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A note on the Jensen inequality for self-adjoint operators, II

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Abstract. This is a continuation of our previous paper. We consider a certain order-like relation for positive operators on a Hilbert space. This relation is defined by using the Jensen inequality with respect to the square-root function. We show that this relation is antisymmetric if the operators are invertible.

1. Introduction.

This is a continuation of our previous paper [**7**]. Let $f(t)$ be a continuous, increasing concave function on the half line $[0, \infty)$ and let A and B be bounded self-adjoint operators on a Hilbert space $\mathfrak{H}$ with an inner product $\langle \cdot, \cdot \rangle$. In the previous paper, we consider the following problem. If A and B satisfy $\langle f(A)\xi, \xi \rangle \leq f(\langle B\xi, \xi \rangle)$ and $\langle f(B)\xi, \xi \rangle \leq f(\langle A\xi, \xi \rangle)$ for any unit vector $\xi \in \mathfrak{H}$, can we conclude $A = B$? This problem was suggested by Professor Bourin [**4**]. In [**7**] we solved this problem affirmatively in the finite-dimensional case. We also dealt with some related problem in the infinite-dimensional case, but we could not get a complete answer. In this paper we consider the case $f(t) = \sqrt{t}$ and we solve this problem affirmatively under the assumption that two positive operators A and B are both invertible.

For two positive operators A and B, we introduce the new relation $A \trianglelefteq B$ defined by $\langle A^{1/2}\xi, \xi \rangle \leq \langle B\xi, \xi \rangle^{1/2}$ for any unit vector $\xi \in \mathfrak{H}$. Using this notation, we can restate the above problem as follows. If A and B satisfy $A \trianglelefteq B$ and $B \trianglelefteq A$, can we conclude $A = B$? We will show that this is true when A and B are both invertible. Here we remark that by [**1**] if $A \trianglelefteq B$ and $A^{-1} \trianglelefteq B^{-1}$, then we have $A = B$. We do not know whether $B \trianglelefteq A$ implies $A^{-1} \trianglelefteq B^{-1}$ or not.

The usual order $A \leq B$ implies $A \trianglelefteq B$ thanks to the Jensen inequality. However the relation $\trianglelefteq$ is not an order relation. Indeed we will construct positive matrices A, B and C such that both $A \trianglelefteq B$ and $B \trianglelefteq C$ hold while $A \trianglelefteq C$ does not hold.

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2. Main result.

Throughout this paper we assume that the readers are familiar with basic notations and results on operator theory. We refer the readers to Conway's book [**5**].

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We denote by $\mathfrak{H}$ a (finite or infinite dimensional) complex Hilbert space and by $B(\mathfrak{H})$ all bounded linear operators on it. The operator norm of $A \in B(\mathfrak{H})$ is denoted by $\|A\|$. The inner product and the norm for two vectors $\xi, \eta \in \mathfrak{H}$ are denoted by $\langle \xi, \eta \rangle$ and $\|\xi\|$ respectively. We denote the defining function for an interval $[a, b)$ by $\chi_{[a,b)}(t)$. We define the absolute value for a bounded linear operator X by $|X| = (X^*X)^{1/2}$.

If two positive operators $A, B \in B(\mathfrak{H})$ satisfy

$$\langle A^{1/2}\xi, \xi \rangle \leq \langle B\xi, \xi \rangle^{1/2}$$

for any unit vector $\xi \in \mathfrak{H}$, we write

$$A \trianglelefteq B.$$

The usual order $A \leq B$ implies that $A \trianglelefteq B$. This is a consequence of the famous Jensen inequality as follows.

$$\langle A^{1/2}\xi, \xi \rangle \leq \langle A\xi, \xi \rangle^{1/2} \leq \langle B\xi, \xi \rangle^{1/2}.$$

Here we remark that the relation $\trianglelefteq$ is not an order relation. Indeed there exist positive matrices A, B and C such that both $A \trianglelefteq B$ and $B \trianglelefteq C$ hold while $A \trianglelefteq C$ does not hold. See Example 2.1.

The following is the main result of this paper.

Theorem 2.1. *Let $A, B \in B(\mathfrak{H})$ be two positive operators such that A is invertible. If they satisfy $A \trianglelefteq B$ and $B \trianglelefteq A$, then we have $A = B$.*

Here we remark that it is hard to remove the assumption of invertibility. See Example 2.1.

