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An integration by parts formula for Feynman path integrals

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Abstract. We are concerned with rigorously defined, by time slicing approximation method, Feynman path integral $\int_{\Omega_{x,y}} F(\gamma)e^{i\nu S(\gamma)}\mathcal{D}(\gamma)$ of a functional $F(\gamma)$, cf. [**13**]. Here $\Omega_{x,y}$ is the set of paths $\gamma(t)$ in $\boldsymbol{R}^d$ starting from a point $y \in \boldsymbol{R}^d$ at time 0 and arriving at $x \in \boldsymbol{R}^d$ at time T, $S(\gamma)$ is the action of γ and $\nu = 2\pi h^{-1}$, with Planck's constant h. Assuming that $p(\gamma)$ is a vector field on the path space with suitable property, we prove the following integration by parts formula for Feynman path integrals:

$$\begin{aligned}\int_{\Omega_{x,y}} & DF(\gamma)[p(\gamma)]e^{i\nu S(\gamma)}\mathcal{D}(\gamma) \\ &= -\int_{\Omega_{x,y}} F(\gamma)\operatorname{Div} p(\gamma)e^{i\nu S(\gamma)}\mathcal{D}(\gamma) - i\nu\int_{\Omega_{x,y}} F(\gamma)DS(\gamma)[p(\gamma)]e^{i\nu S(\gamma)}\mathcal{D}(\gamma).\end{aligned} \tag{1}$$

Here $DF(\gamma)[p(\gamma)]$ and $DS(\gamma)[p(\gamma)]$ are differentials of $F(\gamma)$ and $S(\gamma)$ evaluated in the direction of $p(\gamma)$, respectively, and $\operatorname{Div} p(\gamma)$ is divergence of vector field $p(\gamma)$. This formula is an analogy to Elworthy's integration by parts formula for Wiener integrals, cf. [**1**]. As an application, we prove a semiclassical asymptotic formula of the Feynman path integrals which gives us a sharp information in the case $F(\gamma^*) = 0$. Here γ^* is the stationary point of the phase $S(\gamma)$.

1. Time slicing approximation of Feynman path integral.

Let $[0,T]$, $T > 0$, be an interval. Let $L(t,\dot{x},x) = (1/2)|\dot{x}|^2 - V(t,x)$ be the Lagrangian function with real potential $V(t,x)$, $(t,x) \in [0,T] \times \boldsymbol{R}^d$.

A path γ is a continuous map $\gamma : [0,T] \ni t \to \gamma(t) \in \boldsymbol{R}^d$ starting from $\gamma(0)$ at time 0 and reaching $\gamma(T)$ at time T. In the following, we always assume that $d = 1$ for the sake of simplicity of notation.

We write $\mathcal{X} = L^2([0,T])$. For any $f, g \in \mathcal{X}$ we write $(f,g)_{\mathcal{X}}$ for the inner product of f, g and $\|f\|_{\mathcal{X}}$ for the norm of f in $\mathcal{X}$. Let $\mathcal{H} = H^1([0,T])$ be the

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real L^2-Sobolev space of order 1 equipped with the usual norm $\| \|_{\mathcal{H}}$. For any $x, y \in \boldsymbol{R}$, we write $\mathcal{H}_{x,y} = \{\gamma \in \mathcal{H} : \gamma(0) = y, \gamma(T) = x\}$. $\mathcal{H}_{x,y}$ is an infinite dimensional differentiable manifold. Its tangent space at $\gamma \in \mathcal{H}_{x,y}$ is identified with the Hilbert space $\mathcal{H}_0 = H_0^1([0,T]) = \{\gamma \in \mathcal{H}; \gamma(0) = \gamma(T) = 0\}$ equipped with the inner product

$$(h_1, h_2)_{\mathcal{H}_0} = \int_0^T \frac{d}{dt} h_1(t) \frac{d}{dt} h_2(t)\, dt.$$

We denote the norm in $\mathcal{H}_0$ by$\|h\|_{\mathcal{H}_0}$ for $h \in \mathcal{H}_0$. The cotangent space of $\mathcal{H}_{x,y}$ at γ is identified with $\mathcal{H}_0$ via the inner product of $\mathcal{H}_0$. There are continuous canonical inclusions $\mathcal{H}_0 \subset \mathcal{H} \subset \mathcal{X}$.

The action $S(\gamma)$ of a path $\gamma \in \mathcal{H}$ in the interval $[0,T]$ is the functional on $\mathcal{H}$:

$$S(\gamma) = \int_0^T L\Big(t, \frac{d}{dt}\gamma(t), \gamma(t)\Big) dt. \tag{2}$$

It is Fréchet differentiable and its differential $DS(\gamma)$ of $S(\gamma)$ restricted to $\mathcal{H}_{x,y}$ is a cotangent vector, whose value evaluated at a tangent vector $h \in \mathcal{H}_0$ is

$$DS(\gamma)[h] = \int_0^T \Big(\frac{d}{dt}\gamma(t)\frac{d}{dt}h(t) - \partial_x V(t, \gamma(t))h(t)\Big) dt, \quad \text{for } \forall\ h \in \mathcal{H}_0.$$

A stationary point γ^* of $S(\gamma)$ on $\mathcal{H}_{x,y}$ is the solution of Euler's equation with boundary conditions:

$$\frac{d^2}{dt^2}\gamma(t) + \partial_x V(t, \gamma(t)) = 0, \quad \text{for } 0 < t < T, \tag{3}$$

$$\gamma(0) = y, \quad \gamma(T) = x. \tag{4}$$

The solution γ^* of Euler's equation is called a classical path or a classical orbit starting from $(0, y)$ and arriving at (T, x).

Throughout this paper we always assume the potential $V(t, x)$ has the following properties: For any integer $k \geq 0$ there exists a positive constant v_k such that

$$\big|\partial_x^k V(t, x)\big| \leq v_k (1 + |x|)^{\max\{0, 2-k\}}, \quad \text{for any } x \in \boldsymbol{R}. \tag{5}$$

For the sake of simplicity we assume that $v_0 \leq v_1 \leq v_2 \leq \cdots$.

We fix a positive constant μ so that

$$\mu^2 v_2 < 4 \quad \text{and} \quad \mu v_2 < 1. \tag{6}$$

If $T \leq \mu$, then for any $x, y \in \boldsymbol{R}$ the solution $\gamma^*(t)$ of two points boundary value problem of the Euler's equation (3) exists uniquely and attains the minimum of the action. We write $S(T, 0, x, y) = S(\gamma^*)$, because it is a function of (T, x, y).

Let Δ be an arbitrary division of the interval $[0, T]$ such that

$$\Delta : 0 = T_0 < T_1 < \cdots < T_J < T_{J+1} = T. \tag{7}$$

We set $\tau_j = T_j - T_{j-1}$, $j = 1, 2, \ldots, J+1$, and $|\Delta| = \max\{\tau_j; 1 \leq j \leq J+1\}$.

For $j = 1, 2, \ldots, J$, choose arbitrary point $x_j \in \boldsymbol{R}$ and set $x_0 = y$, $x_{J+1} = x$. We denote by γ_Δ the path such that

$$\gamma_\Delta(T_j) = x_j, \quad j = 0, 1, 2, \ldots, J+1,$$

and

$$\frac{d^2}{dt^2}\gamma(t) + \partial_x V(t, \gamma(t)) = 0, \quad T_{j-1} < t < T_j, \text{ for } j = 1, 2, \ldots, J+1.$$

γ_Δ is a path which may have edges at $t = T_j$, $j = 1, 2, \ldots, J$. We call such a path a piecewise classical path or a piecewise classical path associated with the division Δ. We sometimes express its dependency on $(x_{J+1}, x_J, \ldots, x_0)$ by writing $\gamma_\Delta(t, x_{J+1}, x_J, \ldots, x_0)$ or $\gamma_\Delta(x_{J+1}, x_J, \ldots, x_0)$. It is clear that $\gamma_\Delta \in \mathcal{H}_{x,y}$.

The set $\Gamma(\Delta)$ of all piecewise classical paths associated with the division Δ forms a differentiable manifold of dimension $J+2$, which is embedded in Hilbert space $\mathcal{H}$. The correspondence $\gamma_\Delta \to (x_{J+1}, \ldots, x_0)$ is a global coordinate system of $\Gamma(\Delta)$. We write $\Gamma_{x,y}(\Delta) = \Gamma(\Delta) \cap \mathcal{H}_{x,y}$.

If a functional $F(\gamma)$ of γ is given, $F(\gamma_\Delta)$ is a function of $(x_{J+1}, x_J, \ldots, x_1, x_0)$, which we sometimes write F_Δ as an abbreviation. For example the action $S(\gamma_\Delta)$ of $\gamma_\Delta(x_{J+1}, x_J, \ldots, x_0)$ is a function of $(x_{J+1}, x_J, \ldots, x_1, x_0)$ if Δ is fixed.

$$\begin{aligned} S(\gamma_\Delta(x_{J+1}, x_J, \ldots, x_0)) &= \int_0^T L\Big(t, \frac{d}{dt}\gamma_\Delta(t), \gamma_\Delta(t)\Big)dt \\ &= \sum_{j=1}^{J+1} \int_{T_{j-1}}^{T_j} L\Big(t, \frac{d}{dt}\gamma_\Delta(t), \gamma_\Delta(t)\Big)dt. \end{aligned} \tag{8}$$

Feynman [2] introduced the notion of his path integral

$$\int_{\Omega_{x,y}} F(\gamma)e^{i\nu S(\gamma)}\mathcal{D}(\gamma)$$

by the following formula:

$$\int_{\Omega_{x,y}} F(\gamma)e^{i\nu S(\gamma)}\mathcal{D}(\gamma) = \lim_{|\Delta|\to 0} I[F_\Delta](\Delta;\nu,T,0,x,y), \tag{9}$$

where

$$I[F_\Delta](\Delta;\nu,T,0,x,y) = \prod_{j=1}^{J+1}\left(\frac{\nu}{2\pi i\tau_j}\right)^{1/2}\int_{\boldsymbol{R}^J} F(\gamma_\Delta(x_{J+1},x_J,\ldots,x_1,x_0)) \times e^{i\nu S(\gamma_\Delta(x_{J+1},x_J,\ldots,x_1,x_0))}\prod_{j=1}^{J} dx_j. \tag{10}$$

We call $I[F_\Delta](\Delta;\nu,T,0,x,y)$ time slicing approximation of path integral. Mathematically, the multiple integral on the right hand side of (10) is not absolutely convergent. We consider it as an oscillatory integral, cf. [11], [12].

Following Kumano-go [13] we say that the functional $F(\gamma)$ is F-integrable if the limit on the right hand side of (9) exists. $F(\gamma)\equiv 1$ was proved to be F-integrable, cf. [4], [10] and [6]. More general sufficient conditions for the limit (9) to exist was studied first by [13], cf. also [7].

Now we introduce seminorms which are convenient for us to describe class of functionals $F(\gamma)$ discussed in this paper.

Let $\alpha = (\alpha_{J+1},\alpha_J,\ldots,\alpha_2,\alpha_1,\alpha_0)$ be a multi-index. Then we write $m(\alpha)$ for $\max\{\alpha_j; 0\le j\le J+1\}$. Let $\mathcal{Y}$ be a Banach space equipped with norm $\|\ \|_{\mathcal{Y}}$. Let Δ be a division of $[0,T]$, γ_Δ and $(x_{J+1},x_J,\ldots,x_1,x_0)$ be as before. Assume that the map $G:\Gamma(\Delta)\ni\gamma_\Delta\to G(\gamma_\Delta)\in\mathcal{Y}$ is infinitely differentiable with respect to $(x_{J+1},\ldots,x_0)$. Let K be a non-negative integer, m be a non-negative constant and $X\ge 1$ be a constant. Then we define a seminorm of $G(\gamma_\Delta)$:

$$\|G(\gamma_\Delta)\|_{\{\mathcal{Y};\Delta,m,K,X\}} = \max_{\alpha}\sup_{x}(1+|x_0|+\mathrm{var}(\gamma_\Delta))^{-m}\left\|\left(\prod_{j=0}^{J+1}X^{-|\alpha_j|}\partial_{x_j}^{\alpha_j}\right)G(\gamma_\Delta)\right\|_{\mathcal{Y}}, \tag{11}$$

where $\mathrm{var}(\gamma_\Delta) = \sum_{j=1}^{J+1} |x_j - x_{j-1}|$, max is taken over all multi-indices α with $m(\alpha) \leq K$ and sup is taken over all $(x_{J+1}, \ldots, x_0) \in \boldsymbol{R}^{J+2}$. Moreover if $G(\gamma)$ is defined on $\mathcal{H}$, then we define

$$\|G\|_{\{\mathcal{Y};m,K,X\}} = \sup_{\Delta} \|G\|_{\{\mathcal{Y};\Delta,m,K,X\}}, \tag{12}$$

where sup is taken over all divisions Δ of $[0,T]$. In particular, if $\mathcal{Y} = \boldsymbol{C}$ or $= \boldsymbol{R}$, we simply write $\|G\|_{\{\Delta,m,K,X\}}$ or $\|G\|_{\{m,K,X\}}$.

We usually write an element $h \in \mathcal{H}$ as a function $h(s) \in \mathcal{X}$ of a variable, say, $s \in [0,T]$. We denote this natural embedding by $\tilde{\rho} : \mathcal{H} \to \mathcal{X}$ when we need to emphasize it. We denote the restriction of $\tilde{\rho}$ to $\mathcal{H}_0$ by ρ. The symbol $\rho^* : \mathcal{X} \to \mathcal{H}_0$ expresses the adjoint of ρ.

Suppose that a functional $F(\gamma)$ restricted to $\mathcal{H}_{x,y}$ is Fréchet differentiable at γ. Then $DF(\gamma)$ denotes its differential, which is a cotangent vector $\in \mathcal{H}_0$. $DF(\gamma)[h]$ is the value of $DF(\gamma)$ at the tangent vector $h \in \mathcal{H}_0$, i.e., $DF(\gamma)[h] = (DF(\gamma), h)_{\mathcal{H}_0}$. Moreover, if there exists a density $f_\gamma(s)$ with respect to some positive Borel measure φ on $[0,T]$ such that

$$DF(\gamma)[h] = \int_0^T f_\gamma(s) \rho h(s)\, d\varphi(s), \quad \text{for } \forall h \in \mathcal{H}_0,$$

then we often denote $f_\gamma(s)$ by $\delta F(\gamma)/\delta\gamma(s)$ or $(\delta/\delta\gamma(s))F(\gamma)$.

Definition 1.1. Let m be a non-negative constant. We call $F(\gamma)$ an m-smooth functional if $F(\gamma)$ satisfies all of the following conditions.

F-I: $F(\gamma)$ is an infinitely differentiable map from $\mathcal{H}$ to $\boldsymbol{C}$.

F-2: There exist a positive Borel measure φ in $[0,T]$ such that for any $\gamma \in \mathcal{H}$ the differential $DF(\gamma)$ has its density $\delta F(\gamma)/\delta\gamma(s)$ with respect to φ, that is,

$$DF(\gamma)[h] = \int_0^T \frac{\delta F(\gamma)}{\delta\gamma(s)} \rho h(s)\, d\varphi(s), \quad \text{for } \forall\gamma \in \mathcal{H},\ \forall h \in \mathcal{H}_0.$$

$\delta F(\gamma)/\delta\gamma(s)$ is a continuous function of $s \in [0,T]$ if each $\gamma \in \mathcal{H}$ is fixed.

F-3: The map $\mathcal{H} \ni \gamma \to \delta F(\gamma)/\delta\gamma(s) \in C([0,T])$ is infinitely differentiable, where $C([0,T])$ is the Banach space of continuous functions in $[0,T]$ equipped with the maximum norm $\|f\|_{C([0,T])} = \max_{t\in[0,T]} |f(t)|$ for any $f \in C([0,T])$.

F-4: For any non-negative integer K there are positive constants A_K and X_K such that for any $K = 0, 1, 2, \ldots,$

$$A_K = \|F(\gamma)\|_{\{m,K,X_K\}} + \left\| \frac{\delta F(\gamma)}{\delta\gamma(s)} \right\|_{\{C([0,T]);m,K,X_K\}} < \infty. \tag{13}$$

REMARK 1. Let μ be so small that $v_2\mu^2 < 4$ and $v_2\mu < 1$. If $T \le \mu$, N. Kumano-go gave a fairly large class of Feynman path integrable functionals including those functionals which are m-smooth. See [**13**] and also [**7**].

