_n(p))$ is a complex manifold since the twice iterate of an anti-holomorphic map is holomorphic.

Before going further in our approach, we recall some elements from Linear Algebra ([**2**]). Let $A \in GL(n, \mathbb{C})$. A scalar $\lambda \in \mathbb{C}$ is called a *conjugate-eigenvalue* of A if there exists $z \in \mathbb{C}^n$, $z \neq 0$, such that $\overline{Az} = \lambda z$. Note that the set of conjugate-eigenvalues of A is not discrete but a 'circled' set. That is, if $\overline{Az} = \lambda z$, then $\lambda e^{\theta\sqrt{-1}}$ is also a conjugate-eigenvalue for every $\theta \in \mathbb{R}$, indeed: $\overline{A(e^{-\theta\sqrt{-1}/2} z)} = \lambda e^{\theta\sqrt{-1}} (e^{-\theta\sqrt{-1}/2} z)$.

In the second part of this paper we study the variety of contact Σ around a point of degenerate contact, under some hypotheses.

DEFINITION 1. A holomorphic one-form $\omega = \sum_{j=1}^n f_j(z_j) dz_j$ is called *variable-separated* if $f_j(z) = f_j(z_j)$, $j = 1, \ldots, n$, are holomorphic functions of one complex variable.

If ω is variable-separated, then ω is exact and has a holomorphic first integral $g(z) = \sum_{j=1}^{n} \int^{z_j} f_j(z_j)\, dz_j$. If $0 \in \mathrm{Sing}(\omega)$, then the z_j-axes

$$\Sigma_j = \{(0, \ldots, 0, z_j, 0, \ldots, 0) : z_j \in \mathbb{C}\},\ j = 1, \ldots, n$$

are contained in the variety of contact Σ. It is shown that a point $p \in \Sigma_j \cap \Sigma^0$ with the conjugate-multiplier λ, $\overline{f(p)} = \lambda p$, is a degenerate critical point of $\varphi|L_p$ if and only if there exists a conjugate-eigenvector v of $f'(p)$ belonging to λ, $\overline{f'(p)v} = \lambda v$, such that v is perpendicular to Σ_j (Lemma 2).

Regarding the contact variety we prove:

Theorem 2. *Let $\mathcal{F}(\omega)$ be a holomorphic foliation defined by a variable-separated one-form ω, and suppose that the origin is an isolated singularity of ω. Let $p = (p_1, 0, \ldots, 0) \neq 0$ be a degenerate critical point of the distance function $\varphi|L_p$ with conjugate-multiplier $\lambda_0 := \overline{f_1(p_1)}/p_1 \neq 0$. Suppose the following:*

i) *the space of all conjugate-eigenvectors v of the derivative $f'(p)$ belonging to the conjugate-eigenvalue λ_0 is a real one-dimensional subspace of $0 \times \mathbb{C}_{z_2} \times 0 \times \cdots \times 0$. That is,*

$$|\lambda_0| = |f_2'(0)| \notin \{|f_1'(p_1)|, |f_3'(0)|, \ldots, |f_n'(0)|\}.$$

ii) *the equation $|f_2(z_2)/z_2| = |f_2'(0)|$ defines a local smooth curve in $\mathbb{C} = \mathbb{C}_{z_2}$ passing through the origin $z_2 = 0$.*

iii) *the real line $\{z_2 \in \mathbb{C} : \overline{f_2'(0)z_2} = \lambda_0 z_2\}$ is transverse to the curve $|f_2(z_2)/z_2| = |f_2'(0)|$ at the origin $z_2 = 0$.*

Then there exist a small neighborhood $U \subset \mathbb{C}^n$ of p, a diffeomorphism $\zeta_1 : V \to \zeta_1(V) \subset \mathbb{C}$ of a neighborhood $V \subset \mathbb{C}$ of λ_0 onto a neighborhood of p_1, and a real analytic function $\zeta_2 : V \to \mathbb{C}$, $\zeta_2(\lambda_0) = 0$, such that:

1. *for each $\lambda \in V$, the points*

$$p(\lambda) := (\zeta_1(\lambda), 0, \ldots, 0),$$
$$p_{12}(\lambda) := (\zeta_1(\lambda), \zeta_2(\lambda), 0, \ldots, 0),$$

   are contact points with the conjugate-multiplier λ.
2. *$\Sigma \cap U = \Sigma_1' \cup \Sigma_{12}'$, where $\Sigma_1' = \{p(\lambda) :\ \lambda \in V\}$, $\Sigma_{12}' = \{p_{12}(\lambda) : \lambda \in V\}$.*
3. *$\Lambda \cap U = \Sigma_1' \cap \Sigma_{12}'$, where Λ denotes the set of degenerate contact points of $\mathcal{F}(\omega)$.*
4. *$\Lambda \cap U = \{p(\lambda) : \lambda \in V, |\lambda| = |\lambda_0|\} = \{p_{12}(\lambda) : \lambda \in V, |\lambda| = |\lambda_0|\}$. It is a smooth curve.*
5. *the Morse indices at the points $z = p(\lambda)$, $p_{12}(\lambda)$, for $\lambda \in V$, $|\lambda| \neq |\lambda_0|$, are given by*

$$m(\varphi|L_{p(\lambda)};p(\lambda)) = \begin{cases} m_0 & \text{if } |\lambda| > |\lambda_0| \\ m_0+1 & \text{if } |\lambda| < |\lambda_0|, \end{cases}$$

$$m(\varphi|L_{p_{12}(\lambda)};p_{12}(\lambda)) = \begin{cases} m_0+1 & \text{if } |\lambda| > |\lambda_0| \\ m_0 & \text{if } |\lambda| < |\lambda_0|, \end{cases} \tag{3}$$

where $m_0 = \#\{j = 3, \dots, n : |f_j'(0)| > |\lambda_0|\}$.

