J. Math. Soc. Japan
Vol. 65, No. 2 (2013) pp. 389–409
doi: 10.2969/jmsj/06520389

L^p-bounds for Stein's square functions associated to operators and applications to spectral multipliers

By Peng Chen, Xuan Thinh Duong and Lixin Yan

(Received May 2, 2011)
(Revised Aug.15, 2011)

Abstract. Let (X, d, μ) be a metric measure space endowed with a metric d and a nonnegative Borel doubling measure μ. Let L be a non-negative self-adjoint operator of order m on X. Assume that L generates a holomorphic semigroup e^{-tL} whose kernels $p_t(x, y)$ satisfy Gaussian upper bounds but without any assumptions on the regularity of space variables x and y. Also assume that L satisfies a Plancherel type estimate. Under these conditions, we show the L^p bounds for Stein's square functions arising from Bochner-Riesz means associated to the operator L. We then use the L^p estimates on Stein's square functions to obtain a Hörmander-type criterion for spectral multipliers of L. These results are applicable for large classes of operators including sub-Laplacians acting on Lie groups of polynomial growth and Schrödinger operators with rough potentials.

1. Introduction and main results.

Let (X, d, μ) be a metric measure space endowed with a metric d and a non-negative Borel measure μ satisfying the doubling condition, i.e. there exists a constant $C > 0$ such that for all $x \in X$ and for all $r > 0$,

$$V(x, 2r) \leq CV(x, r) < \infty, \tag{1.1}$$

where $B(x, r) = \{y \in X : d(x, y) < r\}$ and $V(x, r) = \mu(B(x, r))$. In particular, X is a space of homogeneous type. A more general definition and further studies of these spaces can be found in [**9**, Chapter 3].

Note that the doubling property implies the following strong homogeneity property,

---

2010 *Mathematics Subject Classification.* Primary 42B15; Secondary 42B25, 47F05.
Key Words and Phrases. Stein's square function, Bochner-Riesz means, non-negative self-adjoint operator, heat semigroup, space of homogeneous type.
X. T. Duong was partly supported by Australia Research Council (ARC). L. X. Yan was supported by NNSF of China (Grant No. 10925106), Guangdong Province Key Laboratory of Computational Science, and the Fundamental Research Funds for the Central Universities (Grant No. 12lgzs099).

$$V(x, \lambda r) \leq C\lambda^n V(x, r) \tag{1.2}$$

for some $C, n > 0$ uniformly for all $\lambda \geq 1$ and $x \in X$. The smallest value of the parameter n is a measure of the dimension of the space. There also exist constants C and D so that

$$V(y, r) \leq C\Big(1 + \frac{d(x,y)}{r}\Big)^D V(x, r) \tag{1.3}$$

uniformly for all $x, y \in X$ and $r > 0$. Indeed, property (1.3) with $D = n$ is a direct consequence of the triangle inequality for the metric d and the strong homogeneity property (1.2). When X is Ahlfors regular, i.e. $V(x, r) \sim r^n$ uniformly in x, the value D can be taken to be 0.

In this article, we assume that L is a non-negative self-adjoint operator on $L^2(X)$ and that the semigroup e^{-tL}, generated by $-L$ on $L^2(X)$, has the kernel $p_t(x, y)$ which satisfies the following Gaussian upper bound

$$\big|p_t(x,y)\big| \leq \frac{C}{V(x, t^{1/m})} \exp\Big(-\frac{d(x,y)^{m/(m-1)}}{c\, t^{1/(m-1)}}\Big) \tag{1.4}$$

for all $t > 0$, and $x, y \in X$, where C, c and m are positive constants and $m \geq 2$.

Such estimates are typical for elliptic or sub-elliptic differential operators of order m (see for examples, [**10**], [**14**], [**23**], [**25**] and [**32**]).

Since L is a non-negative self-adjoint operator acting on $L^2(X)$, it admits a spectral resolution

$$L = \int_0^\infty \lambda dE(\lambda).$$

For a complex number $\delta = \sigma + i\tau, \sigma > -1$, we define the Bochner-Riesz mean $S_R^\delta(L) = (I - L/R^m)_+^\delta$ of order δ of a function f

$$S_R^\delta(L)f(x) = \int_0^R \Big(1 - \frac{\lambda}{R^m}\Big)^\delta dE(\lambda)f(x), \quad x \in X \tag{1.5}$$

by using the spectral theorem. We then consider the following square function associated to an operator L

$$\mathcal{G}_\delta(L)f(x) = c_{m\delta}\Big(\int_0^\infty \Big|\frac{\partial}{\partial R}S_R^{\delta+1}(L)f(x)\Big|^2 RdR\Big)^{1/2}, \quad x \in X \tag{1.6}$$

where $c_{m\delta} = 1/m(\delta + 1)$.

Note that when L is the Laplacian $-\Delta$ on $\mathbb{R}^n$, the square function $\mathcal{G}_\delta(\Delta)$ is introduced by E. M. Stein in his study of Bochner-Riesz means [**28**]. One can view $\mathcal{G}_\delta(\Delta)$ as a vector-valued singular integral operator associated to the Bochner-Riesz means and it is known that the L^p boundedness of $\mathcal{G}_\sigma(\Delta)$ for $1 < p \le 2$ holds if and only if $\sigma > n(1/p - 1/2) - 1/2$ (see [**20**], [**21**] and [**28**]). However for the range $p > 2$, the condition $\sigma > \max\{1/2, n(1/2 - 1/p)\} - 1$ is known to be necessary and conjectured to be also sufficient. For the dimension $n = 1$ many proofs of the conjecture are known (see [**31**]). The conjecture in two dimensions was proved by Carbery [**5**]. In dimensions $n \ge 3$, there are some partial results, see for instances, for $\sigma > n(1/2 - 1/p) - 1/2$ in [**20**] and [**21**], and for $\sigma > n(1/2 - 1/p) - 1$, $p \in [2(n+2)/n, \infty)$ in [**22**]. The L^p boundedness of the square function $\mathcal{G}_\delta(\Delta)$ has been studied extensively because of its important role in the Bochner-Riesz analysis and we refer the reader to [**5**], [**6**], [**7**], [**20**], [**21**], [**22**], [**28**], [**29**] and [**30**] and the references therein.

In this article, we study and obtain the L^p boundedness of the Stein's square function when the Laplacian is replaced by a non-negative self-adjoint operator L which has upper Gaussian heat kernel bounds and satisfies the so-called Plancherel estimates. Our main result (Theorem 1.1) includes the Laplacian on Euclidean space as a special case but it is also applicable to large classes of operators such as the sub-Laplacians acting on Lie groups of polynomial growth and Schrödinger operators with non-negative potentials. We note that even in the case $L = -\Delta$, the kernel of $\mathcal{G}_\delta(\Delta)$ does not possess enough regularity for the operator $\mathcal{G}_\delta(\Delta)$ to be a standard Calderón-Zygmund operator. In our work, we do not assume any regularity on the space variables of the heat kernels and this results in a much rougher kernel of $\mathcal{G}_\delta(L)$. We will overcome this difficulty by carrying out certain detailed estimates and using the techniques in [**2**], [**3**] and [**14**].

The following theorem is our main result.