2. Divergence operator.

2.1. Some operators of trace class.

We write $\omega = \pi/T$ and for $n = 1, 2, 3, \dots$,

$$e_n(t) = \sqrt{\frac{2}{T}} \sin n\omega t, \quad \varphi_n(t) = \rho\varphi_n(t) = \sqrt{\frac{2}{T}} (n\omega)^{-1} \sin n\omega t. \tag{14}$$

The system $\{e_n, n = 1, 2, 3, \dots\}$ is a complete orthonormal system, c.o.n.s. in short, in $\mathcal{X}$ and $\{\varphi_n, n = 1, 2, 3, \dots\}$ is a c.o.n.s. of $\mathcal{H}_0$. Clearly,

$$\rho\varphi_n = (n\omega)^{-1} e_n, \quad \rho^* e_n = (n\omega)^{-1} \varphi_n.$$
$$\rho\rho^* e_n(t) = (n\omega)^{-2} e_n(t). \tag{15}$$

Let $\mathcal{I}_1$ be the ideal of trace class operators in $\mathcal{X}$ equipped with trace norm $\| \ \|_{\mathcal{I}_1}$ and $\mathcal{I}_2$ be the ideal of Hilbert-Schmidt class operators equipped with norm $\| \ \|_{\mathcal{I}_2}$.

The following Proposition is known.

PROPOSITION 2.1.

1. *$\rho\rho^* \in \mathcal{I}_1$ and $\|\rho\rho^*\|_{\mathcal{I}_1} = \sum_{n=1}^{\infty} (n\omega)^{-2}$.*
2. *$\rho\rho^*$ coincides with the Green operator G_0 of Dirichlet boundary value problem of ordinary differential equation: For all $f \in \mathcal{X}$,*

$$-\frac{d^2}{dt^2} G_0 f(t) = f(t), \tag{16}$$
$$G_0 f(0) = 0, \quad G_0 f(T) = 0.$$

For any $f \in \mathcal{X}$,

$$G_0 f(s) = \int_0^T g_0(s,t) f(t)\, dt,$$

where $g_0(s,t)$ is the Green function

$$g_0(s,t) = \begin{cases} T^{-1}s(T-t) & \text{if } 0 \le s \le t \le T, \\ T^{-1}t(T-s) & \text{if } 0 \le t < s \le T. \end{cases} \tag{17}$$

We have

$$\partial_s g_0(s,t) = \begin{cases} T^{-1}(T-t) & \text{if } 0 \le s < t \le T, \\ -T^{-1}t & \text{if } 0 \le t < s \le T. \end{cases} \tag{18}$$

It is clear that for any $(s,t) \in [0,T] \times [0,T]$,

$$|\partial_s g_0(s,t)| \le 1 \tag{19}$$

and for any $(s,t) \in [0,T] \times [0,T]$,

$$g_0(s,t) = \int_0^s \partial_s g_0(\sigma,t)\, d\sigma.$$

Let $\partial_s G_0$ be the operator in $\mathcal{X}$:

$$\partial_s G_0 f(s) = \int_0^T \partial_s g_0(s,t) f(t) dt, \quad \text{for } f \in \mathcal{X}. \tag{20}$$

Since

$$\int_0^T \int_0^T |\partial_s g_0(s,t)|^2\, ds dt = \frac{T^2}{6},$$

we have

PROPOSITION 2.2. *$\partial_s G_0 \in \mathcal{I}_2$. And $\|\partial_s G_0\|_{\mathcal{I}_2} = T/\sqrt{6}$.*

Let $B : \mathcal{X} \to \mathcal{X}$ be a bounded linear operator. Then we have the following

PROPOSITION 2.3. *$\rho^* B \rho \in \mathcal{I}_1$ and $\rho\rho^* B \in \mathcal{I}_1$. Their traces are equal:*

$$\operatorname{tr} \rho^* B \rho = \operatorname{tr} \rho\rho^* B.$$

Proof is clear.

Propositions 2.2 and 2.3 imply that there exist $k(s,t) \in L^2([0,T]\times[0,T])$ and $h(s,t) \in L^2([0,T]\times[0,T])$ such that for any $f \in \mathcal{X}$,

$$\rho\rho^* Bf(s) = \int_0^T k(s,t)f(t)\,dt, \quad \partial_s G_0 Bf(s) = \int_0^T h(s,t)f(t)\,dt.$$

We shall prove next Lemma.

Lemma 2.4. *For any $s \in [0,T]$ and for almost all $t \in [0,T]$*

$$\int_0^s h(\sigma,t)\,d\sigma = k(s,t).$$

Proof of Lemma. For any $f \in \mathcal{X}$ it is clear that both $\partial_s g_0(s,t)(Bf)(t)$ and $h(s,t)f(t) \in L^1([0,T]\times[0,T])$. Therefore, for any $s \in [0,T]$,

$$\begin{aligned}
&\int_0^T \left(\int_0^s h(\sigma,t)f(t)\,d\sigma\right)dt = \int_0^s \left(\int_0^T h(\sigma,t)f(t)\,dt\right)d\sigma \\
&= \int_0^s \left(\int_0^T \partial_s g_0(\sigma,t)(Bf)(t)\,dt\right)d\sigma = \int_0^T \left(\int_0^s \partial_s g_0(\sigma,t)(Bf)(t)\,d\sigma\right)dt \\
&= \int_0^T g_0(s,t)(Bf)(t)\,dt = \int_0^T k(s,t)f(t)\,dt.
\end{aligned}$$

This proves Proposition 2.4. □

We have

Proposition 2.5. *For almost all $t \in [0,T]$, $k(t,t)$ is well defined and*

$$\int_0^T |k(t,t)|^2\,dt < \infty, \tag{21}$$

$$\operatorname{tr} \rho\rho^* B = \int_0^T k(t,t)\,dt. \tag{22}$$

Proof of Proposition 2.5. For almost $t \in [0,T]$

$$k(t,t) = \int_0^t h(s,t)ds$$

is well-defined because of Lemma 2.4. Inequality (21) is proved in the following way.

$$\int_0^T |k(t,t)|^2\,dt = \int_0^T \left|\int_0^t h(s,t)\,ds\right|^2 dt$$
$$\leq \int_0^T t\int_0^t |h(s,t)|^2\,dsdt \leq T\iint_{[0,T]\times[0,T]} |h(s,t)|^2\,dsdt < \infty.$$

We shall prove (22). Since $\{e_n; n=1,2,3,\dots\}$ is a c.o.n.s. of $\mathcal{X}$, we can write

$$Bf(s) = \sum_{m,n=1}^{\infty} b_{mn}(e_n,f)_{\mathcal{X}}e_m(s). \tag{23}$$

We have

$$\int_0^T k(t,t)dt = \int_0^T\left(\int_0^t h(s,t)\,ds\right)dt = \int_0^T\int_0^T h(s,t)\chi(s,t)\,dsdt,$$

where $\chi(s,t)$ is the characteristic function of the set $\{(s,t)\in\boldsymbol{R}^2 : 0\leq s\leq t, 0\leq t\leq T\}$. Let

$$f_0(s) = \sqrt{\frac{1}{T}}, \quad \text{and} \quad f_m(s) = \sqrt{\frac{2}{T}}\cos(m\omega s) \quad \text{for } m=1,2,3,\dots.$$

Then the system $\{f_0,f_1,f_2,\dots\}$ is a c.o.n.s. of $\mathcal{X}$. Thus $\{f_m(s)\otimes e_n(t) : m = 0,1,2,\dots$ and $n=1,2,3,\dots\}$ is a c.o.n.s. of $L^2([0,T]\times[0,T])$. We have expansions

$$h(s,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m\omega}f_m(s)b_{mn}e_n(t),$$

and

$$\chi(s,t) = \sum_{n=1}^{\infty}\sqrt{2}\frac{(-1)^{n+1}}{n\omega}f_0(s)e_n(t) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m\omega}\delta_{mn}f_m(s)e_n(t).$$

Therefore,

$$\int_0^T \int_0^T h(s,t)\chi(s,t)dsdt = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{m\omega} b_{mn} \frac{1}{m\omega}\delta_{mn} = \sum_{m=1}^{\infty} \frac{1}{(m\omega)^2} b_{mm}$$
$$= \mathrm{tr}\rho\rho^* B.$$

We have proved Proposition 2.5. □

2.2. Divergence of a vector field.

Let $p : \mathcal{H} \ni \gamma \to p(\gamma) \in \mathcal{H}_0$. Then $p(\gamma)$ restricted to $\mathcal{H}_{x,y}$ is a tangent vector field on $\mathcal{H}_{x,y}$. We write as usual $p(\gamma, s) = \rho p(\gamma)(s)$. We have $\partial_s p(\gamma, s) \in \mathcal{X}$.

We use the symbol $\mathcal{L}(\mathcal{X})$ for the Banach space of all bounded linear operators in $\mathcal{X}$ equipped with operator norm.

DEFINITION 2.6 (Admissible vector field). We say that $p(\gamma)$ is an admissible vector field if $p(\gamma)$ has the following properties:

1. There exists a C^1 map $q : \mathcal{H} \to \mathcal{X}$ such that

$$p(\gamma) = \rho^* q(\gamma), \quad \text{for any } \gamma \in \mathcal{H}_{x,y}.$$

2. When we restrict $q(\gamma)$ to $\mathcal{H}_{x,y}$, the Fréchet differential $Dq(\gamma) : \mathcal{H}_0 \ni h \to Dq(\gamma)[h] \in \mathcal{X}$ can be boundedly extended to a bounded linear map $B(\gamma)$ in $\mathcal{X}$, that is, for any $h \in \mathcal{H}_0$,

$$Dq(\gamma)[h] = B(\gamma)\rho h.$$

We often write $\delta q(\gamma)/\delta\gamma$ for $B(\gamma)$.

Let $Dp(\gamma) : \mathcal{H}_0 \to \mathcal{H}_0$ be Fréchet differential of $p(\gamma)$ restricted to $\mathcal{H}_{x,y}$ at $\gamma \in \mathcal{H}_{x,y}$. Then it is clear that for all $h \in \mathcal{H}_0$,

$$Dp(\gamma)[h] = \rho^* B(\gamma)\rho h.$$

That is, for all $h_1, h_2 \in \mathcal{H}_0$,

$$(Dp(\gamma)[h_1], h_2)_{\mathcal{H}_0} = (B(\gamma)\rho h_1, \rho h_2)_{\mathcal{X}}.$$

DEFINITION 2.7 (Divergence of a vector field). Suppose that $p(\gamma)$ is an admissible vector field. We define its divergence $\operatorname{Div} p(\gamma)$ at $\gamma \in \mathcal{H}_{x,y}$ by the following equality:

$$\operatorname{Div} p(\gamma) = \operatorname{tr} \rho^* B(\gamma)\rho = \operatorname{tr} \rho^* \frac{\delta q(\gamma)}{\delta\gamma}\rho.$$

Let $p(\gamma)$ be an admissible vector field. The map $\rho\rho^* B(\gamma)$ is an operator of trace class. We denote its kernel function by $\delta p(\gamma, s)/\delta\gamma(t)$, i.e.,

$$\rho(Dp(\gamma)[h])(s) = \int_0^T \frac{\delta p(\gamma, s)}{\delta\gamma(t)} \rho h(t)\, dt.$$

It is clear that for any $h \in \mathcal{H}_0$,

$$\int_0^T \frac{\delta p(\gamma, s)}{\delta\gamma(t)} \rho h(t)\, dt = Dp(\gamma, s)[h].$$

On account of Proposition 2.5 in the previous subsection we have the following

PROPOSITION 2.8. *Assume $p(\gamma)$ is an admissible vector field. Then*

$$\operatorname{Div} p(\gamma) = \int_0^T \frac{\delta p(\gamma, t)}{\delta\gamma(t)} dt.$$

The notion of admissible vector field defined above is an analogy to infinitesimal version of "admissible transformation" in the case of Wiener integral. cf. [**14**].

3. Statement of main theorem.

DEFINITION 3.1. Let m' be a non-negative number. We say that the vector field $p(\gamma)$ is an m'-admissible vector field if it has all the following properties:

P1: p is an infinitely differentiable map $p : \mathcal{H} \ni \gamma \to p(\gamma) \in \mathcal{H}_0$ of which the restriction to $\mathcal{H}_{x,y}$ is admissible for any fixed $x, y \in \boldsymbol{R}$, that is, there is a C^∞ map $q : \mathcal{H} \to \mathcal{X}$ such that $p(\gamma) = \rho^* q(\gamma)$ for $\gamma \in \mathcal{H}_{x,y}$ and that for all $h \in \mathcal{H}_0$, $Dq(\gamma)[h] = B(\gamma)\rho h$, where $B(\gamma) \in \mathcal{L}(\mathcal{X})$.

P2: The map $\mathcal{H} \ni \gamma \to B(\gamma) \in \mathcal{L}(\mathcal{X})$ is infinitely differentiable. For any non-negative integer K there exists a positive constant Y_K such that

$$B_K = \|q(\gamma)\|_{\{\mathcal{X}; m', K, Y_K\}} + \|B(\gamma)\|_{\{\mathcal{L}(\mathcal{X}); m', K, Y_K\}} < \infty. \tag{24}$$

Let μ be as in (6). Our main theorem is the following

THEOREM 3.2 (Integration by parts). *Let $T \leq \mu$. Suppose that $F(\gamma)$ is an m-smooth functional and that $p(\gamma)$ is an m'-admissible vector field. We further assume that two of $DF(\gamma)[p(\gamma)]$, $F(\gamma) \operatorname{Div} p(\gamma)$ and $F(\gamma)DS(\gamma)[p(\gamma)]$ are F-*

integrable. Then the rest is also F-integrable and the following equality holds.

$$\int_{\Omega_{x,y}} DF(\gamma)[p(\gamma)]e^{i\nu S(\gamma)}\mathcal{D}(\gamma)$$
$$= -\int_{\Omega_{x,y}} F(\gamma)\operatorname{Div} p(\gamma)e^{i\nu S(\gamma)}\mathcal{D}(\gamma) - i\nu\int_{\Omega_{x,y}} F(\gamma)DS(\gamma)[p(\gamma)]e^{i\nu S(\gamma)}\mathcal{D}(\gamma). \tag{25}$$

Let $F(\gamma) \equiv 1$. Then we have the following corollary, which will be used in Section 5.1.

COROLLARY 3.3. *Assume that $p(\gamma)$ is an m'-admissible vector field and that $DS(\gamma)[p(\gamma)]$ is F-integrable. Then* $\operatorname{Div} p(\gamma)$ *is F-integrable and the following equality holds*:

$$\int_{\Omega_{x,y}} DS(\gamma)[p(\gamma)]e^{i\nu S(\gamma)}\mathcal{D}(\gamma) = -(i\nu)^{-1}\int_{\Omega_{x,y}} \operatorname{Div} p(\gamma)e^{i\nu S(\gamma)}\mathcal{D}(\gamma). \tag{26}$$

The following case was proved earlier by N. Kumano-go in [**7**].

REMARK 2. If $p(\gamma, s)$ is independent of γ, i.e., $p(\gamma, s) = h(s)$ then $\operatorname{Div} p(\gamma) = 0$ and the above formula (25) reduces to

$$\int_{\Omega_{x,y}} DF(\gamma)[h]e^{i\nu S(\gamma)}\mathcal{D}(\gamma) = -i\nu\int_{\Omega_{x,y}} F(\gamma)DS(\gamma)[h]e^{i\nu S(\gamma)}\mathcal{D}(\gamma). \tag{27}$$

4. Proof of main theorem.

4.1. Outline of the proof.

Throughout this section Δ denotes an arbitrary division of the interval $[0, T]$ as in Section 1. We use the notation, for example, $(x_{J+1}, x_J, \ldots, x_0)$, γ_Δ and $\alpha = (\alpha_{J+1}, \alpha_J, \ldots, \alpha_2, \alpha_1, \alpha_0)$ etc. as in Section 1. We write

$$N(\Delta) = \prod_{j=1}^{J+1}\left(\frac{\nu}{2\pi i\tau_j}\right)^{1/2},$$

and $y_{\Delta,j} = p(\gamma_\Delta, T_j)$, $j = 0, 1, \ldots, J+1$. Clearly $y_{\Delta,0} = 0 = y_{\Delta,J+1}$. Since definition of oscillatory integral on finite dimensional space $\boldsymbol{R}^J$ implies that

$$\int_{\boldsymbol{R}^J} \sum_{j=1}^{J} \partial_{x_j} \big(F(\gamma_\Delta) y_{\Delta,j} e^{i\nu S(\gamma_\Delta)} \big) \prod_{j=1}^{J} dx_j = 0,$$

we have

$$\begin{aligned} N(\Delta) & \int_{\boldsymbol{R}^J} \sum_{j=1}^{J} \partial_{x_j} (F(\gamma_\Delta)) y_{\Delta,j} e^{i\nu S(\gamma_\Delta)} \prod_{j=1}^{J} dx_j \\ & = -N(\Delta) \int_{\boldsymbol{R}^J} F(\gamma_\Delta) \sum_{j=1}^{J} \partial_{x_j} (y_{\Delta,j}) e^{i\nu S(\gamma_\Delta)} \prod_{j=1}^{J} dx_j \\ & \quad - i\nu N(\Delta) \int_{\boldsymbol{R}^J} F(\gamma_\Delta) \sum_{j=1}^{J} y_{\Delta,j} \partial_{x_j} S(\gamma_\Delta) e^{i\nu S(\gamma_\Delta)} \prod_{j=1}^{J} dx_j. \end{aligned}$$

Our main theorem follows from the above formula if we prove the following lemmas.

