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Visible actions on flag varieties of type D and a generalization of the Cartan decomposition

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Abstract. We give a generalization of the Cartan decomposition for connected compact Lie groups motivated by the work on visible actions of T. Kobayashi [J. Math. Soc. Japan, 2007] for type A group. This paper extends his results to type D group. First, we classify a pair of Levi subgroups (L, H) of a simple compact Lie group G of type D such that $G = LG^{\sigma}H$ where σ is a Chevalley–Weyl involution. This gives the visibility of the L-action on the generalized flag variety G/H as well as that of the H-action on G/L and of the G-action on $(G \times G)/(L \times H)$. Second, we find a generalized Cartan decomposition $G = LBH$ with B in G^{σ} by using the herringbone stitch method which was introduced by Kobayashi in his 2007 paper. Applications to multiplicity-free theorems of representations are also discussed.

1. Introduction and statement of main results.

The aim of this paper is to classify all the pairs of Levi subgroups (L, H) of connected compact simple Lie groups of type D with the following property: $G = LG^{\sigma}H$ where σ is a Chevalley–Weyl involution of G (Definition 2.1). The motivation for considering this kind of decomposition is the theory of *visible actions* on complex manifolds introduced by T. Kobayashi ([**Ko2**]), and the decomposition $G = LG^{\sigma}H$ serves as a basis to generalize the Cartan decomposition to the non-symmetric setting. (We refer to [**He**], [**Ho**], [**Ma2**] and [**Ko4**] and references therein for some aspects of the Cartan decomposition from geometric and group theoretic viewpoints.)

A generalization of the Cartan decomposition for symmetric pairs has been used in various contexts including analysis on symmetric spaces, however, there were no analogous results for non-symmetric cases before Kobayashi's paper [**Ko4**]. Motivated by visible actions on complex manifolds ([**Ko1**], [**Ko2**]), he completely determined the pairs of Levi subgroups

$$(L, H) = (\mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_k), \mathrm{U}(m_1) \times \cdots \times \mathrm{U}(m_l))$$

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of the unitary group $G = \mathrm{U}(n)$ such that the multiplication mapping $L \times \mathrm{O}(n) \times H \to G$ is surjective. Further he developed a method to find a suitable subset B of $\mathrm{O}(n)$ which gives the following decomposition (a generalized Cartan decomposition, see [**Ko4**]):

$$G = LBH.$$

In view of this decomposition theory, we consider the following problems: Let G be a connected compact Lie group, $\mathfrak{t}$ a Cartan subalgebra, and σ a Chevalley–Weyl involution of G with respect to $\mathfrak{t}$.

1) Classify all the pairs of Levi subgroups L and H with respect to $\mathfrak{t}$ such that the multiplication map $\psi : L \times G^\sigma \times H \to G$ is surjective.
2) Find a "good" representative $B \subset G^\sigma$ such that $G = LBH$ in the case ψ is surjective.

We call such a decomposition $G = LBH$ a *generalized Cartan decomposition*. Here we note that the role of the subgroups H and L is symmetric.

The surjectivity of ψ implies that the subgroup L acts on the flag variety G/H in a (strongly) visible fashion (see Definition 5.1). At the same time the H-action on G/L, and the diagonal G-action on $(G\times G)/(L\times H)$ are strongly visible. Then Kobayashi's theory leads us to three multiplicity-free theorems (*triunity* à la [**Ko1**]):

$$\begin{aligned}
&\text{Restriction } G \downarrow L\ : \mathrm{Ind}_H^G(\mathbb{C}_\lambda)|_L,\\
&\text{Restriction } G \downarrow H : \mathrm{Ind}_L^G(\mathbb{C}_\lambda)|_H,\\
&\text{Tensor product} \quad\ : \mathrm{Ind}_H^G(\mathbb{C}_\lambda) \otimes \mathrm{Ind}_L^G(\mathbb{C}_\mu).
\end{aligned}$$

Here $\mathrm{Ind}_H^G(\mathbb{C}_\lambda)$ denotes a holomorphically induced representation of G from a character $\mathbb{C}_\lambda$ of H by the Borel–Weil theorem. See [**Ko1**], [**Ko2**], [**Ko3**] for the general theory on the application of visible actions (including the vector bundle setting), and also Section 5 for the compact simple Lie groups of type D.

In this article, we solve the aforementioned problems for connected compact simple Lie groups G of type D. That is, we give a complete list of the pairs of Levi subgroups that admit generalized Cartan decompositions, by using the herringbone stitch method that Kobayashi introduced in [**Ko4**].

In order to state our main results, we label the Dynkin diagram of type D_n as follows:

Diagram 1.1.

For a subset Π' of the set Π of simple roots, we denote by $\mathrm{L}_{\Pi'}$ the Levi subgroup whose root system is generated by Π'. For example, $\mathrm{L}_{\emptyset}$ is a maximal torus of G and $\mathrm{L}_{\{\alpha_p\}^c} = \mathrm{U}(p) \times \mathrm{SO}(2(n-p))$ for $G = \mathrm{SO}(2n)$ $(1 \leq p \leq n-2)$. Here $(\Pi')^c$ denotes $\Pi \setminus \Pi'$.

THEOREM 1.1. *Let G be a connected compact simple Lie group of type D_n $(n \geq 4)$, σ a Chevalley–Weyl involution, Π', Π'' two proper subsets of Π, and $\mathrm{L}_{\Pi'}$, $\mathrm{L}_{\Pi''}$ the corresponding Levi subgroups. Then the following two conditions on $\{\Pi', \Pi''\}$ are equivalent.*

(i) $G = \mathrm{L}_{\Pi'} \, G^{\sigma} \, \mathrm{L}_{\Pi''}$.
(ii) *One of the conditions below holds up to switch of the factors Π' and Π'' :*

I. $(\Pi')^c = \{\alpha_i\}, \ (\Pi'')^c = \{\alpha_j\}, \quad i \in \{n-1, n\}, \ j \in \{1, 2, 3, n-1, n\},$
II. $(\Pi')^c = \{\alpha_i\}, \ (\Pi'')^c \subset \{\alpha_j, \alpha_k\}, \ i \in \{n-1, n\}, \ j, k \in \{1, n-1, n\},$
III. $(\Pi')^c = \{\alpha_i\}, \ (\Pi'')^c \subset \{\alpha_j, \alpha_k\}, \ i \in \{n-1, n\}, \ j, k \in \{1, 2\},$
IV. $(\Pi')^c = \{\alpha_1\}, \ (\Pi'')^c \subset \{\alpha_j, \alpha_k\}$, *either* j *or* $k \in \{n-1, n\}$,
V. $(\Pi')^c = \{\alpha_i\}, \ (\Pi'')^c \subset \{\alpha_2, \alpha_j\}, \ n = 4, \ (i, j) = (3, 4)$ *or* $(4, 3)$.

Here $G^{\phi} := \{g \in G : \phi(g) = g\}$ for an automorphism ϕ of G. We did not intend to make the above cases I–V be exclusive, that is, there is a small overlap among Cases I, II and III.

As a corollary, we obtain three multiplicity-free theorems for type D groups (see Corollary 5.4 for the restriction to Levi subgroups and Corollary 5.5 for the tensor product representations).

This article is organized as follows. In Section 2, we see that Theorem 1.1 is reduced to the standard Levi subgroups of a matrix group $G = \mathrm{SO}(2n)$ without any loss of generality. In Section 3, we prove that (ii) implies (i). Furthermore, we find explicitly a slice B that gives a generalized Cartan decomposition $G = \mathrm{L}_{\Pi'} \, B \, \mathrm{L}_{\Pi''}$. The converse implication on (ii) $\Rightarrow$ (i) is proved in Section 4 by using the invariant theory for quivers. An application to multiplicity-free representations is discussed in Section 5.

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2. Reduction and matrix realization.

2.1. Reduction.

In this subsection, we show that the surjectivity of $\psi : L \times G^\sigma \times H \to G$ depends on neither the coverings of the group G nor the choice of Cartan subalgebras and Chevalley–Weyl involutions. This consideration reduces a proof of Theorem 1.1 to the case $G = \mathrm{SO}(2n)$.

We firstly recall the definition of a Chevalley–Weyl involution of a connected compact Lie group, and then we show the independence of the coverings.

Definition 2.1. Let G be a connected compact Lie group and σ an involution of G. We call σ a Chevalley–Weyl involution if there exists a maximal torus T of G such that $\sigma(t) = t^{-1}$ for every $t \in T$.

Proposition 2.2. *Let G be a connected compact semisimple Lie group, $\tilde{G}$ its universal covering group, $\phi : \tilde{G} \to G$ the covering homomorphism, and σ (resp. $\tilde{\sigma}$) a Chevalley–Weyl involution with respect to a maximal torus T (resp. $\tilde{T}$) of G (resp. $\tilde{G}$) such that the following diagram commutes.*

$$\begin{array}{ccc} G & \xrightarrow{\sigma} & G \\ {\scriptstyle\phi}\big\uparrow & & {\scriptstyle\phi}\big\uparrow \\ \tilde{G} & \xrightarrow{\tilde{\sigma}} & \tilde{G} \end{array}$$

Then for any subsets Π', Π'' of the set of simple roots Π of the Lie algebra $\mathfrak{g}$ of G, $G = \mathrm{L}_{\Pi'}\, G^\sigma\, \mathrm{L}_{\Pi''}$ holds if and only if $\tilde{G} = \tilde{\mathrm{L}}_{\phi^*\Pi'} \tilde{G}^{\tilde{\sigma}} \tilde{\mathrm{L}}_{\phi^*\Pi''}$ does. Here, ϕ^* denotes the natural induced map from ϕ, $\mathrm{L}_{\Pi'}$ (resp. $\mathrm{L}_{\Pi''}$) the Levi subgroup of G whose root system is generated by Π' (resp. Π''), and $\tilde{\mathrm{L}}_{\phi^*\Pi'}$ (resp. $\tilde{\mathrm{L}}_{\phi^*\Pi''}$) the Levi subgroup of $\tilde{G}$ whose root system is generated by $\phi^*\Pi'$ (resp. $\phi^*\Pi''$).

Proof. Let $Z_{\tilde{G}}$ denote the center of $\tilde{G}$. Assume $G = \mathrm{L}_{\Pi'}\, G^\sigma\, \mathrm{L}_{\Pi''}$. Since $G^\sigma \subset \phi(\tilde{T} \cdot \tilde{G}^{\tilde{\sigma}})$, we have $\phi(\tilde{\mathrm{L}}_{\phi^*\Pi'} \tilde{G}^{\tilde{\sigma}} \tilde{\mathrm{L}}_{\phi^*\Pi''}) = \mathrm{L}_{\Pi'}\, G^\sigma\, \mathrm{L}_{\Pi''} = G$. Then we obtain $\tilde{G} = Z_{\tilde{G}} \cdot (\tilde{\mathrm{L}}_{\phi^*\Pi'} \tilde{G}^{\tilde{\sigma}} \tilde{\mathrm{L}}_{\phi^*\Pi''}) = \tilde{\mathrm{L}}_{\phi^*\Pi'} \tilde{G}^{\tilde{\sigma}} \tilde{\mathrm{L}}_{\phi^*\Pi''}$.

Conversely, assume $\tilde{G} = \tilde{\mathrm{L}}_{\phi^*\Pi'} \tilde{G}^{\tilde{\sigma}} \tilde{\mathrm{L}}_{\phi^*\Pi''}$. Then we have $G = \mathrm{L}_{\Pi'}\, \phi(\tilde{G}^{\tilde{\sigma}})\, \mathrm{L}_{\Pi''}$ because ϕ is surjective. Since $\phi(\tilde{G}^{\tilde{\sigma}}) \subset G^\sigma$, we obtain $G = \mathrm{L}_{\Pi'}\, G^\sigma\, \mathrm{L}_{\Pi''}$. $\square$

Further, we can see that Theorem 1.1 is independent of the choice of Cartan subalgebras and Chevalley–Weyl involutions because any two Cartan subalgebras are conjugate to each other by an inner automorphism, and any two Chevalley–

Weyl involutions of the same Cartan subalgebra $\mathfrak{t}$ are conjugate to each other by the adjoint action of $\exp(\mathfrak{t})$ (see [**Wo**]). For these reasons, we may and do work with the matrix group $\mathrm{SO}(2n)$, and fix a Cartan subalgebra and a Chevalley–Weyl involution as in the next subsection.