Proposition 2.2 (Ando [**2**]). *For two positive operators $A, B \in B(\mathfrak{H})$, the following conditions are equivalent.*

(i) *$A^2 \trianglelefteq B^2$.*
(ii) *$A \leq (1/2t)B^2 + (t/2)$ for any positive number t.*
(iii) *There exists a contraction C satisfying $CB + BC^* = 2A$.*

Proof. The equivalence (i)⇔(ii) is shown in [**1**]. (See also [**7**, Lemma 3.2].)

Suppose that there exists a contraction C satisfying $CB + BC^* = 2A$. Since

$$0 \leq (CB - t)^*(CB - t) = BC^*CB + t^2 - t(CB + BC^*),$$

we see that

$$2tA = t(CB + BC^*) \leq BC^*CB + t^2 \leq B^2 + t^2.$$

Therefore the implication (iii)⇒(ii) holds.

Finally we will show (ii)⇒(iii). We remark that the inequality

$$B^2 + t^2 - 2tA \geq 0$$

holds for any real number t. Thus by the operator-valued Fejer-Riesz theorem ([**8**, Theorem 3.3]) there exist two bounded linear operators X and Y such that

$$B^2 + t^2 - 2tA = (X - tY)^*(X - tY) = X^*X + t^2Y^*Y - t(X^*Y + Y^*X).$$

Therefore we have $B = |X|$, $|Y| = 1$ and $2A = X^*Y + Y^*X$. Here we remark that Y is a contraction because $|Y| = 1$. Take the polar decomposition $X = U|X| = UB$ where U is a partial isometry. Then we get

$$2A = B(U^*Y) + (Y^*U)B.$$

Since U^*Y is a contraction, we are done. □

LEMMA 2.3. *Let c and ϵ be positive numbers such that $\epsilon < c$. Then*

$$2t\lambda - t^2 > 0 \quad \text{and} \quad \frac{\lambda^2}{2t} + \frac{t}{2} - (2t\lambda - t^2)^{1/2} \geq 0$$

for any $c + \epsilon \leq t \leq 2c$ and $c + \epsilon \leq \lambda \leq 2c$. Further there exists a positive number d satisfying

$$\frac{\lambda^2}{2t} + \frac{t}{2} - (2t\lambda - t^2)^{1/2} \leq \frac{d}{2}(t - \lambda)^2 \tag{9}$$

for any $c + \epsilon \leq t \leq 2c$ and $c + \epsilon \leq \lambda \leq 2c$.

PROOF. The proof is same as that of [**7**, Lemma 3.4].
Since $c + \epsilon \leq t \leq 2c$ and $c + \epsilon \leq \lambda \leq 2c$, we have

$$2t\lambda - t^2 = t(2\lambda - t) \geq (c + \epsilon)\{2(c + \epsilon) - 2c\} = 2(c + \epsilon)\epsilon > 0.$$

Next by the arithmetic-geometric mean inequality we have $(\lambda^2/2t) + (t/2) \geq \lambda$ and obviously $\lambda^2 \geq 2t\lambda - t^2$, so that $\lambda \geq (2t\lambda - t^2)^{1/2}$.

Now we set

$$k(t, \lambda) = \frac{d}{2}(t - \lambda)^2 - \frac{\lambda^2}{2t} - \frac{t}{2} + (2t\lambda - t^2)^{1/2}.$$

Then we compute

$$\frac{\partial}{\partial t}k(t,\lambda) = d(t-\lambda) + \frac{\lambda^2}{2t^2} - \frac{1}{2} + \frac{\lambda - t}{(2t\lambda - t^2)^{1/2}}$$

and

$$\frac{\partial^2}{\partial t^2}k(t,\lambda) = d - \frac{\lambda^2}{t^3} + \frac{-(2t\lambda - t^2)^{1/2} - (\lambda - t)^2(2t\lambda - t^2)^{-1/2}}{2t\lambda - t^2}.$$