LEMMA 4.1 (First equality). *There holds the equality*:

$$\sum_{j=1}^{J} y_{\Delta,j} \partial_{x_j} S(\gamma_\Delta) = DS(\gamma_\Delta)[p(\gamma_\Delta)].$$

LEMMA 4.2 (Second equality). *The following equality holds.*

$$\begin{aligned} \lim_{|\Delta| \to 0} \Bigg(& N(\Delta) \int_{\boldsymbol{R}^J} \sum_{j=1}^{J} \partial_{x_j} (F(\gamma_\Delta)) y_{\Delta,j} e^{i\nu S(\gamma_\Delta)} \prod_{j=1}^{J} dx_j \\ & - N(\Delta) \int_{\boldsymbol{R}^J} DF(\gamma_\Delta)[p(\gamma_\Delta)] e^{i\nu S(\gamma_\Delta)} \prod_{j=1}^{J} dx_j \Bigg) = 0. \end{aligned}$$

LEMMA 4.3 (Third equality). *The following equality is true.*

$$\begin{aligned} \lim_{|\Delta| \to 0} \Bigg(& N(\Delta) \int_{\boldsymbol{R}^J} F(\gamma_\Delta) \sum_{j=1}^{J} \partial_{x_j} (y_{\Delta,j}) e^{i\nu S(\gamma_\Delta)} \prod_{j=1}^{J} dx_j \\ & - N(\Delta) \int_{\boldsymbol{R}^J} F(\gamma_\Delta) \operatorname{Div} p(\gamma_\Delta) e^{i\nu S(\gamma_\Delta)} \prod_{j=1}^{J} dx_j \Bigg) = 0. \end{aligned}$$

4.2. Basic facts.

Let Δ be an arbitrary division of $[0,T]$. We use notation in Section 1 such as $(x_{J+1}, x_J, \dots, x_1, x_0)$ and γ_Δ, etc. We summarize some properties of the norm $\| \ \|_{\{\Delta,m,K,X\}}$, etc. here.

Proposition 4.4. *Let $m, m' \geq 0$, $X, X' \geq 1$ be constants and K, K' be non-negative integers.*

1. *If $m \geq m'$, $K \leq K'$ and $X \geq X'$, then for any functional $F(\gamma_\Delta)$ on $\Gamma(\Delta)$*

$$\|F(\gamma_\Delta)\|_{\{\Delta,m,K,X\}} \leq \|F(\gamma_\Delta)\|_{\{\Delta,m',K',X'\}}. \tag{28}$$

2. *For any functionals F, G on $\Gamma(\Delta)$, we have*

$$\begin{aligned} &\|F(\gamma_\Delta)G(\gamma_\Delta)\|_{\{\Delta,m+m',K,X+Y\}} \\ &\quad \leq \|F(\gamma_\Delta)\|_{\{\Delta,m,K,X\}}\|G(\gamma_\Delta)\|_{\{\Delta,m',K,Y\}}. \end{aligned} \tag{29}$$

Proof. (28) is clear. We shall prove (29). Set

$$A_K = \|F(\gamma_\Delta)\|_{\{\Delta,m,K,X\}}, \quad B_K = \|G(\gamma_\Delta)\|_{\{\Delta,m',K,Y\}}.$$

Then, for any multi-index $\alpha = (\alpha_{J+1}, \alpha_J, \dots, \alpha_1, \alpha_0)$ with $m(\alpha) \leq K$,

$$\left| \left(\prod_{j=0}^{J+1} \partial_{x_j}^{\alpha_j} \right) F(\gamma_\Delta) \right| \leq A_K (1 + |x_0| + \mathrm{var}(\gamma_\Delta))^m X^{|\alpha|},$$

$$\left| \left(\prod_{j=0}^{J+1} \partial_{x_j}^{\alpha_j} \right) G(\gamma_\Delta) \right| \leq B_K (1 + |x_0| + \mathrm{var}(\gamma_\Delta))^{m'} Y^{|\alpha|},$$

here and hereafter $|\alpha| = \sum_{j=0}^{J+1} |\alpha_j|$. Leibniz's rule gives

$$\begin{aligned} &\left|\partial_x^\alpha F(\gamma_\Delta)G(\gamma_\Delta)\right| \\ &\quad \leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \left|\partial_x^\beta F(\gamma_\Delta)\right|\left|\partial_x^{\alpha-\beta} G(\gamma_\Delta)\right| \\ &\quad \leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} A_K (1 + |x_0| + \mathrm{var}(\gamma_\Delta))^m X^{|\beta|} B_K (1 + |x_0| + \mathrm{var}(\gamma_\Delta))^{m'} Y^{|\alpha-\beta|} \end{aligned}$$

$$\leq A_K B_K (1+|x_0|+\mathrm{var}(\gamma_\Delta))^{m+m'} \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} X^{|\beta|} Y^{|\alpha-\beta|}$$
$$= A_K B_K (1+|x_0|+\mathrm{var}(\gamma_\Delta))^{m+m'} (X+Y)^{|\alpha|}.$$

This proves (29). □

Corollary 4.5. *If $F(\gamma)$ is an m-smooth functional and $G(\gamma)$ is an m'-smooth functional, then the product $F(\gamma)G(\gamma)$ is $(m+m')$-smooth.*

Proposition 4.6. *Let $\mathcal{Y}, \mathcal{Z}$ be Banach spaces. Let $\mathcal{L}(\mathcal{Y},\mathcal{Z})$ be the Banach space of bounded linear operators from $\mathcal{Y}$ to $\mathcal{Z}$ equipped with the operator norm. Suppose Δ be a division of the interval $[0,T]$ and $F : \Gamma(\Delta) \ni \gamma_\Delta \to F(\gamma_\Delta) \in \mathcal{Y}$ and $R : \Gamma(\Delta) \ni \gamma_\Delta \to R(\gamma_\Delta) \in \mathcal{L}(\mathcal{Y},\mathcal{Z})$ are C^∞ maps.*

1. *If $R(\gamma_\Delta) \equiv R$ does not depend on γ_Δ, then*

$$\|R(\gamma_\Delta)F(\gamma_\Delta)\|_{\{\mathcal{Z};\Delta,m,K,X_K\}} \leq \|R\|_{\mathcal{L}(\mathcal{Y},\mathcal{Z})} \|F(\gamma_\Delta)\|_{\{\mathcal{Y};\Delta,m,K,X_K\}}.$$

2. *In general,*

$$\|R(\gamma_\Delta)F(\gamma_\Delta)\|_{\{\mathcal{Z};\Delta,m+m',K,X_K+Y_K\}}$$
$$\leq \|R(\gamma_\Delta)\|_{\{\mathcal{L}(\mathcal{Y},\mathcal{Z});\Delta,m',K,Y_K\}} \|F(\gamma_\Delta)\|_{\{\mathcal{Y};\Delta,m,K,X_K\}}.$$

If $F(\gamma), R(\gamma)$ are C^∞-map from $\mathcal{H}$ to $\mathcal{Y}$ and $\mathcal{L}(\mathcal{Y},\mathcal{Z})$. Then

$$\|R(\gamma)F(\gamma)\|_{\{\mathcal{Z};m+m',K,X_K+Y_K\}}$$
$$\leq \|R(\gamma)\|_{\{\mathcal{L}(\mathcal{Y},\mathcal{Z});m',K,Y_K\}} \|F(\gamma)\|_{\{\mathcal{Y};m,K,X_K\}}.$$

Proof of Proposition. First part of the proposition is clear. To prove the second part we have only to mimic the proof of Proposition 4.4. □

The following special cases are also useful.

Proposition 4.7. *Besides the assumption of previous proposition, we suppose that $R(\gamma_\Delta)$ depends only on three variables x_{j-1}, x_j, x_{j+1}, i.e., $\partial_{x_k} R(\gamma_\Delta) = 0$ for $k \neq j-1, j, j+1$, and*

$$\|R(\gamma_\Delta)\|_{\{\mathcal{L}(\mathcal{Y},\mathcal{Z});\Delta,m',K,1\}} < \infty.$$

Then

$$\|R(\gamma_\Delta)F(\gamma_\Delta)\|_{\{\mathcal{Z};\Delta,m+m',K,X\}} \le 2^{3K}\|R\|_{\{\mathcal{L}(\mathcal{Y},\mathcal{Z});\Delta,m',K,1\}}\|F(\gamma)\|_{\{\mathcal{Y};\Delta,m,K,X\}}.$$

PROOF OF PROPOSITION 4.7. Let α be a multi-index with $m(\alpha) \le K$. By Leibniz's rule and assumption,

$$\begin{aligned}\partial_x^\alpha R(\gamma_\Delta)F(\gamma_\Delta) &= \sum_{\beta\le\alpha}\binom{\alpha}{\beta}\partial_x^\beta R(\gamma_\Delta)\partial_x^{\alpha-\beta}F(\gamma_\Delta)\\ &= \sum^{*}\binom{\alpha_{j-1}}{\beta_{j-1}}\binom{\alpha_j}{\beta_j}\binom{\alpha_{j+1}}{\beta_{j+1}}\partial_x^{\beta^*}R(\gamma_\Delta)\partial_x^{\alpha-\beta^*}F(\gamma_\Delta),\end{aligned}$$

where $\sum^*$ means summation over only those multi-indices that is of the form $\beta^* = (0,0,\dots,0,\beta_{j-1},\beta_j,\beta_{j+1},0,0,\dots,0)$.

Let $\|R(\gamma_\Delta)\|_{\{\mathcal{L}(\mathcal{Y},\mathcal{Z});\Delta,m',K,1\}} = B_K$ and $\|F(\gamma_\Delta)\|_{\{\mathcal{Y};\Delta,m,K,X\}} = A_K$. Then

$$\begin{aligned}&\left\|\partial_x^\alpha R(\gamma_\Delta)F(\gamma_\Delta)\right\|_{\mathcal{Z}}\\ &\le \sum^{*}\binom{\alpha_{j-1}}{\beta_{j-1}}\binom{\alpha_j}{\beta_j}\binom{\alpha_{j+1}}{\beta_{j+1}}(1+|x_0|+\operatorname{var}(\gamma_\Delta))^{m'}B_K\\ &\quad\times(1+|x_0|+\operatorname{var}(\gamma_\Delta))^m X^{|\alpha|-|\beta^*|}A_K\\ &\le (1+|x_0|+\operatorname{var}(\gamma_\Delta))^{m+m'}\sum^{*}\binom{\alpha_{j-1}}{\beta_{j-1}}\binom{\alpha_j}{\beta_j}\binom{\alpha_{j+1}}{\beta_{j+1}}B_KA_KX^{|\alpha|-|\beta^*|}\\ &\le (1+|x_0|+\operatorname{var}(\gamma_\Delta))^{m+m'}A_KB_KX^{|\alpha|}\left(\sum^{*}\binom{\alpha_{j-1}}{\beta_{j-1}}\binom{\alpha_j}{\beta_j}\binom{\alpha_{j+1}}{\beta_{j+1}}\right)\\ &\le (1+|x_0|+\operatorname{var}(\gamma_\Delta))^{m+m'}A_KB_KX^{|\alpha|}2^{3K},\end{aligned}$$

because $X \ge 1$. Proposition 4.7 has been proved. □

Let $f : \mathcal{H} \ni \gamma \to f(\gamma) \in \mathcal{X}$ and $u(\gamma) = \rho\rho^* f(\gamma)$. We use the symbols $f(\gamma, s)$ and $u(\gamma, s)$ for the functions which represent elements $f(\gamma) \in \mathcal{X}$ and $u(\gamma) \in \mathcal{X}$, respectively. Let $m \ge 0$. Suppose that for any positive integer K there exists a positive X_K such that for any division Δ of $[0, T]$

$$A_K = \|f(\gamma_\Delta)\|_{\{\mathcal{X};\Delta,m,K,X_K\}} < \infty.$$

Then we can apply Proposition 2.1 and have the following facts.

PROPOSITION 4.8. *$u(\gamma) = G_0 f(\gamma)$. $u(\gamma, 0) = u(\gamma, T) = 0$. There hold the* *following estimates*:

$$\sup_{s\in[0,T]} \|u(\gamma_\Delta, s)\|_{\{\Delta,m,K,X_K\}} \le T^{3/2} A_K, \tag{30}$$

$$\sup_{s\in[0,T]} \left\| \frac{d}{ds} u(\gamma_\Delta, s) \right\|_{\{\Delta,m,K,X_K\}} \le T^{1/2} A_K. \tag{31}$$

Proof of Proposition. Proposition 2.1 implies the first part of proposition. Since (19) means $\partial_s G_0 : \mathcal{X} \to C([0,T])$ is a bounded linear map with norm less than $T^{1/2}$, (31) holds. (30) follows easily from this. □

4.3. Proof of the first equality.

We prove Lemma 4.1.

Since $\gamma_\Delta(t)$ is a piecewise classical path with edges at $t = T_j$ for $j = 1, 2, \ldots, J$, integration by parts gives

$$\begin{aligned} DS(\gamma_\Delta)[p(\gamma_\Delta)] &= \int_0^T \frac{d}{dt}\gamma_\Delta(t)\frac{d}{dt}p(\gamma_\Delta, t)dt - \int_0^T \partial_x V(t, \gamma_\Delta(t))p(\gamma_\Delta, t)dt \\ &= \sum_{j=1}^{J+1} \left(\frac{d}{dt}\gamma_\Delta(T_j - 0)p(\gamma_\Delta, T_j) - \frac{d}{dt}\gamma_\Delta(T_{j-1} + 0)p(\gamma_\Delta, T_{j-1}) \right) \\ &= \sum_{j=1}^{J} \partial_{x_j} S(\gamma_\Delta) y_{\Delta,j}. \end{aligned}$$

Lemma 4.1 has been proved. □

4.4. Proof of the second equality.

Let A_K, m, X_K be as in Definition 1.1 and B_K, m', Y_K be as in Definition 3.1. We know

$$\sum_{j=1}^{J} \partial_{x_j} F(\gamma_\Delta) y_{\Delta,j} = \sum_{j=1}^{J} DF(\gamma_\Delta)[\zeta_{\Delta,j}] y_{\Delta,j}, \tag{32}$$

where $\zeta_{\Delta,j}(t) = \partial_{x_j}\gamma_\Delta(t)$, for $t \in [0,T]$, $j = 1, 2, \ldots, J$. The function $\zeta_{\Delta,j}$ is a piecewise smooth curve which may have edges at $t = T_{j-1}$, T_j, T_{j+1}. It is clear that

$$\zeta_{\Delta,j}(s) = 0, \quad \text{for } s \notin (T_{j-1}, T_{j+1}). \tag{33}$$

and for $t \in (T_{j-1}, T_j) \cup (T_j, T_{j+1})$, $\zeta_{\Delta,j}$ satisfies differential equation of Jacobi-field

$$\frac{d^2}{dt^2}\zeta_{\Delta,j}(t) + \partial_x^2 V(t, \gamma_\Delta(t))\zeta_{\Delta,j}(t) = 0, \tag{34}$$

and boundary conditions

$$\zeta_{\Delta,j}(T_{j-1}) = 0, \quad \zeta_{\Delta,j}(T_j) = 1 \quad \zeta_{\Delta,j}(T_{j+1}) = 0. \tag{35}$$

By definition

$$\partial_{x_k}\zeta_{\Delta,j}(t) = 0, \quad \text{for } t \in [0,T], \text{ if } |j-k| > 1, \tag{36}$$

$$\partial_{x_{j-1}}\zeta_{\Delta,j}(t) = 0, \quad \text{for } t \notin [T_{j-1}, T_j], \tag{37}$$

$$\partial_{x_{j+1}}\zeta_{\Delta,j}(t) = 0, \quad \text{for } t \notin [T_j, T_{j+1}]. \tag{38}$$