REMARK 1. Besides the former works [**8**] and [**9**], this paper was motivated by the study of the *Pham polynomial* example in [**9**]. Here we denote by

$$\omega = 3z_1^2 dz_1 + 2z_2 dz_2 + 5z_3^4 dz_3.$$

The z_1-axis Σ_1 contains a circle of degeneracy contact points

$$\{(z_1,0,0) : |f_1(z_1)/z_1| = |f_2'(0)|, z_1 \neq 0\} = \{(z_1,0,0) : |3z_1| = 2\},$$

on which Σ_1 meets another branch $\Sigma_{12} \subset \Sigma$. Nevertheless, the function $f_2(z_2) = 2z_2$ is too 'degenerate' so that the equation $|f_2(z_2)/z_2| = |f_2'(0)|$ does not define a smooth curve in $\mathbb{C}$. In particular, assumption ii) of the above theorem is not satisfied. The conjugate-multiplier $\lambda(p)$ of any $p \in \Sigma_{12}$ has a constant absolute value $|\lambda(p)| = 2$, so the real two dimensional manifold Σ_{12} is not parametrizable by the conjugate-multiplier λ.

An example that satisfies the assumptions i)–iii) is the foliation $\mathcal{F}(\omega)$ defined by

$$\omega = \sum_{j=1}^{3} f_j(z_j)dz_j, \quad f_1(z_1) = z_1^2,\ f_2(z_2) = z_2 + z_2^2,\ f_3(z_3) = 2z_3.$$

The conjugate-multiplier at the degenerate contact point $p = (1,0,0)$ is $\lambda_0 = \overline{f_1(1)}/1 = 1$, which is equal to $f_2'(0) = 1$. The real line $\overline{z_2} = z_2$ is transverse to the curve $|1 + z_2| = 1$ at the origin $z_2 = 0$.

This paper is organized as follows. In Section 2, we prove Theorem 1 and Corollary 1 by using Takagi's Factorization Theorem [**14**]. In Section 3, a real analytic parametrization of the variety of contact Σ by the conjugate-multiplier λ is given by the Implicit Function Theorem. In Section 4, we prove Theorem 2. The idea of the proof is based on the bifurcation in the dynamics of one complex variable, where a neutral fixed point splits into a pair of fixed points: an attracting one and a repelling one.

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2. Dynamics and Morse theory on holomorphic foliations.

Let $\mathcal{F}$ be a codimension one holomorphic foliation in $\mathbb{C}^n$ defined by an integrable holomorphic one-form $\omega = \sum_{j=1}^{n} f_j(z)dz_j$. Consider the distance function from the origin

$\varphi(z) = \sum_{j=1} |z_j|^2$, as in the Introduction.

By applying a unitary coordinate change $z = U\tilde{z}$ to the one-form

$$\omega = \sum_{j=1}^{n} f_j(z) dz_j = {}^t(dz) f(z), \quad f(z) = {}^t(f_1(z), \ldots, f_n(z)),\ dz = {}^t(dz_1, \ldots, dz_n),$$

we have $dz = U\ d\tilde{z}$, and hence $\omega = {}^t(d\tilde{z})\tilde{f}(\tilde{z})$ where $\tilde{f}(\tilde{z}) = {}^tU(f(U\tilde{z}))$. Note that this is compatible with the unitary transformation for the dynamics of an anti-holomorphic mapping $\overline{f(z)} = (\overline{f_1(z)}, \ldots, \overline{f_n(z)})$. In fact, we obtain a mapping

$$\overline{\hat{f}(\tilde{z})} = U^{-1}(\overline{f(U\tilde{z})}), \quad \text{where } \hat{f}(\tilde{z}) = \bar{U}^{-1}(f(U\tilde{z})) = {}^tU(f(U\tilde{z})).$$

A fundamental tool for this unitary transformation is the following classical Linear Algebra result:

Lemma 1 (Takagi's Factorization Theorem, [**9**], [**14**]). *If A is a complex symmetric matrix, i.e., ${}^tA = A$, then there exists a unitary matrix U such that ${}^tUAU = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is a diagonal matrix with nonnegative entries $\lambda_j \geq 0$, $j = 1, \ldots, n$.*

Now we shall prove Theorem 1.