THEOREM 1.1. *Let L be a non-negative self-adjoint operator such that the corresponding heat kernels satisfy Gaussian bounds* (1.4). *Assume that for some $2 \le q \le \infty$ and any $t > 0$ and all Borel functions F such that* $\operatorname{supp} F \subseteq [0, t]$,

$$\int_X \left|K_{F(\sqrt[m]{L})}(x, y)\right|^2 d\mu(x) \le \frac{C}{V(y, t^{-1})} \|F_{(t)}\|_{L^q}^2, \tag{1.7}$$

where $F_{(t)}(\lambda) = F(t\lambda)$. Let $\mathcal{G}_\sigma(L)$ be an operator given in (1.6). *If $p \in (1, \infty)$ and*

$$\sigma > \left(n + 1 - \frac{2}{q}\right)\left|\frac{1}{p} - \frac{1}{2}\right| - \frac{1}{2}, \tag{1.8}$$

then there exists a constant $C > 0$ *such that*

$$C^{-1}\|f\|_{L^p(X)} \le \|\mathcal{G}_\sigma(L)f\|_{L^p(X)} \le C\|f\|_{L^p(X)}.$$

Remark 1.2. Consider the case $1 < p \le 2$ and $\sigma > n(1/p - 1/2) - 1/2$. Assume that condition (1.7) is true with $q = 2$ (which is true for the case of the Laplace operator on $\mathbb{R}^n$). Then our result on the range of p

$$\frac{2n}{n + 2\sigma + 1} < p \le 2$$

gives L^p boundedness of $\mathcal{G}_\sigma(L)$. This range is optimum even for the Laplace operator on $\mathbb{R}^n$.

The Stein's square function can be useful in the study of spectral multipliers of self-adjoint operators. Here, we will use Theorem 1.1 in the proof of the following Hörmander type spectral multiplier theorem.

Theorem 1.3. *Let L be a non-negative self-adjoint operator such that the corresponding heat kernels satisfy Gaussian bounds* (1.4). *Assume that condition* (1.7) *holds for some* $q \in [2, \infty]$. *Let F be a locally absolutely continuous function on* $(0, \infty)$ *and*

$$B := \|F\|_{L^\infty} + \left(\sup_{R>0} R \int_R^{2R} |F'(\lambda)|^2 d\lambda\right)^{1/2} < \infty. \tag{1.9}$$

Then $F(L)$ is bounded on $L^p(X)$ for all $p \in (1, \infty)$ *with* $(n+1-2/q)|1/p - 1/2| < 1/2$. *In addition,*

$$\|F(L)\|_{L^p(X) \to L^p(X)} \le CB$$

with C independent of F.

We note that when L is the Laplacian $-\Delta$ on $\mathbb{R}^n$, condition (1.7) is true with $q = 2$, and the corresponding Theorem 1.3 with $q = 2$ is obtained in [**21**].

This paper is organized as follows. In Section 2, we will state two lemmas concerning kernel estimates of spectral multipliers and $C_c^\infty(\mathbb{R}_+)$ functions, then criteria for L^p boundedness for singular integrals in [**2**], [**3**], which are useful in the sequel. In Section 3 we will prove Theorem 1.1 by using criteria for L^p boundedness, Stein's interpolation theorem and duality theory. We then apply Theorem 1.1 to obtain an important estimate in the proof of Theorem 1.3. In Section 4, we show that our results are applicable to various operators in different settings, including the sub-Laplacians on homogeneous groups and Schrödinger operators with rough potentials.

Throughout, the letter C and c will denote (possibly different) constants that are independent of the essential variables.

2. Some useful estimates on functions and singular integrals.

Let (X, d, μ) be a metric measure space endowed with a distance d and a nonnegative Borel doubling measure μ. Unless otherwise specified in the sequel we always assume that L is a non-negative self-adjoint operator such that the corresponding heat kernels satisfy Gaussian bound (1.4).

We first record a useful auxiliary result, which will be useful in the proof of Theorem 1.1. For a proof, see pp. 453–454, Lemma 4.3 of [**14**].

LEMMA 2.1. *Suppose that L satisfies estimate* (1.7). *Then for any s, $\epsilon > 0$, there exists a constant $C = C(s, \epsilon)$ such that*

$$\int_X \big|K_{F(\sqrt[m]{L})}(x,y)\big|^2 \big(1 + td(x,y)\big)^s d\mu(x) \le \frac{C}{V(y,t^{-1})} \|F_{(t)}\|^2_{W^q_{(s/2)+\epsilon}} \tag{2.1}$$

for all Borel functions F such that $\operatorname{supp} F \subseteq [t/4, t]$, *where $F_{(t)}(\lambda) = F(t\lambda)$ and* $\|F\|_{W^q_s} = \|(I - d^2/dx^2)^{s/2} F\|_{L^q}$.

We call hypothesis (1.7) the Plancherel estimate or the Plancherel condition (see also [**8**], [**14**] and [**15**]). For the standard Laplace operator on Euclidean spaces $\mathbb{R}^n$, condition (1.7) with $q = 2$ is equivalent to $(1, 2)$ Stein-Tomas restriction theorem (which is also the Plancherel estimate of the Fourier transform). For the general operator L, we note that Gaussian bound (1.4) implies estimates (1.7) for $q = \infty$. Indeed, it was proved in Lemma 2.2 of [**14**] that for any Borel function F such that $\operatorname{supp} F \subset [0, R]$,

$$\begin{aligned}\big\|K_{F(\sqrt[m]{L})}(\cdot,y)\big\|^2_{L^2(X)} &= \big\|K_{\overline{F}(\sqrt[m]{L})}(y,\cdot)\big\|^2_{L^2(X)} \\ &\le \frac{C}{V(y,R^{-1})}\|F\|^2_{L^\infty}\end{aligned} \tag{2.2}$$

where $\overline{F}$ denotes the complex conjugate of F. Condition (1.7) holds for large classes of operators including Laplace operators acting on Lie groups of polynomial growth and Schrödinger operators with non-negative potentials (see also Section 4 below). It is also closely related to Strichartz and other dissipative type estimates. For further discussion of condition (1.7), see [**14**].

Now, for a complex number $\delta = \sigma + i\tau$, $\sigma > -1$, recall that the Bochner-Riesz means of order δ are given by $S_R^\delta(L) = (I - L/R^m)_+^\delta$, $R > 0$. Under the Plancherel condition (1.7) with some $q \in [2,\infty]$, it follows by Theorem 3.1 in [**14**] that for $\sigma > (n/2) - (1/q)$, the Bochner-Riesz mean $S_R^\delta(L)$ is of weak type $(1,1)$. Hence by interpolation, it is bounded on $L^p(X)$ for all $1 < p < \infty$. In addition, there exists a constant $C = C(p,\delta) > 0$ such that

$$\big\|S_R^\delta(L)\big\|_{L^p(X)\to L^p(X)} \le C \quad \text{for all } R > 0. \tag{2.3}$$

LEMMA 2.2. *Let ϕ be a function in $C_c^\infty(\mathbb{R}_+)$ supported in $[1/4,1]$. Let $\ell \in \mathbb{Z}$, $m \in 2\mathbb{N}$ and $\delta = \sigma + i\tau$, $\sigma > -1/2$. For $0 < s < \sigma + 1/q$ where $2 \le q \le \infty$, there exist $C, c > 0$ such that*

$$\sup_{\ell\in\mathbb{Z}:\, \ell\le 1} \big\|\phi(\lambda)(1 - 2^{m\ell}\lambda^m)_+^\delta\big\|_{W_s^q(\mathbb{R})} \le Ce^{c|\tau|}. \tag{2.4}$$

PROOF. The proof of Lemma 2.2 is standard. We give a brief argument of this proof for completeness and convenience for the reader.