$\zeta_{\Delta,j}$ is very close to the following piecewise linear function $e_{\Delta,j}$. For $j = 1, 2, \dots, J$

$$e_{\Delta,j}(t) = \begin{cases} 0 & \text{if } t \notin (T_{j-1}, T_{j+1}), \\ \tau_j^{-1}(t - T_{j-1}) & \text{if } t \in [T_{j-1}, T_j], \\ \tau_{j+1}^{-1}(T_{j+1} - t) & \text{if } t \in [T_j, T_{j+1}]. \end{cases}$$

And

$$e_{\Delta,0}(t) = \begin{cases} 0 & \text{if } t \notin (T_0, T_1), \\ \tau_1^{-1}(T_1 - t) & \text{if } t \in [T_0, T_1], \end{cases}$$

$$e_{\Delta,J+1}(t) = \begin{cases} 0 & \text{if } t \notin (T_J, T_{J+1}), \\ \tau_{T+1}^{-1}(t - T_J) & \text{if } t \in [T_J, T_{J+1}]. \end{cases}$$

It is easy to see (cf. for example [**3**], or [**6**]) that for any α, β there exists constant $C_{\alpha\beta}$ such that the following estimate holds: For $j = 1, 2, 3, \dots, J+1$,

$$\left|\partial_{x_{j-1}}^\alpha \partial_{x_j}^\beta (e_{\Delta,j}(t) - \zeta_{\Delta,j}(t))\right| \le C_{\alpha\beta}\tau_j^2 \quad \text{for } t \in [T_{j-1}, T_j] \tag{39}$$

and for $j = 0, 1, 2, \dots, J$,

$$\left|\partial_{x_j}^\alpha \partial_{x_{j+1}}^\beta (e_{\Delta,j}(t) - \zeta_{\Delta,j}(t))\right| \le C_{\alpha\beta}\tau_{j+1}^2 \quad \text{for } t \in [T_j, T_{j+1}]. \tag{40}$$

It is clear that for $t \notin (T_{j-1}, T_j) \cup (T_j, T_{j+1})$

$$e_{\Delta,j}(t) - \zeta_{\Delta,j}(t) = 0. \tag{41}$$

Therefore, for any $K = 0, 1, 2, \dots$ there exists a positive constant C_K independent of Δ such that for any $t \in [0, T]$

$$\|e_{\Delta,j}(t) - \zeta_{\Delta,j}(t)\|_{\{\Delta,0,K,1\}} \le C_K\big(\tau_j^2 \chi_{[T_{j-1},T_j]}(t) + \tau_{j+1}^2 \chi_{[T_j,T_{j+1}]}(t)\big). \tag{42}$$

Here $\chi_{[T_{j-1},T_j]}(t)$ is the characteristic function of the interval $[T_{j-1}, T_j]$.

Remark 3. We can choose constant $C_{\alpha\beta}$ so that it depends only on $v_2, v_3, \dots, v_{|\alpha|+|\beta|+2}$ and does neither depend on Δ nor on x_{j-1}, x_j, x_{j+1}.

The function $e_{\Delta,j}$ is independent of $\{x_j\}_{j=0,1,\dots,J+1}$ and the collection of functions $\{e_{\Delta,j}\}$ is a partition of unity on $[0, T]$, i.e., for any $t \in [0, T]$,

$$\sum_{j=0}^{J+1} e_{\Delta,j}(t) \equiv 1. \tag{43}$$

Using this and the fact that $y_{\Delta,0} = y_{\Delta,J+1} = 0$, we have

$$\begin{aligned} &DF(\gamma_\Delta)[p(\gamma_\Delta)] - \sum_{j=1}^{J} DF(\gamma_\Delta)[y_{\Delta,j}\zeta_{\Delta,j}] \\ &\quad = \sum_{j=0}^{J+1} DF(\gamma_\Delta)[(p(\gamma_\Delta) - y_{\Delta,j})e_{\Delta,j}] + \sum_{j=1}^{J} DF(\gamma_\Delta)[y_{\Delta,j}(e_{\Delta,j} - \zeta_{\Delta,j})]. \end{aligned}$$

In the following we write $Z_K = X_K + Y_K$ and $m_1 = m + m'$ and $N(T, x, y) = (\nu/2\pi i T)^{1/2}(1 + |x| + |y|)^{m_1}$ for brevity. Then Lemma 4.2 follows from the case $\alpha = \beta = 0$ of the next Lemma.

Lemma 4.9.[1] *For any non-negative integers α, β there exist a positive constant C and a positive integer K independent of division Δ, ν and $x, y \in \boldsymbol{R}$ such that*

[1]Statements of Lemma 4.9 and Lemma 4.10 in the first version of manuscript are corrected following kind advice by prof. N. Kumano-go. The author expresses sincere thanks to him.

$$\left|\partial_x^\alpha \partial_y^\beta\left(e^{-i\nu S(\gamma^*)}N(\Delta)\int_{\boldsymbol{R}^J} DF(\gamma_\Delta)\left[\sum_{j=0}^{J+1}(p(\gamma_\Delta)-y_{\Delta,j})e_{\Delta,j}\right]e^{i\nu S(\gamma_\Delta)}\prod_{j=1}^{J} dx_j\right)\right|$$
$$\le C|N(T,x,y)|A_K B_K \varphi([0,T])|\Delta|. \tag{44}$$

$$\left|\partial_x^\alpha \partial_y^\beta\left(e^{-i\nu S(\gamma^*)}N(\Delta)\int_{\boldsymbol{R}^J} \sum_{j=1}^{J} DF(\gamma_\Delta)[y_{\Delta,j}(e_{\Delta,j}-\zeta_{\Delta,j})]e^{i\nu S(\gamma_\Delta)}\prod_{j=1}^{J} dx_j\right)\right|$$
$$\le C|N(T,x,y)|A_K B_K \varphi([0,T])|\Delta|^2. \tag{45}$$

Here $\varphi([0,T])$ *is the measure of the set* $[0,T]$ *with respect to* φ.

We will prove these estimates by means of stationary phase method over a space of large dimension. cf. [**5**], [**13**] and [**8**].

We now begin the proof of (44). Replacing $f(\gamma)$ by $q(\gamma)$ and A_K by B_K of (24), we can apply Proposition 4.8, because $p(\gamma)=\rho\rho^* q(\gamma)$. We have

$$p(\gamma_\Delta,t)-y_{\Delta,j}=\int_{T_j}^{t}\frac{d}{ds}p(\gamma_\Delta,s)\,ds=\int_{T_j}^{t}\partial_s G_0 q(\gamma_\Delta)(s)\,ds.$$

And we obtain by (31), for $t\in[T_{j-1},T_j]$

$$\begin{aligned}\|p(\gamma_\Delta,t)-y_{\Delta,j}\|_{\{\Delta,m',K,Y_K\}} &\le -\int_{T_j}^{t}\left\|\frac{d}{ds}p(\gamma_\Delta,s)\right\|_{\{\Delta,m',K,Y_K\}}ds\\ &\le -\int_{T_j}^{t}T^{1/2}B_K\,ds\le \tau_j T^{1/2}B_K.\end{aligned}$$

Similar estimate holds in the case $t\in[T_j,T_{j+1}]$. Therefore, by Proposition 4.6, there exists a positive constant C which may depend on T but not on Δ, K and j such that

$$\|(p(\gamma_\Delta,t)-y_{\Delta,j})e_{\Delta,j}(t)\|_{\{\Delta,m',K,Y_K\}}\le\begin{cases}CB_K\tau_j & \text{for } t\in[T_{j-1},T_j],\\ CB_K\tau_{j+1} & \text{for } t\in[T_j,T_{j+1}].\end{cases} \tag{46}$$

Writing

$$u(\gamma_\Delta,t)=\sum_{j=0}^{J+1}(p(\gamma_\Delta,t)-y_{\Delta,j})e_{\Delta,j}(t),$$

we have for any fixed $t \in [T_{j-1}, T_j]$,

$$
\begin{aligned}
&\|u(\gamma_\Delta, t)\|_{\{\Delta, m', K, Y_K\}} \\
&\quad = \|(p(\gamma_\Delta, t) - y_{\Delta, j-1}) e_{\Delta, j-1}(t) + (p(\gamma_\Delta, t) - y_{\Delta, j}) e_{\Delta, j}(t)\|_{\{\Delta, m', K, Y_K\}} \\
&\quad \leq 2CB_K \tau_j.
\end{aligned}
$$

Thus,

$$
\|u(\gamma_\Delta)\|_{\{C([0,T]); \Delta, m', K, Y_K\}} \leq 2CB_K |\Delta|.
$$

Since $m_1 = m + m'$, $Z_K = X_K + Y_K$, we have

$$
\begin{aligned}
&\left\| DF(\gamma_\Delta) \left[\sum_{j=0}^{J+1} (p(\gamma_\Delta) - y_{\Delta, j}) e_{\Delta, j} \right] \right\|_{\{\Delta, m_1, K, Z_K\}} \\
&\quad \leq \left\| \int_0^T \frac{\delta F(\gamma_\Delta)}{\delta \gamma(t)} u(\gamma_\Delta, t)\, d\varphi(t) \right\|_{\{\Delta, m_1, K, Z_K\}} \\
&\quad \leq \left\| \frac{\delta F(\gamma_\Delta)}{\delta \gamma} \right\|_{\{L^1([0,T], \varphi); \Delta, m, K, X_K\}} \|u(\gamma_\Delta)\|_{\{C([0,T]); \Delta, m', K, Y_K\}} \\
&\quad \leq CA_K B_K \varphi([0,T]) |\Delta|.
\end{aligned} \tag{47}
$$

In order to apply stationary phase method we need still more information. cf.[**5**], [**13**] and [**8**]. Let Δ be an arbitrary division of interval $[0,T]$ as is given in (7). Let Δ_1 be any division of $[0,T]$ which is coarser than Δ, in other words, Δ be a refinement of Δ_1. Then there is a subset $\{i_1, i_2, \ldots, i_s\}$ of $\{1, 2, 3, \ldots, J\}$ such that division points of Δ_1 are

$$
\Delta_1 : T_0 = T_{i_0} < T_{i_1} < \cdots < T_{i_s} < T_{i_{s+1}} = T_{J+1}. \tag{48}
$$

We set $i_{s+1} = J + 1$ and $i_0 = 0$.

Let $\gamma_{\Delta_1}(t) = \gamma_{\Delta_1}(x_{i_{s+1}}, x_{i_s}, \ldots, x_{i_1}, x_{i_0})(t)$ be an arbitrary piecewise classical path associated with the division Δ_1. We can identify this with the piecewise classical path $\gamma_\Delta \in \Gamma(\Delta)$ with the property $\gamma_\Delta(t) \equiv \gamma_{\Delta_1}(t)$ for any $t \in [0,T]$. We denote this identification map by $\iota : \Gamma(\Delta_1) \to \Gamma(\Delta)$. Let $f : \Gamma(\Delta) \to \boldsymbol{C}$ be a function defined on $\Gamma(\Delta)$. We use the symbol $\iota^* f$ for the pull back of f by ι.

We wish to prove that for any $K = 0, 1, 2, \ldots,$

$$\left\| \iota^* DF(\gamma_\Delta) \left[\sum_{j=0}^{J+1} (p(\gamma_\Delta) - y_{\Delta,j}) e_{\Delta,j} \right] \right\|_{\{\Delta_1, m_1, K, Z_K\}} \leq C A_K B_K \varphi([0,T]) |\Delta|,$$

with positive constant C independent of Δ.

Since $e_{\Delta,j}(t)$ does not depend on x_j, $j = 0, 1, 2, \ldots, J+1$,

$$\iota^* e_{\Delta,j}(t) = e_{\Delta,j}(t), \quad \text{for } t \in [0,T],\ j = 0, 1, \ldots, J+1. \tag{49}$$

and

$$\iota^* p(\gamma_\Delta, t) - y_{\Delta,j} = p(\gamma_{\Delta_1}, t) - p(\gamma_{\Delta_1}, T_j).$$

It is clear that

$$\|\iota^*(p(\gamma_\Delta, t) - y_{\Delta,j}) e_{\Delta,j}\|_{\{\Delta_1, m', K, Y_K\}} \leq \begin{cases} C B_K \tau_j & \text{for } t \in [T_{j-1}, T_j], \\ C B_K \tau_{j+1} & \text{for } t \in [T_j, T_{j+1}]. \end{cases}$$

Therefore, mimicking discussion following (46), we have

$$\left\| \iota^* \sum_{j=0}^{J+1} (p(\gamma_\Delta, t) - y_{\Delta,j}) e_{\Delta,j} \right\|_{\{C([0,T]); \Delta_1, m', K, Y_K\}} \leq 2 C B_K |\Delta|.$$

Clearly,

$$\begin{aligned} \left\| \iota^* \frac{\delta F(\gamma_\Delta)}{\delta \gamma(t)} \right\|_{\{L^1([0,T],\varphi); \Delta_1, m, K, X_K\}} &= \left\| \frac{\delta F(\gamma_{\Delta_1})}{\delta \gamma(t)} \right\|_{\{L^1([0,T],\varphi); \Delta_1, m, K, X_K\}} \\ &\leq A_K \varphi([0,T]). \end{aligned}$$

Therefore, there exists a positive constant C independent of Δ_1, Δ such that

$$\begin{aligned} &\left\| \iota^* DF(\gamma_\Delta) \left[\sum_{j=0}^{J+1} (p(\gamma_\Delta) - y_{\Delta,j}) e_{\Delta,j} \right] \right\|_{\{\Delta_1, m_1, K, Z_K\}} \\ &\quad \leq \left\| \iota^* \frac{\delta F(\gamma_\Delta)}{\delta \gamma(t)} \right\|_{\{L^1([0,T],\varphi); \Delta_1, m, K, X_K\}} \\ &\qquad \times \left\| \iota^* \sum_{j=0}^{J+1} (p(\gamma_\Delta, t) - y_{\Delta,j}) e_{\Delta,j} \right\|_{\{C([0,T]); \Delta_1, m', K, Y_K\}} \end{aligned}$$

$$\leq CA_K B_K \varphi([0,T])|\Delta|. \tag{50}$$

Since we have obtained (47) and (50), we can apply stationary phase method to the oscillatory integral:

$$N(\Delta)\int_{\boldsymbol{R}^J} DF(\gamma_\Delta)\Big[\sum_{j=0}^{J+1}(p(\gamma_\Delta)-y_{\Delta,j})e_{\Delta,j}\Big]e^{i\nu S(\gamma_\Delta)}\prod_{j=1}^{J}dx_j.$$

As a consequence, for any non-negative integers α,β there exist a positive constant C and a positive integer K independent of Δ such that

$$\Big|\partial_x^\alpha\partial_y^\beta\Big(e^{-i\nu S(\gamma^*)}N(\Delta)\int_{\boldsymbol{R}^J} DF(\gamma_\Delta)\Big[\sum_{j=0}^{J+1}(p(\gamma_\Delta)-y_{\Delta,j})(e_{\Delta,j})\Big]e^{i\nu S(\gamma_\Delta)}\prod_{j=1}^{J}dx_j\Big)\Big|$$
$$\leq C|N(T,x,y)|A_K B_K\varphi([0,T])|\Delta|.$$

We have proved (44).

Now we prove (45). By virtue of (39), Proposition 4.7 and (31), there exists a positive constant C_K for each non-negative integer K such that for any fixed $t\in[T_{j-1},T_j]$, $j=1,2,3,\dots,J$,

$$\begin{aligned}&\|y_{\Delta,j}(e_{\Delta,j}(t)-\zeta_{\Delta,j}(t))\|_{\{\Delta,m',K,Y_K\}}\\ &\quad\leq 2^{3K}\|e_{\Delta,j}(t)-\zeta_{\Delta,j}(t)\|_{\{\Delta,0,K,1\}}\|p(\gamma_\Delta,T_j)\|_{\{\Delta,m',K,Y_K\}}\\ &\quad\leq C_K B_K\tau_j^2,\end{aligned}$$

and for $t\in[T_j,T_{j+1}]$, $j=1,2,3,\dots,J$

$$\|y_{\Delta,j}(e_{\Delta,j}(t)-\zeta_{\Delta,j}(t))\|_{\{\Delta,m',K,Y_K\}}\leq C_K B_K\tau_{j+1}^2.$$

Obviously, for $t\notin(T_{j-1},T_{j+1})$

$$\|y_{\Delta,j}(e_{\Delta,j}(t)-\zeta_{\Delta,j}(t))\|_{\{\Delta,m',K,Y_K\}}=0.$$

Therefore,

$$\Big\|\sum_{j=1}^{J}(y_{\Delta,j}(e_{\Delta,j}-\zeta_{\Delta,j}))\Big\|_{\{C([0,T]);\Delta,m',K,Y_K\}}\leq 2C_K B_K|\Delta|^2.$$

This leads to

$$\begin{aligned}
&\Big\| DF(\gamma_\Delta)\Big[\sum_{j=1}^{J}(y_{\Delta,j}(e_{\Delta,j}-\zeta_{\Delta,j}))\Big]\Big\|_{\{\Delta,m_1,K,Z_K\}}\\
&\quad\le \Big\| \int_0^T \frac{\delta F(\gamma_\Delta)}{\delta\gamma(t)}\Big(\sum_{j=1}^{J}(y_{\Delta,j}(e_{\Delta,j}(t)-\zeta_{\Delta,j}(t))\Big)d\varphi(t)\Big\|_{\{\Delta,m_1,K,Z_K\}}\\
&\quad\le \Big\|\frac{\delta F(\gamma_\Delta)}{\delta\gamma(t)}\Big\|_{\{L^1([0,T],\varphi);\Delta,m,K,X_K\}}\Big\|\sum_{j=1}^{J}(y_{\Delta,j}(e_{\Delta,j}-\zeta_{\Delta,j}))\Big\|_{\{C([0,T]);\Delta,m',K,Y_K\}}\\
&\quad\le C_K A_K B_K\varphi([0,T])|\Delta|^2, \qquad (51)
\end{aligned}$$

with some positive constant C_K independent of Δ.