Since $c+\epsilon \le t \le 2c$ and $c+\epsilon \le \lambda \le 2c$, we see that $2t\lambda - t^2 = t(2\lambda - t) \ge (c+\epsilon)\{2(c+\epsilon) - 2c\} = 2(c+\epsilon)\epsilon > 0$. Thus the two-variable function

$$-\frac{\lambda^2}{t^3} + \frac{-(2t\lambda - t^2)^{1/2} - (\lambda - t)^2(2t\lambda - t^2)^{-1/2}}{2t\lambda - t^2}$$

is bounded below on the intervals $c+\epsilon \le t \le 2c$ and $c+\epsilon \le \lambda \le 2c$. Therefore we can find a positive constant d such that $(\partial^2/\partial t^2)k(t,\lambda) > 0$ on the intervals $c+\epsilon \le t \le 2c$ and $c+\epsilon \le \lambda \le 2c$. Then $k(t,\lambda)$ is convex with respect to t. Since $(\partial/\partial t)k(t,\lambda)|_{t=\lambda} = 0$, $k(t,\lambda)$ in t is decreasing for $c+\epsilon \le t \le \lambda$ and increasing for $\lambda \le t \le c$ so that $k(t,\lambda) \ge k(\lambda,\lambda) = 0$. Thus we are done. □

Lemma 2.4. *Let $A, B \in B(\mathfrak{H})$ be positive invertible operators such that $c+\epsilon \le A \le 2c$ for some positive numbers $\epsilon < c$. If they satisfy*

$$(2tA - t^2)^{1/2} \le B \le \frac{A^2}{2t} + \frac{t}{2}$$

for any positive number t on the interval $c+\epsilon \le t \le 2c$, then we have $A = B$.

Proof. The proof is essentially same as that of [**1**], [**6**], [**7**].

First we will show that there exists a positive constant d satisfying

$$\|PBP - (PB^{-1}P)^{-1}\| \le d\|tP - AP\|^2 \tag{1}$$

for any $c+\epsilon \le t \le 2c$ and any spectral projection P of A, where we use $(PB^{-1}P)^{-1}$ to denote the inverse of $PB^{-1}P$ on $P\mathfrak{H}$. In the following we use commutativity of A and P without any particular mention.

By assumption we have two inequalities

$$(2tA - t^2)^{1/2} \le B \le \frac{A^2}{2t} + \frac{t}{2} \tag{2}$$

and

$$2t(A^2 + t^2)^{-1} \le B^{-1} \le (2tA - t^2)^{-1/2}. \tag{3}$$

Here we remark that $(2tA - t^2)^{-1/2}$ is a bounded operator because $2tA - t^2 = t(2A - t)$

and $2A \geq 2(c+\epsilon) > 2c \geq t > 0$. On the other hand we have

$$(2tA - t^2)^{1/2} \leq A \leq \frac{A^2}{2t} + \frac{t}{2}. \tag{4}$$

By the inequalities (2) and (4), we see that

$$\pm(AP - PBP) \leq \frac{(AP)^2}{2t} + \frac{t}{2}P - (2tAP - t^2P)^{1/2}$$

and hence

$$\|AP - PBP\| \leq \left\| \frac{(AP)^2}{2t} + \frac{t}{2}P - (2tAP - t^2P)^{1/2} \right\|. \tag{5}$$

By the inequality (3) we have

$$2t(A^2 + t^2)^{-1}P \leq PB^{-1}P \leq (2tA - t^2)^{-1/2}P$$

and hence

$$(2tAP - t^2P)^{1/2} \leq (PB^{-1}P)^{-1} \leq \frac{(AP)^2}{2t} + \frac{t}{2}P. \tag{6}$$

By the inequalities (4) and (6) we have

$$\pm(AP - (PB^{-1}P)^{-1}) \leq \frac{(AP)^2}{2t} + \frac{t}{2}P - (2tAP - t^2P)^{1/2}$$

and hence

$$\|AP - (PB^{-1}P)^{-1}\| \leq \left\| \frac{(AP)^2}{2t} + \frac{t}{2}P - (2tAP - t^2P)^{1/2} \right\|. \tag{7}$$

By the inequalities (5) and (7) we get

$$\|PBP - (PB^{-1}P)^{-1}\| \leq 2\left\| \frac{(AP)^2}{2t} + \frac{t}{2}P - (2tAP - t^2P)^{1/2} \right\|. \tag{8}$$

By the inequality (8) and Lemma 2.3 we have shown the inequality (1).