Observe that for every $\ell \le -1$, the function $\phi(\lambda)(1 - 2^{m\ell}\lambda^m)_+^\delta = \phi(\lambda)(1 - 2^{m\ell}\lambda^m)^\delta$ is in $C_0^\infty([1/2,1])$, and then estimate (2.4) holds. To complete the proof, it suffices to consider the cases $\ell = 0, 1$. For the case $\ell = 0$, we note that for any Sobolev space $W_s^q(\mathbb{R})$, if k is integer greater than s, then

$$\begin{aligned}\big\|\phi(\lambda)(1-\lambda^m)_+^\delta\big\|_{W_s^q} &\le \big\|(1-\lambda^2)_+^\delta\big\|_{W_s^q}\|(1+\lambda^2+\cdots+\lambda^{m-2})^\delta\|_{C^k[1/4,1]}\|\phi(\lambda)\|_{C^k}\\ &\le C(1+|\tau|)^{[s]+1}\big\|(1-\lambda^2)_+^\delta\big\|_{W_s^q}.\end{aligned} \tag{2.5}$$

Let $\mathcal{F}$ denote the Fourier transform. Since $q \in [2,\infty]$, it follows from Hausdorff-Young inequality ([**29**, p. 583]), that

$$\begin{aligned}\big\|(1-\lambda^2)_+^\delta\big\|_{W_s^q} &= \big\|(I - d^2/dx^2)^{s/2}(1-\lambda^2)_+^\delta\big\|_{L^q}\\ &\le C\big\|(1+t^2)^{s/2}\mathcal{F}((1-\lambda^2)_+^\delta)(t)\big\|_{L^p},\end{aligned} \tag{2.6}$$

where $q^{-1} + p^{-1} = 1$ and $1 \le p \le 2$. For this purpose we recall the following well-known facts in the theory of Bessel function ([**28**, p. 106]),

$$\begin{aligned}\mathcal{F}\big((1-\lambda^2)_+^\delta\big)(t) &= \int_{-1}^1 (1-\lambda^2)^\delta e^{-it\lambda}d\lambda\\ &= \pi\Gamma(\delta+1)J_{\delta+(1/2)}(t)t^{-\delta-(1/2)},\end{aligned} \tag{2.7}$$

where

$$J_\zeta(t) = \frac{2}{\pi}\frac{t^\zeta}{\Gamma(\zeta+1/2)}\int_0^1 (1-u^2)^{\zeta-1/2}\cos(ut)du, \quad \mathrm{Re}(\zeta) > -\frac{1}{2}, \tag{2.8}$$

$$|J_{\xi+i\eta}(t)| \le \begin{cases} C_\xi e^{\pi|\eta|}|t|^{-1/2}, & |t|\ge 1,\ \xi\ge 0 \\ C_\xi e^{(1/2)\pi|\eta|}|t|^{\xi}, & |t|>0,\ \xi\ge 0. \end{cases} \tag{2.9}$$

Since $\sigma > s - 1/q$, from (2.5), (2.6), (2.7), (2.8) and (2.9) it can be verified that

$$\left\|\phi(\lambda)(1-\lambda^m)_+^\delta\right\|_{W_s^q} \le C_\sigma(1+|\tau|)^{[s]+1}e^{c|\tau|} \le Ce^{c'|\tau|}.$$

The similar argument above holds for the case $\ell = 1$, and then estimate (2.4) is proved. Hence, the proof of Lemma 2.2 is complete. $\square$

In the following, we will often just use B for $B(x_B, r_B)$. Denote by M the Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{B\ni x}\frac{1}{V(B)}\int_B |f(y)|d\mu(y),$$

where B ranges over all open balls containing x. Also given $\lambda > 0$, we will write λB for the $\lambda-$dilated ball, which is the ball with the same center as B and with radius $r_{\lambda B} = \lambda r_B$. We set

$$U_1(B) := 4B, \quad \text{and} \quad U_j(B) := 2^{j+1}B\backslash 2^jB \ \text{ for } \ j = 2, 3, \ldots. \tag{2.10}$$

We now state the following criteria for L^p boundedness for singular integrals in [**2**], [**3**], [**4**], which will be useful in the proof of Theorem 1.1.

Proposition 2.3. *Let T be a sublinear operator which is bounded on $L^2(X)$. Let $\{A_r\}_{r>0}$ be a family of linear operators acting on $L^2(X)$. Assume that for $j \ge 2$*

$$\left(\int_{U_j(B)} \left|T(I-A_{r_B})f\right|^2 d\mu\right)^{1/2} \le Cg(j)V(2^{j+1}B)^{-1/2}\int_B |f|d\mu \tag{2.11}$$

and for $j \ge 1$

$$\left(\int_{U_j(B)} \left|A_{r_B}f\right|^2 d\mu\right)^{1/2} \le Cg(j)V(2^{j+1}B)^{-1/2}\int_B |f|d\mu \tag{2.12}$$

for all ball B with r_B the radius of B and all f supported in B. If $\sum g(j) < \infty$, then T is of weak type $(1,1)$, with a bound depending only on its $(2,2)$ norm, on the constant in (1.1), and on the constants C in (2.11) and (2.12), hence bounded on $L^p(X)$ for $1 < p < 2$.

Proof. For the proof of Proposition 2.3, we refer it to Theorem 5.11, [**4**]; Theorem 1.1, [**2**]. □

Proposition 2.4. *Let T be a sublinear operator which is bounded on $L^2(X)$. Let $\{A_r\}_{r>0}$ be a family of linear operators acting on $L^2(X)$. Assume*

$$\left(\frac{1}{V(B)}\int_B \left|T(I-A_{r_B})f\right|^2 d\mu\right)^{1/2} \le CM\left(|f|^2\right)^{1/2}(x) \tag{2.13}$$

and

$$\left\|TA_{r_B}f\right\|_{L^\infty(B)} \le CM\left(|Tf|^2\right)^{1/2}(x) \tag{2.14}$$

for all $f \in L^2(X)$, all $x \in X$ and all ball $B \ni x$, r_B being the radius of B. If $2 < p < \infty$ and $Tf \in L^p(X)$ when $f \in L^p(X)$, then T is strong type (p,p), and its operator norm is bounded by a constant depending only on its $(2,2)$ norm, on the constant in (1.1), on p and on the constants C in (2.13) and (2.14).

Proof. For the proof of Proposition 2.4, we refer it to Theorem 2.1, [**3**]. □

3. Proofs of main results.

3.1. Proof for the L^p bounds on Stein's square functions.

We now show Theorem 1.1 by considering the following three cases.

Case 1: We first show that for every $\delta = \sigma + i\tau$, $\sigma > -1/2$, there exists a positive constant B_σ such that for every $f \in L^2(X)$,

$$\|\mathcal{G}_\delta(L)f\|_{L^2(X)} = B_\sigma \|f\|_{L^2(X)}. \tag{3.1}$$

Let us prove (3.1). For every $R > 0$ and $\lambda > 0$, we recall that $S_R^\delta(\lambda) = (1-\lambda/R^m)_+^\delta$, and set

$$F_R^\delta(\lambda) = c_{m\delta} R \frac{\partial}{\partial R} S_R^{\delta+1}(\lambda) \tag{3.2}$$

with $c_{m\delta} = 1/m(\delta+1)$. It follows from the spectral theory in [**33**] that for any

$f \in L^2(X)$,

$$\begin{aligned}
\|\mathcal{G}_\delta(L)f\|_{L^2(X)} &= \left\{ \int_0^\infty \langle \overline{F_R^\delta}(L)\, F_R^\delta(L) f, f\rangle \frac{dR}{R} \right\}^{1/2} \\
&= \left\{ \left\langle \int_0^\infty |F_R^\delta|^2(L) \frac{dR}{R} f, f \right\rangle \right\}^{1/2} \\
&= \left\{ \int_{\lambda^{1/m}}^\infty \left(1 - \frac{\lambda}{R^m}\right)^{2\sigma} \frac{\lambda^2}{R^{2m+1}} dR \right\}^{1/2} \|f\|_{L^2(X)} \\
&= B_\sigma \|f\|_{L^2(X)}, \qquad (3.3)
\end{aligned}$$

where

$$B_\sigma^2 = \int_{\lambda^{1/m}}^\infty \left(1 - \frac{\lambda}{R^m}\right)^{2\sigma} \frac{\lambda^2}{R^{2m+1}} dR = \int_1^\infty s^{-(2m+1)}(1 - s^{-m})^{2\sigma} ds < \infty$$

and the above integral converges if $\sigma > -1/2$. This proves (3.1).