Proof of Theorem 1. Let $p \in \Sigma^0$. By taking a unitary coordinate transformation if necessary, we may assume that $p = (p_1, 0, \ldots, 0)$, $p_1 > 0$. Multiplying ω by a local holomorphic function if necessary, we may assume that ω has a first integral g, $\omega = dg$, in a neighborhood of p. Since p is a contact point, we have a Taylor expansion

$$g(z) = c_0 + c_1(z_1 - p_1) + \frac{c_{11}}{2}(z_1 - p_1)^2 + \sum_{j=2}^{n} c_{1j}(z_1 - p_1)z_j + \frac{1}{2}\sum_{j,k=2}^{n} c_{jk} z_j z_k + h.o.t.$$

in a neighborhood of p, where $h.o.t.$ stands for higher order terms, i.e., of degree greater than 2. We may assume that $c_0 = 0$ and $c_{jk} = c_{kj}$, $j, k = 2, \ldots, n$. Lemma 1 implies that by a unitary transformation on $(z_2, \ldots, z_n)$, we obtain

$$g(z) = c_1(z_1 - p_1) + \frac{c_{11}}{2}(z_1 - p_1)^2 + \sum_{j=2}^{n} \tilde{c}_{1j}(z_1 - p_1)z_j + \frac{1}{2}\sum_{j=2}^{n} b_j z_j^2 + h.o.t..$$

We may further assume that $b_j/c_1 \geq 0$, $j = 2, \ldots, n$.

The leaf L passing through the point p is defined by the equation $g(z) = 0$. It is written as the graph of a function

$$z_1 = z_1(z_2, \ldots, z_n) = p_1 - \sum_{j=2}^{n} \frac{b_j}{2c_1} z_j^2 + h.o.t., \tag{4}$$

in a neighborhood of p. Denote by $z_j = x_j + \sqrt{-1}\, y_j$. The distance function $\varphi|L$ is written as

$$
\begin{aligned}
|z_1|^2 + \sum_{j=2}^{n} |z_j|^2 &= \Bigl(p_1^2 - (p_1/c_1) \sum_{j=2}^{n} b_j \bigl(x_j^2 - y_j^2\bigr) + h.o.t.\Bigr) + \sum_{j=2}^{n} \bigl(x_j^2 + y_j^2\bigr) \\
&= p_1^2 + \sum_{j=2}^{n} (1 - p_1 b_j/c_1) x_j^2 + \sum_{j=2}^{n} (1 + p_1 b_j/c_1) y_j^2 + h.o.t..
\end{aligned}
$$

Thus the distance function $\varphi|L$ is non-degenerate at p if and only if

$$
p_1 b_j / c_1 \neq 1, \qquad j = 2, \dots, n. \tag{5}
$$

If this is the case, the Morse index is given by

$$
m(\varphi|L; p) = \#\{2 \leq j \leq n : p_1 b_j / c_1 > 1\}.
$$

On the other hand, the hyperplane H_p is given by $\{(z_1, \dots, z_n) : z_1 = p_1\}$. The one-form $\omega = dg = \sum_{j=1}^{n} f_j(z)\, dz_j$ is written as

$$
\begin{aligned}
f_1(z) &= c_1 + c_{11}(z_1 - p_1) + \sum_{j=2}^{n} \tilde{c}_{1j} z_j + h.o.t. \\
f_k(z) &= \tilde{c}_{1k}(z_1 - p_1) + b_k z_k + h.o.t., \qquad k = 2, \dots, n.
\end{aligned}
$$

Remark that the conjugate-multiplier at p is equal to $\overline{c_1}/p_1$. The local dynamics $\overline{F_p}$ in (2) is written as the complex conjugate of the mapping

$$
F_p : (p_1 : z_2 : \cdots : z_n) \mapsto \Bigl(c_1 + \sum_{j=2}^{n} \tilde{c}_{1j} z_j + h.o.t. : b_2 z_2 + h.o.t. : \cdots : b_n z_n + h.o.t.\Bigr).
$$

Thus the point $\pi_n(p) = (1 : 0 : \cdots : 0)$ is a hyperbolic fixed point of $\overline{F_p}$ if and only if (5) holds. The complex dimension of the unstable manifold of $\pi_n(p)$ is given as $\dim_{\mathbb{C}} W^u(\pi_n(p)) = \#\{j = 2, \dots, n : p_1 b_j / c_1 > 1\}$. This completes the proof. $\square$

Corollary 1 is an immediate consequence of Theorem 1 and of the paragraph that precedes this theorem.

The following lemma is a generalization of Propositions 12–14 of [**9**], where we considered the Pham polynomial and computed the Morse index $m(\varphi|L_p; p)$ on the z_j-axis, $j = 1, 2, 3$.

