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Blow-up problems in the strained vorticity dynamics and critical exponents

Dedicated to the memory of the late Professor Tatsuyuki Nakaki

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Abstract. Two partial differential equations are studied from the viewpoint of critical exponents. They are equations for a scalar unknown of one spatial variable, and produce self-similar solutions of the Navier-Stokes equations. Global existence and blow-up are examined for them, and the critical exponent separating them is determined.

1. Introduction.

One of the difficult problems on the Navier-Stokes equations is the one about the existence global-in-time or blow-up in-finite-time of the solutions in three dimensions. It is well-known that this is a very difficult problem, and there have been proposed models which mimic the essence of the 3D mechanism and, at the same time, are feasible for mathematical analysis. Perhaps, the oldest of such is the Burgers equation: $u_t + uu_x = \nu u_{xx}$, where the subscripts imply differentiation. But we know now that this equation is too simple. Fujita's introduction of the nonlinear heat equation $u_t = \triangle u + u^{1+\alpha}$, where $\triangle$ denotes the Laplace operator, was motivated similarly, see [**5**]. Although it and its sister equations have their own merit and many mathematicians have developed the theory to an amazing extent ([**28**], [**29**]), we now know that such equations are quite different from the Navier-Stokes equations. In fact, both the Fujita equation and the Burgers equation ignore the non-locality which appears in some form or other in differential equations for incompressible fluid flow. Constantin, Lax and Majda [**3**] proposed a simple model for 3D vortex dynamics, where non-local nature of vorticity-velocity relation is cleverly incorporated. However, both Fujita's equation and the Constantin-Lax-Majda model ignore the role of a convection term, which we now know is important, see [**11**], [**26**]. The author therefore wishes to

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have an equation which is mathematically accessible and, at the same time, is closer to the Navier-Stokes equations than those models are.

We shall consider in this paper the following nonlinear nonlocal equations: The first one is

$$\omega_t = \|\omega(t)\|_p^\alpha (x\omega_x + \omega) + \omega_{xx} \quad (-\infty < x < \infty), \tag{1}$$

with the initial condition $\omega(0,x) = \omega_0(x)$. Here, the norm is the L^p-norm:

$$\|g\|_p = \left(\int_{\mathbb{R}} |g(x)|^p \,\mathrm{d}x \right)^{1/p}$$

if $p \in [1,\infty)$, and $\|g\|_\infty = \text{ess sup}_{x\in\mathbb{R}} |g(x)|$ if $p = \infty$. The symbol $\omega(t)$ denotes $\omega(t,\cdot)$, which is a function of x only with t being frozen. The exponent α can be any non-negative constant. The second equation is:

$$\omega_t = \|\omega(t)\|_p^\alpha (r\omega_r + 2\omega) + \frac{1}{r}(r\omega_r)_r \quad (0 \le r < \infty), \tag{2}$$

The L^p-norm here is:

$$\|g\|_p = \left(2\pi \int_0^\infty |g(r)|^p r \mathrm{d}r \right)^{1/p}$$

and $\|g\|_\infty = \text{ess sup}_{0\le r<\infty} |g(r)|$. Namely it is an L^p-norm in $\mathbb{R}^2$ for axisymmetric functions.

Our equations are more complicated than those of Fujita type or the Constantin-Lax-Majda equation. However, ours have a merit that their solutions are exact solutions (of infinite energy) of the Navier-Stokes equations. We now explain how we derived these equations.

We first note that the Navier-Stokes equations for incompressible viscous fluid are written as $\boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p + \nu\triangle\boldsymbol{u}$, where $\boldsymbol{u}$ and p denotes the velocity and the pressure, respectively. $\nu > 0$ is the kinematic viscosity, which we assume to be a positive constant. The following equation for $\omega : [0,\infty)\times\mathbb{R}^2 \to \mathbb{R}$ was proposed in [**24**].

$$\omega_t + (-\gamma_1(t)x + u)\omega_x + (-\gamma_2(t)y + v)\omega_y - (\gamma_1(t) + \gamma_2(t))\omega = \nu\triangle\omega, \tag{3}$$

supplemented by

$$u(t,x,y) = \frac{-1}{2\pi}\int_{\mathbb{R}^2}\frac{y-\eta}{(x-\xi)^2+(y-\eta)^2}\omega(t,\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta,$$
$$v(t,x,y) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{x-\xi}{(x-\xi)^2+(y-\eta)^2}\omega(t,\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta.$$

In (3), γ_1 and γ_2 may be arbitrary if they are a function of t only. These equations are derived by assuming the following form of the vector field of the Navier-Stokes equations:

$$\boldsymbol{u} = \big(-\gamma_1(t)x + u(t,x,y),\ -\gamma_2(t)y + v(t,x,y),\ (\gamma_1(t)+\gamma_2(t))z\big), \tag{4}$$

where $\boldsymbol{u}$ is the velocity field, t denotes time, and (x,y,z) denotes a point in three dimensional space $\mathbb{R}^3$. ω, the vorticity, is given by $\omega = v_x - u_y$. The reader may find that substitution of (4) reduces the Navier-Stokes equations to the equation (3) without any approximation. Thus the solutions of (3) yield exact solutions of the Navier-Stokes equations via (4) (and a suitable form of the pressure).

Here the reader may well wonder why this kind of ansatz is introduced and what history this equation possesses. It is related to papers [**7**], [**13**], [**20**], [**21**], [**23**], and we will explain them in due course.