Case 2: We will prove that for every $p \in (1, \infty)$ satisfying $\sigma > (n+1-(2/q))|(1/p)-(1/2)|-(1/2)$, there exists a constant $C = C(p) > 0$ such that for every $f \in L^p(X)$,

$$\|\mathcal{G}_\sigma(L)f\|_{L^p(X)} \leq C\|f\|_{L^p(X)}. \qquad (3.4)$$

To prove (3.4), we need some preliminary results. Let $\phi \in C_c^\infty(0,\infty)$ be a non-negative function satisfying $\operatorname{supp}\phi \subseteq [1/4, 1]$ and $\sum_{\ell=-\infty}^\infty \phi(2^{-\ell}\lambda) = 1$ for any $\lambda > 0$. Let F_R^δ be a function given in (3.2). Since $\operatorname{supp} F_R^\delta(\lambda^m) \subset [0, R]$ and $\operatorname{supp}\phi \subseteq [1/4, 1]$, we have that for every $\lambda > 0$,

$$F_R^\delta(\lambda^m) = \sum_{\ell=-\infty}^{\infty} \phi(2^{-\ell}\lambda/R) F_R^\delta(\lambda^m) = \sum_{\ell=-\infty}^{1} \phi(2^{-\ell}\lambda/R) F_R^\delta(\lambda^m). \qquad (3.5)$$

This decomposition implies that the sequence $\sum_{\ell=-N}^1 \phi(2^{-\ell}\sqrt[m]{L}/R) F_R^\delta(L)$ converges strongly in $L^2(X)$ to $F_R^\delta(L)$ (see for instance, Reed and Simon [**24**, Theorem VIII.5]). For every $\ell \leq 1$ and $r > 0$, we set for $\lambda > 0$,

$$F_{R,\ell,r}^\delta(\lambda) = \phi(2^{-\ell}\lambda/R) F_R^\delta(\lambda^m)(1 - e^{-(r\lambda)^m}). \qquad (3.6)$$

We may write

$$F_R^\delta(L)(I-e^{-r^mL})f=\lim_{N\to\infty}\sum_{\ell=-N}^{1}F^\delta_{R,\ell,r}(\sqrt[m]{L})f,$$

where the sequence converges strongly in $L^2(X)$.

For a ball B, we let r_B be the radius of B. For every $j=2,3,\dots$, we recall that $U_j(B)=2^{j+1}B\backslash 2^jB$ is defined in (2.10). Let $K_{F^\delta_{R,\ell,r_B}(\sqrt[m]{L})}(x,y)$ be the kernel of the operator $F^\delta_{R,\ell,r_B}(\sqrt[m]{L})$. Then the following result holds.

Lemma 3.1. *Suppose that $F^\delta_{R,\ell,r_B}(\sqrt[m]{L})$ are defined as above. Let $\sigma>(n/2)-(1/q)$ with some $q\in[2,\infty]$ and let $n/2<s<\sigma+(1/q)$ and $m+(n/2)-s>0$. Then there exists a constant $C>0$ such that*

$$\begin{aligned}\int_0^\infty\int_{U_j(B)}&\big(\big|K_{F^\delta_{R,\ell,r_B}(\sqrt[m]{L})}(x,y)\big|^2+\big|K_{F^\delta_{R,\ell,r_B}(\sqrt[m]{L})}(y,x)\big|^2\big)d\mu(x)\frac{dR}{R}\\&\le\frac{Ce^{c|\tau|}}{V(2^{j+1}B)}\big(2^{(2m-1)\ell}2^{-j}+2^{(2m+n-2s)\ell}2^{j(n-2s)}\big),\quad j=2,3,\dots\end{aligned}\tag{3.7}$$

Proof of Lemma 3.1. Since L is a non-negative self-adjoint operator on $L^2(X)$, we have that $K_{F^\delta_{R,\ell,r_B}(\sqrt[m]{L})}(y,x)=K_{\overline{F}^\delta_{R,\ell,r_B}(\sqrt[m]{L})}(x,y)$. So we only estimate (3.7) for the kernel $K_{F^\delta_{R,\ell,r_B}(\sqrt[m]{L})}(x,y)$ since $K_{F^\delta_{R,\ell,r_B}(\sqrt[m]{L})}(y,x)$ satisfies the similar estimate as $K_{F^\delta_{R,\ell,r_B}(\sqrt[m]{L})}(x,y)$.

Note that $\operatorname{supp}F^\delta_{R,\ell,r_B}(\lambda)\subset[2^\ell R/4,2^\ell R]$. We use Lemma 2.1 and Hölder's inequality to obtain that for every $y\in B$ and every $s>0$,

$$\begin{aligned}\int_{U_j(B)}&\big|K_{F^\delta_{R,\ell,r_B}(\sqrt[m]{L})}(x,y)\big|^2d\mu(x)\\&\le\int_X\big|K_{F^\delta_{R,\ell,r_B}(\sqrt[m]{L})}(x,y)\big|^2(1+2^\ell Rd(x,y))^{2s}d\mu(x)\big(2^jr_B2^\ell R\big)^{-2s}\\&\le\frac{C}{V(y,(2^\ell R)^{-1})}\big(2^jr_B2^\ell R\big)^{-2s}\big\|F^\delta_{R,\ell,r_B}(2^\ell R\lambda)\big\|^2_{W^q_s}.\end{aligned}\tag{3.8}$$

Now for any Sobolev space $W^q_s(\mathbb{R})$, if k is an integer greater than s, then

$$\begin{aligned}\big\|F^\delta_{R,\ell,r_B}&(2^\ell R\lambda)\big\|_{W^q_s}\\&\le C\big\|(2^\ell\lambda)^m\phi(\lambda)(1-2^{m\ell}\lambda^m)^\delta_+\big\|_{W^q_s}\big\|(1-e^{-(2^\ell Rr_B)^m\lambda^m})\big\|_{C^k[1/4,1]}\\&\le C2^{m\ell}\big\|\phi(\lambda)(1-2^{m\ell}\lambda^m)^\delta_+\big\|_{W^q_s}\min\{1,(2^\ell Rr_B)^m\}.\end{aligned}\tag{3.9}$$

Note that for all $y \in B$, by (1.2),

$$\begin{aligned}\frac{1}{V(y,(2^\ell R)^{-1})} &\le \frac{V(y,2^{j+2}r_B)}{V(y,(2^\ell R)^{-1})V(2^{j+1}B)}\\ &\le \frac{C}{V(2^{j+1}B)}\max\{1,(2^j r_B 2^\ell R)^n\}. \qquad (3.10)\end{aligned}$$