LEMMA 2. *Let $\omega = \sum_{j=1}^{n} f_j(z_j)\, dz_j$ be a variable-separated one-form, and suppose that the origin is an isolated singularity of ω. Let $p = (p_1, 0, \dots, 0) \in \Sigma^0$, and L_p the* *leaf of $\mathcal{F}(\omega)$ passing through p. Denote the conjugate-multiplier at p by $\lambda_0 := \overline{f_1(p_1)}/p_1$. Then p is a non-degenerate critical point of the distance function $\varphi|L_p$ if and only if*

$$|\lambda_0| \notin \{|f_j'(0)| : j = 2, \dots, n\}.$$

If this is the case, then the Morse index at p is given as

$$m(\varphi|L_p; p) = \#\{j = 2, \dots, n : |f_j'(0)| > |\lambda_0|\}.$$

Proof. The mapping $f(z) = (f_1(z_1), \dots, f_n(z_n))$ on the hyperplane H_p is written as

$$f(p_1, z_2, \dots, z_n) = \big(f_1(p_1), f_2'(0)z_2 + h.o.t., \dots, f_n'(0)z_n + h.o.t.\big).$$

Thus the point $\pi_n(p) = (1 : 0 : \cdots : 0)$ is a hyperbolic fixed point of the mapping $\overline{F_p} = \pi_n \circ \overline{f} \circ (\pi_n|H_p)^{-1}$ if and only if $|p_1 f_j'(0)/f_1(p_1)| \neq 1$, $j = 2, \dots, n$. If this is the case, the dimension of the unstable manifold of $\pi_n(p)$ is given as $\#\{j = 2, \dots, n : |p_1 f_j'(0)/f_1(p_1)| > 1\}$. The proof completes by applying Theorem 1. $\square$

In particular cases of variable-separated one-forms, the Morse index of a contact point $p \in \Sigma_1$ is obtained as follows.

- If $f_1(z_1) = c_1 z_1 + c_2 z_1^2 + h.o.t.$, $c_1 \neq 0$, i.e. the head term of $f_1(z_1)$ is linear, then $m(\varphi|L_p; p) = \#\{j = 2, \dots, n : |f_j'(0)| > |c_1|\}$ if $|p_1| \neq 0$ is small.
- If $f_1(z_1) = c_2 z_1^2 + c_3 z_1^3 + h.o.t.$, i.e. the head term of $f_1(z_1)$ is quadratic or more, then $m(\varphi|L_p; p) = n - 1$ for small $|p_1| \neq 0$.
- If $f_1(z_1)$ is a polynomial of degree ≥ 2, then $m(\varphi|L_p; p) = 0$ if $|p_1|$ is large.

3. Real analytic parametrization of the variety of contact.

The following lemma is found in [**9**].

Lemma 3. *Let $A \in GL(n, \mathbb{C})$. For $\lambda \in \mathbb{C}$, the following conditions are equivalent*:

1. *$|\lambda|^2$ is an eigenvalue of $\bar{A}A$.*
2. *λ is a conjugate-eigenvalue of A.*
3. *The $\mathbb{R}$-linear transformation $z \mapsto \overline{Az} - \lambda z$ in $\mathbb{C}^n$ is not invertible. That is,* $\mathrm{Ker}(z \mapsto \overline{Az} - \lambda z) \neq 0$.

The following Implicit Function Theorem gives us a local (real analytic) parametrization of Σ^0 by the conjugate-multiplier λ.

Lemma 4 (Implicit Function Theorem). *Let $z_0 \in \Sigma \setminus \{0\}$. Suppose that the conjugate-multiplier λ_0 of z_0 is not a conjugate-eigenvalue of the derivative $f'(z_0)$. That is, we suppose $\overline{f(z_0)} = \lambda_0 z_0$, and $\overline{f'(z_0)v} \neq \lambda_0 v$ for any $v \in T_{z_0}\mathbb{C}^n$, $v \neq 0$. Then there exists a neighborhood $V_0 \subset \mathbb{C}$ of λ_0, a neighborhood $U_0 \subset \mathbb{C}^n$ of z_0, and a real analytic map $\phi : V_0 \to U_0$ of real rank 2, such that*

1. $\phi(\lambda_0) = z_0$, *and*
2. *for each* $\lambda \in V_0$ *and* $z \in U_0$, λ *is a conjugate-multiplier of* z *if and only if* $z = \phi(\lambda)$.

Proof. Consider the real analytic function $F : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$, $F(z, \lambda) := \overline{f(z)} - \lambda z$. We have

$$dF = \overline{f'(z)dz} - \lambda dz - z d\lambda,$$

and $F(z_0, \lambda_0) = 0$. The $\mathbb{R}$-linear operator

$$\overline{f'(z_0)\, dz} - \lambda_0\, dz : T_{z_0}\mathbb{C}^n \to T_0\mathbb{C}^n$$

is invertible because λ_0 is not a conjugate-eigenvalue of $f'(z_0)$. The real analytic Implicit Function Theorem implies that there exist a neighborhood $U_0 \times V_0$ of (z_0, λ_0) and a real analytic function $\phi : V_0 \to U_0$, such that for each $(z, \lambda) \in U_0 \times V_0$, $F(z, \lambda) = 0$ if and only if $z = \phi(\lambda)$. □

Next we give some examples where Σ^0 has real dimension two but does not have a natural parametrization by the conjugate-multiplier.