If $\gamma_1 \equiv \gamma_2$, there exist axisymmetric solutions, where $\omega = \omega(t,r)$ with $r = \sqrt{x^2+y^2}$. With $\gamma(t) = \gamma_1(t) = \gamma_2(t)$, the vorticity satisfies the following equation:

$$\omega_t - \gamma(t)(r\omega_r + 2\omega) = \nu\frac{1}{r}(r\omega_r)_r \quad (0 \le r < \infty). \tag{5}$$

If γ is a constant, then (5) has been known for many decades, originally due to Burgers, see [**2**], [**9**], [**14**], [**19**], [**20**]. However, at this stage, $\gamma(t)$ may be an arbitrary function of t. In spite of the arbitrariness of γ, (4) gives us an exact solution of the 3D Navier-Stokes equations.

If we start with the following ansatz

$$\boldsymbol{u} = (-\gamma(t)x, v(t,x,z), \gamma(t)z),$$

then we obtain

$$v_t - \gamma(t)(xv_x - zv_z) = \nu\triangle v + \tilde{\gamma}(t),$$

where $\tilde{\gamma}(t)$ is an unknown function of t only. Rather than discussing this equation in general, we restrict ourselves to the case where v depends only on x. Then ω, which is defined by $\omega = v_x$ satisfies

$$\omega_t - \gamma(t)(x\omega_x + \omega) = \nu\omega_{xx} \quad (-\infty < x < \infty). \tag{6}$$

Equations (5) and (6) can be solved with respect to ω, once γ is prescribed. Since the vector field $(-ax, -by, (a+b)z)$, where a and b are constants, is called a straining flow, we call γ_1 and γ_2 a strain rate. "The strained vorticity dynamics" in the title of the present paper comes from this. The straining flow naturally appears if we integrate the Poisson equation for the pressure $-\triangle p = \sum_{1\le i,j\le 3} u_{i,x_j} u_{j,x_i}$, which is derived from the Navier-Stokes equations. There are many techniques to incorporate the straining flow, by which we can simulate 3D fluid motion: See [**7**], [**23**], for instance. However, there is no way of specifying $\gamma(t)$ without resorting to a kind of hypothesis. The case where $\gamma(t) = \gamma_0$ is a constant is the classical one: [**2**], [**9**], [**14**]. Kambe [**12**], [**13**] realized that the strain rate γ may depend on t and he derived equations similar to ours. He then studied some examples. Moffatt [**20**] considered the case where $\gamma(t) = a/(T-t)$ with positive constants a and T. Both Kambe and Moffatt, however, assumed that γ is a prescribed function of t. Accordingly, [**24**] seems to be the first to treat the case where γ depends on the unknown ω on non-local fashion.

In the present paper we consider the following model

$$\gamma(t) = \mu\|\omega(t)\|_p^\alpha, \tag{7}$$

where $\mu > 0$ and $\alpha \ge 0$ are constant, and $\|\cdot\|_p$ denotes the L^p-norm. The constant $\nu > 0$ and $\mu > 0$ can be assumed to be unity, by which we do not lose generality since we can normalize them to unity by rescaling t, r (or x), and ω. We thus get to equations (1) and (2). (7) is a generalization of [**24**] in the sense that [**24**] considered only the case of $\alpha = 1$. If $\alpha = 1$, the hypothesis (7) has a physical motivation, for which we refer the reader to [**24**].

The case of $\alpha = 1$, $1 \le p < \infty$ for (2) was considered in [**24**], where the global existence was proved. For (1) they proved that a solution exists globally in time for all $p \in [1, \infty]$. The case of $\alpha = 1$, $p = \infty$ for (2) was left unanswered by [**24**] but was solved by [**22**], where it was proved that the solution blows up in finite time if and only if $\|\omega(0)\|_1 > 1$.

In the present paper we consider both (1) and (2) for general α, and determine the parameter range of (α, p) for which blow-up occurs.

Roughly speaking, our conclusion is as follows:

- Set $\alpha^* = 2p/(p-1)$ for (1) and $\alpha^* = p/(p-1)$ for (2). Then if $0 \le \alpha < \alpha^*$, all the solutions exist in $0 \le t < \infty$. If $\alpha^* \le \alpha < \infty$, a solution blows up in finite time if its initial data is large.

In this sense, we call $\alpha^* = 2p/(p-1)$ or $\alpha^* = p/(p-1)$ a critical exponent. Critical exponent is used in many senses in many equations, see [**15**]. In the present case, we may say that a larger p induces blow-up with smaller α, which hints that the L^∞-norm of the vorticity is the most relevant to a blow-up. This may have a relation to the Beale-Kato-Majda theorem on the possible blow-up of solutions of the Euler equations, in which blow-up occurs if and only if a certain integral of the L^∞-norm of the vorticity is infinite, see [**1**].

This paper is organized as follows. After preparing some preliminary materials in Section 2, we study the equation (1) in Section 3. Section 4 is devoted to the study of (2). Some special solutions are considered in Section 5. The local existence theorems will be proved in Section 6.

2. Preliminary.

Throughout the present paper, we assume that $\omega_0 \in L^1 \cap L^p$ with some $p \in (1, \infty]$. We first state a theorem on local existence about (1):

THEOREM 1. *Suppose that* $1 < p < \infty$. *For all* $\omega_0 \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ *there exist a* $T > 0$ *depending only on* $\|\omega_0\|_1$ *and* $\|\omega_0\|_p$ *such that there exists a unique solution in* $C([0,T]; L^1 \cap L^p)$ *which satisfies* (1) *in the classical sense in* $0 < t < T$.

THEOREM 2. *For all* $\omega_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *there exist a* $T > 0$ *depending only on* $\|\omega_0\|_1$ *and* $\|\omega_0\|_\infty$ *such that there exists a unique solution of* (1) *in* $0 \le t \le T$. *The solution belongs to* $C([0,T]; L^1 \cap L^p)$ *for all* $p < \infty$.