Hence by (3.8), (3.9) and (3.10), we have that for every $y \in B$ and every $s > 0$,

$$\begin{aligned}&\int_{U_j(B)} \left|K_{F^\delta_{R,\ell,r_B}(\sqrt[m]{L})}(x,y)\right|^2 d\mu(x)\\ &\le \frac{C2^{2m\ell}}{V(2^{j+1}B)}\max\{1,(2^j r_B 2^\ell R)^n\}\min\{1,(2^\ell R r_B)^{2m}\}\\ &\quad\times \left(2^j r_B 2^\ell R\right)^{-2s}\left\|\phi(\lambda)(1-2^{m\ell}\lambda^m)^\delta_+\right\|^2_{W^q_s}. \qquad (3.11)\end{aligned}$$

We now use (3.11) to estimate (3.7). One may write

$$\begin{aligned}&\int_0^\infty \int_{U_j(B)} \left|K_{F^\delta_{R,\ell,r_B}(\sqrt[m]{L})}(x,y)\right|^2 d\mu(x)\frac{dR}{R}\\ &\le \left(\int_0^{2^{-(j+\ell)}r_B^{-1}} + \int_{2^{-(j+\ell)}r_B^{-1}}^\infty\right)\int_{U_j(B)} \left|K_{F^\delta_{R,\ell,r_B}(\sqrt[m]{L})}(x,y)\right|^2 d\mu(x)\frac{dR}{R}\\ &= I + II.\end{aligned}$$

For the term I, we note that $0 < R < 2^{-(j+\ell)}r_B^{-1}$, and then $\max\{1,(2^j r_B 2^\ell R)^n\} \le 1$. Let $s = 1/2$ in (3.11). It follows by Lemma 2.2 that $\|\phi(\lambda)(1-2^{m\ell}\lambda^m)^\delta_+\|_{W^q_s} < Ce^{c|\tau|}$. Also, since $\ell \le 1$, this implies that $\min\{1,(2^\ell R r_B)^{2m}\} \le C\min\{1,(Rr_B)^{2m}\}$. Those facts give

$$\begin{aligned}I &\le \frac{Ce^{c|\tau|}}{V(2^{j+1}B)}\int_0^{2^{-(j+\ell)}r_B^{-1}} 2^{2m\ell}\min\{1,(Rr_B)^{2m}\}\left(2^j r_B 2^\ell R\right)^{-1}\frac{dR}{R}\\ &\le \frac{Ce^{c|\tau|}}{V(2^{j+1}B)}2^{(2m-1)\ell}2^{-j}\int_0^\infty \min\{1,(Rr_B)^{2m}\}(Rr_B)^{-1}\frac{dR}{R}\\ &\le \frac{Ce^{c|\tau|}}{V(2^{j+1}B)}2^{(2m-1)\ell}2^{-j}.\end{aligned}$$

Consider the term II. Since $\sigma > (n/2)-(1/q)$, we can choose s in (3.11) satisfying $n/2 < s < \sigma + (1/q)$ and $m + (n/2) - s > 0$. By Lemma 2.2 again, we have that $\|\phi(\lambda)(1-2^{m\ell}\lambda^m)_+^\delta\|_{W_s^q} < Ce^{c|\tau|}$. One obtains

$$\begin{aligned} II &\le \frac{Ce^{c|\tau|}}{V(2^{j+1}B)} \int_0^\infty 2^{2m\ell} \min\{1, (Rr_B)^{2m}\}\big(2^j r_B 2^\ell R\big)^{n-2s} \frac{dR}{R} \\ &\le \frac{Ce^{c|\tau|}}{V(2^{j+1}B)} 2^{(2m+n-2s)\ell} 2^{j(n-2s)} \int_0^\infty \min\{1, (Rr_B)^{2m}\}(Rr_B)^{n-2s} \frac{dR}{R} \\ &\le \frac{Ce^{c|\tau|}}{V(2^{j+1}B)} 2^{(2m+n-2s)\ell} 2^{j(n-2s)}. \end{aligned}$$

Combining estimates I and II, we have therefore proved (3.7) for the kernel $K_{F^\delta_{R,\ell,\,r_B}(\sqrt[m]{L})}(x,y)$, and then the proof of Lemma 3.1 is finished.

BACK TO THE PROOF OF THEOREM 1.1. We now begin to prove (3.4) by considering the following three sub-cases.

SUBCASE (2.1). We first apply Proposition 2.3 to show that if $\sigma > (n/2) - (1/q)$, then $\mathcal{G}_\delta(L)$ is of weak type $(1,1)$, and bounded on $L^p(X)$ for $1 < p \le 2$.

Let B be a ball with r_B the radius of B and all f supported in B. We let $T = \mathcal{G}_\delta(L)$ and $A_{r_B} = e^{-r_B^m L}$ in Proposition 2.3. From the definition of $\mathcal{G}_\delta(L)$ and (3.6), we use the Minkowski inequality to obtain that for every $j \ge 2$,

$$\begin{aligned} &\big\|\mathcal{G}_\delta(L)(I - e^{-r_B^m L})f\big\|_{L^2(U_j(B))} \\ &\qquad \le \sum_{\ell \le 1} \bigg(\int_0^\infty \!\! \int_{U_j(B)} \big|F^\delta_{R,\ell,r_B}(\sqrt[m]{L})f\big|^2 d\mu \frac{dR}{R}\bigg)^{1/2}. \end{aligned} \tag{3.12}$$

By Hölder's inequality and Lemma 3.1,

$$\begin{aligned} &\int_0^\infty \!\! \int_{U_j(B)} \big|F^\delta_{R,\ell,r_B}(\sqrt[m]{L})f\big|^2 d\mu \frac{dR}{R} \\ &\quad \le \sup_{y\in B} \int_0^\infty \int_{U_j(B)} \big|K_{F^\delta_{R,\ell,r_B}(\sqrt[m]{L})}(x,y)\big|^2 d\mu(x) \frac{dR}{R} \|f\|^2_{L^1(X)} \\ &\quad \le \frac{Ce^{c|\tau|}}{V(2^{j+1}B)} \big(2^{(2m-1)\ell}2^{-j} + 2^{(2m+n-2s)\ell}2^{j(n-2s)}\big)\|f\|^2_{L^1(X)}. \end{aligned} \tag{3.13}$$

Note that $n/2 < s < \sigma + (1/q)$ and $m + (n/2) - s > 0$. Putting (3.13) into (3.12), a simple calculation shows that for every $j \geq 2$,

$$\bigg(\int_{U_j(B)} \big|\mathcal{G}_\delta(L)(I - e^{-r_B^m L})f\big|^2 d\mu\bigg)^{1/2} \leq Ce^{c|\tau|} g(j) V(2^{j+1}B)^{-1/2} \|f\|_{L^1(X)} \quad (3.14)$$

with $\sum g(j) = \sum (2^{-j/2} + 2^{j(n/2-s)}) < \infty$. This verifies estimate (2.11) to $T = \mathcal{G}_\delta(L)$ and $A_{r_B} = e^{-r_B^m L}$.

Note also that by the Gaussian bounds (1.4), estimate (2.12) always holds for $A_{r_B} = e^{-r_B^m L}$. Therefore the operator $\mathcal{G}_\delta(L)$ is of weak type $(1,1)$, hence by interpolation,

$$\|\mathcal{G}_\delta(L)f\|_{L^p(X)} \leq C_{p,\nu} e^{c|\tau|} \|f\|_{L^p(X)} \quad (3.15)$$

for $1 < p \leq 2$ and $\delta = \nu + i\tau$, $\nu > (n/2) - (1/q)$.