2.2. Matrix realization.

Throughont this article, we realize $G = \mathrm{SO}(2n)$ as a matrix group as follows:

$$G := \{g \in \mathrm{SL}(2n, \mathbb{C}) : {}^t g J_{2n} g = J_{2n},\ {}^t \overline{g} g = I_{2n}\}, \tag{2.2.1}$$

where J_m is defined by

$$J_m := \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & O & & & \\ & & \iddots & & \\ & & & O & \\ 1 & & & & \end{pmatrix} \in \mathrm{GL}(m, \mathbb{R}).$$

Then, the corresponding Lie algebra of G forms

$$\mathfrak{g} := \{X \in \mathfrak{sl}(2n, \mathbb{C}) : {}^t X J_{2n} + J_{2n} X = O,\ {}^t \overline{X} + X = O\}.$$

We take a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ as diagonal matrices:

$$\mathfrak{t} = \bigoplus_{1 \le i \le n} \mathbb{R}\sqrt{-1} H_i,$$

where $H_i := \mathrm{E}_{i,i} - \mathrm{E}_{2n+1-i, 2n+1-i}$.

We define

$$\sigma : G \to G, \quad g \mapsto \bar{g}, \tag{2.2.2}$$

where $\bar{g}$ denotes the complex conjugate of $g \in G$. The differential of σ is denoted by the same letter. This involutive automorphism σ is a Chevalley–Weyl involution with respect to $\mathfrak{t}$.

We let $\{\varepsilon_i\}_{1 \le i \le n} \subset (\mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C})^*$ be the dual basis of $\{H_i\}_{1 \le i \le n}$. Then we define a set of simple roots $\Pi := \{\alpha_1, \dots, \alpha_n\}$ by

$$\alpha_i := \varepsilon_i - \varepsilon_{i+1} \ (1 \le i \le n-1), \quad \alpha_n := \varepsilon_{n-1} + \varepsilon_n.$$

Let $n = n_1 + \cdots + n_k$ be a partition of n with $n_1, \ldots, n_{k-1} > 0$ and $n_k \ge 0$. We put

$$s_i := \sum_{1 \le p \le i} n_p \quad (1 \le i \le k-1),$$
$$\Pi' := \Pi \setminus \{\alpha_{s_i} \in \Pi : 1 \le i \le k-1\},$$

and denote by $\mathrm{L}_{\Pi'}$ the Levi subgroup whose root system is generated by Π'. In the matrix realization, $\mathrm{L}_{\Pi'}$ takes the form:

$$\begin{aligned} \mathrm{L}_{\Pi'} &= \mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_{k-1}) \times \mathrm{SO}(2n_k) \\ &= \left\{ \begin{pmatrix} A_1 & & & & & & \\ & \ddots & & & & & \\ & & A_{k-1} & & & & \\ & & & B & & & \\ & & & & J_{n_{k-1}} \overline{A_{k-1}} J_{n_{k-1}}^{-1} & & \\ & & & & & \ddots & \\ & & & & & & J_{n_1} \overline{A_1} J_{n_1}^{-1} \end{pmatrix} : A_i \in \mathrm{U}(n_i), B \in \mathrm{SO}(2n_k) \right\}. \end{aligned} \tag{2.2.3}$$

Here, we note that the pair $(G, \mathrm{L}_{\Pi'})$ forms a symmetric pair if and only if $\Pi \setminus \Pi' = \{\alpha_1\}$, $\{\alpha_n\}$ or $\{\alpha_{n-1}\}$. For a later purpose, we give explicit involutions τ_1, μ and μ^ξ of G of which the connected component of fixed point subgroups are $\mathrm{L}_{\{\alpha_1\}^c}$, $\mathrm{L}_{\{\alpha_n\}^c}$ and $\mathrm{L}_{\{\alpha_{n-1}\}^c}$.

$$\mathrm{L}_{\{\alpha_1\}^c} = (G^{\tau_1})_0, \qquad \tau_1 : G \to G, \qquad g \mapsto I_{1,2(n-1),1} g I_{1,2(n-1),1}, \tag{2.2.4}$$

$$\mathrm{L}_{\{\alpha_n\}^c} = G^{\mu}, \qquad \mu : G \to G, \qquad g \mapsto I_{n,n} g I_{n,n}, \tag{2.2.5}$$

$$\mathrm{L}_{\{\alpha_{n-1}\}^c} = G^{\mu^\xi}, \qquad \mu^\xi = \xi \circ \mu \circ \xi : G \to G \quad \text{(see (2.2.7))}, \tag{2.2.6}$$

where K_0 denotes the connected component of K containing the identity element for a Lie group K, and $I_{1,2(n-1),1}$ and $I_{n,n}$ are defined by $I_{1,2(n-1),1} := \mathrm{diag}(1, -1, \ldots, -1, 1)$ and $I_{n,n} := \mathrm{diag}(\underbrace{1, \ldots, 1}_{n}, \underbrace{-1, \ldots, -1}_{n})$.

The Dynkin diagram of type D_n has an outer automorphism of order two, which switches α_{n-1} and α_n. This outer automorphism is induced from the following involution of G.

$$\xi : G \to G, \qquad x \to g_\xi x g_\xi^{-1}, \tag{2.2.7}$$

$$\text{where}\quad g_\xi := \begin{array}{c} \begin{matrix} & & & \overset{n}{\vee} & \overset{n+1}{\vee} & & & \end{matrix} \\ \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 0 & 1 & & & \\ & & & 1 & 0 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix} \end{array}.$$

Note that for any $\Pi' \subset \Pi$, $\xi(\mathrm{L}_{\Pi'})$ coincides with $\mathrm{L}_{\Pi''}$ where Π'' is obtained from Π' by replacing α_{n-1} with α_n. We also note that ξ preserves G^σ.

To obtain a generalized Cartan decomposition by the herringbone stitch method ([**Ko4**]), we will use an involutive automorphism τ_p of G ($1 \leq p \leq n-1$) given by

$$\tau_p : G \to G, \qquad g \mapsto I_{p,2(n-p),p}\, g\, I_{p,2(n-p),g} \tag{2.2.8}$$

where $I_{p,2(n-p),p} := \mathrm{diag}(\underbrace{1,\ldots,1}_{p}, \underbrace{-1,\ldots,-1}_{2(n-p)}, \underbrace{1,\ldots,1}_{p})$. Then the connected component of the fixed point subgroup G^{τ_p} is given by

$$\begin{aligned} &\mathrm{SO}(2p) \times \mathrm{SO}(2(n-p)) \\ &\quad := \left\{ \left(\begin{array}{c|c|c} A & & B \\ \hline & S & \\ \hline C & & D \end{array} \right) : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SO}(2p),\ S \in \mathrm{SO}(2(n-p)) \right\}. \end{aligned} \tag{2.2.9}$$

3. Generalized Cartan decomposition.

In this section, we give a proof of the implication (ii) $\Rightarrow$ (i) in Theorem 1.1. The idea is to use the herringbone stitch method that reduces unknown decompositions for non-symmetric pairs to the known Cartan decomposition for symmetric pairs. For this, we divide the proof to six cases (Subsections 3.1–3.6).

3.1. Decomposition for the symmetric case (Case I-1).

In this subsection we recall a well-known fact on the Cartan decomposition for the symmetric case ([**Ho**, Theorem 6.10], [**Ma3**, Theorem 1]) and deal with Case I with $i, j \in \{n-1, n\}$.

Fact 3.1. *Let K be a connected compact Lie group with Lie algebra $\mathfrak{k}$ and two involutions τ, τ' $(\tau^2 = (\tau')^2 = \mathrm{id})$. Let H and H' be subgroups of K such that*

$$(K^{\tau})_0 \subset H \subset K^{\tau} \quad \text{and} \quad (K^{\tau'})_0 \subset H' \subset K^{\tau'}.$$

We take a maximal abelian subspace $\mathfrak{b}$ in

$$\mathfrak{k}^{-\tau,-\tau'} := \{X \in \mathfrak{k} : \tau(X) = \tau'(X) = -X\}$$

and write B for the connected abelian subgroup with Lie algebra $\mathfrak{b}$.

Suppose that $\tau\tau'$ is semisimple on the center $\mathfrak{z}$ of $\mathfrak{k}$. Then we have

$$K = HBH'.$$

We shall apply Fact 3.1 to Case I with $i, j \in \{n-1, n\}$ in Theorem 1.1. Let

$$(\Pi')^c = \Pi \setminus \Pi' = \{\alpha_n\}, \quad (\Pi'')^c = \Pi \setminus \Pi'' = \{\alpha_{n-1}\}. \tag{3.1.1}$$

(See Diagram 1.1 for the label of the Dynkin diagram.) Then, both $(G, \mathrm{L}_{\Pi'})$ and $(G, \mathrm{L}_{\Pi''})$ are symmetric pairs with μ and $\mu^{\xi} = \xi \circ \mu \circ \xi$ the corresponding involutions respectively (see (2.2.5) and (2.2.7) for the definitions of μ and ξ). We take maximal abelian subspaces $\mathfrak{b} \subset \mathfrak{g}^{-\mu}$ and $\mathfrak{b}' \subset \mathfrak{g}^{-\mu,-\mu^{\xi}}$ as follows:

$$\begin{aligned} \mathfrak{b} &:= \bigoplus_{1 \le i \le [n/2]} \mathbb{R}(\mathrm{E}_{2i-1,2n-2i+1} - \mathrm{E}_{2i,2n-2i+2} - \mathrm{E}_{2n-2i+1,2i-1} + \mathrm{E}_{2n-2i+2,2i}), \\ \mathfrak{b}' &:= \mathfrak{b} \cap \xi(\mathfrak{b}). \end{aligned} \tag{3.1.2}$$

We note that both $\mathfrak{b}$ and $\mathfrak{b}'$ are contained in $\mathfrak{g}^{\sigma}$ where σ is the complex conjugation (2.2.2). Using Fact 3.1, we obtain the following proposition.

Proposition 3.2 (Generalized Cartan decomposition). *Let $G = \mathrm{SO}(2n)$ and $\mathrm{L}_{\Pi'}$, $\mathrm{L}_{\Pi''}$ be as in* (3.1.1)*, and define $B := \exp(\mathfrak{b})$, $B' := \exp(\mathfrak{b}')$ for $\mathfrak{b}$, $\mathfrak{b}'$ as in* (3.1.2)*. Then we have the following three decompositions of G.*

$$\begin{aligned} G &= \mathrm{L}_{\Pi'}\, B\, \mathrm{L}_{\Pi'} \\ &= \mathrm{L}_{\Pi''}\, \xi(B)\, \mathrm{L}_{\Pi''} \\ &= \mathrm{L}_{\Pi'}\, B'\, \mathrm{L}_{\Pi''}\, . \end{aligned}$$

3.2. Decomposition for Case I-2.

In this subsection, we deal with the following case:

$$(\Pi')^c = \{\alpha_i\}, \quad (\Pi'')^c = \{\alpha_3\} \ (i = n-1 \text{ or } n).$$

Since ξ switches the role of $n-1$ and n, $G = \mathrm{L}_{\Pi'}\, G^\sigma\, \mathrm{L}_{\Pi''}$ holds for $i = n$ if and only if so does for $i = n-1$ (see (2.2.7) for the definition of ξ). Thus, we may and do assume $i = n$ without loss of generality, and put

$$\begin{aligned} L &:= \mathrm{L}_{\{\alpha_n\}^c} (= \mathrm{U}(n)), \\ H &:= \mathrm{L}_{\{\alpha_3\}^c} (= \mathrm{U}(3) \times \mathrm{SO}(2n-6)), \end{aligned} \tag{3.2.1}$$

for simplicity. We also note that the equality $G = LG^\sigma H$ follows for $n = 4$ from Case II in Theorem 1.1. (See Subsection 3.3.)