Since we obviously have $L^1 \cap L^\infty \hookrightarrow L^1 \cap L^p$, Theorem 2 is a consequence of Theorem 1. A similar theorem holds true for (2) if we replace $L^p(\mathbb{R})$ with $L^p(\mathbb{R}^2)$. Theorem 1 will be proved in Section 6. We remark that, as is often employed, $L^1 \cap L^p$ is regarded as a Banach space with $\|f\|_{L^1} + \|f\|_{L^p}$.

We next note that the positivity is preserved in the sense that $\omega_0(x) \ge 0$ everywhere implies $\omega(t,x) \ge 0$ for all t and x for (1). Similarly, $\omega_0(r) \ge 0$ everywhere implies that $\omega(t,r) \ge 0$ for all t and r for (2). See [**22**], [**24**]. Accordingly, for the sake of simplicity, *we henceforth consider only those initial data which are nonnegative everywhere.*

We next note that for all $1 \le q < \infty$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \omega(t,x)^q \mathrm{d}x = (q-1)\|\omega(t)\|_p^\alpha \int_{\mathbb{R}} \omega(t,x)^q \mathrm{d}x - q(q-1)\int_{\mathbb{R}} \omega(t,x)^{q-2}\omega_x(t,x)^2 \mathrm{d}x \tag{8}$$

for (1), and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \omega(t,r)^q r\mathrm{d}r$$
$$= (2q-2)\|\omega(t)\|_p^\alpha \int_0^\infty \omega(t,r)^q r\mathrm{d}r - q(q-1)\int \omega(t,r)^{q-2}\omega_r(t,r)^2 r\mathrm{d}r \qquad (9)$$

for (2). These can be verified by integration by parts. In particular, if we set $q = 1$, we see that the total vorticity is preserved;

$$\int_{-\infty}^\infty \omega(t,x)\,\mathrm{d}x \equiv \int_{-\infty}^\infty \omega_0(x)\,\mathrm{d}x, \quad \int_0^\infty \omega(t,r)r\,\mathrm{d}r \equiv \int_0^\infty \omega_0(r)r\,\mathrm{d}r$$

for (1) and (2), respectively. We have therefore obtained an important proposition that $\|\omega(t)\|_1 \equiv \|\omega_0\|_1$.

We will use the L^p-L^q estimate: Let $\Omega = \Omega(t,x)$ be the solution of the heat equation:

$$\Omega_t = \triangle\Omega \quad (0 < t,\ x \in \mathbb{R}^d).$$

We then have for $q \le p$

$$\|\Omega(t)\|_p \le (4\pi t)^{-d/2(1/q-1/p)}\|\Omega(0)\|_q. \qquad (10)$$

For proof, see [**8**] or [**28**], for instance.

3. Equation (1).

Here we consider (1) and examine global existence and blow-up. Since $\|\omega(t)\|_1$ is preserved, the solution exists globally in time if we have proved that $\|\omega(t)\|_p$ is locally bounded. By this we mean that for all $T > 0$ the solution satisfies that $\|\omega(t)\|_p \le c(T)$ $(0 \le t \le T)$ if a solution exists in $0 \le t \le T$. Here $c(T)$ denotes a positive constant which depends only on T, $\|\omega_0\|_1$, and $\|\omega_0\|_p$.

We now prove

Theorem 3. *Suppose that $1 < p < \infty$. If $0 \le \alpha < 2p/(p-1)$, the solution exists globally in time for all non-negative $\omega_0 \in L^1 \cap L^p$. If $2p/(p-1) \le \alpha$ then a solution with a sufficiently large $\|\omega_0\|_p$ blows up in finite time.*

Proof. We start with a particular case of (8):

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \omega(t,x)^p \mathrm{d}x = (p-1)\|\omega(t)\|_p^{\alpha+p} - p(p-1)\int_{\mathbb{R}} \omega(t,x)^{p-2}\omega_x(t,x)^2 \mathrm{d}x$$
$$\leq (p-1)\|\omega(t)\|_p^{\alpha+p}.$$

Accordingly,

$$\|\omega(t)\|_p^p \leq \|\omega(0)\|_p^p \exp\left((p-1)\int_0^t \|\omega(s)\|_p^\alpha \, \mathrm{d}s\right). \tag{11}$$

We therefore obtain boundedness of $\|\omega(t)\|_p$ if

$$\int_0^t \|\omega(s)\|_p^\alpha \, \mathrm{d}s$$

is bounded by some function of the initial data ω_0.

Suppose that the solution ω exists in $0 \leq t \leq T$ and set

$$\sigma(t) = \exp\left(\int_0^t \|\omega(s)\|_p^\alpha \, \mathrm{d}s\right). \tag{12}$$

Define $\tau = \tau(t)$ by

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \sigma^2, \quad \tau(0) = 0.$$

Note that $t \leftrightarrow \tau$ is one-to-one. We finally define Ω by

$$\omega(t,x) = \sigma\Omega(\tau, \sigma x) \quad \text{or} \quad \Omega(\tau,\xi) = \frac{1}{\sigma}\omega\left(t, \frac{\xi}{\sigma}\right). \tag{13}$$

Then Ω satisfies the one-dimensional linear heat equation:

$$\Omega_\tau = \Omega_{\xi\xi}, \quad \Omega(0,\xi) = \omega_0(\xi). \tag{14}$$

This beautiful trick is due to Lundgren [**16**], whence we call it Lundgren's trick. It was effectively used by [**12**], [**13**]. Note also that

$$\int_{\mathbb{R}} \omega(t,x)^p \mathrm{d}x = \sigma^p \int_{\mathbb{R}} \Omega(\tau,\sigma x)^p \mathrm{d}x = \sigma^{p-1}\int_{\mathbb{R}} \Omega(\tau,\xi)^p \mathrm{d}\xi.$$