Subcase (2.2). We now apply Proposition 2.4 to show that if $\sigma > (n/2) - (1/q)$, then $\mathcal{G}_\delta(L)$ is bounded on $L^p(X)$ for $2 \leq p < \infty$.

Recall that Proposition 2.4 applies if $T = \mathcal{G}_\delta(L)$ is assumed to act on $L^p(X)$. If we set $\mathcal{G}_{\delta,\epsilon}(L)f(x) = c_{m\delta}(\int_\epsilon^{1/\epsilon} |\partial_R S_R^{\delta+1}(L)f(x)|^2 R dR)^{1/2}$ for $0 < \epsilon < 1$, then for $f \in L^2(X)$ we have $\|\mathcal{G}_{\delta,\epsilon}(L)f\|_{L^2(X)} \leq B_\sigma \|f\|_{L^2(X)}$ and $\mathcal{G}_{\delta,\epsilon}(L)f \to \mathcal{G}_\delta(L)f$ in $L^2(X)$ as $\epsilon \to 0$ while $\|\mathcal{G}_{\delta,\epsilon}(L)f\|_{L^p(X)} \leq C_\epsilon \|f\|_{L^p(X)}$ for $f \in L^p(X)$ (this follows from (2.3)). As the application of Proposition 2.4 to $\mathcal{G}_{\delta,\epsilon}(L)$ gives us a uniform bound with respect to ϵ, a limiting argument yields the L^p-boundedness of $\mathcal{G}_\delta(L)$ on $L^2(X) \cap L^p(X)$, hence on $L^p(X)$. Henceforth, we ignore this approximation step and our goal is now to establish (2.13) and (2.14) for $\mathcal{G}_\delta(L)$.

Let $f \in L^2(X)$. Take a ball B with radius r_B and a point y in B. Decompose f as $f_1 + f_2 + f_3 + \cdots$ with $f_j = f\chi_{U_j(B)}$. By the Minkowski inequality,

$$\big\|\mathcal{G}_\delta(L)(I - e^{-r_B^m L})f\big\|_{L^2(B)} \leq \sum_{j\geq 1} \big\|\mathcal{G}_\delta(L)(I - e^{-r_B^m L})f_j\big\|_{L^2(B)}. \quad (3.16)$$

For $j = 1$, we use the L^2 boundedness of $\mathcal{G}_\delta(L)(I - e^{-r_B^m L})$:

$$\big\|\mathcal{G}_\delta(L)(I - e^{-r_B^m L})f_1\big\|_{L^2(B)} \leq C\big\|f\big\|_{L^2(4B)} \leq CV(4B)^{1/2}\big(M(|f|^2)\big)^{1/2}(y). \quad (3.17)$$

For $j \geq 2$ we use the Minkowski inequality and the Cauchy-Schwarz inequality to write

$$\begin{aligned}
&\big\|\mathcal{G}_\delta(L)(I-e^{-r_B^m L})f_j\big\|_{L^2(B)}\\
&\quad\le \sum_{\ell\le 1}\Big(\int_0^\infty\int_B \big|F^\delta_{R,\ell,r_B}(\sqrt[m]{L})f_j(x)\big|^2 d\mu(x)\frac{dR}{R}\Big)^{1/2}\\
&\quad\le V(B)^{1/2}\|f_j\|_{L^2(X)}\sum_{\ell\le 1}\Big\{\sup_{x\in B}\int_0^\infty\int_{U_j(B)}\big|K_{F^\delta_{R,\ell,r_B}(\sqrt[m]{L})}(x,y)\big|^2 d\mu(y)\frac{dR}{R}\Big\}^{1/2}.
\end{aligned}\tag{3.18}$$

Now we apply Lemma 3.1 again to (3.18), and a simple calculation shows that for every $j\ge 2$,

$$\begin{aligned}
&\big\|\mathcal{G}_\delta(L)(I-e^{-r_B^m L})f_j\big\|_{L^2(B)}\\
&\qquad\le Ce^{c|\tau|}V(B)^{1/2}\big(2^{-j/2}+2^{j(n/2-s)}\big)\big(M(|f|^2)\big)^{1/2}(y),
\end{aligned}\tag{3.19}$$

which, together with estimates (3.16) and (3.17), verifies condition (2.13) in Proposition 2.4 to $T=\mathcal{G}_\delta(L)$ and $A_{r_B}=e^{-r_B^m L}$.

Next we have to check (2.14) in Proposition 2.4. From the Gaussian bounds (1.4) of the heat kernel $p_{r_B^m}(x,y)$ of $e^{-r_B^m L}$ and the commutativity of the semigroup, we have that for any $x\in B$,

$$\begin{aligned}
&\big|F^\delta_R(\sqrt[m]{L})\big(e^{-r_B^m L}f\big)(x)\big|\\
&\quad=\big|e^{-r_B^m L}\big(F^\delta_R(\sqrt[m]{L})f\big)(x)\big|\\
&\quad\le C\sum_{j\ge 1}e^{-c2^{jm/(m-1)}}2^{nj}\Big(\frac{1}{V(2^{j+1}B)}\int_{2^{j+1}B}\big|F^\delta_R(\sqrt[m]{L})f(y)\big|^2 d\mu(y)\Big)^{1/2},
\end{aligned}$$

which, together with the Minkowski inequality, gives

$$\begin{aligned}
&\big|\mathcal{G}_\delta(L)(e^{-r_B^m L}f)(x)\big|\\
&\quad\le C\sum_{j\ge 1}e^{-c2^{jm/(m-1)}}2^{nj}\Big(\int_0^\infty\frac{1}{V(2^{j+1}B)}\int_{2^{j+1}B}\big|F^\delta_R(\sqrt[m]{L})f(y)\big|^2 d\mu(y)\frac{dR}{R}\Big)^{1/2}.
\end{aligned}$$

Exchanging the sum and integral, the latter is equal to

$$C\sum_{j\ge 1}e^{-c2^{jm/(m-1)}}2^{nj}\Big(\frac{1}{V(2^{j+1}B)}\int_{2^{j+1}B}\big|\mathcal{G}_\delta(L)f\big|^2 d\mu\Big)^{1/2}$$

which is controlled by $C(M(|\mathcal{G}_\delta(L)|^2))^{1/2}(y)$ for any $y \in B$. This verifies estimate (2.14). Therefore, it follows by interpolation that

$$\|\mathcal{G}_\delta(L)f\|_{L^p(X)} \leq C_{p,\nu}e^{c|\tau|}\|f\|_{L^p(X)} \tag{3.20}$$

for $2 \leq p < \infty$ and $\delta = \nu + i\tau$, $\nu > (n/2) - (1/q)$.

Subcase (2.3). Now we can use the Stein's interpolation theorem ([**34**, p. 100]) to prove (3.4). Let H be a Hilbert space of functions on $(0, \infty)$ whose inner product is defined by $\langle f_R, g_R\rangle = \int_0^\infty f_R \overline{g}_R R^{-1} dR$. Let $1 < p < \infty$ and let $L^p_{\mathbb{C}}(X)$ and $L^p_H(X)$ be the spaces of L^p integrable functions with values in $\mathbb{C}$ and H, respectively. Recall that $F^\delta_R(\lambda) = c_{m\delta}R(\partial/\partial R)S^{\delta+1}_R(\lambda)$ is defined in (3.2). It follows from (3.15) and (3.20) that if $\sigma > (n/2) - (1/q)$, then

$$F^\delta_R(L)f = c_{m\delta}R\frac{\partial}{\partial R}S^{\delta+1}_R(L)f \tag{3.21}$$

can be seen as a linear operator mapping boundedly from $L^p_{\mathbb{C}}(X)$ into $L^p_H(X)$ for all $1 < p < \infty$. Moreover, we have that $\|F^\delta_R(L)f\|_{L^p_H(X)} = \|\mathcal{G}_\delta(L)f\|_{L^p(X)}$.