Example 1. Let $\omega = \sum_{j=1}^n \lambda_j z_j\, dz_j$ be a diagonal linear one-form, where $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$. Then the variety of contacts $\Sigma = \cup_{j=1}^n \Sigma_j$ consists of the z_j-axes Σ_j, $j = 1, \dots, n$, and hence $\Sigma \setminus 0$ is a manifold of real dimension two. However, if we denote by $f(z) = (\lambda_1 z_1, \dots, \lambda_n z_n)$, the conjugate multiplier of a contact point $p = (0, \dots, re^{\theta\sqrt{-1}}, \dots, 0) \in \Sigma_k$ is equal to $\lambda_k e^{-2\theta\sqrt{-1}}$, which is a conjugate-eigenvalue of the constant diagonal matrix $f'(p) = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$.

Example 2. Let $\omega = dg$, where $g(z_1, z_2, z_3) = (z_1 - 1) + (z_1 - 1)^2 + (z_1 - 1)z_2 + z_2^2 + 2z_3^2$ on $\mathbb{C}^3$. Then $p = (1, 0, 0) \in \Sigma^0$ is a contact point with the conjugate-multiplier $\lambda = 1$. The leaf L_p passing through p is written locally by the graph of the function

$$z_1 = 1 - z_2^2 - 2z_3^2 + h.o.t.,$$

thus p is a non-degenerate critical point of the distance function $\varphi|L_p$. Proposition 5.1 of [**8**] implies that Σ^0 is locally a manifold of real dimension two which is transverse to the foliation $\mathcal{F}$.

However, we have

$$f(z) = (1 + 2(z_1 - 1) + z_2, (z_1 - 1) + 2z_2, 4z_3),$$

so $f'(p) = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$. By an orthogonal transformation $P = 1/\sqrt{2}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$, we have ${}^tP \cdot f'(p) \cdot P = \mathrm{diag}(3, 1, 4)$. Thus the conjugate-multiplier $\lambda = 1$ at p is equal to a conjugate-eigenvalue of $f'(p)$, and the Implicit Function Theorem (Lemma 4) is not applicable.

If $p \in \Sigma^0$ is a non-degenerate critical point of $\varphi|L_p$, then there exists a neighborhood $U \subset \mathbb{C}^n$ of p such that $U \cap \Sigma^0$ is a real analytic manifold of real dimension two which is transverse to the foliation $\mathcal{F}(\omega)$ ([**8**, Proposition 5.1]). The following is an example that Σ^0 is locally a real two dimensional manifold, but not transverse to $\mathcal{F}(\omega)$ at a point of degenerate contact $p \in \Sigma^0$.

Example 3. Let $\omega = dg$, $g(z_1, z_2, z_3) = (z_1 - 1) - (z_1 - 1)^2 + 2(z_1 - 1)z_2 + (1/2)z_2^2 + 2z_3^2$. Then $p = (1, 0, 0) \in \Sigma^0$ is a contact point with the conjugate-multiplier $\lambda = 1$. The leaf L_p passing through p is written locally as the graph of the function

$$z_1 = z_1(z_2, z_3) = 1 - \frac{1}{2}z_2^2 - 2z_3^2 + h.o.t.,$$

thus p is a degenerate critical point of the distance function $\varphi|L_p$. However, we have

$$f(z) = (1 - 2(z_1 - 1) + 2z_2, 2(z_1 - 1) + z_2, 4z_3),$$

so $f'(p) = \left(\begin{smallmatrix} -2 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 4 \end{smallmatrix}\right)$. By a unitary transformation $P = 1/\sqrt{5}\left(\begin{smallmatrix} 1 & 2\sqrt{-1} & 0 \\ 2 & -\sqrt{-1} & 0 \\ 0 & 0 & \sqrt{5} \end{smallmatrix}\right)$, we have ${}^tP \cdot f'(p) \cdot P = \mathrm{diag}(2, 3, 4)$. Thus the conjugate-multiplier $\lambda = 1$ at p is not a conjugate-eigenvalue of $f'(p)$, and the Implicit Function Theorem (Lemma 4) implies that Σ^0 is locally a manifold of real dimension two. The tangent spaces of $U \cap \Sigma^0$ and the leaf L_p at p are written as $T_p(U \cap \Sigma^0) = \{(t\sqrt{-1}, s - t\sqrt{-1}, 0) : s, t \in \mathbb{R}\}$, $T_p(L_p) = 0 \times \mathbb{C}^2$, respectively, so we have $T_p(U \cap \Sigma^0) + T_p(L_p) \neq T_p\mathbb{C}^3$. Thus $U \cap \Sigma^0$ is not transverse to the foliation $\mathcal{F}(\omega)$.

4. Morse index on the local branches of the variety of contacts.

In this section we prove Theorem 2. Let $\omega = \sum_{j=1}^n f_j(z_j)\, dz_j$ be a variable-separated one-form, and suppose that the origin is an isolated singularity of ω. We moreover assume conditions i)–iii) in Theorem 2. Denote by $\ell_j(z_j) = \overline{f_j(z_j)}/z_j$, $j = 1, \ldots, n$.