Namely, $\|\omega(t)\|_p = \sigma^{1-1/p}\|\Omega(\tau)\|_p$. Now we have

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \sigma\|\omega(t)\|_p^\alpha = \sigma^{1+\alpha(p-1)/p}\|\Omega(\tau)\|_p^\alpha.$$

Therefore

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} = \sigma^{-1+\alpha(p-1)/p}\|\Omega(\tau)\|_p^\alpha, \tag{15}$$

where $\sigma = \sigma(t(\tau))$ is now regarded as a function of τ. In what follows, we employ a convention that $\sigma(\tau)$ implies $\sigma(t(\tau))$.

We first consider the case where $2p/(p-1) < \alpha$. We set

$$\kappa = -2 + \frac{\alpha(p-1)}{p},$$

which is positive. (15) is then written as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\sigma^{-\kappa} = -\kappa\|\Omega(\tau)\|_p^\alpha.$$

After integrating, we have

$$\frac{1}{\sigma(\tau)^\kappa} = 1 - \kappa\int_0^\tau \|\Omega(s)\|_p^\alpha \,\mathrm{d}s. \tag{16}$$

If the initial data is so large that

$$\kappa\int_0^\infty \|\Omega(s)\|_p^\alpha \,\mathrm{d}s > 1, \tag{17}$$

then the right hand side of (16) vanishes at some $\tau = \tau_0 < \infty$. Thus, if a solution exists, it must hold that $\tau < \tau_0$. By $\mathrm{d}t/\mathrm{d}\tau = \sigma^{-2} \le 1$, we have $t \le \tau$. Therefore the solution blows up before t exceeds τ_0. If

$$\kappa\int_0^\infty \|\Omega(s)\|_p^\alpha \,\mathrm{d}s < 1, \tag{18}$$

then there exists a constant σ_0 such that $\sigma(\tau) \le \sigma_0$ for all $0 \le \tau < \infty$. By the definition, we have $\sigma \ge 1$. We see therefore that $1 \le \sigma(\tau) \le \sigma_0$ for all τ. This implies that $\sigma(t) < \sigma_0$ for all t. The solution therefore exists in $0 \le t < \infty$ thanks to (11) and (12).

Before we proceed further, we consider on what conditions (17) or (18) is satisfied. First let us note that if the initial data ω_0 is replaced by $\lambda\omega_0$ with a positive constant λ then $\|\Omega(\tau)\|_p$ is replaced by $\lambda\|\Omega(\tau)\|_p$. In this sense, large solutions satisfy (17) and small solutions do (18). But this argument does not give us quantitative knowledge. We therefore use the L^p-L^q estimate (10), in which d is equal to 1 in the present case. We then have

$$\|\Omega(\tau)\|_p \le \|\omega_0\|_p, \quad \|\Omega(\tau)\|_p \le c_0 \tau^{-(p-1)/(2p)}\|\omega_0\|_1.$$

This gives us

$$\begin{aligned}\int_0^\infty \|\Omega(s)\|_p^\alpha \,\mathrm{d}s = \int_0^1 + \int_1^\infty &\le \|\omega_0\|_p^\alpha + c_0^\alpha \|\omega_0\|_1^\alpha \int_1^\infty s^{-\alpha(p-1)/(2p)}\,\mathrm{d}s \\ &= \|\omega_0\|_p^\alpha + \frac{c_0^\alpha\|\omega_0\|_1^\alpha}{\alpha(p-1)/(2p)-1}.\end{aligned}$$

Namely,

$$\kappa\int_0^\infty \|\Omega(s)\|_p^\alpha \,\mathrm{d}s \le \kappa\|\omega_0\|_p^\alpha + 2c_0^\alpha\|\omega_0\|_1^\alpha.$$

Therefore, $\kappa\|\omega_0\|_p^\alpha + 2c_0^\alpha\|\omega_0\|_1^\alpha < 1$ is a sufficient condition for the global existence.

On the other hand, it is not difficult to construct an initial data such that

$$\|\Omega(\tau)\|_p \ge M\tau^{-(p-1)/(2p)} \quad (1 \le \tau < \infty), \tag{19}$$

where M can be taken arbitrarily large. The author does not know, except for constructing ad hoc examples, a systematic theory to derive a lower bound such as (19). There are many functions which satisfy (19). For instance, $\omega_0(\xi) = Me^{-c\xi^2}$, where M and c are positive constants, is one of them. In fact, in this special case, Ω is explicitly written as $\Omega(\tau,\xi) = M/\sqrt{1+4c\tau}e^{-c\xi^2/(1+4c\tau)}$ and we have

$$\|\Omega(\tau)\|_p = M\left(\frac{\pi}{cp}\right)^{1/(2p)}(1+4c\tau)^{-(p-1)/(2p)}. \tag{20}$$

Note that if $\omega_0(x) \ge Me^{-c\xi^2}$, then the equality sign in (20) is replaced by $\ge$. Therefore there are many functions satisfying (19). Now if we have an initial data satisfying (19), then

$$\kappa \int_0^\infty \|\Omega(s)\|_p^\alpha \,\mathrm{d}s \geq \kappa \int_1^\infty M^\alpha s^{-\alpha(p-1)/(2p)} \mathrm{d}s \geq 2M^\alpha,$$

which may be taken as large as we wish.