The Stein's interpolation theorem is valid for H-valued L^p-spaces and apply it between the inequalities (3.1) and (3.15), and (3.1) and (3.20) to get that if $p \in (1, \infty)$ and $\sigma > (n + 1 - (2/q))|(1/p) - (1/2)| - 1/2$, then

$$\|\mathcal{G}_\sigma(L)f\|_{L^p(X)} = \|F^\sigma_R(L)f\|_{L^p_H(X)} \leq C\|f\|_{L^p(X)}. \tag{3.22}$$

The desired estimate (3.4) follows readily, and the proof of Case 2 is complete.

Case 3: Finally, we show that for every $p \in (1, \infty)$ satisfying $\sigma > (n + 1 - (2/q))|(1/p) - (1/2)| - 1/2$, there exists a constant $C = C(p) > 0$ such that for every $f \in L^p(X)$,

$$\|f\|_{L^p(X)} \leq C\|\mathcal{G}_\sigma(L)f\|_{L^p(X)}. \tag{3.23}$$

To prove it, we need a suitable version of the Calderón reproducing formula. By L^2-functional calculus for every $f \in L^2(X)$ one can write

$$f = B_\sigma^{-2}\int_0^\infty \left(F^\sigma_R(L)\right)^2 f\frac{dR}{R}$$

with the integral converging in $L^2(X)$.

Estimate (3.23) then follows from (3.22) and the duality method. Hence, the proof of Theorem 1.1 is complete. □

3.2. Proof for the Hörmander type spectral multiplier theorem. Observe that

$$\sup_{R>0} R\int_R^{2R} |F'(\lambda)|^2 d\lambda \sim \sup_{R>0} R\int_R^{2R} |G'(\lambda)|^2 d\lambda,$$

where $G(\lambda) = F(\sqrt[m]{\lambda})$. For this reason, we can replace $F(L)$ by $F(\sqrt[m]{L})$ in the proof. Notice that $F(\lambda) = F(\lambda) - F(0) + F(0)$ and hence

$$F(\sqrt[m]{L}) = (F(\cdot) - F(0))(\sqrt[m]{L}) + F(0)I.$$

Replacing F by $F - F(0)$, we may assume in the sequel that $F(0) = 0$. Let $f \in L^2(X)$ and $h = F(\sqrt[m]{L})f$. Recall that $\mathcal{G}_0(L)$ is the square function in (1.6). The general idea of the proof is to show that

$$\mathcal{G}_0(L)h(x) \le CB\mathcal{G}_0(L)f(x), \quad x \in X. \tag{3.24}$$

Then in view of (3.24) and Theorem 1.1 the following norm inequalities prove the theorem:

$$\begin{aligned}\big\|F(\sqrt[m]{L})f\big\|_{L^p(X)} &= \|h\|_{L^p(X)} \le C\|\mathcal{G}_0(L)h\|_{L^p(X)}\\ &\le CB\|\mathcal{G}_0(L)f\|_{L^p(X)} \le CB\|f\|_{L^p(X)}\end{aligned}$$

for all $p \in (1,\infty)$ with $(n+1-(2/q))|(1/p)-(1/2)| < 1/2$.

Let us prove (3.24). For simplicity, we denote by $dE_\lambda = dE_{\sqrt[m]{L}}(\lambda)$. With the notation as in the proof of Theorem 1.1, we have

$$F_R^0(L)h(x) = \int_0^R \frac{\lambda^m}{R^m} F(\lambda) dE_\lambda f(x),$$

and by integration by parts to obtain

$$\int_0^R \frac{\lambda^m}{R^m} F(\lambda) dE_\lambda f(x) = F(\lambda)\int_0^\lambda \frac{t^m}{R^m} dE_t f(x)\Big|_0^R - \int_0^R F'(\lambda)\int_0^\lambda \frac{t^m}{R^m} dE_t f(x) d\lambda.$$

Observe that by (1.7),

$$\Big| \int_0^\lambda \frac{t^m}{R^m} dE_t f(x) \Big| \le \frac{C}{\sqrt{V(x,\lambda^{-1})}} \big\| (\lambda^m t^m / R^m) \big\|_{L^2([0,\lambda])} \|f\|_{L^2(X)}.$$

Since F is bounded, we obtain

$$\lim_{\lambda \to 0} F(\lambda) \int_0^\lambda \frac{t^m}{R^m} dE_t f(x) = 0,$$

which yields

$$F_R^0(L)h(x) = F(R) F_R^0(L) f(x) - \int_0^R F'(\lambda) \frac{\lambda^m}{R^m} F_\lambda^0(L) f(x) d\lambda.$$

By the Schwarz inequality the last integral is, in absolute value, dominated by

$$\left(\frac{1}{R} \int_0^R |F'(\lambda)|^2 \lambda^2 d\lambda \right)^{1/2} \left(\frac{1}{R^{2m-1}} \int_0^R \big| F_\lambda^0(L) f(x) \big|^2 \lambda^{2m-2} d\lambda \right)^{1/2}.$$

Divide $(0,R)$ into the intervals of the form $(R/2^{j+1}, R/2^j)$ and dominate λ^2 by $R^2/2^{2j}$ in each interval. Then the first integral is bounded by

$$\sum_{j=0}^\infty \frac{1}{2^{j-1}} \frac{R}{2^{j+1}} \int_{R/2^{j+1}}^{R/2^j} |F'(\lambda)|^2 d\lambda \le 4 \sup_{R>0} R \int_R^{2R} |F'(\lambda)|^2 d\lambda.$$

Therefore

$$\begin{aligned} &\mathcal{G}_0(L)h(x) \\ &\quad \le \|F\|_{L^\infty} \left(\int_0^\infty \frac{F_R^0(L) f(x)}{R} dR \right)^{1/2} \\ &\qquad + 2 \left(\sup_{R>0} R \int_R^{2R} |F'(\lambda)|^2 d\lambda \right)^{1/2} \left(\int_0^\infty \big| F_\lambda^0(L) f(x) \big|^2 \lambda^{2m-2} d\lambda \int_\lambda^\infty \frac{dR}{R^{2m}} \right)^{1/2} \\ &\quad \le 4B \mathcal{G}_0(L) f(x). \end{aligned}$$