By the well-known formula known as Schur complement, we have

$$(PB^{-1}P)^{-1} = PBP - PBP^{\perp}(P^{\perp}BP^{\perp})^{-1}P^{\perp}BP$$

and hence

$$PBP-(PB^{-1}P)^{-1}=PBP^{\perp}(P^{\perp}BP^{\perp})^{-1}P^{\perp}BP \tag{9}$$

with $P^{\perp}=1-P$. Therefore by inequality (1) and (9) we see that

$$\|PBP^{\perp}(P^{\perp}BP^{\perp})^{-1}P^{\perp}BP\|\leq d\|tP-AP\|^{2}. \tag{10}$$

Then by the inequality (10) we compute

$$\begin{aligned}\|P^{\perp}BP\|^{2}&=\|(P^{\perp}BP^{\perp})^{1/2}(P^{\perp}BP^{\perp})^{-1/2}P^{\perp}BP\|^{2}\\&\leq\|B\|\cdot\|(P^{\perp}BP^{\perp})^{-1/2}P^{\perp}BP\|^{2}\\&=\|B\|\cdot\|PBP^{\perp}(P^{\perp}BP^{\perp})^{-1}P^{\perp}BP\|\\&\leq d\|B\|\cdot\|tP-AP\|^{2}\end{aligned}$$

and hence

$$\|P^{\perp}BP\|^{2}\leq d\|B\|\cdot\|tP-AP\|^{2}. \tag{11}$$

For each integer n, let P_i $(i=1,2,\ldots,n-1)$ be the spectral projections of A corresponding to the interval $[c+\epsilon+\{(i-1)\{2c-(c+\epsilon)\}/n\},\ c+\epsilon+\{i\{2c-(c+\epsilon)\}/n\})$ and let P_n be the spectral projections of A corresponding to the interval $[2c-\{(2c-(c+\epsilon))/n\},2c]$. Then we have $\sum_i P_i=1$ and

$$\|t_iP_i-AP_i\|\leq\frac{c-\epsilon}{n} \tag{12}$$

where $t_i=c+\epsilon+\{(i-1)\{2c-(c+\epsilon)\}/n\}$. By the inequalities (11) and (12) we see that

$$\begin{aligned}\left\|\sum_{i=1}^{n}P_i^{\perp}BP_i\right\|^{2}&=\left\|\left\{\sum_{i=1}^{n}P_i^{\perp}BP_i\right\}\left\{\sum_{j=1}^{n}P_jBP_j^{\perp}\right\}\right\|\\&=\left\|\sum_{i=1}^{n}P_i^{\perp}BP_iBP_i^{\perp}\right\|\\&\leq\sum_{i=1}^{n}\|P_i^{\perp}BP_iBP_i^{\perp}\|\\&=\sum_{i=1}^{n}\|P_i^{\perp}BP_i\|^{2}\\&\leq\sum_{i=1}^{n}d\|B\|\cdot\|t_iP_i-AP_i\|^{2}\end{aligned}$$

$$\leq \sum_{i=1}^{n} d\|B\| \cdot \frac{(c-\epsilon)^2}{n^2} = d\|B\| \cdot \frac{(c-\epsilon)^2}{n}$$

and hence

$$\left\| \sum_{i=1}^{n} P_i^{\perp} B P_i \right\|^2 \leq d\|B\| \cdot \frac{(c-\epsilon)^2}{n}. \tag{13}$$

Since

$$A - B = \sum_{i=1}^{n} (AP_i - P_i B P_i) + \sum_{i=1}^{n} P_i^{\perp} B P_i,$$

by (13) we see that

$$\|A - B\| \leq \left\| \sum_{i=1}^{n} (AP_i - P_i B P_i) \right\| + \left\| \sum_{i=1}^{n} P_i^{\perp} B P_i \right\|$$

$$\leq \sup_i \|AP_i - P_i B P_i\| + \left(d\|B\| \cdot \frac{(c-\epsilon)^2}{n} \right)^{1/2}.$$

On the other hand by (5) and Lemma 2.3 we have

$$\|AP_i - P_i B P_i\| \leq \frac{d}{2} \|tP_i - AP_i\|^2 \leq \frac{d}{2} \left(\frac{c-\epsilon}{n} \right)^2.$$