First, let us take a symmetric subgroup $G'G'' = (G^{\tau_6})_0$ containing H where $G' := \mathrm{SO}(6) \times I_{2n-6}$ and $G'' := I_6 \times \mathrm{SO}(2n-6) (\subset H)$ (see (2.2.8) for the definition of τ_6). We define a maximal abelian subspace $\mathfrak{b}'$ of $\mathfrak{g}^{-\tau_6,-\mu}$ by

$$\mathfrak{b}' := \begin{cases} \displaystyle\bigoplus_{1 \le j \le 3} \mathbb{R}(\mathrm{E}_{j,n+j} - \mathrm{E}_{n+j,j} - \mathrm{E}_{n+1-j,2n+1-j} + \mathrm{E}_{2n+1-j,n+1-j}) & (n \ge 6), \\ \displaystyle\bigoplus_{1 \le j \le 2} \mathbb{R}(\mathrm{E}_{j,n+j} - \mathrm{E}_{n+j,j} - \mathrm{E}_{n+1-j,2n+1-j} + \mathrm{E}_{2n+1-j,n+1-j}) & (n = 5). \end{cases} \tag{3.2.2}$$

Then we give a decomposition of G by using Fact 3.1 as follows.

$$G = L \exp(\mathfrak{b}')(G'G''). \tag{3.2.3}$$

Second, we consider the centralizer of $\mathfrak{b}'$. We define an abelian subgroup T'' by $T'' := \exp(\mathfrak{t}'')$ where

$$\mathfrak{t}'' := \begin{cases} \bigoplus_{1\le i\le 3} \mathbb{R}\sqrt{-1}(\mathrm{E}_{i,i} - \mathrm{E}_{2n+1-i,2n+1-i} - \mathrm{E}_{n+1-i,n+1-i} + \mathrm{E}_{n+i,n+i}) & (n \ge 6), \\ \bigoplus_{1\le i\le 2} \mathbb{R}\sqrt{-1}(\mathrm{E}_{i,i} - \mathrm{E}_{11-i,11-i} - \mathrm{E}_{6-i,6-i} + \mathrm{E}_{5+i,5+i}) \\ \qquad\qquad \oplus \mathbb{R}\sqrt{-1}(\mathrm{E}_{3,3} - \mathrm{E}_{8,8}) & (n = 5). \end{cases}$$

A simple matrix computation shows that $\mathfrak{b}'$ commutes with $\mathfrak{t}''$.

Lemma 3.3. *$Z_G(\mathfrak{b}') \supset T''$.*

Third, we consider the double coset decomposition of G' by $(G')^{\mu}$ and a maximal torus $T' := G' \cap \exp(\mathfrak{t})$ of G', which consists of diagonal matrices. For this, we decompose the Lie algebra $\mathfrak{g}'$ of G' as follows.

$$\mathfrak{g}' = (\mathfrak{g}')^{\mu} \oplus (\mathfrak{g}')^{-\mu}.$$

It is easy to see that $(\mathfrak{g}')^{-\mu}$ is rewritten as

$$(\mathfrak{g}')^{-\mu} = \bigcup_{g \in T'} \mathrm{Ad}(g)(\mathfrak{g}')^{-\mu,\sigma}.$$

Then we can find that the exponential mapping

$$\exp : \bigcup_{g \in T'} \mathrm{Ad}(g)(\mathfrak{g}^{-\mu,\sigma}) \to G'/(G')^{\mu}$$

is surjective. Thus we have

$$G' = T' \exp(\mathfrak{g}^{-\mu,\sigma})(G')^{\mu}. \tag{3.2.4}$$

We are ready to give a proof of a generalized Cartan decomposition for Case I with $(i, j) = (n, 3)$.

Proposition 3.4 (Generalized Cartan decomposition). *Let $G = \mathrm{SO}(2n)$ and L, H be as in* (3.2.1). *We set $B := \exp(\mathfrak{b}')\exp(\mathfrak{g}^{-\mu,\sigma})$ (see* (3.2.2) *for the definition of $\mathfrak{b}'$). Then we have*

$$G = LBH.$$

Proof. In the following proof, we use the herringbone stitch method introduced by Kobayashi ([**Ko4**]).

$$
\begin{aligned}
G &= L\exp(\mathfrak{b}')(G'G'') && \text{by (3.2.3)}\\
&= L\exp(\mathfrak{b}')T'\exp(\mathfrak{g}^{-\mu,\sigma})(G')^{\mu}G'' && \text{by (3.2.4)}\\
&= L\exp(\mathfrak{b}')T'\exp(\mathfrak{g}^{-\mu,\sigma})H && \text{by } (G')^{\mu}G'' = H. \qquad (3.2.5)
\end{aligned}
$$

Since T' and T'' satisfy $T'\exp(\mathfrak{g}^{-\mu,\sigma})H = T''\exp(\mathfrak{g}^{-\mu,\sigma})H$, we can continue the decomposition (3.2.5) as follows.

$$
\begin{aligned}
(3.2.5) &= L\exp(\mathfrak{b}')T''\exp(\mathfrak{g}^{-\mu,\sigma})H\\
&= LT''\exp(\mathfrak{b}')\exp(\mathfrak{g}^{-\mu,\sigma})H && \text{by Lemma 3.3}\\
&= L\exp(\mathfrak{b}')\exp(\mathfrak{g}^{-\mu,\sigma})H\\
&= LBH.
\end{aligned}
$$

□

Here is a herringbone stitch which we have used for $L\backslash G/H$ in Case I with $i = n, j = 3$.

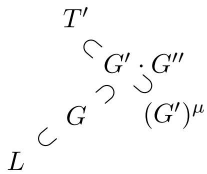

Figure 3.2.

3.3. Decomposition for Case II.

In this subsection we deal with the following case:

$$(\Pi')^c = \{\alpha_i\},\quad (\Pi'')^c = \{\alpha_j, \alpha_k\}\ (i \in \{n-1, n\},\ j \neq k \text{ and } j, k \in \{1, n-1, n\}).$$

Since ξ (see (2.2.7) for the definition of ξ) switches the role of $n-1$ and n, and $\mathrm{L}_{\{\alpha_1,\alpha_n\}^c}$ is conjugate to $\mathrm{L}_{\{\alpha_{n-1},\alpha_n\}^c}$ by an element of G^{σ} where σ is the complex conjugation (2.2.2), $G = \mathrm{L}_{\Pi'}\, G^{\sigma}\, \mathrm{L}_{\Pi''}$ holds for $(i,j,k) = (n,1,n)$ if and only if so does for each of the other triples (i,j,k). Thus, we may and do assume $(i,j,k) = (n,1,n)$ without any loss of generality, and put

$$
\begin{aligned}
L &:= \mathrm{L}_{\{\alpha_n\}^c}(= \mathrm{U}(n)), \qquad (3.3.1)\\
H &:= \mathrm{L}_{\{\alpha_1,\alpha_n\}^c}(= \mathrm{U}(1)\times \mathrm{U}(n-1)),
\end{aligned}
$$

for simplicity. The goal of this subsection is to prove

$$G = L \exp(\mathfrak{b}') DH, \tag{3.3.2}$$

where the subspace $\mathfrak{b}'$ and the subset D are defined by

$$\mathfrak{b}' := \bigoplus_{1 \leq i \leq [n/2]} \mathbb{R}(\mathrm{E}_{2i-1,2n-2i+1} - \mathrm{E}_{2i,2n-2i+2} - \mathrm{E}_{2n-2i+1,2i-1} + \mathrm{E}_{2n-2i+2,2i}), \tag{3.3.3}$$

$$D := D_1 D_2 \cdots D_{[(n-1)/2]} \tag{3.3.4}$$

for $D_j := \exp(\mathbb{R}(\mathrm{E}_{2j-1,2j+1} - \mathrm{E}_{2j+1,\ 2j-1} - \mathrm{E}_{2n-2j,2n-2j+2} + \mathrm{E}_{2n-2j+2,2n-2j}))$ $(1 \leq j \leq [(n-1)/2])$. This subspace $\mathfrak{b}'$ is a maximal abelian subspace of $\mathfrak{g}^{-\mu}$.

As the first step to the goal, we use Proposition 3.2 and then obtain

$$G = L \exp(\mathfrak{b}') L. \tag{3.3.5}$$

Second, we consider the centralizer of $\mathfrak{b}'$. We omit details of the proof of the following lemma since it follows from a simple matrix computation.

Lemma 3.5. $Z_G(\mathfrak{b}') \supset K := \begin{cases} (\mathrm{SU}(2))^m & (n = 2m), \\ (\mathrm{SU}(2))^m \times \mathrm{U}(1) & (n = 2m+1). \end{cases}$
Here, we realize the subgroup K as block diagonal matrices in G.

Third, we consider the double coset decomposition of L by K and H.

Lemma 3.6. *$L = KDH$.*

Proof. The following proof is due to [**Sa3**]. Let us identify L/H with $\mathbb{C}P^n$ in the natural way. Here, we note that $D \cdot H/H$ is identified with a subset

$$\left\{ [z_1 : \cdots : z_n] \in \mathbb{C}P^n : z_k \in \mathbb{R}\ (1 \leq k \leq n) \text{ and } z_{2l} = 0 \left(1 \leq l \leq \left[\frac{n}{2} \right] \right) \right\}$$

of $\mathbb{C}P^n$. We shall show the equality $K \cdot D \cdot H/H = L/H$ for two cases $n = 2m$ and $n = 2m+1$ separately.

- Case 1: $n = 2m$. Since the SU(2)-action on S^3 is transitive, for any $[z_1 : \cdots : z_{2m}] \in \mathbb{C}P^n$, there exists $g = (g_1, \ldots, g_m) \in K$ such that

$$\begin{aligned} g \cdot [z_1 : \cdots : z_{2m}] &= [g_1 \cdot (z_1 : z_2) : \cdots : g_m \cdot (z_{2m-1} : z_{2m})] \\ &= \left[\left(\sqrt{|z_1|^2 + |z_2|^2} : 0\right) : \cdots : \left(\sqrt{|z_{2m-1}|^2 + |z_{2m}|^2} : 0\right)\right] \\ &\in D \cdot H/H. \end{aligned}$$

Thus, we obtain $K \cdot D \cdot H/H = L/H$.

- Case 2: $n = 2m + 1$. As similar to the case $n = 2m$, for any $[z_1 : \cdots : z_{2m} : z_{2m+1}] \in \mathbb{C}P^n$, we can find an element $h = (h_1, \ldots, h_m)$ of the commutator subgroup $K_{ss} = [K, K]$ satisfying

$$h \cdot [z_1 : \cdots : z_{2m}] = \left[\left(\sqrt{|z_1|^2 + |z_2|^2} : 0\right) : \cdots : \left(\sqrt{|z_{2m-1}|^2 + |z_{2m}|^2} : 0\right)\right].$$

We then put $\theta := arg(z_{2m+1})$ and $g := (h, e^{-\sqrt{-1}\theta}) \in K$, and obtain

$$\begin{aligned} & g \cdot [z_1 : \cdots : z_{2m} : z_{2m+1}] \\ &= \left[\left(\sqrt{|z_1|^2 + |z_2|^2} : 0\right) : \cdots : \left(\sqrt{|z_{2m-1}|^2 + |z_{2m}|^2} : 0\right) : |z_{2m+1}|\right] \\ &\quad \in D \cdot H/H. \end{aligned}$$

Hence we have $K \cdot D \cdot H/H = L/H$. □

We are ready to give a proof of a generalized Cartan decomposition (3.3.2).

Proposition 3.7 (Generalized Cartan decomposition). *Let* $G = \mathrm{SO}(2n)$ *and* L*,* H *be as in* (3.3.1). *We put* $B := \exp(\mathfrak{b}')D$ (*see* (3.3.3) *and* (3.3.4) *for the definitions of* $\mathfrak{b}'$ *and* D). *Then we have* $G = LBH$.