Let Δ_1 be any division of $[0,T]$ which is coarser than Δ. Now we discuss the pull back of $DF(\gamma_\Delta)[\sum_{j=1}^{J}(y_{\Delta,j}(e_{\Delta,j}-\zeta_{\Delta,j}))]$. The pull back $\iota^*\zeta_{\Delta,j}$ vanishes outside (T_{j-1},T_{j+1}) and satisfies differential equation of Jacobi field and boundary value:

$$\begin{gathered}
\frac{d^2}{dt^2}\iota^*\zeta_{\Delta,j}(t)+\partial_x^2V(t,\gamma_{\Delta_1}(t))\iota^*\zeta_{\Delta,j}(t)=0,\quad t\in(T_{j-1},T_j)\cup(T_j,T_{j+1}),\\
\iota^*\zeta_{\Delta,j}(T_{j-1})=\iota^*\zeta_{\Delta,j}(T_{j+1})=0,\quad\text{and}\quad \iota^*\zeta_{\Delta,j}(T_j)=1.
\end{gathered}$$

Therefore, the estimates (39), (40) and (42) replaced $\zeta_{\Delta,j}$ by $\iota^*\zeta_{\Delta,j}$ hold with the same constants $C_{\alpha,\beta}$ and C_K. We have clearly

$$e_{\Delta,j}(t)-\iota^*\zeta_{\Delta,j}(t)=0\quad\text{if } t\notin[T_{j-1},T_{j+1}]. \qquad (52)$$

And

$$\|e_{\Delta,j}(t)-\iota^*\zeta_{\Delta,j}(t)\|_{\{\Delta,0,K,1\}}\le\begin{cases}C_K\tau_j^2 & \text{for } t\in[T_{j-1},T_j],\\ C_K\tau_{j+1}^2 & \text{for } t\in[T_j,T_{j+1}].\end{cases} \qquad (53)$$

Thus we can obtain in the same way as in (51)

$$\Big\|\iota^*DF(\gamma_\Delta)\Big[\sum_{j=1}^{J}(y_{\Delta,j}(e_{\Delta,j}-\zeta_{\Delta,j}))\Big]\Big\|_{\{\Delta_1,m',K,Z_K\}}\le C_K A_K B_K\varphi([0,T])|\Delta|^2, \qquad (54)$$

with some positive constant C_K independent of Δ.

It follows from (54), (51) and stationary phase method that for any non-negative integers α and β, there exists a positive integer K and a positive constant C such that

$$\left|\partial_x^\alpha \partial_y^\beta \left(e^{-i\nu S(\gamma^*)} N(\Delta) \int_{\boldsymbol{R}^J} \sum_{j=1}^J DF(\gamma_\Delta)[y_{\Delta,j}(e_{\Delta,j} - \zeta_{\Delta,j})] e^{i\nu S(\gamma_\Delta)} \prod_{j=1}^J dx_j \right)\right|$$
$$\leq C|N(T,x,y))| A_K B_K \varphi([0,T])|\Delta|^2.$$

This proves (45). We have proved Lemma 4.9. Therefore, proof of Lemma 4.2 has been completed. □

4.5. Proof of the third equality.

Let $B(\gamma) \in \mathcal{L}(\mathcal{X})$ be as in Definition 3.1. We can use Propositions 2.3, 2.5, 2.8 and Lemma 2.4. Let us denote the kernel function of $\rho\rho^* B(\gamma) = G_0 B(\gamma)$ by $k(\gamma, s, t)$ and that of $\partial_s G_0 B(\gamma)$ by $h(\gamma, s, t)$. We know

$$k(\gamma, s, t) = \int_0^s h(\gamma, \sigma, t)\, d\sigma, \quad \text{for almost all } t \in [0,T], \tag{55}$$

$$\operatorname{Div} p(\gamma) = \int_0^T k(\gamma, t, t)\, dt,$$

$$\int_Q |h(\gamma, s, t)|^2\, dsdt \leq \|\partial_s G_0\|_{\mathcal{I}_2}^2 \|B(\gamma)\|_{\mathcal{L}(\mathcal{X})}^2. \tag{56}$$

Here and hereafter we write $Q = [0,T] \times [0,T]$.

Inequality (56) and inequality (24) in Definition 3.1 for $p(\gamma)$ implies that for any division Δ of $[0,T]$

$$\|h(\gamma_\Delta, s, t)\|_{\{L^2(Q);\Delta,m',K,Y_K\}} \leq B_K \|\partial_s G_0\|_{\mathcal{I}_2}. \tag{57}$$

It is clear from (55) that for almost all $t \in [0,T]$, $k(\gamma, s, t)$ is continuous in s. Since the range of G_0 is in $\mathcal{H}_0$, we have

$$k(\gamma, 0, t) = k(\gamma, T, t) = 0 \quad \text{for almost all } t \in [0,T]. \tag{58}$$

We know

$$\partial_{x_j} y_{\Delta,j} = Dy_{\Delta,j}[\partial_{x_j}\gamma_\Delta] = D(\rho\rho^* q(\gamma_\Delta)(T_j))[\zeta_{\Delta,j}] = \int_0^T k(\gamma_\Delta, T_j, t)\zeta_{\Delta,j}(t)\, dt.$$

Using partition of unity $\{e_{\Delta,j}\}$ again and (58), we have

$$\begin{aligned}
&\operatorname{Div} p(\gamma_\Delta) - \sum_{j=1}^{J} \partial_{x_j} y_{\Delta,j} \\
&= \sum_{j=0}^{J+1} \int_0^T k(\gamma_\Delta, t, t) e_{\Delta,j}(t)\, dt - \sum_{j=1}^{J} \int_0^T k(\gamma_\Delta, T_j, t) \zeta_{\Delta,j}(t)\, dt \\
&= \sum_{j=0}^{J+1} \int_0^T (k(\gamma_\Delta, t, t) - k(\gamma_\Delta, T_j, t)) e_{\Delta,j}(t)\, dt \\
&\quad + \sum_{j=1}^{J} \int_0^T k(\gamma_\Delta, T_j, t)(e_{\Delta,j}(t) - \zeta_{\Delta,j}(t)) dt.
\end{aligned}$$

Lemma 4.3 follows from the case $\alpha = \beta = 0$ of the next Lemma.

LEMMA 4.10. *For any non-negative integers α, β there exist a positive constant C and a positive integer K independent of Δ such that*

$$\begin{aligned}
&\Bigg| \partial_x^\alpha \partial_y^\beta \Bigg(e^{-i\nu S(\gamma^*)} N(\Delta) \int_{\boldsymbol{R}^J} F(\gamma_\Delta) e^{i\nu S(\gamma_\Delta)} \\
&\qquad \times \sum_{j=0}^{J+1} \int_0^T (k(\gamma_\Delta, t, t) - k(\gamma_\Delta, T_j, t)) e_{\Delta,j}(t)\, dt \prod_{j=1}^{J} dx_j \Bigg) \Bigg| \\
&\le C |N(T, x, y)| \|\partial_s G_0\|_{\mathcal{I}_2} A_K B_K T^{1/2} |\Delta|^{1/2}, \qquad (59)
\end{aligned}$$

and

$$\begin{aligned}
&\Bigg| \partial_x^\alpha \partial_y^\beta \Bigg(e^{-i\nu S(\gamma^*)} N(\Delta) \int_{\boldsymbol{R}^J} F(\gamma_\Delta) e^{i\nu S(\gamma_\Delta)} \\
&\qquad \times \int_0^T \sum_{j=1}^{J} k(\gamma_\Delta, T_j, t)(e_{\Delta,j}(t) - \zeta_{\Delta,j}(t))\, dt \prod_{j=1}^{J} dx_j \Bigg) \Bigg| \\
&\le C |N(T, x, y)| A_K B_K \|\partial_s G_0\|_{\mathcal{I}_2} T^{3/2} |\Delta|^{3/2}. \qquad (60)
\end{aligned}$$

PROOF OF LEMMA 4.10. We begin with the proof of (59). Using (55), we have

$$\begin{aligned}
&\int_0^T (k(\gamma_\Delta, t, t) - k(\gamma_\Delta, T_j, t)) e_{\Delta,j}(t)\, dt \\
&\quad = \int_0^T \int_{T_j}^t h(\gamma_\Delta, s, t) e_{\Delta,j}(t)\, dsdt \\
&\quad = -\int_{Q_j^-} h(\gamma_\Delta, s, t) e_{\Delta,j}(t)\, dsdt + \int_{Q_j^+} h(\gamma_\Delta, s, t) e_{\Delta,j}(t)\, dsdt,
\end{aligned}$$

where Q_j^- is the triangle $\{(s,t) \in Q; t \leq s \leq T_j, T_{j-1} \leq t \leq T_j\}$ and $Q_j^+ = \{(s,t) \in Q; T_j \leq s \leq t, T_j \leq t \leq T_{j+1}\}$. We denote characteristic functions of Q_j^- and Q_j^+ by $\chi_j^-(s,t)$ and $\chi_j^+(s,t)$, respectively. Then

$$\begin{aligned}
&\sum_{j=0}^{J+1} \int_0^T (k(\gamma_\Delta, t, t) - k(\gamma_\Delta, T_j, t))\, e_{\Delta,j}(t)\, dt \\
&\quad = \int_Q \left(\sum_{j=0}^{J+1} (\chi_j^+(s,t) - \chi_j^-(s,t)) e_{\Delta,j}(t) h(\gamma_\Delta, s, t) \right) dsdt \\
&\quad = (\chi(\Delta), h(\gamma_\Delta))_{L^2(Q)},
\end{aligned}$$

here $\chi(\Delta) \in L^2(Q)$ is the function $\chi(\Delta, s, t) = \sum_{j=0}^{J+1} (\chi_j^+(s,t) - \chi_j^-(s,t)) e_{\Delta,j}(t)$ and $(\ ,\ )_{L^2(Q)}$ is the inner product in the space $L^2(Q)$. $\chi(\Delta)$ does not depend on $(x_{J+1}, x_J, \ldots, x_1, x_0)$. Its norm $\|\chi(\Delta)\|_{L^2(Q)}$ is majorized as

$$\begin{aligned}
\|\chi(\Delta)\|_{L^2(Q)}^2 &= \sum_{j=0}^{J+1} \int_{Q_j^- \cup Q_j^+} e_{\Delta,j}(t)^2\, dsdt \\
&\leq \sum_{j=1}^{J} \frac{1}{2} \big(\tau_j^2 + \tau_{j+1}^2 \big) + \frac{1}{2} \big(\tau_1^2 + \tau_{J+1}^2 \big) \leq |\Delta| T.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left\| \sum_{j=0}^{J+1} \int_0^T (k(\gamma_\Delta, t, t) - k(\gamma_\Delta, T_j, t)) e_{\Delta,j}(t)\, dt \right\|_{\{\Delta, m', K, Y_K\}} \\
&\quad \leq |\Delta|^{1/2} T^{1/2} \|h(\gamma_\Delta, s, t)\|_{\{L^2(Q); \Delta, m', K, Y_k\}} \leq \|\partial_s G_0\|_{\mathcal{I}_2} B_K T^{1/2} |\Delta|^{1/2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left\| \sum_{j=0}^{J+1} \int_0^T (k(\gamma_\Delta, t, t) - k(\gamma_\Delta, T_j, t)) e_{\Delta,j}(t)\, dt F(\gamma_\Delta) \right\|_{\{\Delta, m_1, K, Z_K\}} \\
&\quad \le \left\| \sum_{j=0}^{J+1} \int_0^T (k(\gamma_\Delta, t, t) - k(\gamma_\Delta, T_j, t)) e_{\Delta,j}(t)\, dt \right\|_{\{\Delta, m', K, Y_K\}} \|F(\gamma_\Delta)\|_{\{\Delta, m, K, X_K\}} \\
&\quad \le \|\partial_s G_0\|_{\mathcal{I}_2} A_K B_K T^{1/2} |\Delta|^{1/2}. \qquad (61)
\end{aligned}$$

Let Δ_1 be an arbitrary division of $[0, T]$ coarser than Δ and $\iota : \Gamma(\Delta_1) \to \Gamma(\Delta)$ be the embedding. Then we obtain that

$$\begin{aligned}
&\left\| \iota^* \sum_{j=0}^{J+1} \int_0^T (k(\gamma_\Delta, t, t) - k(\gamma_\Delta, T_j, t)) e_{\Delta,j}(t)\, dt \right\|_{\{\Delta_1, m', K, Y_K\}} \\
&\quad = \left\| (\chi(\Delta, s, t), h(\gamma_{\Delta_1}, s, t))_{L^2(Q)} \right\|_{\{\Delta_1, m', K, Y_K\}} \\
&\quad \le \|\partial_s G_0\|_{\mathcal{I}_2} \|B(\gamma_{\Delta_1})\|_{\{\mathcal{L}(\mathcal{X}); \Delta_1, m', K, Y_K\}} |\Delta|^{1/2} T^{1/2} \\
&\quad \le \|\partial_s G_0\|_{\mathcal{I}_2} B_K T^{1/2} |\Delta|^{1/2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left\| \iota^* \sum_{j=0}^{J+1} \int_0^T (k(\gamma_\Delta, t, t) - k(\gamma_\Delta, T_j, t)) e_{\Delta,j}(t)\, dt F(\gamma_\Delta) \right\|_{\{\Delta_1, m_1, K, Z_K\}} \\
&\quad \le \left\| \sum_{j=0}^{J+1} \int_0^T (k(\gamma_{\Delta_1}, t, t) - k(\gamma_{\Delta_1}, T_j, t)) e_{\Delta,j}(t)\, dt \right\|_{\{\Delta_1, m', K, Y_K\}} \\
&\qquad \times \|F(\gamma_{\Delta_1})\|_{\{\Delta_1, m, K, X_K\}} \\
&\quad \le \|\partial_s G_0\|_{\mathcal{I}_2} A_K B_K T^{1/2} |\Delta|^{1/2}. \qquad (62)
\end{aligned}$$

(59) follows from (61), (62) and stationary phase method.