Thus we get

$$\|A - B\| \leq \frac{d}{2} \left(\frac{c-\epsilon}{n} \right)^2 + \left(d\|B\| \cdot \frac{(c-\epsilon)^2}{n} \right)^{1/2}.$$

By tending $n \to \infty$ we see that $A = B$. □

LEMMA 2.5. *Let $A, B \in B(\mathfrak{H})$ be positive operators satisfying $A \trianglelefteq B$. If A is invertible, then B is also invertible.*

PROOF. By assumption, there exists a positive number c which satisfies $c \leq A$. Then we have

$$c^{1/2} \langle \xi, \xi \rangle \leq \langle A^{1/2} \xi, \xi \rangle \leq \langle B\xi, \xi \rangle^{1/2}$$

for any unit vector $\xi \in \mathfrak{H}$. Therefore B is invertible. □

LEMMA 2.6. *Let A be a positive operator and let C be a contraction. If they satisfy $CA + AC^* = 2A$, then we have $CP = P$ where P is the range projection of A.*

Proof. This is a kind of triangle equality. The proof is implicitly contained in [**3**]. By assumption we have $(C-1)A = A(1-C^*)$. This means that the operator $(C-1)A$ is skew-adjoint. Therefore the spectrum $\sigma((C-1)A)$ is contained in $i\mathbb{R}$. On the other hand we see that $\sigma((C-1)A) \cup \{0\} = \sigma(A^{1/2}(C-1)A^{1/2}) \cup \{0\}$, and by [**3**, Lemma 2.2] we have $\sigma(A^{1/2}(C-1)A^{1/2}) \cap i\mathbb{R} = \{0\}$. Therefore we conclude that $\sigma((C-1)A) = \{0\}$. Since $(C-1)A$ is skew-adjoint, we see that $(C-1)A = 0$. □

Proof of Theorem 2.1. By Lemma 2.5 we see that both A and B are invertible. It is enough to show that two relations $A^2 \trianglelefteq B^2$ and $B^2 \trianglelefteq A^2$ ensure that $A = B$ for positive invertible operators A and B.

By Proposition 2.2 we have two inequalities

$$A \leq \frac{B^2}{2t} + \frac{t}{2} \tag{14}$$

and

$$B \leq \frac{A^2}{2t} + \frac{t}{2} \tag{15}$$

for any positive number t. Since A is positive invertible, there exists a positive number c satisfying $A > c$. Let ϵ be a positive number with $\epsilon < c$. It follows from (14) and Lemma 2.3

$$0 \leq 2tA - t^2 \leq B^2$$

for any $c + \epsilon \leq t \leq 2c$. Then since the map $X \longmapsto X^{1/2}$ is order-preserving in the cone of positive operators, we have from (15)

$$(2tA - t^2)^{1/2} \leq B \leq \frac{A^2}{2t} + \frac{t}{2}$$

for any $c + \epsilon \leq t \leq 2c$. Let $P = \chi_{[c+\epsilon, 2c]}(A)$. Then we have

$$(2tAP - t^2P)^{1/2} \leq PBP \leq \frac{(AP)^2}{2t} + \frac{t}{2}P$$

and $(c+\epsilon)P \leq AP \leq 2cP$. Therefore by Lemma 2.4 we have $AP = PBP$. By Proposition 2.2 there exists a contraction D such that

$$DA + AD^* = 2B$$

and hence

$$PDPA + APD^*P = 2PBP = 2AP.$$

Then by Lemma 2.6 we see that $PDP = P$. Since

$$P = PD^*PDP \leq PD^*DP \leq P,$$

we have $PD^*(1-P)DP = 0$, i.e., $(1-P)DP = 0$ and hence $DP = PDP+(1-P)DP = P$. By the same argument we see that $PD = P$. Therefore we have

$$2BP = (DA + AD^*)P = DPA + AD^*P = 2AP$$

and hence $BP = PB$. Since ϵ is arbitrary, we have

$$A\chi_{(c,2c]}(A) = B\chi_{(c,2c]}(A) = \chi_{(c,2c]}(A)B.$$

Since $\chi_{(c,2c]}(A)$ commutes with B, so does $1 - \chi_{(c,2c]}(A) = \chi_{(2c,\infty)}(A)$. Then, the second characterization in Proposition 2.2 clearly guarantees that the positive invertible operators $A\chi_{(2c,\infty)}(A)$ and $B\chi_{(2c,\infty)}(A)$ on $\chi_{(2c,\infty)}(A)\mathfrak{H}$ satisfy