Proof.

$$\begin{aligned} G &= L\exp(\mathfrak{b}')L && \text{by (3.3.5)} \\ &= L\exp(\mathfrak{b}')KDH && \text{by Lemma 3.6} \\ &= LK\exp(\mathfrak{b}')DH && \text{by Lemma 3.5} \\ &= LBH. \end{aligned}$$

□

Here is a herringbone stitch which we have used for $L\backslash G/H$ in Case II with $(i, j, k) = (n, 1, n)$.

$$\begin{array}{ccccccc}
 & & K & & & & \\
 & & & \subset & & & \\
 & & & & L & & \\
 & & & \supset & & \supset & \\
 & & G & & & & H \\
 & \subset & & & & & \\
L & & & & & &
\end{array}$$

Figure 3.3.

3.4. Decomposition for Case III.

In this subsection we deal with the following case:

$$(\Pi')^c = \{\alpha_i\}, \quad (\Pi'')^c = \{\alpha_1, \alpha_2\} \ (i = n-1 \text{ or } n).$$

As in the beginning of Subsection 3.2, we may and do assume $i = n$ without loss of generality, and put

$$\begin{aligned} L &:= \mathrm{L}_{\{\alpha_n\}^c} (= \mathrm{U}(n)), \\ H &:= \mathrm{L}_{\{\alpha_1,\alpha_2\}^c} (= \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{SO}(2n-4)), \end{aligned} \tag{3.4.1}$$

for simplicity. This subsection aims for showing

$$G = L \exp(\mathfrak{b}') \exp(\mathfrak{b}'') H, \tag{3.4.2}$$

where the subspaces $\mathfrak{b}'$ and $\mathfrak{b}''$ are defined by

$$\mathfrak{b}' := \bigoplus_{i=1,2} \mathbb{R}\,(\mathrm{E}_{i,n+i} - \mathrm{E}_{n+i,i} - \mathrm{E}_{n+1-i,2n+1-i} + \mathrm{E}_{2n+1-i,n+1-i}), \tag{3.4.3}$$

$$\begin{aligned} \mathfrak{b}'' := {} & \mathbb{R}(\mathrm{E}_{1,2} - \mathrm{E}_{2,1} - \mathrm{E}_{2n-1,2n} + \mathrm{E}_{2n,2n-1}) \\ & \oplus \mathbb{R}(\mathrm{E}_{1,2n-1} - \mathrm{E}_{2n-1,1} - \mathrm{E}_{2,2n} + \mathrm{E}_{2n,2}). \end{aligned} \tag{3.4.4}$$

First, we take a symmetric subgroup $(G^{\tau_4})_0 = G'G''$ containing H where $G' := \mathrm{SO}(4) \times I_{2n-4}$ and $G'' := I_4 \times \mathrm{SO}(2n-4) (\subset H)$. In light that $\mathfrak{b}'$ is a maximal abelian subspace of $\mathfrak{g}^{-\tau_4,-\mu}$, we see from Fact 3.1 that

$$G = L \exp(\mathfrak{b}')(G'G''). \tag{3.4.5}$$

Next we consider the double coset decomposition of G' by a symmetric subgroup T' defined by $T' := (G')^{\tau_1}_0$. The point here is that T' satisfies $T'G'' = H$. Applying Fact 3.1 to $(G', \tau_1|_{G'}, \tau_1|_{G'})$, we have

$$G' = T' \exp(\mathfrak{b}'')T'. \tag{3.4.6}$$

We are ready to give a proof of a generalized Cartan decomposition (3.4.2) by using the herringbone stitch method.

Proposition 3.8 (Generalized Cartan decomposition). *Let $G = \mathrm{SO}(2n)$ and L, H be as in* (3.4.1). *We put $B := \exp(\mathfrak{b}')\exp(\mathfrak{b}'')$ (see* (3.4.3) *and* (3.4.4) *for the definitions of $\mathfrak{b}'$ and $\mathfrak{b}''$). Then we have $G = LBH$.*

Proof.

$$\begin{aligned} G &= L\exp(\mathfrak{b}')(G'G'') && \text{by (3.4.5)} \\ &= L\exp(\mathfrak{b}')(T'\exp(\mathfrak{b}'')T')G'' && \text{by (3.4.6)} \\ &= L\exp(\mathfrak{b}')T'\exp(\mathfrak{b}'')H && \text{by } T'G'' = H. \end{aligned} \tag{3.4.7}$$

We define

$$T'' := \exp\left(\bigoplus_{i=1,2} \mathbb{R}\sqrt{-1}((\mathrm{E}_{i,i} - \mathrm{E}_{2n+1-i,2n+1-i}) - (\mathrm{E}_{n+1-i,n+1-i} - \mathrm{E}_{n+i,n+i}))\right).$$

Then T' and T'' satisfy the following equality:

$$T'\exp(\mathfrak{b}'')H = T''\exp(\mathfrak{b}'')H,$$

and T'' centralizes $\mathfrak{b}'$. From this, we can continue the decomposition as follows.

$$\begin{aligned} (3.4.7) &= L\exp(\mathfrak{b}')T''\exp(\mathfrak{b}'')H \\ &= LT''\exp(\mathfrak{b}')\exp(\mathfrak{b}'')H \\ &= LBH. \end{aligned}$$

□

Here is a herringbone stitch which we have used for $L\backslash G/H$ in Case III with $i = n$.

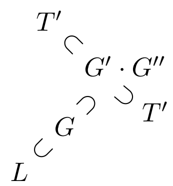

Figure 3.4.

3.5. Decomposition for Case IV.

In this subsection we deal with the following case:

$$(\Pi')^c = \{\alpha_1\}, \quad (\Pi'')^c = \{\alpha_j, \alpha_k\} \ (1 \leq j \leq n \text{ and } k = n-1 \text{ or } n).$$

As in the beginning of Subsection 3.2, we may and do assume $k = n$ without loss of generality, and put

$$\begin{aligned} L &:= \mathrm{L}_{\{\alpha_1\}^c}(= \mathrm{U}(1) \times \mathrm{SO}(2n-2)), \\ H &:= \mathrm{L}_{\{\alpha_j, \alpha_n\}^c}(= \mathrm{U}(j) \times \mathrm{U}(n-j)), \end{aligned} \tag{3.5.1}$$

for simplicity. The goal of this subsection is to prove

$$G = L \exp(\mathfrak{b}') \exp(\mathfrak{b}'') H, \tag{3.5.2}$$

where the subspaces $\mathfrak{b}'$ and $\mathfrak{b}''$ are defined by

$$\mathfrak{b}' := \bigoplus_{i=1,2} \mathbb{R}(\mathrm{E}_{1,n+i-1} - \mathrm{E}_{n+i-2,1} - \mathrm{E}_{n+2-i,2n} + \mathrm{E}_{2n,n+2-i}), \tag{3.5.3}$$

$$\begin{aligned} \mathfrak{b}'' := {} & \mathbb{R}(\mathrm{E}_{1,2n+1-j} - \mathrm{E}_{2n+1-j,1} - \mathrm{E}_{j,2n} + \mathrm{E}_{2n,j}) \\ & \oplus \mathbb{R}(\mathrm{E}_{j+1,n+1} - \mathrm{E}_{n+1,j+1} - \mathrm{E}_{n,2n-j} + \mathrm{E}_{2n-j,n}). \end{aligned} \tag{3.5.4}$$

Then $\mathfrak{b}'$ and $\mathfrak{b}''$ are maximal abelian subspaces of $\mathfrak{g}^{-\tau_1, -\tau_j}$ and $(\mathfrak{g}^{\tau_j})^{-(\tau_1\tau_{n-1}), -\mu}$ respectively. We apply Fact 3.1 to (G, τ_1, τ_j), and then obtain

$$G = L \exp(\mathfrak{b}')(G^{\tau_j})_0. \tag{3.5.5}$$

Next we consider the double coset decomposition of $(G^{\tau_j})_0$ by H and a subgroup L' of $G^{\tau_j} \cap L$ given by

$$L' := I_2 \times \mathrm{SO}(2j-2) \times \mathrm{SO}(2(n-j)-2) \times I_2$$

$$= \left\{ \begin{pmatrix} 1 & & & & & & & 0 \\ & A & & & & & B & \\ & & E & & & F & & \\ & & & 1 & 0 & & & \\ & & & 0 & 1 & & & \\ & & G & & & H & & \\ & C & & & & & D & \\ 0 & & & & & & & 1 \end{pmatrix} : \begin{array}{l} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SO}(2j-2), \\ \begin{pmatrix} E & F \\ G & H \end{pmatrix} \in \mathrm{SO}(2(n-j)-2) \end{array} \right\}.$$

The point here is that L' centralizes $\mathfrak{b}'$. Applying Fact 3.1 to $((G^{\tau_j})_0, \tau_1\tau_{n-1}, \mu)$, we get

$$(G^{\tau_j})_0 = (G^{\tau_j,(\tau_1\tau_{n-1})})_0 \exp(\mathfrak{b}'')G^{\tau_j,\mu}. \tag{3.5.6}$$

Further, it is easy to see $(G^{\tau_j,(\tau_1\tau_{n-1})})_0 \exp(\mathfrak{b}'')G^{\tau_j,\mu} = L' \exp(\mathfrak{b}'')G^{\tau_j,\mu}$. Thus we have

$$(G^{\tau_j})_0 = L' \exp(\mathfrak{b}'')G^{\tau_j,\mu}. \tag{3.5.7}$$

We are ready to give a proof of a generalized Cartan decomposition (3.5.2) by using the herringbone stitch method.

PROPOSITION 3.9 (Generalized Cartan decomposition). *Let $G = \mathrm{SO}(2n)$ and L, H be as in* (3.5.1), *and put $B := \exp(\mathfrak{b}')\exp(\mathfrak{b}'')$* (*see* (3.5.3) *and* (3.5.4) *for the definitions of $\mathfrak{b}'$ and $\mathfrak{b}''$*). *Then we have $G = LBH$.*

PROOF.

$$\begin{aligned} G &= L\exp(\mathfrak{b}')(G^{\tau_j})_0 && \text{by (3.5.5)} \\ &= L\exp(\mathfrak{b}')(L'\exp(\mathfrak{b}'')G^{\tau_j,\mu}) && \text{by (3.5.7)} \\ &= L\exp(\mathfrak{b}')(L'\exp(\mathfrak{b}'')H) && \text{by } G^{\tau_j,\mu} = H \\ &= LL'\exp(\mathfrak{b}')\exp(\mathfrak{b}'')H && \text{by } L' \subset Z_G(\mathfrak{b}') \\ &= LBH. \end{aligned}$$

□

Here is a herringbone stitch which we have used for $L\backslash G/H$ in Case IV with $k = n$.

$$\begin{array}{ccccc} L' & & & & \\ & \subset & & & \\ & & (G^{\tau_j})_0 & & \\ & \supset & & \supset & \\ & G & & & H \\ \subset & & & & \\ L & & & & \end{array}$$

Figure 3.5.