We now consider the case where $\alpha = 2p/(p-1)$. (15) is now

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} = \sigma \|\Omega(\tau)\|_p^{2p/(p-1)},$$

whence

$$\log \sigma(\tau) = \int_0^\tau \|\Omega(s)\|_p^{2p/(p-1)} \,\mathrm{d}s.$$

This shows the local boundedness of $\sigma(\tau)$ as a function of τ. But this does not by itself imply global existence, since at this stage there remains a possibility that t tends to a finite limit as $\tau \to \infty$. That is, global existence is guaranteed only if $t \mapsto \sigma(t)$ is locally bounded, which is not the same as the local boundedness of $\tau \mapsto \sigma(\tau)$. In fact, since $\mathrm{d}t/\mathrm{d}\tau = \sigma(\tau)^{-2}$, blow-up occurs if (and only if)

$$\int_0^\infty \frac{\mathrm{d}\tau}{\sigma(\tau)^2} < \infty.$$

Accordingly, if $\sigma(\tau)$ grows sufficiently rapidly as $\tau \to \infty$, then $\lim_{\tau\to\infty} t(\tau) < \infty$ and the solution blows up. For instance, if we have ω_0 for which (19) holds, then we have

$$\log \sigma(\tau) \geq \log \sigma(\tau_0) + \int_{\tau_0}^\tau M^{2p/(p-1)} s^{-1} \,\mathrm{d}s \geq M^{2p/(p-1)} \log \tau + \text{constant}.$$

And we have a blow-up if $M^{2p/(p-1)} > 1/2$.

In this way we can obtain blow-up solutions, if the initial data is large. Although we do not obtain a sharp criterion for sufficiency, we may be content with this rough observation and the fact that there exist special self-similar blow-up solutions, which will be given in Section 5 below.

We finally consider the case where $0 \leq \alpha < 2p/(p-1)$. We come back to (16), where κ is now negative. Since (10) gives us

$$\|\Omega(\tau)\|_p \leq c\tau^{-(p-1)/(2p)} \|\omega_0\|_1,$$

we obtain

$$\sigma(\tau)^{-\kappa} = 1 - \kappa \int_0^\tau \|\Omega(s)\|_p^\alpha \, \mathrm{d}s \leq c + c\tau^{1-(p-1)\alpha/(2p)}.$$

Namely,

$$\sigma(\tau) \leq c + c\tau^{1/2},$$

which implies that

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sigma(\tau)^2} \geq \frac{c'}{1+\tau}.$$

Therefore $t \geq c' \log(1+\tau)$. Consequently there exist positive constant A and B such that $\sigma(t) = \sigma(\tau) \leq c + c\tau^{1/2} \leq Ae^{Bt}$. This proves the global existence. □

We now consider the case where $p = \infty$:

$$\omega_t = \|\omega(t)\|_\infty^\alpha (x\omega_x + \omega) + \omega_{xx} \quad (-\infty < x < \infty). \tag{21}$$

The proof is carried out in a parallel fashion. We define Ω by (13) and assume that

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \sigma^2,\ \tau(0) = 0; \quad \frac{\mathrm{d}\sigma}{\mathrm{d}t} = \sigma\|\omega(t)\|_\infty^\alpha,\ \sigma(0) = 1. \tag{22}$$

Then Ω satisfies $\Omega_\tau = \Omega_{\xi\xi}$. Also we have

$$\|\omega(t)\|_\infty = \sigma\|\Omega(\tau)\|_\infty, \quad \frac{\mathrm{d}\sigma}{\mathrm{d}t} = \sigma^{\alpha+1}\|\Omega(\tau)\|_\infty^\alpha. \tag{23}$$

Accordingly,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} = \sigma^{\alpha-1}\|\Omega\|_\infty^\alpha \quad \text{or} \quad \sigma^{1-\alpha}\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} = \|\Omega\|_\infty^\alpha,$$

which yields

$$\sigma(t)^{2-\alpha} = 1 + (2-\alpha)\int_0^\tau \|\Omega(s)\|_\infty^\alpha \, \mathrm{d}s. \tag{24}$$

In the case of $p = \infty$, Ω can be estimated rather easily. In fact, we have

$$\|\Omega(\tau)\|_\infty \le \frac{1}{\sqrt{4\pi\tau}}\|\omega_0\|_1, \quad \|\Omega(\tau)\|_\infty \le \|\omega_0\|_\infty, \tag{25}$$

which can be derived easily by

$$\Omega(\tau,\xi) = \frac{1}{\sqrt{4\pi\tau}}\int_{\mathbb{R}} \omega_0(\eta)\exp\biggl(-\frac{(\xi-\eta)^2}{4\tau}\biggr)\mathrm{d}\eta. \tag{26}$$

Although (11) is not available in the present case, we see from (23) that the local boundedness of σ implies that of ω, since $\|\Omega(\tau)\|_\infty$ is bounded by $\|\omega_0\|_\infty$.

With these formulas in hand, we can prove

THEOREM 4. *Suppose that $0 \le \alpha < 2$, that $\omega_0 \ge 0$, and that $\omega_0 \in L^1 \cap L^\infty$. Then the solution exists globally in time. Suppose that $2 \le \alpha$ and $0 \le \omega_0 \in L^1 \cap L^\infty$. Then the solution blows up in finite time if ω_0 is large.*

The same method as was used in the case of $p < \infty$ can be used in the case of $p = \infty$. In fact interpretation of the proof above into the present case is straightforward. And we may safely omit the proof of this theorem.