Next we shall prove (60). We denote the characteristic function of the interval $[0, T_j]$ by $\chi_{[0,T_j]}(s)$. Then

$$\begin{aligned}
&\int_0^T k(\gamma_\Delta, T_j, t)(e_{\Delta,j}(t) - \zeta_{\Delta,j}(t))\, dt \\
&\quad = \big(\chi_{[0,T_j]}(s)(e_{\Delta,j}(t) - \zeta_{\Delta,j}(t)), h(\gamma_\Delta, s, t)\big)_{L^2(Q)}.
\end{aligned}$$

Since $\partial_{x_k}\chi_{[0,T_j]}(s)(e_{\Delta,j}(t)-\zeta_{\Delta,j}(t))=0$ for $k\neq j-1,j,j+1$, Proposition 4.7 leads us to

$$\begin{aligned}&\big\|(\chi_{[0,T_j]}(s)(e_{\Delta,j}(t)-\zeta_{\Delta,j}(t)),h(\gamma_\Delta,s,t))_{L^2(Q)}\big\|_{\{\Delta,m',K,Y_K\}}\\&\quad\leq 2^{3K}\|\chi_{[0,T_j]}(s)(e_{\Delta,j}(t)-\zeta_{\Delta,j}(t))\|_{\{L^2(Q);\Delta,0,K,1\}}\|h(\gamma_\Delta,s,t)\|_{\{L^2(Q);\Delta,m',K,Y_K\}}\\&\quad\leq 2^{3K}C_K\|\partial_sG_0\|_{\mathcal{I}_2}B_K\big(\tau_j^2+\tau_{j+1}^2\big)(\tau_j+\tau_{j+1})^{1/2}T^{1/2},\end{aligned}$$

with some positive constant C_K. Therefore,

$$\begin{aligned}&\bigg\|\sum_{j=1}^J\int_0^T k(\gamma_\Delta,T_j,t)(e_{\Delta,j}(t)-\zeta_{\Delta,j}(t))dt\bigg\|_{\{\Delta,m',K,Y_K\}}\\&\quad\leq\sum_{j=1}^J\bigg\|\int_0^T k(\gamma_\Delta,T_j,t)(e_{\Delta,j}(t)-\zeta_{\Delta,j}(t))dt\bigg\|_{\{\Delta,m',K,Y_K\}}\\&\quad\leq\sum_{j=1}^J 2^{3K}C_K\|\partial_sG_0\|_{\mathcal{I}_2}B_K\big(\tau_j^2+\tau_{j+1}^2\big)(\tau_j+\tau_{j+1})^{1/2}T^{1/2}\\&\quad\leq C_K\|\partial_sG_0\|_{\mathcal{I}_2}B_KT^{3/2}|\Delta|^{3/2},\end{aligned}$$

here and hereafter we denote various positive constants which are different from place to place but may depend on K by the same symbol C_K. Consequently,

$$\begin{aligned}&\bigg\|F(\gamma_\Delta)\sum_{j=1}^J\int_0^T k(\gamma_\Delta,T_j,t)(e_{\Delta,j}(t)-\zeta_{\Delta,j}(t))dt\bigg\|_{\{\Delta,m_1,K,Z_K\}}\\&\quad\leq\|F(\gamma_\Delta)\|_{\{\Delta,m,K,X_K\}}\bigg\|\sum_{j=1}^J\int_0^T k(\gamma_\Delta,T_j,t)(e_{\Delta,j}(t)-\zeta_{\Delta,j}(t))dt\bigg\|_{\{\Delta,m',K,Y_K\}}\\&\quad\leq C_K\|\partial_sG_0\|_{\mathcal{I}_2}A_KB_KT^{3/2}|\Delta|^{3/2}.\end{aligned}\tag{63}$$

Let Δ_1 be any division of $[0,T]$ coarser than Δ. Then we shall prove similar estimate for the pull-back

$$\begin{aligned}&\iota^*F(\gamma_\Delta)\sum_{j=1}^J\int_0^T k(\gamma_\Delta,T_j,t)(e_{\Delta,j}(t)-\zeta_{\Delta,j}(t))dt\\&\qquad=F(\gamma_{\Delta_1})\sum_{j=1}^J\int_0^T k(\gamma_{\Delta_1},T_j,t)(e_{\Delta,j}(t)-\iota^*\zeta_{\Delta,j}(t))dt.\end{aligned}$$

Since the estimate (53) holds, we have

$$\left\| \iota^* F(\gamma_\Delta) \sum_{j=1}^{J} \int_0^T k(\gamma_\Delta, T_j, t)(e_{\Delta,j}(t) - \zeta_{\Delta,j}(t))dt \right\|_{\{\Delta, m_1, K, Z_K\}} \\ \leq C_K \|\partial_s G_0\|_{\mathcal{I}_2} A_K B_K T^{3/2} |\Delta|^{3/2}. \tag{64}$$

Using (63) and (64), we can apply stationary phase method. As a result, for any non-negative integers α, β, there exist a positive integer K and a positive constant C such that

$$\left| \partial_x^\alpha \partial_y^\beta \left(e^{-i\nu S(\gamma^*)} N(\Delta) \int_{\boldsymbol{R}^J} F(\gamma_\Delta) e^{i\nu S(\gamma_\Delta)} \right.\right. \\ \left.\left. \times \sum_{j=1}^{J} \int_0^T k(\gamma_\Delta, T_j, t)(e_{\Delta,j}(t) - \zeta_{\Delta,j}(t))dt \prod_{j=1}^{J} dx_j \right) \right| \\ \leq C |N(T, x, y)| \|\partial_s G_0\|_{\mathcal{I}_2} A_K B_K T^{3/2} |\Delta|^{3/2}. \tag{65}$$

We have proved (60). Lemma 4.10 has been proved. □

Therefore, Lemma 4.3 is proved. □

We have completed proof of our main Theorem 3.2. □

5. Application to semiclassical asymptotic behaviour of Feynman path integrals.

5.1. A sharper asymptotic formula.

We always assume $T < \mu$. Let $F(\gamma)$ be an m-smooth functional. Then semiclassical asymptotic formula was proved by Kumano-go [**13**].

$$\int_{\Omega_{x,y}} F(\gamma) e^{i\nu S(\gamma)} \mathcal{D}(\gamma) \\ = \left(\frac{\nu}{2\pi i T} \right)^{1/2} D(T, 0, x, y)^{-1/2} e^{i\nu S(\gamma^*)} \big(F(\gamma^*) + \nu^{-1} r(\nu, T, 0, x, y) \big), \tag{66}$$

where γ^* is the classical path connecting (T, x) and $(0, y)$ in time-space and $D(T, 0, x, y)$ is Van Vleck-Morette determinant, cf. [**15**], and also [**6**].

If $F(\gamma^*) = 0$, then the main term of the right hand side of (66) vanishes. What happens in that case? Even in this case integration by parts formula enables us to get a sharper information if the following additional assumption is satisfied.

ASSUMPTION 5.1. We assume $F(\gamma)$ has all of the following properties:

1. $F(\gamma)$ is a real valued m-smooth functional. For fixed γ, $DF(\gamma)[h] = \int_0^T (\delta F(\gamma)/\delta\gamma(s))\rho h(s)\, ds$ for any $h \in \mathcal{H}_0$ and $\delta F(\gamma)/\delta\gamma(s) \in \mathcal{X}$ as a function of s, which we write $\delta F(\gamma)/\delta\gamma$. The map $\mathcal{H} \ni \gamma \to \delta F(\gamma)/\delta\gamma \in \mathcal{X}$ is a C^∞ map. There exists a C^∞ map $\mathcal{H} \ni \gamma \to A(\gamma) \in \mathcal{L}(\mathcal{X})$ such that for any $h \in \mathcal{H}_0$,
$$D\frac{\delta F(\gamma)}{\delta\gamma}[h] = A(\gamma)\rho h.$$
2. For any $K = 0, 1, 2, \ldots$, there exist positive constants A_K and X_K such that
$$A_K = \left\| \frac{\delta F(\gamma)}{\delta\gamma} \right\|_{\{\mathcal{X};m,K,X_K\}} + \|A(\gamma)\|_{\{\mathcal{L}(\mathcal{X});m,K,X_K\}} < \infty. \tag{67}$$

We often use symbol $\delta^2 F(\gamma)/\delta\gamma(s)\delta\gamma(t)$ for the integral kernel of $A(\gamma)$, if it exists, i.e., for any $f, g \in \mathcal{X}$
$$(A(\gamma)f, g)_{\mathcal{X}} = \int_0^T \int_0^T \frac{\delta^2 F(\gamma)}{\delta\gamma(s)\delta\gamma(t)} f(s)g(t)\, dsdt.$$

Suppose that $F(\gamma)$ satisfies Assumption 5.1 and $F(\gamma^*) = 0$. Then for any $\gamma \in \mathcal{H}_{x,y}$, $\gamma - \gamma^* \in \mathcal{H}_0$ and
$$F(\gamma) = \int_0^1 DF(\gamma_\theta)[\gamma - \gamma^*]\, d\theta = (\rho(\gamma - \gamma^*), \zeta(\gamma))_{\mathcal{X}},$$
where $\gamma_\theta = \theta\gamma + (1-\theta)\gamma^*$, $0 \le \theta \le 1$, $(\ ,\ )_{\mathcal{X}}$ is the inner product in $\mathcal{X}$ and $\zeta(\gamma) \in \mathcal{X}$ is the following function of t
$$\zeta(\gamma, t) = \int_0^1 \left.\frac{\delta F(\gamma)}{\delta\gamma(t)}\right|_{\gamma=\gamma_\theta} d\theta. \tag{68}$$

On the other hand, the fact $DS(\gamma^*) = 0$ implies that for all $h \in \mathcal{H}_0$,
$$\begin{aligned} DS(\gamma)[h] &= DS(\gamma)[h] - DS(\gamma^*)[h] \\ &= (\gamma - \gamma^*, h)_{\mathcal{H}_0} - \big(\tilde{W}(\gamma)\rho(\gamma - \gamma^*), \rho h\big)_{\mathcal{X}}. \end{aligned} \tag{69}$$

Here $(\ ,\ )_{\mathcal{H}_0}$ is the inner product in Hilbert space $\mathcal{H}_0$ and $\tilde{W}(\gamma)$ is the multiplication operator $\mathcal{X} \ni h(s) \to \tilde{W}(\gamma, s)h(s) \in \mathcal{X}$ with

$$\tilde{W}(\gamma, s) = \int_0^1 \partial_x^2 V(s, \gamma_\theta(s))\, d\theta. \tag{70}$$

It is clear that

$$\sup_{s\in[0,T],\gamma\in\mathcal{H}} \left|\tilde{W}(\gamma, s)\right| \le v_2. \tag{71}$$

Now we can state our results in this section. Some of proofs are left to the next subsection. We begin with

Proposition 5.2. *If $T \le \mu$, $I - \tilde{W}(\gamma)\rho\rho^*$ is an invertible operator in $\mathcal{X}$.*

$$\left\|(I - \tilde{W}(\gamma)\rho\rho^*)^{-1}\right\|_{\mathcal{L}(\mathcal{X})} \le \left(1 - \frac{T^2}{8} v_2\right)^{-1}.$$

Proposition 5.2 enables us to introduce the following vector field, which is the key tool for our purpose.

$$p(\gamma) = \rho^*\bigl(I - \tilde{W}(\gamma)\rho\rho^*\bigr)^{-1}\zeta(\gamma). \tag{72}$$

Then

Proposition 5.3. *Suppose that $F(\gamma)$ satisfies Assumption 5.1 and $F(\gamma^*) = 0$. Then the following equality holds:*

$$DS(\gamma)[p(\gamma)] = F(\gamma).$$

This implies that $DS(\gamma)[p(\gamma)]$ is F-integrable.

Proof of Proposition 5.3. Since $p(\gamma) \in \mathcal{H}_0$, Equality (69) gives

$$\begin{aligned} DS(\gamma)[p(\gamma)] &= \bigl(\gamma - \gamma^*, \rho^*(I - \tilde{W}(\gamma)\rho\rho^*)^{-1}\zeta(\gamma)\bigr)_{\mathcal{H}_0} \\ &\quad - \bigl(\tilde{W}(\gamma)\rho(\gamma - \gamma^*), \rho\rho^*(I - \tilde{W}(\gamma)\rho\rho^*)^{-1}\zeta(\gamma)\bigr)_{\mathcal{X}} \\ &= \bigl(\rho(\gamma - \gamma^*), (I - \tilde{W}(\gamma)\rho\rho^*)^{-1}\zeta(\gamma)\bigr)_{\mathcal{X}} \\ &\quad - \bigl(\rho(\gamma - \gamma^*), \tilde{W}(\gamma)\rho\rho^*(I - \tilde{W}(\gamma)\rho\rho^*)^{-1}\zeta(\gamma)\bigr)_{\mathcal{X}} \end{aligned}$$

$$= (\rho(\gamma - \gamma^*), \zeta(\gamma))_{\mathcal{X}}$$
$$= F(\gamma),$$

because $\tilde{W}(\gamma)$ is a self-adjoint operator. Proposition 5.3 has been proved. □

As a consequence, we have

PROPOSITION 5.4. *Under the same assumption as in Proposition* 5.3 *the following equality holds*:

$$\int_{\Omega_{x,y}} F(\gamma)e^{i\nu S(\gamma)}\mathcal{D}(\gamma) = \int_{\Omega_{x,y}} DS(\gamma)[p(\gamma)]e^{i\nu S(\gamma)}\mathcal{D}(\gamma). \tag{73}$$

Note that both sides of (73) have definite meaning by virtue of Proposition 5.3.

We can show the following fact:

PROPOSITION 5.5. *If* $F(\gamma)$ *satisfies Assumption* 5.1 *and* $F(\gamma^*) = 0$, *then* $p(\gamma)$ *defined by* (72) *is an* m*-admissible vector field.*

Once Proposition 5.5 is proved, the next theorem follows easily from Corollary 3.3 and Proposition 5.4.

THEOREM 5.6. *Suppose that* $F(\gamma)$ *satisfies the Assumption* 5.1 *with some* $m \geq 0$. *Suppose further that* $F(\gamma^*) = 0$. *Let* $\zeta(\gamma)$ *and* $p(\gamma)$ *be as above. Then* $\operatorname{Div} p(\gamma)$ *is* F*-integrable and*

$$\int_{\Omega_{x,y}} F(\gamma)e^{i\nu S(\gamma)}\mathcal{D}(\gamma) = -(i\nu)^{-1}\int_{\Omega_{x,y}} \operatorname{Div} p(\gamma)e^{i\nu S(\gamma)}\mathcal{D}(\gamma). \tag{74}$$

Applying Kumano-go's theorem of semiclassical asymptotics, c.f. [**13**], to (74), we have the following theorem.

THEOREM 5.7. *Under the same assumption as in Theorem* 5.6 *the following asymptotic formula holds*:

$$\int_{\Omega_{x,y}} F(\gamma)e^{i\nu S(\gamma)}\mathcal{D}(\gamma) = \left(\frac{\nu}{2\pi i T}\right)^{1/2} D(T,0,x,y)^{-1/2}e^{i\nu S(\gamma^*)}$$
$$\times \big(-(i\nu)^{-1}\operatorname{Div} p(\gamma^*) + \nu^{-2}r(\nu,T,0,x,y)\big).$$

Here the remainder term $r(\nu,T,0,x,y)$ has the following property: For any non-negative integers α,β there exists a positive constant $C_{\alpha\beta}$ such that

$$\left|\partial_x^\alpha\partial_y^\beta r(\nu,T,0,x,y)\right| \le C_{\alpha\beta}(1+|x|+|y|)^m.$$

We now calculate $\operatorname{Div} p(\gamma^*)$. We write $G_{\gamma^*}=\rho\rho^*(I-\tilde W(\gamma^*)\rho^*\rho)^{-1}=G_0(I-\tilde W(\gamma^*)G_0)^{-1}$. Since $\gamma^*_\theta=\gamma^*$, we have $\tilde W(\gamma^*,t)=\partial_x^2V(t,\gamma^*(t))$. Thus $G_{\gamma^*}=G_0(I-\partial_x^2V(t,\gamma^*(t))G_0)^{-1}$. We know that G_{γ^*} is an operator of trace class. Let $G_\gamma(s,t)$ denote the Green function of the differential equation of Jacobi field at γ:

$$-\left(\frac{d^2}{dt^2}+\partial_x^2V(t,\gamma(t))\right)u(t)=f(t),\quad u(0)=0=u(T). \tag{75}$$

Then it is easy to see that the kernel function of G_{γ^*} is nothing but $G_{\gamma^*}(s,t)$.