3.6. Decomposition for Case V.
In this subsection, we deal with the following case for $G = \mathrm{SO}(8)$:

$$(\Pi')^c = \{\alpha_i\}, \quad (\Pi'')^c = \{\alpha_2, \alpha_j\} \ ((i, j) = (3, 4) \text{ or } (4, 3)).$$

We may assume $(i,j)=(4,3)$ without any loss of generality since $\xi(\mathrm{L}_{\{\alpha_4\}^c})=\mathrm{L}_{\{\alpha_3\}^c}$ and $\xi(\mathrm{L}_{\{\alpha_2,\alpha_3\}^c})=\mathrm{L}_{\{\alpha_2,\alpha_4\}^c}$. For simplicity, we put

$$\begin{aligned} L &:= \mathrm{L}_{\{\alpha_4\}^c}(=\mathrm{U}(4)), \qquad (3.6.1)\\ H &:= \mathrm{L}_{\{\alpha_2,\alpha_3\}^c}(=\xi(\mathrm{U}(2)\times\mathrm{U}(2))). \end{aligned}$$

The goal of this subsection is to prove

$$G = L\exp(\mathfrak{a})\xi(B''B')H, \tag{3.6.2}$$

where the subspace $\mathfrak{a}$ and the subgroups B', B'' are defined by

$$\mathfrak{a} := \mathbb{R}(\mathrm{E}_{1,7}-\mathrm{E}_{2,8}-\mathrm{E}_{7,1}+\mathrm{E}_{8,2}), \tag{3.6.3}$$

$$B' := \exp(\mathbb{R}(\mathrm{E}_{1,4}-\mathrm{E}_{4,1}-\mathrm{E}_{5,8}+\mathrm{E}_{8,5})\oplus\mathbb{R}(\mathrm{E}_{2,3}-\mathrm{E}_{3,2}-\mathrm{E}_{6,7}+\mathrm{E}_{7,6})), \tag{3.6.4}$$

$$B'' := \exp(\mathbb{R}(\mathrm{E}_{1,3}-\mathrm{E}_{3,1}-\mathrm{E}_{6,8}+\mathrm{E}_{8,6})). \tag{3.6.5}$$

Then $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{g}^{-\mu,-\mu^{\xi}}$.

First, we decompose G by using Proposition 3.2 as follows.

$$G = L\exp(\mathfrak{a})\xi(L). \tag{3.6.6}$$

Next, we recall a generalized Cartan decomposition for type A group ([**Ko4**, Theorem 3.1]). We set $H' := \mathrm{SU}(2)\times\mathrm{U}(1)\times\mathrm{U}(1)\subset L$ which is realized as block diagonal matrices and $T := \left\{\begin{pmatrix} e^{\sqrt{-1}\theta}I_4 & O\\ O & e^{-\sqrt{-1}\theta}I_4\end{pmatrix}\in L : \theta\in\mathbb{R}\right\}$. Then we have

Lemma 3.10 ([**Ko4**, Theorem 3.1]). $L=(H'T)B''B'\xi(H)$.

Further, we can see that $L=(H'T)B''B'\xi(H)=H'B''B'\xi(H)$ since T is the center of L, and thus we have the following decomposition of $\xi(L)$.

$$\xi(L) = H'\xi(B''B')H. \tag{3.6.7}$$

Here, we note $\xi(H')=H'$.

We are ready to give a proof of a generalized Cartan decomposition (3.6.2).

Proposition 3.11 (Generalized Cartan decomposition). *Let $G=\mathrm{SO}(8)$ and L,H be as in* (3.6.1). *Put $B := \exp(\mathfrak{a})\xi(B''B')$ (see* (3.6.3), (3.6.4) *and* (3.6.5) *for the definitions of $\mathfrak{a}$, B' and B''). Then we have $G=LBH$.*

PROOF.

$$\begin{aligned} G &= L\exp(\mathfrak{a})\xi(L) && \text{by (3.6.6)} \\ &= L\exp(\mathfrak{a})(H'\xi(B''B')H) && \text{by (3.6.7)} \\ &= LH'\exp(\mathfrak{a})\xi(B''B')H && \text{by } H' \subset Z_G(\mathfrak{a}) \\ &= LBH. \end{aligned}$$

□

Here is a herringbone stitch which we have used for $L\backslash G/H$ in Case V.

$$\begin{array}{ccccccc} & & H' & & & & \\ & & & \subset & & & \\ & & & & \xi(L) & & \\ & & & \supset & & \supset & \\ & & G & & & & H \\ & \subset & & & & & \\ L & & & & & & \end{array}$$

Figure 3.6.

4. Application of invariant theory.

In this section, we prove that (i) implies (ii) in Theorem 1.1. The idea of the proof is to use invariants of quivers. Although Lemmas 4.1, 4.2 and 4.3 are parallel to [**Ko4**, Lemmas 6.1, 6.2 and 6.3] respectively, we give proofs of these lemmas for the sake of completeness. This section could be read independently of Section 3 which gives a proof on the opposite implication of (ii) $\Rightarrow$ (i) in Theorem 1.1.

4.1. Invariants of quivers.

Let $\sigma : \mathrm{M}(N,\mathbb{C}) \to \mathrm{M}(N,\mathbb{C})$ be the complex conjugation with respect to $\mathrm{M}(N,\mathbb{R})$.

LEMMA 4.1 (c.f. [**Ko4**, Lemma 6.1]). *Let $G \subset \mathrm{GL}(N,\mathbb{C})$ be a σ-stable subgroup, $R \in \mathrm{M}(N,\mathbb{R})$, and L a subgroup of G. If there exists $g \in G$ such that*

$$\mathrm{Ad}(L)(\mathrm{Ad}(g)R) \cap \mathrm{M}(N,\mathbb{R}) = \emptyset, \tag{4.1.1}$$

then $G \neq LG^{\sigma}G_R$. Here $G_R := \{h \in G : hRh^{-1} = R\}$.

PROOF. Let us observe that $\mathrm{Ad}(G^{\sigma}G_R)R = \mathrm{Ad}(G^{\sigma})R \subset \mathrm{M}(N,\mathbb{R})$. Then, the condition (4.1.1) implies $\mathrm{Ad}(Lg)R \cap \mathrm{Ad}(G^{\sigma}G_R)R = \emptyset$, and thus $Lg \cap G^{\sigma}G_R = \emptyset$. Therefore we have $g \notin LG^{\sigma}G_R$. □

We return to the case $G = \mathrm{SO}(2n)$. Let $k, r \geq 2$ be integers. We fix a partition $n = n_1 + \cdots + n_k$ of a positive integer n with $n_1, \ldots, n_{k-1} > 0$ and $n_k \geq 0$, and consider a loop $i_0 \to i_1 \to \cdots \to i_r$ such that

$$i_s \in \begin{cases} \{1, \ldots, 2k-1\} & (n_k \neq 0), \\ \{1, \ldots, k-1, k+1, \ldots, 2k-1\} & (n_k = 0), \end{cases}$$

and $i_0 = i_r$, $i_{s-1} \neq i_s$ $(1 \leq s \leq r)$. Correspondingly to this loop, we define a non-linear mapping

$$A_{i_0 \cdots i_r} : \mathrm{M}(2n, \mathbb{C}) \to \begin{cases} \mathrm{M}(n_{i_0}, \mathbb{C}) & (i_0 = i_r \neq k) \\ \mathrm{M}(2n_k, \mathbb{C}) & (i_0 = i_r = k) \end{cases}$$

as follows: Let $P \in \mathrm{M}(2n, \mathbb{C})$, and we write P as $(P_{ij})_{1 \leq i,j \leq 2k-1}$ in the block matrix form corresponding to the partition $2n = n_1 + \cdots + n_{k-1} + 2n_k + n_{k-1} + \cdots + n_1$ of $2n$ such that

$$P_{ij} \in \begin{cases} \mathrm{M}(n_i, n_j; \mathbb{C}) & (i, j \neq k), \\ \mathrm{M}(2n_k, n_j; \mathbb{C}) & (i = k, j \neq k), \\ \mathrm{M}(n_i, 2n_k; \mathbb{C}) & (i \neq k, j = k), \\ \mathrm{M}(2n_k, \mathbb{C}) & (i = j = k), \end{cases} \tag{4.1.2}$$

where $n_{2k-i} := n_i$ $(1 \leq i \leq k)$. We define $\tilde{P}_{ij}$ and $A_{i_0 \cdots i_r}(P)$ by

$$\tilde{P}_{ij} := \begin{cases} P_{ij} & (i + j \leq 2k), \\ J_{n_i} {}^t P_{2k-j, 2k-i} J_{n_j} & (i + j > 2k,\ i, j \neq k), \\ J_{2n_k} {}^t P_{2k-j, k} J_{n_j} & (i = k, j > k), \\ J_{n_i} {}^t P_{k, 2k-i} J_{2n_k} & (i > k, j = k). \end{cases}$$

$$A_{i_0 \cdots i_r}(P) := \tilde{P}_{i_0 i_1} \tilde{P}_{i_1 i_2} \cdots \tilde{P}_{i_{r-1} i_r}.$$

Then for any $l = (l_1, \ldots, l_{k-1}, l_k) \in L := \mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_{k-1}) \times \mathrm{SO}(2n_k)$ (see (2.2.3) in Section 2 for the realization as matrices), a direct computation shows

$$(\widetilde{\mathrm{Ad}(l)P})_{ij} = l_i \tilde{P}_{ij} l_j^{-1} \tag{4.1.3}$$

where $l_s \in \mathrm{U}(n_s)$ $(1 \leq s \leq k-1)$, $l_k \in \mathrm{SO}(2n_k)$. The equation (4.1.3) leads us to the following lemma (c.f. [**Ko4**, Lemma 6.2]):

LEMMA 4.2. *If there exists a loop $i_0 \to i_1 \to \cdots \to i_r$ such that at least one of the coefficients of the characteristic polynomial* $\det(\lambda I_{n_{i_0}} - A_{i_0 \cdots i_r}(P))$ *is not real, then*

$$\mathrm{Ad}(L)P \cap \mathrm{M}(2n, \mathbb{R}) = \emptyset.$$

PROOF. From (4.1.3), we can see that the characteristic polynomial of $A_{i_0 \cdots i_r}(P)$ is invariant under the conjugation by L. Therefore if there exists $l \in L$ such that $\mathrm{Ad}(l)P \in \mathrm{M}(2n, \mathbb{R})$ and thus the characteristic polynomial of $A_{i_0 \cdots i_r}(\mathrm{Ad}(l)P)$ is real, then that of $A_{i_0 \cdots i_r}(P)$ is also a real polynomial. By contraposition, our lemma holds. $\square$

By using Lemmas 4.1 and 4.2, we obtain the next lemma (c.f. [**Ko4**, Lemma 6.3]):

LEMMA 4.3. *Let $n = n_1 + \cdots + n_k$ be a partition and $L = \mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_{k-1}) \times \mathrm{SO}(2n_k)$ the corresponding Levi subgroup of $\mathrm{SO}(2n)$. Let us suppose that R is a block diagonal matrix*:

$$R := \begin{pmatrix} R_1 & & & \\ & R_2 & & \\ & & \ddots & \\ & & & R_{2k-1} \end{pmatrix},$$

where R_s, $R_{2k-s} \in \mathrm{M}(n_s, \mathbb{R})$ $(1 \le s \le k-1)$, and $R_k \in \mathrm{M}(2n_k, \mathbb{R})$ (the last condition makes sense when $n_k \neq 0$).

If there exist $X \in \mathfrak{so}(2n)$ and a loop $i_0 \to \cdots \to i_r$ such that

$$\det\left(\lambda I_{n_{i_0}} - A_{i_0 \cdots i_r}([X, R])\right) \notin \mathbb{R}[\lambda],$$

then the multiplication map $L \times G^\sigma \times G_R \to G$ is not surjective. Here, $[X, R] := XR - RX$.