4. Equation (2).

Here we consider (2). For this equation we modify the definition of Ω as

$$\omega(t,r) = \sigma^2\Omega(\tau,\sigma r),$$

where $\tau = \tau(t)$, $\sigma = \sigma(t)$. If they satisfy

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \sigma\|\omega(t)\|_p^\alpha,\ \sigma(0) = 1, \quad \frac{\mathrm{d}\tau}{\mathrm{d}t} = \sigma^2,\ \tau(0) = 0,$$

then $\Omega = \Omega(\tau,\rho)$ satisfies

$$\frac{\partial\Omega}{\partial\tau} = \frac{\partial^2\Omega}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\Omega}{\partial\rho}.$$

Namely Ω satisfies the axi-symmetric heat equation in $\mathbb{R}^2$.

Note that

$$\|\omega(t)\|_p = \sigma^{2-2/p}\|\Omega(\tau)\|_p,$$

whence

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \sigma^{1+2\alpha(p-1)/p}\|\Omega(\tau)\|_p^\alpha, \quad \frac{\mathrm{d}\tau}{\mathrm{d}t} = \sigma^2.$$

This implies that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} = \sigma^{-1+2\alpha(p-1)/p}\|\Omega(\tau)\|_p^\alpha.$$

We now prove

THEOREM 5. *Suppose that $1 < p < \infty$ and $p/(p-1) \leq \alpha$. Then a solution blows up if the initial data is large. If $0 \leq \alpha < p/(p-1)$, the solution exists globally in time.*

PROOF. Set

$$\lambda = \frac{2\alpha(p-1)}{p} - 2,$$

which is positive if $\alpha > p/(p-1)$. We then have

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\sigma(\tau)^{-\lambda} = -\lambda\|\Omega(\tau)\|_p^\alpha \quad \text{or} \quad \frac{1}{\sigma(\tau)^\lambda} = 1 - \lambda\int_0^\tau \|\Omega(s)\|_p^\alpha \,\mathrm{d}s.$$

The conclusion follows in the same way as in the preceding section. Namely the solution blows up if

$$\lambda\int_0^\infty \|\Omega(s)\|_p^\alpha \,\mathrm{d}s > 1,$$

and exists globally in time if the inequality of the opposite direction holds true.

If $\alpha = p/(p-1)$, then

$$\log\sigma(\tau) = \int_0^\tau \|\Omega(s)\|_p^{p/(p-1)} \,\mathrm{d}s.$$

Note also that if $\omega_0(r) = Me^{-cr^2}$ then

$$\Omega(\tau,\rho) = \frac{M}{1+4c\tau}\exp\left(-\frac{c\rho^2}{1+4c\tau}\right),$$

which gives us

$$\|\Omega(\tau)\|_p \sim \tau^{-(p-1)/p} \quad \text{as} \quad \tau \to \infty.$$

The rest of the proof goes just as in the previous section.

We finally assume that $a < p/(p-1)$. The L^p-L^q estimates (10) is now

$$\|\Omega(\tau)\|_p \le (4\pi\tau)^{-(p-1)/p} \|\omega_0\|_1.$$

Hence we obtain

$$\sigma(\tau)^{-\lambda} = 1 - \lambda \int_0^\tau \|\Omega(s)\|_p^\alpha \, \mathrm{d}s \le c + c \int_1^\tau s^{-(p-1)\alpha/p} \, \mathrm{d}s.$$

The rest of the proof is the same as before. □

In quite an analogous way, we obtain

Theorem 6. *Suppose that* $p = \infty$. *Then, if* $1 \le \alpha$, *the solution blows up if* ω_0 *is large. If* $0 \le \alpha < 1$, *all the solution exist globally in time.*

We remark that this theorem in the special case of $\alpha = 1$ was proved in [**22**], where a necessary and sufficient condition for blow-up was computed to be $\|\omega_0\|_1 > 1$. In general cases a concrete expression for the necessary and sufficient condition in terms of ω_0 is difficult for us to obtain. This is because we need some estimates from below of $\|\Omega(\tau)\|_p$.

5. Some special solutions.

5.1. Steady-state.

Here we compute certain steady-states. Although we do not study in this paper asymptotic behavior of solutions as $t \to \infty$, the information on steady-states will be helpful in the future study.

Steady-states of (1) is characterized by

$$\|\omega\|_p^\alpha (x\omega' + \omega) + \omega'' = 0 \quad (-\infty < x < \infty) \quad \left(' = \frac{\mathrm{d}}{\mathrm{d}x}\right). \tag{27}$$

Since $\|\omega\|_p^\alpha$ is a constant, we set $2\mu = \|\omega\|_p^\alpha$. Then,

$$2\mu(x\omega' + \omega) + \omega'' = 0 \quad (-\infty < x < \infty). \tag{28}$$

This equation was considered in [**25**] in a context of self-similar solution of the Navier-Stokes equations. Its general solution is (see [**25**, page 182]):

$$\omega(x) = Ae^{-\mu x^2} + Be^{-\mu x^2} \int_0^x e^{\mu y^2}\,\mathrm{d}y,$$

where A and B are constants. Note that the second term in the right hand side decays in the order of $|x|^{-1}$ as $|x| \to \infty$. Therefore this function belongs to L^p but not to L^1. We therefore set $B = 0$.