Lemma 5. *Let $f_1(z)$ be a holomorphic function of one complex variable defined in a neighborhood U of the origin $0 \in \mathbb{C}$. Let $p_1 \in U \setminus 0$. If $|f_1'(p_1)| \neq |f_1(p_1)/p_1|$, the function $\ell_1(z) := \overline{f_1(z)}/z$ is a local diffeomorphism of a neighborhood $W_1 \ni p_1$ in $\mathbb{C}$ onto a neighborhood $V_1 \ni \ell_1(p_1)$ in $\mathbb{C}$.*

Proof. This is a direct proof of a special case of the Implicit Function Theorem (Lemma 4) for the case of dimension one. Denote by $z = x + \sqrt{-1}\, y$, $f_1(z) = u + \sqrt{-1}\, v$ and $\ell_1(z) = \mu + \sqrt{-1}\, \nu$. We have

$$\ell_1(z) = \mu + \sqrt{-1}\, \nu = \frac{u - \sqrt{-1}\, v}{x + \sqrt{-1}\, y}.$$

The Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$, imply that the Jacobian determinant of the mapping $(\mu, \nu) = (\mu(x, y), \nu(x, y))$ is given by

$$\mu_x\nu_y - \mu_y\nu_x = \frac{|\ell_1(z)|^2 - |f_1'(z)|^2}{|z|^2}. \qquad \square$$

Denote the inverse function of ℓ_1 by $\zeta_1 : V_1 \to W_1$.

The function $\ell_2(z_2) = \overline{f_2(z_2)}/z_2$ is not a local diffeomorphism of a neighborhood of the origin $z_2 = 0$. By blowing-up, we consider a Möbius strip coordinate

$$M = \mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z})/\sim, \quad (r,\theta) \sim (-r, \theta+\pi),$$

with

$$\chi : M \to \mathbb{C}, \quad \chi(r,\theta) = re^{\sqrt{-1}\,\theta}.$$

Suppose that we have $\overline{f_2'(0)v} = \lambda_0 v$ for $v = e^{\sqrt{-1}\theta_0}$, $\theta_0 \in \mathbb{R}$.

Lemma 6. *The function $\lambda = \ell_2 \circ \chi(r,\theta) = \ell_2(re^{\sqrt{-1}\,\theta})$ is a local diffeomorphism of a small neighborhood $W_2 \subset M$ of $(r,\theta) = (0,\theta_0)$ onto a small neighborhood $V_2 \subset \mathbb{C}$ of λ_0.*

Proof. Let $f_2(z_2) = c_1 z_2 + c_2 z_2^2 + h.o.t.$. The partial derivatives of the function

$$\lambda := \ell_2(re^{\sqrt{-1}\,\theta}) = \overline{c_1}e^{-2\sqrt{-1}\theta} + r\overline{c_2}e^{-3\sqrt{-1}\theta} + h.o.t.$$

at $(r,\theta) = (0,\theta_0)$ are given as

$$\left.\frac{\partial\lambda}{\partial r}\right|_{(r,\theta)=(0,\theta_0)} = \overline{c_2}e^{-3\sqrt{-1}\,\theta_0}, \qquad \left.\frac{\partial\lambda}{\partial\theta}\right|_{(r,\theta)=(0,\theta_0)} = -2\overline{c_1}\sqrt{-1}e^{-2\sqrt{-1}\,\theta_0}. \tag{6}$$

Since $f_2(z_2)/z_2 = c_1 + c_2 z_2 + h.o.t.$, the curve $|f_2(z_2)/z_2| = |c_1|$ has a tangent line $\mathrm{Re}(c_2 z_2/c_1) = 0$ at the origin $z_2 = 0$. By the transversality assumption iii) in Theorem 2, we obtain $\mathrm{Re}(c_2 e^{\sqrt{-1}\theta_0}/c_1) \neq 0$. This implies that the two partial derivatives in (6) are linearly independent over $\mathbb{R}$, so that the function $\lambda = \ell_2(re^{\sqrt{-1}\,\theta})$ is a local diffeomorphism of a neighborhood of $(r,\theta) = (0,\theta_0)$. $\square$

Let $V \subset V_1 \cap V_2$ be a small neighborhood of $\lambda_0 = \overline{f_1(p_1)}/p_1$. Let $\rho_2 = (\ell_2 \circ \chi)^{-1} : V \to W_2$ be the inverse function given by Lemma 6. We have

$$\ell_2(\chi(\rho_2(\lambda))) = \lambda, \quad \lambda \in V.$$

Let $\zeta_2 = \chi \circ \rho_2$. It is a local 'inverse' function of ℓ_2, which maps $V \ni \lambda_0$ onto a neighborhood of the origin by blowing-down.