PROOF. Let us set $P(\varepsilon) := \mathrm{Ad}(\exp(\varepsilon X))R$. It suffices to show

$$\det\left(\lambda I_{n_{i_0}} - A_{i_0 \cdots i_r}(P(\varepsilon))\right) \notin \mathbb{R}[\lambda]$$

for some $\varepsilon > 0$. We set $Q := [X, R]$. The matrix $P(\varepsilon)$ depends real analytically on ε, and we have

$$P(\varepsilon) = R + \varepsilon Q + O(\varepsilon^2),$$

as ε tends to 0. In particular, if $i \neq j$ then the (i,j)-block of matrix $P_{ij}(\varepsilon) \in \mathrm{M}(n_i, n_j; \mathbb{C})$ satisfies

$$P_{ij}(\varepsilon) = \varepsilon Q_{ij} + O(\varepsilon^2) \quad \text{as } \varepsilon \text{ tends to } 0.$$

Then, we have

$$\begin{aligned} \det\bigl(\lambda I_{n_{i_0}} - A_{i_0\cdots i_r}(P(\varepsilon))\bigr) &= \det\bigl(\lambda I_{n_{i_0}} - \varepsilon^r \tilde{Q}_{i_0 i_1}\cdots\tilde{Q}_{i_{r-1}i_r} + O(\varepsilon^{r+1})\bigr) \\ &= \det\bigl(\lambda I_{n_{i_0}} - \varepsilon^r A_{i_0\cdots i_r}(Q) + O(\varepsilon^{r+1})\bigr) \\ &= \sum_{s=0}^{n_{i_0}} \lambda^{n_{i_0}-s}\varepsilon^{sr} h_s(\varepsilon), \end{aligned} \tag{4.1.4}$$

where $h_s(\varepsilon)$ $(0 \leq s \leq n_{i_0})$ are real analytic functions of ε such that

$$\det\bigl(\lambda I_{n_{i_0}} - A_{i_0\cdots i_r}(Q)\bigr) = \sum_{s=0}^{n_{i_0}} \lambda^{n_{i_0}-s} h_s(0).$$

From our assumption, this polynomial has complex coefficients, namely, there exists s such that $h_s(0) \notin \mathbb{R}$. It follows from (4.1.4) that $\det(\lambda I_{n_{i_0}} - A_{i_0\cdots i_r}(P(\varepsilon))) \notin \mathbb{R}[\lambda]$ for any sufficiently small ε. Hence, we have shown the lemma. $\square$

4.2. Necessary conditions for $G = LG^\sigma H$.

Throughout this subsection, we set

$$\begin{aligned} (G, L, H) = (&\mathrm{SO}(2n),\ \mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_{k-1}) \times \mathrm{SO}(2n_k), \\ &\qquad \mathrm{U}(m_1) \times \cdots \times \mathrm{U}(m_{l-1}) \times \mathrm{SO}(2m_l)), \end{aligned}$$

where $n = n_1 + \cdots + n_k = m_1 + \cdots + m_l$ with $n_i, m_j > 0$ $(1 \leq i \leq k-1,\ 1 \leq j \leq l-1)$ and $n_k,\ m_l \geq 0$. We give necessary conditions on (L, H) (resp. $(L, \xi(H))$) under which $G = LG^\sigma H$ (resp. $G = LG^\sigma \xi(H)$) holds. We divide the proof into six cases (Propositions 4.4–4.9).

Proposition 4.4. *$G \neq LG^\sigma H$ if one of the following two conditions is satisfied.*

$$k \geq 4,\ m_1 = 1. \tag{4.2.1}$$

$$k \geq 3,\ n_k \neq 0,\ m_1 = 1. \tag{4.2.2}$$

Proposition 4.5. *$G \neq LG^\sigma H$ if n_k, $m_l \neq 0$, n_1, $m_1 \geq 2$.*

Proposition 4.6. *$G \neq LG^\sigma H$ if $k = 2$, $n_1 \geq 4$, $n_2 \geq 2$, $m_l = 0$.*

Proposition 4.7. *$G \neq LG^\sigma H$ if $k = 3$, $\max\{n_1, n_2\} \geq 2$, $n_3 \neq 0$, $m_l = 0$.*

Proposition 4.8. *$G \neq LG^\sigma H$ if $k = 3$, n_1, $n_2 \geq 2$, $n_3 = m_l = 0$.*

Proposition 4.9. *$G \neq LG^\sigma \xi(H)$ if $n \geq 5$, $k = 3$, n_1, $n_2 \geq 2$, $n_k = m_l = 0$.*

Proof of Proposition 4.4. We note that the following two inclusive relations reduce a proof of Proposition 4.4 to the case $k = 3$, $l = 2$, $n_3 \neq 0$ and $m_1 = 1$:

$$L \subset \begin{cases} \mathrm{U}(n_1) \times \mathrm{U}(n_2 + \cdots + n_{k-2}) \times \mathrm{SO}(2(n_{k-1} + n_k)) & (k \geq 4), \\ \mathrm{U}(n_1) \times \mathrm{U}(n_2 + \cdots + n_{k-1}) \times \mathrm{SO}(2n_k) & (k \geq 3,\ n_k \neq 0), \end{cases}$$
$$H \subset \mathrm{U}(1) \times \mathrm{SO}(2(m_2 + \cdots + m_l)).$$

We shall show that $G \neq LG^\sigma H$ if $k = 3$, $l = 2$, $n_3 \neq 0$ and $m_1 = 1$. Under this condition, (G, L, H) takes the form:

$$(G, L, H) = (\mathrm{SO}(2n), \mathrm{U}(n_1) \times \mathrm{U}(n_2) \times \mathrm{SO}(n_3), \mathrm{U}(1) \times \mathrm{SO}(2n-2)).$$

Let $1 \to 2 \to 5 \to 3 \to 1$ be a loop and define a diagonal matrix R by $R := \mathrm{diag}(1, 0, \ldots, 0, -1)$. Then, G_R coincides with H. We fix $u \in \mathbb{C}$ and define $X = (X_{ij})_{1 \leq i,j \leq 5} \in \mathfrak{so}(2n)$ in the block matrix form corresponding to the partition $2n = n_1 + n_2 + 2n_3 + n_2 + n_1$ as (4.1.2):

$$X_{12} := \begin{pmatrix} & -u \\ O & \end{pmatrix} \in \mathrm{M}(n_1, n_2; \mathbb{C}), \quad X_{14} := \begin{pmatrix} -1 & \\ & O \end{pmatrix} \in \mathrm{M}(n_1, n_2; \mathbb{C}),$$

$$X_{31} := \begin{pmatrix} 1 & \\ & O \\ 1 & \end{pmatrix} \in \mathrm{M}(2n_3, n_1; \mathbb{C}).$$

We define the block entries X_{11}, X_{15}, X_{22}, X_{23}, X_{24}, X_{32}, X_{33}, X_{34}, X_{42}, X_{43}, X_{44}, X_{51} and X_{55} to be zero matrices. The remaining block entries are automatically determined by the definition (2.2.1) of $G = \mathrm{SO}(2n)$. Then, $Q := [X, R]$ has the following block entries.

$$Q_{12} = \begin{pmatrix} O & u \\ & \end{pmatrix} \in \mathrm{M}(n_1, n_2; \mathbb{C}), \quad Q_{14} = \begin{pmatrix} 1 & \\ & O \end{pmatrix} \in \mathrm{M}(n_1, n_2; \mathbb{C}),$$

$$Q_{31} = \begin{pmatrix} 1 & \\ & O \\ 1 & \end{pmatrix} \in \mathrm{M}(2n_3, n_1; \mathbb{C}).$$

By a simple matrix computation, we have

$$A_{12531}(Q) = Q_{12} J_{n_2} {}^t Q_{14} J_{n_1} J_{n_1} {}^t Q_{31} J_{2n_3} Q_{31} = \begin{pmatrix} 2u & \\ & O \end{pmatrix},$$

and thus the characteristic polynomial $\det(\lambda I_{n_1} - A_{12531}(Q)) = \lambda^{n_1} - 2u\lambda^{n_1 - 1}$ is not defined over $\mathbb{R}$ if u is not real. By using Lemma 4.3, we obtain $G \neq LG^\sigma H$. $\square$

Proof of Proposition 4.5. We can reduce a proof of Proposition 4.5 to the case $k = l = 2$, $m_1 \geq n_1 \geq 2$ and n_2, $m_2 \neq 0$ because the following two inclusive relations hold:

- $\mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_{k-1}) \times \mathrm{SO}(2n_k)$ is contained in

$$\begin{cases} \mathrm{U}(n_1 + n_2) \times \mathrm{SO}(2(n_3 + \cdots + n_k)) & (k \geq 4), \\ \mathrm{U}(n_1) \times \mathrm{SO}(2(n_2 + \cdots + n_k)) & (n_k \neq 0,\ n_1 \geq 2), \end{cases}$$

- $\mathrm{U}(m_1) \times \cdots \times \mathrm{U}(m_{l-1}) \times \mathrm{SO}(2m_l) \subset \mathrm{U}(m_1) \times \mathrm{SO}(2(m_2 + \cdots + m_l))$.

We shall show $G \neq LG^\sigma H$ in the case $k = l = 2$, $m_1 \geq n_1 \geq 2$ and n_2, $m_2 \neq 0$. Let $1 \to 2 \to 3 \to 1$ be a loop and define a diagonal matrix R by

$$R := \mathrm{diag}\big(-1, \overbrace{1, \ldots, 1}^{n_1 - 2}, -1, \overbrace{1, \ldots, 1}^{m_1 - n_1}, 0, \ldots, 0, \overbrace{-1, \ldots, -1}^{m_1 - n_1}, 1, \overbrace{-1, \ldots, -1}^{n_1 - 2}, 1\big).$$

Then, G_R is conjugate to H by an element of G^σ. We fix $u \in \mathbb{C}$ and define $X = (X_{ij})_{1 \leq i,j \leq 3} \in \mathfrak{so}(2n)$ in the block matrix form corresponding to the partition $2n = n_1 + 2n_2 + n_1$ as (4.1.2):

$$X_{12} := \begin{pmatrix} O & \begin{matrix} u & 0 \\ & \\ 0 & 1 \end{matrix} & O \end{pmatrix} \in \mathrm{M}(n_1, 2n_2; \mathbb{C}), \ \ X_{31} := \begin{pmatrix} 1 & & \\ & O & \\ & & -1 \end{pmatrix} \in \mathrm{M}(n_1, \mathbb{C}).$$

We define the block entries X_{11}, X_{22} and X_{33} to be zero matrices. The remaining block entries of X are determined automatically by (2.2.1). Then $Q := [X, R]$ has the following block entries.

$$Q_{12} = \begin{pmatrix} O & \begin{matrix} u & 0 \\ & \\ 0 & 1 \end{matrix} & O \end{pmatrix}, \quad Q_{31} = \begin{pmatrix} -2 & & \\ & O & \\ & & 2 \end{pmatrix}.$$

A simple matrix computation shows

$$A_{1231}(Q) = Q_{12} J_{2n_2} {}^t Q_{12} J_{n_1} Q_{31} = \begin{pmatrix} -2u & & \\ & O & \\ & & 2u \end{pmatrix},$$

and we find that $\det(\lambda I_{n_1} - A_{1231}(Q)) = \lambda^{n_1} - 4u^2 \lambda^{n_1 - 2} \notin \mathbb{R}[\lambda]$ if $u^2 \notin \mathbb{R}$. By Lemma 4.3, we have proved $G \neq LG^\sigma H$. $\square$

Proof of Proposition 4.6. Clearly H is contained in $\mathrm{U}(n)$ under the condition of Proposition 4.6. Hence, it is enough to prove the following:

$$G \neq LG^\sigma H \quad \text{if} \quad k = l = 2,\ n_1 \geq 4,\ n_2 \geq 2,\ m_2 = 0. \tag{4.2.3}$$

Let $1 \to 2 \to 3 \to 1$ be a loop and define a diagonal matrix R as follows.

$$R := \mathrm{diag}(\overbrace{1, \dots, 1}^{n_1 - 2}, -1, -1, -1, -1, \overbrace{1, \dots, 1}^{n_2 - 2}, \overbrace{-1, \dots, -1}^{n_2 - 2}, 1, 1, 1, 1, \overbrace{-1, \dots, -1}^{n_1 - 2}).$$

Then G_R is conjugate to H by an element of G^σ. Let us fix $u \in \mathbb{C}$ and define $X = (X_{ij})_{1 \leq i,j \leq 3} \in \mathfrak{so}(2n)$ in the block matrix form corresponding to the partition $2n = n_1 + 2n_2 + n_1$ as (4.1.2):

$$X_{12} := \begin{pmatrix} u & & & & \\ & 1 & & & \\ & & O & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \quad X_{31} := \begin{pmatrix} & & & -1 & \\ & & & & 1 \\ & & O & & \\ -1 & & & & \\ & 1 & & & \end{pmatrix},$$

where $X_{12} \in \mathrm{M}(n_1, 2n_2; \mathbb{C})$, $X_{31} \in \mathrm{M}(n_1, \mathbb{C})$. We define the block entries X_{11}, X_{22}

and X_{33} to be zero matrices. The remaining block entries of X are automatically determined by (2.2.1). Then $Q := [X, R]$ has the following block entries:

$$Q_{12} = \begin{pmatrix} -2u & & & & \\ & -2 & & & \\ & & O & & \\ & & & 2 & \\ & & & & 2 \end{pmatrix}, \quad Q_{31} = \begin{pmatrix} & & & 2 & \\ & & & & -2 \\ & & O & & \\ -2 & & & & \\ & 2 & & & \end{pmatrix}.$$