Calculation shows:

Theorem 5.8. *Under the same assumption as in Theorem* 5.7

$$\begin{aligned}\operatorname{Div} p(\gamma^*)&=\frac12\int_0^T\int_0^T\frac{\delta}{\delta\gamma(s)}\left(G_{\gamma^*}(s,t)\frac{\delta F(\gamma^*)}{\delta\gamma(t)}\right)dsdt\\&=\frac12\int_0^T\int_0^T\frac{\delta G_{\gamma^*}(s,t)}{\delta\gamma(s)}\frac{\delta F(\gamma^*)}{\delta\gamma(t)}\,dsdt+\frac12\operatorname{tr}G_{\gamma^*}A(\gamma^*).\end{aligned} \tag{76}$$

If in addition the operator $A(\gamma^*)$ has the integral kernel $\delta^2F(\gamma^*)/\delta\gamma(s)\delta\gamma(t)$, then

$$\begin{aligned}\operatorname{Div} p(\gamma^*)&=\frac12\int_0^T\int_0^T\frac{\delta G_{\gamma^*}(s,t)}{\delta\gamma(s)}\frac{\delta F(\gamma^*)}{\delta\gamma(t)}\,dsdt\\&\quad+\frac12\int_0^T\int_0^T G_{\gamma^*}(s,t)\frac{\delta^2F(\gamma^*)}{\delta\gamma(s)\delta\gamma(t)}\,dsdt.\end{aligned}$$

Example 5.9 (Semiclassical limit of covariance matrix). For any $a(s,t)\in C([0,T]\times[0,T])$ we set

$$F(\gamma)=\int_0^T\int_0^T(\gamma(s)-\gamma^*(s))(\gamma(t)-\gamma^*(t))a(s,t)\,dsdt.$$

Then

$$\operatorname{Div} p(\gamma^*) = \frac{1}{2}\int_0^T\int_0^T G_{\gamma^*}(s,t)a(s,t)\,dsdt. \tag{77}$$

Therefore, we have semiclassical asymptotic formula

$$\begin{aligned}
&\int_{\Omega_{x,y}}\left(\int_0^T\int_0^T(\gamma(s)-\gamma^*(s))(\gamma(t)-\gamma^*(t))a(s,t)\,dsdt\right)e^{i\nu S(\gamma)}\mathcal{D}(\gamma)\\
&=\left(\frac{\nu}{2\pi iT}\right)^{1/2}D(T,0,x,y)^{-1/2}e^{i\nu S(\gamma^*)}\\
&\quad\times\left(-(i\nu)^{-1}\left(\int_0^T\int_0^T G_{\gamma^*}(s,t)a(s,t)\,dsdt\right)+\nu^{-2}r(\nu,T,0,x,y)\right).
\end{aligned} \tag{78}$$

Here the remainder term $r(\nu,T,0,x,y)$ has the following property: For any non-negative integers α,β there exists a positive constant $C_{\alpha\beta}$ such that

$$\left|\partial_x^\alpha\partial_y^\beta r(\nu,T,0,x,y)\right| \le C_{\alpha\beta}(1+|x|+|y|)^2.$$

This means that semiclassical limit of covariance matrix of Feynman path integral equals $-(i\nu)^{-1}G_{\gamma^*}(s,t)$ after suitable normalization.

Proofs of Propositions 5.2, 5.5 and Theorem 5.8 will be given in the next subsection.

5.2. Proof of a sharper asymptotic formula.

For any index $1\le p\le\infty$ and $f\in L^p([0,T])$, we write $\|f\|_{L^p}$ the norm of f in $L^p([0,T])$. Since $|\tilde W(\gamma)(t)|\le v_2$ for any $\gamma\in\mathcal{H}$ and $t\in[0,T]$,

$$\|\tilde W(\gamma)f\|_{L^p}\le v_2\|f\|_{L^p},\quad(1\le p\le\infty).$$

We use the Green operator G_0 defined by (16) in Section 2.1. Since the kernel function $g_0(s,t)$ of G_0 is given by (17), the following Lemma holds.

Lemma 5.10. *Let p be $1\le p\le\infty$. It is clear that for any $f\in C([0,T])$,*

$$\|G_0f\|_{L^p}\le\frac{T^2}{8}\|f\|_{L^p},\quad \|G_0f\|_{C([0,T])}\le\frac{1}{4}\sqrt{\frac{T^3}{3}}\|f\|_{\mathcal{X}},\quad \|G_0f\|_{C([0,T])}\le\frac{T}{4}\|f\|_{L^1}.$$

$$\|\partial_sG_0f\|_{L^p}\le\frac{T}{2}\|f\|_{L^p},\quad \|\partial_sG_0f\|_{C([0,T])}\le\sqrt{\frac{T}{3}}\|f\|_{\mathcal{X}},\quad \|\partial_sG_0f\|_{C([0,T])}\le\|f\|_{L^1}.$$

In order to prove Proposition 5.2, we have only to prove the next proposition, because $\rho\rho^* = G_0$ in $\mathcal{X}$.

PROPOSITION 5.11. *Under the assumption that* $T \le \mu$ *the operator* $(I - \tilde{W}(\gamma)G_0)$ *is invertible in* $\mathcal{X}$. *We have*

$$\left\|(I - \tilde{W}(\gamma)G_0)^{-1} f\right\|_{\mathcal{X}} \le c_0 \|f\|_{\mathcal{X}},$$

where

$$c_0 = \left(1 - \frac{v_2 T^2}{8}\right)^{-1}.$$

PROOF. Using Lemma 5.10, we have

$$\left\|\tilde{W}(\gamma)G_0 f\right\|_{\mathcal{X}} \le \frac{v_2 T^2}{8} \|f\|_{\mathcal{X}}.$$

Since $T < \mu$, we have $v_2 T^2/8 < 1/2$. Thus $(I - \tilde{W}(\gamma)G_0)^{-1}$ exists and

$$\left\|(I - \tilde{W}(\gamma)G_0)^{-1} f\right\|_{\mathcal{X}} \le c_0 \|f\|_{\mathcal{X}}.$$

Proposition is proved. □

The crucial fact in this section is following

PROPOSITION 5.12. *For any* $K = 0, 1, 2, \ldots,$ *there exists a constant* $Y_K \ge 1$ *independent of* γ *such that*

$$\left\|(I - \tilde{W}(\gamma)G_0)^{-1}\right\|_{\{\mathcal{L}(\mathcal{X});0,K,Y_K\}} \le c_0.$$

PROOF. Let Δ be an arbitrary division of the interval $[0, T]$, i.e.,

$$\Delta : 0 = T_0 < T_1 < T_2 < \cdots < T_J < T_{J+1} = T.$$

We use the notation in Section 1, for example, $(x_{J+1}, x_J, \ldots, x_1, x_0)$ and γ_Δ, etc. It is clear that

$$\partial_{x_j} \tilde{W}(\gamma_\Delta, t) = \zeta_{\Delta,j}(t) \int_0^1 \partial_x^3 V(\theta\gamma_\Delta(t) + (1-\theta)\gamma^*(t))\theta \, d\theta, \tag{79}$$

where $\zeta_{\Delta,j}(t) = \partial_{x_j}\gamma_\Delta(t)$. By (33)

$$\partial_{x_j}\tilde{W}(\gamma_\Delta, t) = 0, \quad \text{for } t \notin [T_{j-1}, T_{j+1}]. \tag{80}$$

If $|j-k| \geq 2$, then $\zeta_{\Delta,j}(t)\zeta_{\Delta,k}(t) \equiv 0$ by (33) and $\partial_{x_k}\zeta_{\Delta,j}(t) \equiv 0$ by (36). Thus we have

$$\partial_{x_j}\partial_{x_k}\tilde{W}(\gamma_\Delta, t) = 0 \quad \text{for any } t \in [0,T] \text{ if } |k-j| \geq 2. \tag{81}$$

We know from estimates (39) and (40) that there exists a positive constant $C_{\alpha,\beta}$ independent of Δ and of j such that for $j = 0,1,2,\dots,J+1$ if $\alpha \geq 1$

$$\left|\partial_{x_j}^{\alpha}\partial_{x_{j+1}}^{\beta}\zeta_{\Delta,j}(t)\right| \leq C_{\alpha,\beta}\chi_{[T_{j-1},T_{j+1}]}(t), \quad \left|\partial_{x_j}^{\alpha}\partial_{x_{j-1}}^{\beta}\zeta_{\Delta,j}(t)\right| \leq C_{\alpha,\beta}\chi_{[T_{j-1},T_{j+1}]}(t).$$

Here $\chi_{[T_{j-1},T_{j+1}]}(t)$ is the characteristic function of the interval $[T_{j-1}, T_{j+1}]$. Hence, for any positive integer K there exists a positive constant C_K independent of Δ such that as far as $0 < \alpha_j \leq K$, $\alpha_{j+1} \leq K$, $\alpha_{j-1} \leq K$ and $t \in [0,T]$

$$\left|\partial_{x_j}^{\alpha_j}\partial_{x_{j+1}}^{\alpha_{j+1}}\tilde{W}(\gamma_\Delta, t)\right| + \left|\partial_{x_j}^{\alpha_j}\partial_{x_{j-1}}^{\alpha_{j-1}}\tilde{W}(\gamma_\Delta, t)\right| \leq C_K\ \chi_{[T_{j-1},T_{j+1}]}(t). \tag{82}$$

The constant C_K depends on $v_3, v_4, \dots, v_{2K+2}$ but not on $v_j, j \geq 2K+3$.

For any $f \in C([0,T])$ we write

$$u(\gamma_\Delta, t) = (I - \tilde{W}(\gamma_\Delta, t)G_0)^{-1}f(t).$$

Proposition 5.12 follows from Proposition 5.11 and the next lemma, which we shall prove by induction on the order relation "$<$" among multi-indices. Let $\alpha = (\alpha_{J+1}, \alpha_J, \dots, \alpha_1, \alpha_0)$ and $\beta = (\beta_{J+1}, \beta_J, \dots, \beta_2, \beta_1, \beta_0)$ be multi-indices. Recall that $\alpha > \beta$ if and only if $\alpha_j \geq \beta_j$ for $j = 0,1,2,\dots,J+1$ and $\alpha \neq \beta$. $\beta < \alpha$ is equivalent to $\alpha > \beta$

LEMMA 5.13. *Let C_K be as in* (82) *and c_0 be as in Proposition* 5.12. *Set $Y_0 = 1$ and for any positive integer $K \geq 1$ define Y_K by*

$$Y_K = \max\{Y_{K-1}, 2^{2K-1}3^{-1/2}c_0C_KT^2\}. \tag{83}$$

Then for any multi-index α we have

$$\left\|\partial_x^{\alpha}u(\gamma_\Delta)\right\|_{\mathcal{X}} \leq Y_{m(\alpha)}^{|\alpha|}\|u(\gamma_\Delta)\|_{\mathcal{X}}. \tag{84}$$

Proof. In the case $\alpha = 0$, (84) is obviously true. Let multi-index α be such as $|\alpha| \geq 1$. Suppose that the inequality (84) for any β with $\beta < \alpha$ is true, i.e.,

$$\left\|\partial_x^\beta u(\gamma_\Delta)\right\|_{\mathcal{X}} \leq Y_{m(\beta)}^{|\beta|} \|u(\gamma_\Delta)\|_{\mathcal{X}}, \quad \text{if } \beta < \alpha. \tag{85}$$

We shall prove (84) using (85). Obviously,

$$\partial_x^\alpha\big((I - \tilde{W}(\gamma_\Delta, t)G_0)u(\gamma_\Delta, t)\big) = \partial_x^\alpha f(t) = 0.$$

We set $g(t) = (I - \tilde{W}(\gamma_\Delta)G_0)\partial_x^\alpha u(\gamma_\Delta, t)$. Using Leibnitz' rule, we have

$$g(t) = \sum_{0\leq\beta<\alpha} \binom{\alpha}{\beta} \partial_x^{\alpha-\beta} \tilde{W}(\gamma_\Delta, t)G_0 \partial_x^\beta u(\gamma_\Delta, t).$$

Since $\beta < \alpha$, induction hypothesis implies $\partial_x^\beta u(\gamma_\Delta) \in \mathcal{X}$ on the right hand side of the above equality. Hence $G_0\partial_x^\beta u(\gamma_\Delta) \in \mathcal{H}_0$. Thus $g(T) = g(0) = 0$.

If $t \neq 0$, then there exists some $j \in \{J+1, J, \dots, 2, 1\}$ such that $t \in (T_{j-1}, T_j]$. We know from (80) for any $t \in [T_{j-1}, T_j]$

$$\partial_{x_k} \tilde{W}(\gamma_\Delta, t) = 0 \quad t \in [T_{j-1}, T_j] \text{ if } k \neq j \text{ and } k \neq j-1.$$

Hence for any $t \in (T_{j-1}, T_j]$

$$g(t) = \sum_{0\leq\beta^*<\alpha}^{*} \binom{\alpha_{j-1}}{\beta_{j-1}} \binom{\alpha_j}{\beta_j} \partial_x^{\alpha-\beta^*} \tilde{W}(\gamma_\Delta, t)G_0 \partial_x^{\beta^*} u(\gamma_\Delta, t),$$

here sum $\sum_{0\leq\beta^*<\alpha}^*$ is taken over all these $\beta^* = (\beta_{J+1}, \beta_J, \dots, \beta_2, \beta_1, \beta_0) < \alpha$ such that $\beta_k = 0$ unless $k = j$ or $k = j-1$, i.e., $\beta^* = (0, 0, \dots, 0, \beta_j, \beta_{j-1}, 0, \dots, 0)$.

We write $K = m(\alpha)$. By induction hypothesis $\partial_x^{\beta^*} u(\gamma_\Delta) \in \mathcal{X}$. As a result of this, Proposition 5.10 and (85),

$$\left\|G_0 \partial_x^{\beta^*} u(\gamma_\Delta)\right\|_{C([0,T])} \leq \frac{1}{4}\sqrt{\frac{T^3}{3}} \left\|\partial_x^{\beta^*} u(\gamma_\Delta)\right\|_{\mathcal{X}} \leq \frac{1}{4}\sqrt{\frac{T^3}{3}} Y_K^{|\alpha|-1} \|u(\gamma_\Delta)\|_{\mathcal{X}}.$$

It follows from this and (82) that

$$\left|\partial_x^{\alpha-\beta^*} \tilde{W}(\gamma_\Delta, t)G_0 \partial_x^{\beta^*} u(\gamma_\Delta, t)\right| \leq \frac{1}{4}\sqrt{\frac{T^3}{3}} C_K Y_K^{|\alpha|-1} \|u(\gamma_\Delta)\|_{\mathcal{X}}.$$

Therefore, for any $t \in (T_{j-1}, T_j]$

$$|g(t)| \le \sum_{0\le\beta^*<\alpha}^{*} \binom{\alpha_{j-1}}{\beta_{j-1}}\binom{\alpha_j}{\beta_j}\frac{1}{4}\sqrt{\frac{T^3}{3}}C_K Y_K^{|\alpha|-1}\|u(\gamma_\Delta)\|_{\mathcal{X}}$$
$$\le 2^{2K-2}\sqrt{\frac{T^3}{3}}C_K Y_K^{|\alpha|-1}\|u(\gamma_\Delta)\|_{\mathcal{X}}.$$

Since the right hand side of this inequality does not depend on j, we have

$$|g(t)| \le 2^{2K-2}\sqrt{\frac{T^3}{3}}C_K Y_K^{|\alpha|-1}\|u(\gamma_\Delta)\|_{\mathcal{X}}, \quad \text{for any } t\in[0,T].$$

Consequently we have

$$\|g\|_{\mathcal{X}} \le 2^{2K-2}3^{-1/2}T^2 C_K Y_K^{|\alpha|-1}\|u(\gamma_\Delta)\|_{\mathcal{X}}. \tag{86}$$

We use Proposition 5.11 and definition (83) of Y_K, and we obtain

$$\left\|\partial_x^\alpha u(\gamma_\Delta,t)\right\|_{\mathcal{X}} \le c_0 2^{2K-2}3^{-1/2}T^2 C_K Y_K^{|\alpha|-1}\|u(\gamma_\Delta)\|_{\mathcal{X}}$$
$$\le Y_K^{|\alpha|}\|u(\gamma_\Delta)\|_{\mathcal{X}}.$$

Inequality (84) for α is proved. Induction process is over. Lemma 5.13 has been proved. □

Proof of Proposition 5.12 has been completed. □

Now we begin proof of Proposition 5.5. Let us recall definition (72):

$$p(\gamma) = \rho^* q(\gamma),$$
$$q(\gamma) = \left(I - \tilde{W}(\gamma)\rho\rho^*\right)^{-1}\zeta(\gamma),$$
$$\zeta(\gamma) = \int_0^1 \frac{\delta F(\gamma)}{\delta\gamma}\bigg|_{\gamma=\theta\gamma+(1-\theta)\gamma^*} d\theta.$$

We shall prove that $p(\gamma)$ has property P1 of Definition 3.1 of m-admissibility.

Since $F(\gamma)$ is m-smooth and satisfies (67), we know that $\delta F(\gamma)/\delta\gamma \in \mathcal{X}$ and $\mathcal{H} \ni \gamma \to \delta F(\gamma)/\delta\gamma \in \mathcal{X}$ is an infinitely differentiable map. This implies that $\zeta(\gamma) \in \mathcal{X}$ and that the map: $\mathcal{H} \ni \gamma \to \zeta(\gamma) \in \mathcal{X}$ is also an infinitely differentiable map. Obviously, $\gamma \to \tilde{W}(\gamma)$ is also an infinitely differentiable map from $\mathcal{H}$ to $C^k([0,T])$ for any $k = 0, 1, 2, \ldots$. Therefore, $q(\gamma) \in \mathcal{X}$ and $\mathcal{H} \ni \gamma \to q(\gamma) \in \mathcal{X}$ is an infinitely differentiable map.

Let Y_K be the constant in (83) and A_K, X_K be as in (67) of Assumption 5.1.