Let $H_{p(\lambda)} = \{(z_1, \dots, z_n) : z_1 = p(\lambda)\}$ be a hyperplane. We first consider the mapping $\overline{F_{p(\lambda)}} = \pi_n \circ \overline{f} \circ (\pi_n|H_{p(\lambda)})^{-1}$ of a neighborhood of $\pi_n(p(\lambda))$ in $\mathbb{CP}^{n-1}$ to $\mathbb{CP}^{n-1}$, which can also be regarded as an approximation of the mapping $\overline{F_{p_{12}(\lambda)}}$. It is written as

$$\overline{F_{p(\lambda)}} : (\zeta_1(\lambda) : z_2 : \cdots : z_n) \mapsto (\overline{f_1(\zeta_1(\lambda)))} : \overline{f_2(z_2)} : \cdots : \overline{f_n(z_n)})$$
$$= (\zeta_1(\lambda) : \overline{f_2(z_2)}/\lambda : \cdots : \overline{f_n(z_n)}/\lambda),$$

since $\overline{f_1(\zeta_1(\lambda)))} = \lambda\zeta_1(\lambda)$. For each $\lambda \in V$ fixed, we denote by

$$\phi_j(z_j) = \overline{f_j(z_j)}/\lambda, \qquad j = 2, \ldots, n.$$

Lemma 7. *In a neighborhood of the origin $z_2 = 0$, the anti-holomorphic function $\phi_2(z_2) = \overline{f_2(z_2)}/\lambda$ has two fixed points $z_2 = 0$ and $z_2 = \zeta_2(\lambda)$. If $|\lambda| < |\lambda_0|$ then $z_2 = 0$ is a repelling fixed point, and $z_2 = \zeta_2(\lambda)$ is an attracting fixed point. If $|\lambda| > |\lambda_0|$ then $z_2 = 0$ is attracting and $z_2 = \zeta_2(\lambda)$ is repelling.*

Proof. Let $f_2(z_2) = \sum_{\ell=1}^{\infty} c_\ell z_2^\ell$. Then the twice iterate of ϕ_2 is a holomorphic function

$$\phi_2(\phi_2(z_2)) = |c_1/\lambda|^2 z_2 + |\lambda|^{-2}(\overline{c_1}c_2 + c_1^2\overline{c_2\lambda^{-1}})z_2^2 + h.o.t..$$

The transversality assumption ii) in Theorem 2 implies that $c_2^2\overline{c_1}/\lambda_0 c_1^2 \notin \mathbb{R}_-$ (see Lemma 6). Thus we have $\overline{c_1}c_2 + c_1^2\overline{c_2\lambda_0^{-1}} \neq 0$, so the coefficient of z_2^2 in $\phi_2(\phi_2(z_2))$ does not vanish if $\lambda \in V$ is close to λ_0. This implies that the function ϕ_2 has at most two periodic points of period two in a small neighborhood of the origin $z_2 = 0$. They are in fact the fixed points of ϕ_2, i.e., the origin $z_2 = 0$ itself and $z_2 = \zeta_2(\lambda)$.

Let $\lambda = \lambda_0 e^{\sqrt{-1}\theta}/(1+t)$, where $\theta, t \in \mathbb{R}$. Note that $|c_1/\lambda|^2 = (1+t)^2$. By solving the equation of z_2

$$1 = (1+t)^2 + |\lambda|^{-2}(\overline{c_1}c_2 + c_1^2\overline{c_2\lambda^{-1}})z_2 + h.o.t.,$$

we obtain

$$\zeta_2(\lambda) = -2|\lambda_0|^2(\overline{c_1}c_2 + c_1^2\overline{c_2\lambda_0^{-1}})^{-1}\, t + o(t).$$

The derivative of ϕ_2^2 is

$$(\phi_2^2)'(z_2) = |c_1/\lambda|^2 + 2|\lambda|^{-2}(\overline{c_1}c_2 + c_1^2\overline{c_2\lambda^{-1}})z_2 + h.o.t.,$$

thus we have

$$(\phi_2^2)'(0) = (1+t)^2, \qquad (\phi_2^2)'(\zeta_2(\lambda)) = 1 - 2t + o(t).$$

By taking the square roots, the absolute values of the dynamical multipliers of the function ϕ_2 at the fixed points $z_2 = 0$ and $\zeta_2(\lambda)$ are $1+t$ and $1-t+o(t)$, respectively. Thus $z_2 = 0$ is repelling and $z_2 = \zeta_2(\lambda)$ is attracting if $t > 0$; $z_2 = 0$ is attracting and $z_2 = \zeta_2(\lambda)$ is repelling if $t < 0$. This completes the proof. $\square$

LEMMA 8. *Let $\lambda \in V$, $|\lambda| \neq |\lambda_0|$. The points $\pi_n(p(\lambda))$, $\pi_n(p_{12}(\lambda))$ are hyperbolic fixed points of the mapping $\overline{F_{p(\lambda)}}$. The complex dimensions of their unstable manifolds are given as follows*:

$$\dim_{\mathbb{C}} W^u(\overline{F_{p(\lambda)}}; \pi_n(p(\lambda))) = \begin{cases} m_0 & \text{if } |\lambda| > |\lambda_0| \\ m_0 + 1 & \text{if } |\lambda| < |\lambda_0|. \end{cases}$$

$$\dim_{\mathbb{C}} W^u(\overline{F_{p(\lambda)}}; \pi_n(p_{12}(\lambda))) = \begin{cases} m_0 + 1 & \text{if } |\lambda| > |\lambda_0| \\ m_0 & \text{if } |\lambda| < |\lambda_0|. \end{cases}$$

PROOF. The absolute values of dynamical multiplier of the function $\phi_2(z_2)$ at the fixed points $z_2 = 0$ and $z_2 = \zeta_2(\lambda)$ are obtained in Lemma 7.