The solution of (27) is obtained if the following condition is satisfied:

$$\int_{\mathbb{R}} \left(Ae^{-\mu x^2}\right)^p \mathrm{d}x = (2\mu)^{p/\alpha}.$$

This is written as

$$A = 2^{1/\alpha} \left(\frac{p}{\pi}\right)^{1/(2p)} \mu^{(2p+\alpha)/(2p\alpha)}. \tag{29}$$

Therefore for any given $\mu > 0$ there exists a unique $A > 0$ satisfying $2\mu = \|\omega\|_p^\alpha$. We have thus obtained a family of steady-states parameterized by $\mu \in (0, \infty)$. The results are summarized as follows:

THEOREM 7. *For all $0 \le \alpha < \infty$ and all $1 < p \le \infty$, (1) and (2) have a family of steady-states.*

PROOF. The argument above proves the case for (1) if $1 < p < \infty$. If $p = \infty$, we should put $A = (2\mu)^{1/\alpha}$. If $\alpha = 0$, then $\mu = 1/2$ and A is arbitrary. As for (2), the equation to be solved is:

$$2\mu(r\omega' + 2\omega) + r^{-1}(r\omega')' = 0 \quad (0 \le r < \infty) \quad \left(2\mu = \|\omega\|_p^\alpha,\ ' = \frac{\mathrm{d}}{\mathrm{d}r}\right). \tag{30}$$

This equation admits a solution $\omega = Ae^{-\mu r^2}$ if

$$A = 2^{1/\alpha} \left(\frac{p}{\pi}\right)^{1/p} \mu^{(p+\alpha)/(p\alpha)}. \qquad \square$$

If only steady-states are considered, then $\|\omega\|_p^\alpha$ is a constant. Therefore the classical steady-state of Burgers in [**2**] satisfies (30). Our solutions above, $\omega =$

$Ae^{-\mu r^2}$, are nothing but the Burgers vortices in a disguised form. The Burgers vortices are stable. This was proved by Giga and Kambe [**9**] in a special case, and later by Gallay and Wayne [**6**] in a general case. But this stability is meant in the framework of the equation (5) where $\gamma(t)$ is a given constant, namely, (2) with $\alpha = 0$. If we consider the stability of the solution in the framework of (1) or (2), then this poses a new problem.

It is known that there are steady-states of (3) without axisymmetry, which was first demonstrated numerically by Robinson and Saffman [**27**]. A mathematical proof of existence is recently discovered by Maekawa [**17**], [**18**]. [**18**] is remarkable in that the author proves existence of steady-states in a strain in which any asymmetry of the external strain can be assumed. Stability of the asymmetric solutions are proved by [**17**]. In this regard, a study of (3) in a general γ_1 and γ_2 would offer interesting problems.

Although we do not study the stability of our steady-states, the following consideration may be interesting to the reader. *Our solution* $Ae^{-\mu x^2}$ *with* (29) *is unstable if* $2p/(p-1) < \alpha$. In fact, if $\omega_0(x) = Me^{-\mu x^2}$, then $\Omega(\tau, \xi) = M/\sqrt{1+4\mu\tau}\exp(-\mu\xi^2/(1+4\mu\tau))$. An elementary computation shows that

$$\kappa \int_0^\infty \|\Omega(s)\|_p^\alpha \, \mathrm{d}s = \frac{M^\alpha}{A^\alpha}.$$

Therefore, for all $M > A$ the solution corresponding to $\omega_0(x) = Me^{-\mu x^2}$ blows up in finite time. If $M < A$, then σ is bounded as is proved in the proof of Theorem 3. On the other hand, Ω tends to zero. Hence $\omega(t,x) = \sigma\Omega(\tau, \sigma x)$ tends to zero if $M < A$. This also suggests that $\omega \equiv 0$ is stable if $2p/(p-1) < \alpha$.

If $0 \le \alpha < 2p/(p-1)$, the steady-state may be stable, as the following reasoning suggests. Suppose that $\omega_0(x) = Me^{-\mu x^2}$ with M slightly different from A. Note that

$$\sigma(\tau)^{-\kappa} = 1 - \kappa \int_0^\tau \|\Omega(s)\|_p^\alpha \, \mathrm{d}s = 1 - \left(\frac{M}{A}\right)^\alpha + \left(\frac{M}{A}\right)^\alpha (1+4\mu\tau)^{-\kappa/2},$$

where $\kappa = -2 + \alpha(p-1)/p < 0$. Therefore

$$\lim_{\tau\to\infty} \frac{\sigma(\tau)}{\sqrt{\tau}} = \sqrt{4\mu}\left(\frac{M}{A}\right)^{-\alpha/\kappa} (:= \sigma_0).$$

This implies that

$$\omega(t,x)=\frac{\sigma(\tau)}{\sqrt{4\pi\tau}}\int_{\mathbb{R}} Me^{-\mu y^2}\exp\bigg(-\frac{(\sigma(\tau)x-y)^2}{4\tau}\bigg)\,\mathrm{d}y\to\frac{M\sigma_0}{2\sqrt{\mu}}\exp\bigg(-\frac{\sigma_0^2}{4}x^2\bigg).$$

as $\tau\to\infty$.

5.2. Self-similar blow-up.

We now look for a solution of (1) of the following form:

$$\omega(t,x)=(T-t)^{-\delta}\phi\bigg(\frac{x}{(T-t)^s}\bigg), \tag{31}$$

where $T>0$, $\delta>0$ and s are parameters to be chosen. A solution of this form is called a self-similar blow-up solution. By substituting (31) into (1), we see that δ and s must be chosen as

$$\delta=\frac{1}{\alpha}+\frac{1}{2p},\quad s=\frac{1}{2}. \tag{32}$$

To prove this, set $\xi=x/(T-t)^s$. Then ϕ satisfies

$$(T-t)^{2s-1}(\delta\phi(\xi)+s\xi\phi'(\xi))=(T-t)^{2s+\alpha s/p-\delta\alpha}\|\phi\|_p^\alpha(\xi\phi'+\phi(\xi))+\phi''(\xi).$$