By a simple matrix computation, we have

$$A_{1231}(Q) = Q_{12} J_{2n_2} {}^t Q_{12} J_{n_1} Q_{31} = 8 \begin{pmatrix} & & & -u & \\ & & & & 1 \\ & & O & & \\ 1 & & & & \\ & -u & & & \end{pmatrix},$$

and thus $\det(\lambda I_{n_1} - A_{1231}(Q)) = \lambda^{n_1 - 4}(\lambda^2 + 64u)^2 \notin \mathbb{R}[\lambda]$ if $u \notin \mathbb{R}$. From Lemma 4.3, we obtain $G \neq L G^\sigma H$. $\square$

PROOF OF PROPOSITION 4.7. For Proposition 4.7, it is enough to show that

$$G \neq L G^\sigma H \quad \text{if} \quad k = 3,\ l = 2,\ n_2 \geq 2,\ n_3 \neq 0,\ m_2 = 0. \tag{4.2.4}$$

Under this condition, (G, L, H) takes the form:

$$(G, L, H) = (\mathrm{SO}(2n), \mathrm{U}(n_1) \times \mathrm{U}(n_2) \times \mathrm{SO}(2n_3), \mathrm{U}(n)) \ (n_2 \geq 2).$$

Let $1 \to 2 \to 3 \to 4 \to 1$ be a loop and define a diagonal matrix R as follows.

$$R := \mathrm{diag}(\overbrace{1, \ldots, 1}^{n_1+n_2-1}, -1, -1, \overbrace{1, \ldots, 1}^{n_3-1}, \overbrace{-1, \ldots, -1}^{n_3-1}, 1, 1, \overbrace{-1, \ldots, -1}^{n_1+n_2-1}).$$

We note that G_R is conjugate to H by an element of G^σ. Let us fix $u \in \mathbb{C}$ and define $X = (X_{ij})_{1 \leq i,j \leq 5} \in \mathfrak{so}(2n)$ in the block matrix form corresponding to the partition $2n = n_1 + n_2 + 2n_3 + n_2 + n_1$ as (4.1.2):

$$X_{12} := \begin{pmatrix} & & 1 \\ O & & \end{pmatrix} \in \mathrm{M}(n_1, n_2; \mathbb{C}), \quad X_{23} := \begin{pmatrix} u & & \\ & O & \\ & & 1 \end{pmatrix} \in \mathrm{M}(n_2, 2n_3; \mathbb{C}),$$

$$X_{41} := \begin{pmatrix} & O \\ -1 & \end{pmatrix} \in \mathrm{M}(n_2, n_1; \mathbb{C}).$$

We define the block entries X_{11}, X_{13}, X_{15}, X_{22}, X_{24}, X_{31}, X_{33}, X_{35}, X_{42}, X_{44}, X_{51}, X_{53} and X_{55} to be zero matrices. The remaining block entries of X are automatically determined by (2.2.1). Then, $Q := [X, R]$ has the following block entries:

$$Q_{12} = \begin{pmatrix} & & -2 \\ O & & \end{pmatrix}, \quad Q_{23} = \begin{pmatrix} -2u & & \\ & O & \\ & & 2 \end{pmatrix}, \quad Q_{41} = \begin{pmatrix} & O \\ -2 & \end{pmatrix}.$$

A simple matrix computation shows

$$A_{12341}(Q) = Q_{12} Q_{23} J_{2n_3} {}^t Q_{23} J_{n_2} Q_{41} = \begin{pmatrix} -16u & \\ & O \end{pmatrix},$$

and thus we obtain $\det(\lambda I_{n_1} - A_{12341}(Q)) = \lambda^{n_1} + 16u\lambda^{n_1 - 1} \notin \mathbb{R}[\lambda]$ if $u \notin \mathbb{R}$. From Lemma 4.3, we have proved $G \neq LG^\sigma H$. $\square$

Proof of Proposition 4.8. Under the condition of Proposition 4.8, H is contained in $\mathrm{U}(n)$. Hence it is enough to show the following:

$$G \neq LG^\sigma H \quad \text{if} \quad k = 3,\ l = 2,\ n_1, n_2 \geq 2,\ n_3 = m_2 = 0.$$

Let $1 \to 4 \to 2 \to 5 \to 1$ be a loop and define R by

$$R := \mathrm{diag}(\overbrace{1, \ldots, 1}^{n}, \overbrace{-1, \ldots, -1}^{n}).$$

Then, $G_R = H$. Let us fix $u \in \mathbb{C}$ and define $X = (X_{ij})_{1 \leq i,j \leq 5} \in \mathfrak{so}(2n)$ in the block matrix form corresponding to the partition $2n = n_1 + n_2 + 2n_3 + n_2 + n_1$ as (4.1.2):

$$X_{14} := \begin{pmatrix} -u & & -1 \\ & O & \\ -1 & & -1 \end{pmatrix} \in \mathrm{M}(n_1, n_2; \mathbb{C}), \quad X_{42} := \begin{pmatrix} -1 & & \\ & O & \\ & & 1 \end{pmatrix} \in \mathrm{M}(n_2, \mathbb{C}),$$

$$X_{51} := \begin{pmatrix} -1 & & \\ & \text{\huge O} & \\ & & 1 \end{pmatrix} \in \mathrm{M}(n_1, \mathbb{C}).$$

We define X_{11}, X_{12}, X_{21}, X_{22}, X_{44}, X_{45}, X_{54} and X_{55} to be zero matrices. The remaining block entries of X are automatically determined by (2.2.1). Here we note that none of the block entries X_{13}, X_{23}, X_{31}, X_{33}, X_{34}, X_{35}, X_{43} and X_{53} exists since $n_3 = 0$. Then $Q := [X, R]$ has the following block entries:

$$Q_{14} = \begin{pmatrix} 2u & & 2 \\ & \text{\huge O} & \\ 2 & & 2 \end{pmatrix}, \quad Q_{42} = \begin{pmatrix} -2 & & \\ & \text{\huge O} & \\ & & 2 \end{pmatrix}, \quad Q_{51} = \begin{pmatrix} -2 & & \\ & \text{\huge O} & \\ & & 2 \end{pmatrix}.$$

A simple matrix computation shows

$$A_{14251}(Q) = Q_{14} Q_{42} J_{m_2} {}^t\!Q_{14} J_{m_1} Q_{51} = \begin{pmatrix} 16(u-1) & & \\ & \text{\huge O} & \\ & & 16(u-1) \end{pmatrix},$$

and thus, $\det(\lambda I_{n_1} - A_{14251}(Q)) = \lambda^{n_1 - 2}(\lambda - 16(u-1))^2 \notin \mathbb{R}[\lambda]$ if $u \notin \mathbb{R}$. From Lemma 4.3, we get $G \neq LG^\sigma H$. □

PROOF OF PROPOSITION 4.9. We may assume $n_2 \geq n_1$. It suffices to show that $G \neq LG^\sigma \xi(H)$ if $k = 3$, $l = 2$, $n_1 \geq 2$, $n_2 \geq 3$ and $n_3 = m_2 = 0$. Let $1 \to 4 \to 2 \to 5 \to 1$ be a loop and define R by $R := \mathrm{diag}(\overbrace{1, \ldots, 1}^{n-1}, -1, 1, \overbrace{-1, \ldots, -1}^{n-1})$. Then, $G_R = \xi(H)$. Let us fix $u \in \mathbb{C}$ and define $X = (X_{ij})_{1 \leq i,j \leq 5} \in \mathfrak{so}(2n)$ in the block matrix form corresponding to the partition $2n = n_1 + n_2 + 2n_3 + n_2 + n_1$ as (4.1.2):

$$X_{14} := \begin{pmatrix} 0 & -u & & -1 \\ & & \text{\huge O} & \\ 0 & -1 & & -1 \end{pmatrix} \in \mathrm{M}(n_1, n_2; \mathbb{C}),$$

$$X_{42} := \begin{pmatrix} 0 & & 0 & 0 \\ -1 & & 0 & 0 \\ & \text{\huge O} & & \\ 0 & & 1 & 0 \end{pmatrix} \in \mathrm{M}(n_2, \mathbb{C}), \quad X_{51} := \begin{pmatrix} -1 & & \\ & \text{\huge O} & \\ & & 1 \end{pmatrix} \in \mathrm{M}(n_1, \mathbb{C}).$$

The remaining block entries of X are defined in the same way as in the proof of Proposition 4.8. Then $Q := [X, R]$ has the following block entries.

$$Q_{14} = \begin{pmatrix} 0 & 2u & & 2 \\ & & O & \\ 0 & 2 & & 2 \end{pmatrix}, \quad Q_{42} = \begin{pmatrix} 0 & & 0 & 0 \\ -2 & & 0 & 0 \\ & O & & \\ 0 & & 2 & 0 \end{pmatrix}, \quad Q_{51} = \begin{pmatrix} -2 & & \\ & O & \\ & & 2 \end{pmatrix}.$$

By a simple matrix computation, we have

$$A_{14251}(Q) = Q_{14} Q_{42} J_{m_2} {}^t Q_{14} J_{m_1} Q_{51} = \begin{pmatrix} 16(u-1) & & \\ & O & \\ & & 16(u-1) \end{pmatrix},$$

and find that $\det(\lambda I_{n_1} - A_{14251}(Q)) = \lambda^{n_1 - 2}(\lambda - 16(u-1))^2 \notin \mathbb{R}[\lambda]$ if $u \notin \mathbb{R}$. From Lemma 4.3, we have proved $G \neq L G^\sigma \xi(H)$. $\square$

4.3. Completion of the proof of Theorem 1.1.

We complete the proof of the implication (i) $\Rightarrow$ (ii) in Theorem 1.1 (Proposition 4.10) by using Propositions 4.4–4.9. For a given partition $n = n_1 + \cdots + n_k$ with $n_1, \ldots, n_{k-1} > 0$ and $n_k \geq 0$, we have a Levi subgroup $\mathrm{L}_{\Pi'} = \mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_{k-1}) \times \mathrm{SO}(2n_k)$ of $\mathrm{SO}(2n)$, which is associated to the subset

$$\Pi' := \Pi \setminus \left\{ \alpha_i \in \Pi : i = \sum_{s=1}^{j} n_s, \ 1 \leq j \leq k-1 \right\}$$

of the set of simple roots Π (see Diagram 1.1 for the label of the Dynkin diagram).

Proposition 4.10. *Let G be the special orthogonal group* $\mathrm{SO}(2n)$ $(n \geq 4)$, *σ a Chevalley–Weyl involution, Π', Π'' subsets of the set of simple roots Π, and $\mathrm{L}_{\Pi'}$, $\mathrm{L}_{\Pi''}$ the corresponding Levi subgroups. Then we have*

$$G \neq \mathrm{L}_{\Pi'} \, G^\sigma \, \mathrm{L}_{\Pi''} \tag{4.3.1}$$

if one of the following conditions up to switch of Π' and Π'' is satisfied $(1 \leq i, j, k \leq n)$:

(I) *Either $(\Pi')^c$ or $(\Pi'')^c$ contains more than two elements.*
(II) *Both $(\Pi')^c$ and $(\Pi'')^c$ contain two elements.*
(III) *Both $(\Pi')^c$ and $(\Pi'')^c$ contain some simple root other than $\alpha_1, \alpha_{n-1}, \alpha_n$.*

(IV) $\#(\Pi')^c = 2$, $(\Pi'')^c = \{\alpha_i\}$, *and* $i \notin \{1, n-1, n\}$.
(V) $\#(\Pi')^c = 2$, $(\Pi'')^c = \{\alpha_1\}$, *and* $(\Pi')^c$ *contains neither* α_{n-1} *nor* α_n.
(VI) $(\Pi')^c = \{\alpha_i\}$, $(\Pi'')^c = \{\alpha_j\}$, $i \notin \{1, 2, 3, n-1, n\}$, $j \in \{n-1, n\}$.
(VII) $(n \geq 5)$ $(\Pi')^c = \{\alpha_i, \alpha_j\}$, $(\Pi'')^c = \{\alpha_k\}$, $i \neq j$, $k \in \{n-1, n\}$, *and* $(i, j) \neq (1, 2), (1, n-1), (1, n), (n-1, n)$.
(VIII) $(n = 4)$ $(\Pi')^c = \{\alpha_i, \alpha_2\}$, $(\Pi'')^c = \{\alpha_i\}$, $i \in \{3, 4\}$.