$$\{A\chi_{(2c,\infty)}(A)\}^2 \trianglelefteq \{B\chi_{(2c,\infty)}(A)\}^2$$

and

$$\{B\chi_{(2c,\infty)}(A)\}^2 \trianglelefteq \{A\chi_{(2c,\infty)}(A)\}^2.$$

Since $A\chi_{(2c,\infty)}(A) \geq 2c\chi_{(2c,\infty)}(A)$, by the same argument we see that

$$A\chi_{(2c,4c]}(A) = B\chi_{(2c,4c]}(A) = \chi_{(2c,4c]}(A)B.$$

Therefore by repeating this argument we have $A = B$. □

LEMMA 2.7. *For any operator X, we have*

$$\operatorname{Re} X \leq \frac{1}{2t}|X|^2 + \frac{t}{2}$$

for any positive number t.

PROOF. Since

$$0 \leq (X - t)^*(X - t) = |X|^2 + t^2 - 2t \operatorname{Re} X,$$

we are done. □

EXAMPLE 2.1. First we will show that there exist 2×2 positive matrices A, B and C such that both $A^2 \trianglelefteq B^2$ and $B^2 \trianglelefteq C^2$ hold while $A^2 \trianglelefteq C^2$ does not hold.

We set

$$X = \begin{pmatrix} \sqrt{2} & 1 \\ 0 & \sqrt{2} \end{pmatrix}, \qquad A = \operatorname{Re} X = \begin{pmatrix} \sqrt{2} & 1/2 \\ 1/2 & \sqrt{2} \end{pmatrix} \geq 0$$

and

$$B = |X| = \frac{1}{3} \begin{pmatrix} 4 & \sqrt{2} \\ \sqrt{2} & 5 \end{pmatrix}.$$

By Lemma 2.7 and Proposition 2.2 we have $A^2 \trianglelefteq B^2$. Next we set

$$Y = \frac{1}{3} \begin{pmatrix} 4 & 2\sqrt{2} \\ 0 & 5 \end{pmatrix}$$

and $C = |Y|$. Since

$$\operatorname{Re} Y = \frac{1}{3} \begin{pmatrix} 4 & \sqrt{2} \\ \sqrt{2} & 5 \end{pmatrix} = B,$$

we have $B^2 \trianglelefteq C^2$. Suppose that $A^2 \trianglelefteq C^2$. Then by Proposition 2.2 we have

$$A \leq \frac{1}{2t} C^2 + \frac{t}{2}$$

for any positive number t. Let $E = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Then we see that $EAE = \sqrt{2}E$ and $E((1/2t)C^2 + (t/2))E = ((1/2t)(16/9) + (t/2))E$. Therefore we have

$$\sqrt{2} \leq \frac{8}{9t} + \frac{t}{2}$$

for any positive number t. This is impossible because the minimal value of the scalar on the right hand side is $4/3$ while $4/3 < \sqrt{2}$.

Next we show that $(A+\epsilon)^2 \trianglelefteq (B+\epsilon)^2$ is not valid for any positive number ϵ. If this were the case, then we would have

$$E(A+\epsilon)E = (\sqrt{2}+\epsilon)E \leq \frac{1}{2t} E(B+\epsilon)^2 E + \frac{t}{2} E = \left(\frac{9\epsilon^2 + 24\epsilon + 18}{18t} + \frac{t}{2} \right) E$$

for any positive number t. Since the minimal value of the scalar on the right hand side is $\sqrt{9\epsilon^2 + 24\epsilon + 18}/3$, we have

$$(\sqrt{2}+\epsilon)^2 = \epsilon^2 + 2\sqrt{2}\epsilon + 2 \leq \left(\frac{\sqrt{9\epsilon^2 + 24\epsilon + 18}}{3} \right)^2 = \epsilon^2 + \frac{8}{3}\epsilon + 2.$$

This is obviously wrong because $2\sqrt{2} > 8/3$.

It is unclear if the invertibility assumption can be dropped in the main theorem. At least our example shows that the standard trick of adding $\epsilon 1$ to A, B does not work.

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