This is independent of t if and only if

$$2s-1=0\quad\text{and}\quad 2s+\frac{\alpha s}{p}-\delta\alpha=0,$$

by which we obtain (32). The equation for ϕ becomes

$$\bigg(2\mu-\frac{1}{2}\bigg)\xi\phi'+(2\mu-\delta)\phi+\phi''=0\quad(-\infty<\xi<\infty), \tag{33}$$

where $\mu=1/2\|\phi\|_p^\alpha$. If $\delta=1/2$, then this equation is the same as (28) with 2μ being replaced by $2\mu-1/2$. We do not know solvability of (33) for a general δ. However, it can be written as

$$\bigg(2\mu-\frac{1}{2}\bigg)(\xi\phi)'+\bigg(\frac{1}{2}-\delta\bigg)\phi+\phi''(\xi)=0.$$

Since we are looking for a solution which is everywhere positive and decays to zero as $|x|\to\infty$, integration of this equation forces us to put $\delta=1/2$. We therefore assume this.

Now we have $(1/\alpha) + (1/2p) = 1/2$, i.e., $\alpha = 2p/(p-1)$, which is exactly the critical exponent. Then (33), which is now

$$\left(2\mu - \frac{1}{2}\right)(\xi\phi' + \phi) + \phi'' = 0 \quad (-\infty < \xi < \infty),$$

admits a solution of the form $\phi(\xi) = A\exp(-(\mu - 1/4)\xi^2)$, where $A > 0$ and $\mu > 1/4$ are constants. This becomes a solution of our problem if and only if $2\mu = \|\omega\|_p^\alpha$, i.e.,

$$A = \left(\frac{p}{\pi}\right)^{1/(p\alpha)} \left(\mu - \frac{1}{4}\right)^{1/(p\alpha)} (2\mu)^{1/\alpha}.$$

If $p = \infty$, then

$$\mu = \frac{1}{2}A^\alpha.$$

Similarly we can obtain a self-similar blow-up solution of (2) if $\alpha = p/(p-1)$. If we set

$$\omega(t,r) = \frac{1}{T-t}\phi\left(\frac{r}{\sqrt{T-t}}\right),$$

then the equation becomes:

$$\left(2\mu - \frac{1}{2}\right)(\rho\phi' + 2\phi) + \rho^{-1}(\rho\phi')' = 0 \quad (0 < \rho < \infty) \quad \left(\rho = \frac{r}{\sqrt{T-t}}\right).$$

The solution is:

$$\phi(\rho) = A\exp\left(-\left(\mu - \frac{1}{4}\right)\rho^2\right), \quad A = \left(\frac{p}{\pi}\right)^{1/p} \left(\mu - \frac{1}{4}\right)^{1/p} (2\mu)^{1/\alpha}.$$

We thus have

Theorem 8. *Suppose that $\alpha = 2p/(p-1)$ or $\alpha = p/(p-1)$ for (1) and (2), respectively. Then there exists a family of self-similar blow-up solutions of the following form*:

$$\omega(t,x) = \frac{A}{\sqrt{T-t}} \exp\biggl(- \Bigl(\mu - \frac{1}{4}\Bigr) \frac{x}{\sqrt{T-t}} \biggr),$$
$$\omega(t,r) = \frac{A}{T-t} \exp\biggl(- \Bigl(\mu - \frac{1}{4}\Bigr) \frac{r}{\sqrt{T-t}} \biggr),$$

respectively for (1) *and* (2).

6. Proof of the local existence theorems.

Theorem 1 can be proved by the usual successive iteration method. But here we prove it by Lundgren's trick. We suppose that $p < \infty$. Let $\omega_0 \in L^1 \cap L^p$. A unique solution of (14) exists and belongs to $C([0,T]; L^1 \cap L^p)$ for any $T > 0$. $\|\Omega(\tau)\|_p$ is a continuous function of τ in $[0,T]$. We then consider the following ordinary differential equation.

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} = \sigma^{-1+\alpha(p-1)/p} \|\Omega(\tau)\|_p^\alpha, \quad \sigma(0) = 1. \tag{34}$$

This has a unique solution, say $\sigma = \sigma(\tau)$ $(0 \le \tau \le T_1)$, where $T_1 > 0$ depends only on ω_0. Note that $\sigma(\tau) \ge 1$ for all τ. We then define t by

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sigma(\tau)^2}, \quad t(0) = 0. \tag{35}$$

Since $\sigma \ge 1$, we have $t \le \tau$ and t can be defined in $0 \le \tau \le T_1$. Then we define ω by

$$\omega(t,x) = \sigma \Omega(\tau, \sigma x). \tag{36}$$

It is easy to verify that ω is a solution of (1) in $0 \le t \le T_2 = t(T_1)$ and belongs to $C([0,T_2]; L^1 \cap L^p)$.

Uniqueness is seen by tracing this process conversely: if ω is a classical solution of (1) in $C([0,T]; L^1 \cap L^p)$, then we define σ and τ, by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \sigma \|\omega(t)\|_p^\alpha,\ \sigma(0) = 1; \quad \frac{\mathrm{d}\tau}{\mathrm{d}t} = \sigma(t)^2,\ \tau(0) = 0.$$

We then define Ω by

$$\Omega(\tau, \xi) = \frac{1}{\sigma} \omega\biggl(\tau, \frac{\xi}{\sigma}\biggr).$$

Since Ω is a solution of the linear heat equation with initial data in $L^1 \cap L^p$, it is uniquely determined. We now invert the function $t \mapsto \tau$ in a neighborhood of 0. Then $t = t(\tau)$ and σ may be considered to be a function of τ. They satisfy (34) and (35). Apparently, these ordinary differential equations admit a unique solution near $(\sigma(0), t(0)) = (1, 0)$, which implies that ω is unique. □

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