Proof. We note the following:

(1) The role of $\mathrm{L}_{\Pi'}$ and $\mathrm{L}_{\Pi''}$ is symmetric.
(2) $G \neq \mathrm{L}_{\Pi'} G^\sigma \mathrm{L}_{\Pi''}$ holds if and only if $G \neq \xi(\mathrm{L}_{\Pi'}) G^\sigma \xi(\mathrm{L}_{\Pi''})$ does.

First, we can see that (I) implies (4.3.1) by combining (4.2.1) of Proposition 4.4 with Propositions 4.5 and 4.7. Second, Proposition 4.5 implies that (4.3.1) holds under each of the conditions (II), (III) and (IV). Third, we can see the condition (V) implies (4.3.1) by using (4.2.2) of Proposition 4.4. Fourth, we can also see that the condition (VI) implies (4.3.1) by Proposition 4.6. Fifth, by combining Proposition 4.7 with Propositions 4.8 and 4.9, we can see that (4.3.1) holds under the condition (VII). Finally, it follows from Proposition 4.8 that the condition (VIII) implies (4.3.1). □

5. Application to representation theory.

In this section, we shall see our generalized Cartan decomposition leads to three multiplicity-free representations by using the framework of visible actions ("triunity" à la [**Ko1**]). The concept of (strongly) visible actions on complex manifolds was introduced by T. Kobayashi. Let us recall the definition ([**Ko2**]).

Definition 5.1. We say a biholomorphic action of a Lie group G on a complex manifold D is *strongly visible* if the following two conditions are satisfied:

(1) There exists a real submanifold S such that (we call S a "slice")

$$D' := G \cdot S \text{ is an open subset of } D.$$

(2) There exists an antiholomorphic diffeomorphism σ of D' such that

$$\sigma|_S = \mathrm{id}_S,$$
$$\sigma(G \cdot x) = G \cdot x \text{ for any } x \in D'.$$

Definition 5.2. In the above setting, we say the action of G on D is S-visible. This terminology will be used also if S is just a subset of D.

Let G be a compact Lie group and L, H its Levi subgroups. Then G/L, G/H and $(G \times G)/(L \times H)$ are complex manifolds. If the triple (G, L, H) satisfies $G = LG^{\sigma}H$, the following three group-actions are all strongly visible:

$$L \curvearrowright G/H$$
$$H \curvearrowright G/L$$
$$\mathrm{diag}(G) \curvearrowright (G \times G)/(L \times H).$$

The following theorem ([**Ko3**]) leads us to multiplicity-free representations:

THEOREM 5.3. *Let G be a Lie group and $\mathcal{V}$ a G-equivalent Hermitian holomorphic vector bundle on a connected complex manifold D. If the following three conditions from* (1) *to* (3) *are satisfied, then any unitary representation that can be embedded in the vector space $\mathcal{O}(D, \mathcal{V})$ of holomorphic sections of $\mathcal{V}$ decomposes multiplicity-freely*:

(1) *The action of G on D is S-visible. That is, there exists a subset $S \subset D$ satisfying the conditions given in Definition* 5.1. *Further, there exists an automorphism $\hat{\sigma}$ of G such that $\sigma(g \cdot x) = \hat{\sigma}(g) \cdot \sigma(x)$ for any $g \in G$ and $x \in D'$.*
(2) *For any $x \in S$, the fiber $\mathcal{V}_x$ at x decomposes as the multiplicity free sum of irreducible unitary representations of the isotropy subgroup G_x. Let $\mathcal{V}_x = \bigoplus_{1 \le i \le n(x)} \mathcal{V}_x^{(i)}$ denote the irreducible decomposition of $\mathcal{V}_x$.*
(3) *σ lifts to an antiholomorphic automorphism $\tilde{\sigma}$ of $\mathcal{V}$ and satisfies $\tilde{\sigma}(\mathcal{V}_x^{(i)}) = \mathcal{V}_x^{(i)}$ $(1 \le i \le n(x))$ for each $x \in S$.*

Although our application is limited to finite dimensional representations, it is noteworthy that this theorem works for both compact and non-compact complex manifolds, for both finite and infinite dimensional representations, and for both discrete and continuous spectra. See, for example, [**Ko1**] and [**Ko6**]. [**Ko1**] deals with finite dimensional representations whereas the latter deals with infinite dimensional representations (not necessarily highest weight modules).

Now we return to the case where G is a connected compact Lie group of type D_n. The fundamental weights $\omega_1, \ldots, \omega_n$ with respect to the set of simple roots $\alpha_1, \ldots, \alpha_n$ are given as follows (see Diagram 1.1 for the label of the Dynkin diagram).

$$\begin{aligned}\omega_i &= \alpha_1 + 2\alpha_2 + \cdots + (i-1)\alpha_{i-1} + i(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{l-2}), \\ &\quad + \frac{1}{2}i(\alpha_{n-1} + \alpha_n) \quad (1 \le i < n-1),\end{aligned}$$

$$\omega_{n-1} = \frac{1}{2}\left(\alpha_1 + 2\alpha_2 + \cdots + (n-2)\alpha_{n-2} + \frac{1}{2}n\alpha_{n-1} + \frac{1}{2}(n-2)\alpha_n\right),$$
$$\omega_n = \frac{1}{2}\left(\alpha_1 + 2\alpha_2 + \cdots + (n-2)\alpha_{n-2} + \frac{1}{2}(n-2)\alpha_{n-1} + \frac{1}{2}n\alpha_n\right).$$

By using the Borel–Weil theory together with Theorem 5.3 and our generalized Cartan decompositions, we obtain the following Corollaries 5.4 and 5.5.

COROLLARY 5.4 (Corollary of Theorem 1.1). *If the pair* (L, λ) *is an entry in the Tables* 5-1, 5-2 *or* 5-3, *then the restriction* $\pi_\lambda|_L$ *of the irreducible representation* π_λ *of* G *with highest weight* λ *to* L *decomposes multiplicity-freely.*

maximal parabolic type

Levi subgroup L	*highest weight* λ
$\mathrm{L}_{\{\alpha_l\}^c}$	$a\omega_1$, $a\omega_2$, $a\omega_3$, $a\omega_{n-2}$, $a\omega_l$
$\mathrm{L}_{\{\alpha_1\}^c}$	$a\omega_i$
$\mathrm{L}_{\{\alpha_2\}^c}$, $\mathrm{L}_{\{\alpha_3\}^c}$	$a\omega_l$
$\mathrm{L}_{\{\alpha_j\}^c}$	$a\omega_1$

Table 5-1.

non-maximal parabolic type

Levi subgroup L	*highest weight* λ
$\mathrm{L}_{\{\alpha_n\}^c}$, $\mathrm{L}_{\{\alpha_{n-1}\}^c}$	$a\omega_1 + b\omega_l$, $a\omega_1 + b\omega_2$, $a\omega_{n-2} + b\omega_l$
$\mathrm{L}_{\{\alpha_1\}^c}$	$a\omega_i + b\omega_l$
$\mathrm{L}_{\{\alpha_1,\alpha_2\}^c}$, $\mathrm{L}_{\{\alpha_1,\alpha_n\}^c}$, $\mathrm{L}_{\{\alpha_1,\alpha_{n-1}\}^c}$, $\mathrm{L}_{\{\alpha_{n-1},\alpha_n\}^c}$	$a\omega_l$
$\mathrm{L}_{\{\alpha_j,\alpha_n\}^c}$, $\mathrm{L}_{\{\alpha_j,\alpha_{n-1}\}^c}$	$a\omega_1$

Table 5-2.

Here, $l = n-1$ *or* n *and* i, j, a, b *are integers satisfying* $1 \le i, j \le n$ *and* $0 \le a, b$. *The following Table* 5-3 *is only for* $n = 4$ $((i,j) = (3,4)$ *or* $(4,3))$:

non-maximal parabolic type

L	λ
$\mathrm{L}_{\{\alpha_i\}^c}$	$a\omega_2 + b\omega_j$
$\mathrm{L}_{\{\alpha_2,\alpha_i\}^c}$	$a\omega_j$

Table 5-3.

Corollary 5.5 (Corollary of Theorem 1.1). *The tensor product representation $\pi_\lambda \otimes \pi_\mu$ of any two irreducible representations π_λ, π_μ of G with highest weights (λ, μ) listed in the below* Tables 5-4 *or* 5-5 *decomposes as a multiplicity-free sum of irreducible representations of G.*

maximal parabolic type

pair of highest weights (λ, μ)
$(a\omega_k,\ b\omega_1)$,
$(a\omega_k,\ b\omega_2)$,
$(a\omega_k,\ b\omega_3)$,
$(a\omega_k,\ b\omega_{n-2})$,
$(a\omega_k,\ b\omega_l)$,
$(a\omega_1,\ b\omega_i)$

Table 5-4.

non-maximal parabolic type

n	*pair of highest weights* (λ, μ)
$n \geq 4$	$(a\omega_k,\ b\omega_{n-2} + c\omega_l)$,
	$(a\omega_k,\ b\omega_1 + c\omega_l)$,
	$(a\omega_k,\ b\omega_1 + c\omega_2)$,
	$(a\omega_1,\ b\omega_i + c\omega_l)$
$n = 4$	$(a\omega_4,\ b\omega_2 + c\omega_3)$,
	$(a\omega_3,\ b\omega_2 + c\omega_4)$

Table 5-5.

Here, $k, l \in \{n-1, n\}$, $1 \leq i \leq n$ and a, b, c are arbitrary non-negative integers.

We note that the condition (2) of Theorem 5.3 is automatically satisfied since the fiber of a holomorphic vector bundle is one-dimensional in the setting of the Borel–Weil Theory. We also note that we can take the complex conjugation as σ in Theorem 5.3.

Remark 5.6. P. Littelmann ([**Li2**]) classified all the pairs of maximal parabolic subgroups $(P_\omega, P_{\omega'})$ of any simple Lie group G over any algebraically closed field of characteristic zero such that the corresponding tensor products $n\omega \otimes m\omega'$ (n and m are arbitrary non-negative integers) decomposes multiplicity-freely where ω and ω' are fundamental weights. (His classification is exactly Table 5-4 and does not include Table 5-5 in the D_n case.) Moreover, he found the branching rules of $n\omega \otimes m\omega'$ and the restriction of $n\omega$ to the maximal Levi subgroup $L_{\omega'}$ of $P_{\omega'}$ for any pair (ω, ω') that admits a G-spherical action on $G/P_\omega \times G/P_{\omega'}$ by using the generalized Littlewood–Richardson rule ([**Li1**]). From his formula, we can immediately see that such restriction $n\omega|_{L_{\omega'}}$ is also multiplicity-free and obtain the same list as Table 5-1 (but not Tables 5-2 and 5-3) in the D_n case.

Remark 5.7. J. R. Stembridge ([**St2**]) gave a complete list of pairs of highest weights with the corresponding tensor product representation multiplicity-free for any complex simple Lie algebra. His method is combinatorial. He also classified multiplicity-free restrictions to Levi subalgebras for all complex simple Lie algebras. Our approach has given a geometric proof of a part of his work based on generalized Cartan decompositions.

We have listed an application of Theorem 5.3 only for the line bundle case. As in [**Ko1**] for type A groups, we think there is a good room for a generalization to the vector bundle case also for type D groups.

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