LEMMA 5.14. *There exists a positive constant $c(m, v_2)$ depending on m, v_2 such that for any $K = 0, 1, 2, \ldots,$*

$$\|q(\gamma)\|_{\{\mathcal{X};m,K,X_K+Y_K\}} \leq c(m, v_2)A_K < \infty.$$

PROOF. By virtue of Proposition 4.6 and Proposition 5.12,

$$\begin{aligned} &\|q(\gamma)\|_{\{\mathcal{X};m,K,X_K+Y_K\}} \\ &\quad \leq \left\|(I - \tilde{W}(\gamma)G_0)^{-1}\right\|_{\{\mathcal{L}(\mathcal{X});0,K,Y_K\}} \|\zeta(\gamma)\|_{\{\mathcal{X};m,K,X_K\}} \leq c_0 \|\zeta(\gamma)\|_{\{\mathcal{X};m,K,X_K\}}. \end{aligned}$$

By definition of $\zeta(\gamma)$

$$\|\zeta(\gamma)\|_{\{\mathcal{X};m,K,X_K\}} = \left\| \int_0^1 \frac{\delta F(\gamma_\theta)}{\delta \gamma} \, d\theta \right\|_{\{\mathcal{X};m,K,X_K\}} \leq \int_0^1 \left\| \frac{\delta F(\gamma_\theta)}{\delta \gamma} \right\|_{\{\mathcal{X};m,K,X_K\}} d\theta.$$

If $\gamma \in \mathcal{H}_{x,y}$, then $\gamma_\theta \in \mathcal{H}_{x,y}$ for any $\theta \in [0,1]$. Let Δ be an arbitrary division of the interval $[0,T]$

$$\Delta : 0 = T_0 < T_1 < T_2 < \cdots < T_J < T_{J+1} = T.$$

We use the notation in Section 1, for example, $(x_{J+1}, x_J, \ldots, x_1, x_0)$ and γ_Δ, etc. We write $\gamma_{\Delta,\theta} = \theta\gamma_\Delta + (1-\theta)\gamma^*$, for $0 \leq \theta \leq 1$. Then

$$\left\| \partial_x^\alpha \frac{\delta F(\gamma_{\Delta,\theta})}{\delta \gamma} \right\|_{\mathcal{X}} \leq \theta^{|\alpha|}(1 + |x_0| + \mathrm{var}(\gamma_{\Delta,\theta}))^m X_{m(\alpha)}^{|\alpha|} A_{m(\alpha)}.$$

Since there exists some positive constant $c(v_2)$ depending on v_2 such that

$$\mathrm{var}(\gamma^*) \leq c(v_2)(1 + |x_{J+1}| + |x_0|),$$

we have

$$\mathrm{var}(\gamma^*) \leq 2c(v_2)(1 + |x_0| + \mathrm{var}(\gamma)), \quad \text{for any } \gamma \in \mathcal{H}_{x,y}. \tag{87}$$

Thus

$$
\begin{aligned}
(1 + |x_0| + \mathrm{var}(\gamma_{\Delta,\theta})) &\leq (1 + |x_0| + \mathrm{var}(\gamma_\Delta) + \mathrm{var}(\gamma^*)) \\
&\leq (1 + 2c(v_2))(1 + |x_0| + \mathrm{var}(\gamma_\Delta)). \qquad (88)
\end{aligned}
$$

Therefore,

$$
\left\| \partial_x^\alpha \frac{\delta F(\gamma_{\Delta,\theta})}{\delta\gamma} \right\|_{\mathcal{X}} \leq (1 + 2c(v_2))^m (1 + |x_0| + \mathrm{var}(\gamma_\Delta))^m X_{m(\alpha)}^{|\alpha|} A_{m(\alpha)}.
$$

Thus

$$
\left\| \frac{\delta F(\gamma_\theta)}{\delta\gamma} \right\|_{\{\mathcal{X};m,K,X_K\}} \leq (1 + 2c(v_2))^m A_K.
$$

Therefore,

$$
\|\zeta(\gamma)\|_{\{\mathcal{X};m,K,X_K\}} \leq (1 + 2c(v_2))^m A_K.
$$

Consequently, we have, by virtue of Proposition 5.12,

$$
\|q(\gamma)\|_{\{\mathcal{X};m,K,X_K+Y_K\}} \leq c_0(1 + 2c(v_2))^m A_K.
$$

Lemma 5.14 is now proved. □

Next we calculate $Dq(\gamma)[h]$ for $h \in \mathcal{H}_0$. By definition of $q(\gamma)$

$$
\begin{aligned}
Dq(\gamma)[h] &= \big(I - \tilde{W}(\gamma)G_0\big)^{-1} \big(D\tilde{W}(\gamma)[h]G_0(I - \tilde{W}(\gamma)G_0)^{-1}\zeta(\gamma) + D\zeta(\gamma)[h]\big) \\
&= \big(I - \tilde{W}(\gamma)G_0\big)^{-1} \big(D\tilde{W}(\gamma)[h]\rho\rho^* q(\gamma) + D\zeta(\gamma)[h]\big). \qquad (89)
\end{aligned}
$$

Since $\tilde{W}(\gamma)$ is the multiplication operator: $f(s) \to \int_0^1 \partial_x^2 V(\gamma_\theta(s), s)\, d\theta f(s)$,

$$
D\tilde{W}(\gamma)[h](s)\rho\rho^* q(\gamma)(s) = U_1(\gamma, s)\rho\rho^* q(\gamma)(s)\rho h(s), \qquad (90)
$$

where

$$
U_1(\gamma, s) = \int_0^1 \partial_x^3 V(\gamma_\theta(s), s)\theta\, d\theta.
$$

Since

$$|U_1(\gamma, s)| \leq v_3, \tag{91}$$

the map $f(s) \to U_1(\gamma, s)\rho\rho^* q(\gamma) f(s)$ is a bounded linear map in $\mathcal{X}$, which depends smoothly on γ.

On the other hand, we have

$$D\zeta(\gamma)[h] = \tilde{A}(\gamma)\rho h, \tag{92}$$

where

$$\tilde{A}(\gamma) = \int_0^1 \theta A(\gamma_\theta)\, d\theta. \tag{93}$$

It follows from (89) and (92) that

$$Dq(\gamma)[h] = B(\gamma)\rho h, \tag{94}$$

here $B(\gamma)$ is given by

$$B(\gamma) f = \big(I - \tilde{W}(\gamma) G_0\big)^{-1} \big(U_1(\gamma)\rho\rho^* q(\gamma) + \tilde{A}(\gamma)\big) f, \quad \text{for any } f \in \mathcal{X}. \tag{95}$$

It is clear that $B(\gamma) \in \mathcal{L}(\mathcal{X})$ and it is infinitely differentiable with respect $\gamma \in \mathcal{H}$. Therefore, we have proved that the vector field $p(\gamma)$ has property P1.

We shall prove $p(\gamma)$ has property P2.

LEMMA 5.15. *For any $K = 0, 1, 2, \ldots$ let $Z_K = X_K + 2Y_K$. Then for each K, there exists positive constant C_K such that*

$$\|B(\gamma)\|_{\{\mathcal{L}(\mathcal{X}); m, K, Z_K\}} \leq C_K A_K.$$

PROOF. Using Proposition 4.6 and Proposition 5.12,

$$\begin{aligned}
&\|B(\gamma)\|_{\{\mathcal{L}(\mathcal{X}); m, K, Z_K\}} \\
&\quad \leq \big\|(I - \tilde{W}(\gamma) G_0)^{-1}\big\|_{\{\mathcal{L}(\mathcal{X}); 0, K, Y_K\}} \big\|U_1(\gamma)\rho\rho^* q(\gamma) + \tilde{A}(\gamma)\big\|_{\{\mathcal{L}(\mathcal{X}); m, K, X_K + Y_K\}} \\
&\quad \leq c_0 \big(\|U_1(\gamma)\rho\rho^* q(\gamma)\|_{\{\mathcal{L}(\mathcal{X}); m, K, X_K + Y_K\}} + \|\tilde{A}(\gamma)\|_{\{\mathcal{L}(\mathcal{X}); m, K, X_K + Y_K\}}\big).
\end{aligned} \tag{96}$$

Map $\mathcal{X} \ni f \to U_1(\gamma)\rho\rho^* q(\gamma) f \in \mathcal{X}$ is the multiplication of two functions.

Let Δ be an arbitrary division of $[0, T]$ and γ_Δ be an arbitrary piecewise classical path. Then we have

$$\partial_{x_j} U_1(\gamma_\Delta, s) = \int_0^1 \zeta_{\Delta,j}(s)\big(\partial_x^4 V(s, \gamma_{\Delta,\theta}(s))\theta^2\big)d\theta.$$

Therefore, if $|j-k| \geq 2$, then for any $s \in [0,T]$

$$\partial_{x_j}\partial_{x_k} U_1(\gamma_\Delta, s) = 0.$$

In just the same way as (82), for any $K = 1, 2, 3, \dots$ there exists a positive constant C_K such that we have

$$\big|\partial_{x_j}^{\alpha_j}\partial_{x_{j+1}}^{\alpha_{j+1}} U_1(\gamma_\Delta, s)\big| \leq C_K \chi_{[T_{j-1},T_{j+1}]}(s), \tag{97}$$

if $0 < \alpha_j \leq K$ and $\alpha_{j+1} \leq K$. Let $s \in [0,T]$. Then we may assume $s \in [T_j, T_{j+1}]$ with some j.

$$\begin{aligned}
&\partial_x^\alpha(U_1(\gamma_\Delta, s)\rho\rho^* q(\gamma_\Delta)(s)) \\
&\quad = \sum_\beta^{*} \binom{\alpha_j}{\beta_j}\binom{\alpha_{j+1}}{\beta_{j+1}} \partial_{x_j}^{\beta_j}\partial_{x_{j+1}}^{\beta_{j+1}} U_1(\gamma_\Delta, s)\partial_x^{\alpha-\beta}\rho\rho^* q(\gamma_\Delta)(s),
\end{aligned}$$

where $\sum_\beta^*$ means the sum over those $\beta = (0, 0, \dots, 0, \beta_j, \beta_{j+1}, 0, 0, \dots, 0)$. We set $C_0 = v_3$ as in (91) and $C_K, K = 1, 2, 3$ be as in (97). If $m(\alpha) \leq K$, then (91) and (97) give

$$\begin{aligned}
&\big|\partial_x^\alpha(U_1(\gamma_\Delta, s)\rho\rho^* q(\gamma_\Delta)(s))\big| \\
&\quad \leq \sum_\beta^{*} \binom{\alpha_j}{\beta_j}\binom{\alpha_{j+1}}{\beta_{j+1}} \big|\partial_{x_j}^{\beta_j}\partial_{x_{j+1}}^{\beta_{j+1}} U_1(\gamma_\Delta, s)\big|\big|\partial_x^{\alpha-\beta}\rho\rho^* q(\gamma_\Delta)(s)\big| \\
&\quad \leq \sum_\beta^{*} \binom{\alpha_j}{\beta_j}\binom{\alpha_{j+1}}{\beta_{j+1}} C_K \big|\partial_x^{\alpha-\beta}\rho\rho^* q(\gamma_\Delta)(s)\big| \\
&\quad \leq 2^{2K} C_K (1 + |x_0| + \mathrm{var}(\gamma_\Delta))^m (X_K + Y_K)^{|\alpha|} \|\rho\rho^* q(\gamma_\Delta)\|_{\{C([0,T]);\Delta,m,K,X_K+Y_K\}}.
\end{aligned}$$

The right hand side is independent of j. Using Lemma 5.14, we have proved

$$\begin{aligned}
&\|U_1(\gamma_\Delta)\rho\rho^* q(\gamma_\Delta)\|_{\{\mathcal{L}(\mathcal{X});\Delta,m,K,X_K+Y_K\}} \\
&\quad \le 2^{2K} C_K \|\rho\rho^* q(\gamma_\Delta)\|_{\{C([0,T]);\Delta,m,K,X_K+Y_K\}} \\
&\quad \le 2^{2K-2} C_K \frac{T^{3/2}}{\sqrt{3}} c_0 c(m,v_2) A_K. \qquad (98)
\end{aligned}$$

Next we discuss $\tilde{A}(\gamma_\Delta)$. We have, by (93) and (88),

$$\begin{aligned}
\|\tilde{A}\|_{\{\mathcal{L}(\mathcal{X});\Delta,m,K,X_K\}} &\le \int_0^T \theta \|A(\gamma_\theta)\|_{\{\mathcal{L}(\mathcal{X});\Delta,m,K,X_K\}}\, d\theta \\
&\le A_K(1+2c(v_2))^m. \qquad (99)
\end{aligned}$$

Thus it follows from (96), (98) and (99) that

$$\|B(\gamma)\|_{\{\mathcal{L}(\mathcal{X});m,K,Z_K\}} \le C_K A_K$$

with some positive constant C_K. We have proved Lemma 5.15. □

We have proved that $p(\gamma)$ has property P2. Therefore it is m-admissible, i.e. we have proved Proposition 5.5.

Now we can apply integration by parts formula of Theorem 3.2. Thus we have proved Theorem 5.6.

Applying Kumano-go's result in [**13**] to Theorem 5.6, we obtain Theorem 5.7. We have proved the sharp asymptotic formula up to the explicit expression of $\operatorname{Div} p(\gamma^*)$.

Now we calculate $\operatorname{Div} p(\gamma^*)$ to prove Theorem 5.8. For that purpose we have to calculate kernel function of $\rho\rho^* B(\gamma^*)$.

If $\gamma = \gamma^*$, then $\gamma^*_\theta = \gamma^*$, for any $\theta \in [0,1]$. Then

$$\rho\rho^* B(\gamma^*) = G_{\gamma^*} U_1(\gamma^*) G_{\gamma^*} \zeta(\gamma^*) + G_{\gamma^*} \tilde{A}(\gamma^*), \qquad (100)$$

and

$$\begin{aligned}
\zeta(\gamma^*, t) &= \int_0^1 \frac{\delta F(\gamma^*)}{\delta\gamma(t)}\, d\theta = \frac{\delta F(\gamma^*)}{\delta\gamma(t)}, \\
\tilde{W}(\gamma^*, t) &= \partial_x^2 V(t, \gamma^*(t)),
\end{aligned}$$

$$U_1(\gamma^*, t) = \int_0^T \partial_x^3 V(t, \gamma^*(t))\theta\, d\theta = \frac{1}{2}\partial_x^3 V(t, \gamma^*(t)),$$

$$\tilde{A}(\gamma^*) = \int_0^1 \theta A(\gamma^*)\, d\theta = \frac{1}{2}A(\gamma^*).$$

Hence for any $f \in \mathcal{X}$

$$\begin{aligned} &G_{\gamma^*}U_1(\gamma^*)G_{\gamma^*}\zeta(\gamma^*)f(s) \\ &= \int_0^T G_{\gamma^*}(s,t)\frac{1}{2}\partial_x^3 V(t,\gamma^*(t))f(t)\int_0^T G_{\gamma^*}(t,t_1)\frac{\delta F(\gamma^*)}{\delta\gamma(t_1)}\, dt_1 dt \\ &= \frac{1}{2}\int_0^T\int_0^T \frac{\delta}{\delta\gamma(t)}(G_\gamma(s,t_1))\Big|_{\gamma=\gamma^*} f(t)\frac{\delta F(\gamma^*)}{\delta\gamma(t_1)}\, dt_1 dt. \end{aligned}$$

Therefore,

$$\begin{aligned} &\operatorname{tr}\, G_{\gamma^*}U_1(\gamma^*)G_{\gamma^*}\zeta(\gamma^*) \\ &= \frac{1}{2}\int_0^T G_{\gamma^*}(s,s)\partial_x^3 V(s,\gamma^*(s))\int_0^T G_{\gamma^*}(s,t_1)\frac{\delta F(\gamma^*)}{\delta\gamma(t_1)}\, dt_1 ds \\ &= \frac{1}{2}\int_0^T\int_0^T \frac{\delta}{\delta\gamma(s)}(G_{\gamma^*}(s,t_1))\frac{\delta F(\gamma^*)}{\delta\gamma(t_1)}\, dt_1 ds. \end{aligned}$$

Therefore,

$$\operatorname{Div} p(\gamma^*) = \frac{1}{2}\int_0^T\int_0^T \frac{\delta}{\delta\gamma(s)}(G_\gamma(s,t_1))\Big|_{\gamma=\gamma^*}\frac{\delta F(\gamma^*)}{\delta\gamma(t_1)}\, dt_1 ds + \frac{1}{2}\operatorname{tr} G_{\gamma^*}A(\gamma^*).$$

(76) of Theorem 5.8 has been proved.

The rest of Theorem 5.8 follows from this.

Theorem 5.8 has been proved. □

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