For $j = 3, \ldots, n$, the dynamical multiplier of the function $\phi_j(z_j)$ at the origin $z_j = 0$ has the absolute value $|f_j'(0)/\lambda|$, which is close to $|f_j'(0)/\lambda_0|$. This completes the proof. □

Consider a small line segment

$$\{\lambda_0 e^{\sqrt{-1}\,\theta}/(1+t) : -\epsilon < t < \epsilon\} \subset V$$

with θ fixed. We are going to find a good unitary transformation in computing the Morse index at $p_{12}(\lambda)$ for $\lambda = \lambda_0 e^{\sqrt{-1}\,\theta}/(1+t)$, $t \neq 0$. Let

$$q = q(\lambda) := (\|p_{12}(\lambda)\|, 0, \ldots, 0).$$

Let $H_q = \{(w_1, \ldots, w_n) : w_1 = \|p_{12}(\lambda)\|\}$ be a hyperplane. Let $P := A \oplus I_{n-2}$ be a unitary matrix, where I_{n-2} denotes the identity matrix of size $n-2$, and

$$A := (a_{j,\ell}(\lambda))_{j,\ell=1,2} = \frac{1}{\|p_{12}(\lambda)\|} \begin{pmatrix} \zeta_1(\lambda) & -\overline{\zeta_2(\lambda)} \\ \zeta_2(\lambda) & \overline{\zeta_1(\lambda)} \end{pmatrix}.$$

Then we have $p_{12}(\lambda) = P(q(\lambda))$. Note that

$$|a_{j,\ell}(\lambda)| = \begin{cases} 1 + O(t^2) & \text{if } j = \ell \\ O(t) & \text{if } j \neq \ell \end{cases} \tag{7}$$

as $t \to 0$ for $\lambda = \lambda_0 e^{\sqrt{-1}\theta}/(1+t)$ with θ fixed.

Let $f^P(w) = ({}^t P \circ f \circ P)(w)$. The mapping $\overline{F_q^P} := \pi_n \circ \overline{f^P} \circ (\pi_n | H_q)^{-1}$ of a neighborhood of $\pi_n(q)$ in $\mathbb{CP}^{n-1}$ to $\mathbb{CP}^{n-1}$ is the complex conjugate of the mapping

$$F_q^P : (\|p_{12}(\lambda)\| : w_2 : \cdots : w_n) \mapsto \left(f_1^P : \cdots : f_n^P\right)$$

where

$$f_\nu^P = \begin{cases} \sum_{j=1}^2 a_{j\nu} f_j(\zeta_j(\lambda) + a_{j2}w_2), & \nu = 1, 2 \\ f_\nu(w_\nu) & \nu = 3, \dots, n. \end{cases}$$

Note that we have

$$\big(f_1^P, f_2^P, \dots, f_n^P\big)|_{(w_2,\dots,w_n)=0} = (\overline{\lambda}\|p_{12}(\lambda)\|, 0, \dots, 0)$$

since $f_j(\zeta_j(\lambda)) = \overline{\lambda\zeta_j(\lambda)}$, $j = 1, 2$. Let $\psi_\nu(w_2, \dots, w_n) = \|p_{12}(\lambda)\| f_\nu^P / f_1^P$, $\nu = 2, \dots, n$, so that the mapping F_q^P is written as

$$F_q^P(\|p_{12}(\lambda)\| : w_2 : \cdots : w_n) = (\|p_{12}(\lambda)\| : \psi_2(w) : \cdots : \psi_n(w)).$$

We have $\psi_\nu(0) = 0$, $\nu = 2, \dots, n$. The absolute values of the partial derivatives $\partial\psi_\nu/\partial w_\mu$ at $(w_2, \dots, w_n) = 0$ are given as

$$\left| \frac{\partial \psi_\nu}{\partial w_\mu}\bigg|_{(w_2,\dots,w_n)=0} \right| = \begin{cases} 1 - t + O(t^2) & \nu = \mu = 2 \\ |f_\nu'(0)/\overline{\lambda}| & 3 \le \nu = \mu \le n \\ 0 & \text{otherwise} \end{cases}$$

since we have

$$\frac{\partial \psi_2}{\partial w_2}\bigg|_{(w_2,\dots,w_n)=0} = \sum_{j=1}^{2} a_{j2}a_{j2} f_j'(\zeta_j(\lambda))/\overline{\lambda}$$

with the estimates (7) and $|f_2'(\zeta_2(\lambda))/\overline{\lambda}| = 1 - t + O(t^2)$ (see the proof of Lemma 7). Thus we have

$$\dim_{\mathbb{C}} W^u\big(\overline{F_{q(\lambda)}^P}; \pi_n(q(\lambda))\big) = \begin{cases} m_0 & \text{if } t > 0 \\ m_0 + 1 & \text{if } t < 0, \end{cases}$$

which gives (3). This completes the proof of Theorem 2. $\square$

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