Estimate (3.24) follows readily. The proof of Theorem 1.3 is complete. □

4. Applications.

4.1. Sub-Laplacians on homogeneous groups.

Let $\boldsymbol{G}$ be a Lie group of polynomial growth and let $X_1, \ldots, X_k$ be a system of left-invariant vector fields on $\boldsymbol{G}$ satisfying the Hörmander condition. We define the sub-Laplace operator L acting on $L^2(\boldsymbol{G})$ by the formula

$$L = -\sum_{i=1}^{k} X_i^2. \tag{4.1}$$

If $B(x,r)$ is the ball define by the distance associated with system $X_1, \ldots, X_k$ (see e.g. Chapter III.4, [**32**]), then there exist natural numbers $n_0, n_\infty \geq 0$ such that $V(x,r) \sim r^{n_0}$ for $r \leq 1$ and $V(x,r) \sim r^{n_\infty}$ for $r > 1$ (see e.g. Chapter III.2, [**32**]). Note that this implies that doubling condition (1.2) holds with the doubling dimension $n = \max\{n_0, n_\infty\}$. Note also that one can take $D = 0$ in the estimates (1.3). We call $\boldsymbol{G}$ a homogeneous group if there exists a family of dilations on $\boldsymbol{G}$. A family of dilations on a Lie group $\boldsymbol{G}$ is a one-parameter group $(\tilde{\delta}_t)_{t>0}$ $(\tilde{\delta}_t \circ \tilde{\delta}_t = \tilde{\delta}_{ts})$ of automorphisms of $\boldsymbol{G}$ determined by

$$\tilde{\delta}_t Y_j = t^{n_j} Y_j, \tag{4.2}$$

where $Y_1, \ldots, Y_\ell$ is a linear basis of Lie algebra of $\boldsymbol{G}$ and $n_j \geq 1$ for $1 \leq j \leq \ell$ (see [**16**]). We say that an operator L defined by (4.1) is homogeneous if $\tilde{\delta}_t X_i = t X_i$ for $1 \leq i \leq k$ and the system $X_1, \ldots, X_k$ satisfies the Hörmander condition. Then for the sub-Riemannian geometry corresponding to the system $X_1, \ldots, X_k$ one has $n_0 = n_\infty = \sum_{j=1}^{\ell} n_j$ (see [**16**]). Hence the doubling dimension is equal to $n = n_0 = n_\infty$ (see e.g., [**8**], [**11**], [**16**], [**19**]).

Proposition 4.1. *Let L be the homogeneous sub-Laplacian defined by the formula* (4.1) *acting on a homogeneous group $\boldsymbol{G}$. Then the results of Theorems* 1.1 *and* 1.3 *hold for $q = 2$.*

Proof. It is well know that the heat kernel corresponding to the operator L satisfies Gaussian bound (1.4). It is also not difficult to check that for some constant $C > 0$

$$\left\|F(\sqrt{L})\right\|^2_{L^2(X)\to L^\infty(X)} = C \int_0^\infty |F(t)|^2 t^{n-1} dt.$$

See for example equation (7.1) of [**14**] or [**8**, Proposition 10]. It follows from the above equality that the operator L satisfies estimate (1.7) with $q = 2$. By Theorems 1.1 and 1.3, Proposition 4.1 follows readily. □

Corollary 4.2. *Let L be a positive definite self-adjoint left invariant operator on a homogeneous group $\boldsymbol{G}$. Suppose that the operator L is homogeneous of order m, i.e., $\tilde{\delta}_t L = t^m L$ and that*

$$\left|K_{\exp(-L)}(x,y)\right| = \left|K_{\exp(-L)}(e, x^{-1}y)\right| \le C \exp\left(-c|x^{-1}y|^{m/(m-1)}\right), \tag{4.3}$$

where C, c are positive constants and $|\cdot|$ is a homogeneous norm on $\boldsymbol{G}$. Then the results of Theorems 1.1 *and* 1.3 *hold for $q = 2$.*

Proof. From Corollary 7.1 of [**14**], we know that the operator L satisfies the Plancherel estimate (1.7) with $q = 2$. By Theorems 1.1 and 1.3, Corollary 4.2 follows readily. □

Proposition 4.1 can be extended to "quasi-homogeneous" operators acting on homogeneous groups, see [**26**] and [**14**].

Proposition 4.3. *Let L be a group invariant operator acting on a Lie group $\boldsymbol{G}$ of polynomial growth defined by* (4.1). *Then the results of Theorems* 1.1 *and* 1.3 *hold for $q = \infty$ and the doubling dimension $n = \max\{n_0, n_\infty\}$.*

Proof. It is well known that the heat kernel corresponding to the operator L satisfies Gaussian bound (1.4) so that the operator L satisfies estimate (1.7) for $q = \infty$ (see also Lemma 2.2 of [**14**]). Hence the results of Theorems 1.1 and 1.3 hold for $q = \infty$ and the doubling dimension $n = \max\{n_0, n_\infty\}$. □

4.2. Schrödinger operators with rough potentials.

Consider the Shrödinger operator $-\Delta + V$ on $\mathbb{R}^3$, where Δ is the standard Laplace operator and $V(x) \ge 0$ is a compactly supported function satisfying

$$\int_{\mathbb{R}^6} \frac{V(x)V(y)}{|x-y|^2} dxdy < \infty \quad \text{and} \quad \sup_{x\in\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{V(y)}{|x-y|} dy < 4\pi. \tag{4.4}$$

For the Schrödinger operator in this setting, estimate (1.7) holds for $q = 2$ (see Theorem 7.15, [**14**]). We then have the following result.

Proposition 4.4. *Assume that $L = -\Delta + V$ where Δ is the standard Laplace operator acting on $\mathbb{R}^3$ and $V(x) \ge 0$ is a compactly supported function satisfying condition* (4.4). *Then the operator L satisfies estimate* (1.7) *for $q = 2$, and hence the results of Theorems* 1.1 *and* 1.3 *hold for $q = 2$.*

Proof. This result is a consequence of Theorem 7.15 of [**14**] and Theorems 1.1 and 1.3. □

Acknowledgements. The authors would like to thank the referee for several useful suggestions.

References

[1] G. Alexopoulos, Spectral multipliers on Lie groups of polynomial growth, Proc. Amer. Math. Soc., **120** (1994), 973–979.

[2] P. Auscher, On Necessary and Sufficient Conditions for L^p-Estimates of Riesz Transforms Associated to Elliptic Operators on $\mathbb{R}^n$ and Related Estimates, Mem. Amer. Math. Soc., **186**, Amer. Math. Soc., Providence, RI, 2007.

[3] P. Auscher, T. Coulhon, X. T. Duong and S. Hofmann, Riesz transform on manifolds and heat kernel regularity, Ann. Sci. École Norm. Sup. (4), **37**, (2004), 911–957.

[4] F. Bernicot and J. Zhao, New abstract Hardy spaces, J. Funct. Anal., **255** (2008), 1761–1796.

[5] A. Carbery, The boundedness of the maximal Bochner-Riesz operator on $L^4(\mathbb{R}^2)$, Duke Math. J., **50** (1983), 409–416.

[6] A. Carbery, G. Gasper and W. Trebels, Radial Fourier multipliers of $L^p(\mathbb{R}^2)$, Proc. Nat. Acad. Sci. U.S.A., **81** (1984), 3254–3255.

[7] M. Christ, On almost everywhere convergence of Bochner-Riesz means in higher dimensions, Proc. Amer. Math. Soc., **95** (1985), 16-20.

[8] M. Christ, L^p bounds for spectral multipliers on nilpotent groups, Trans. Amer. Math. Soc., **328** (1991), 73–81.

[9] R. R. Coifman and G. Weiss, Analyse Harmonique Non-Commutative sur Certains Espaces Homogènes, Lecture Notes in Math., **242**, Springer-Verlag, Berlin, New York, 1971.

[10] E. B. Davies, Heat Kernels and Spectral Theory, Cambridge Tracts in Math., **92**, Cambridge University Press, 1989.

[11] L. De Michele and G. Mauceri, H^p multipliers on stratified groups, Ann. Mat. Pura Appl. (4), **148** (1987), 353–366.

[12] H. Dappa, A Marcinkiewicz criterion for L^p-multipliers, Pacific J. Math., **111** (1984), 9–21.

[13] X. T. Duong and A. McIntosh, Singular integral operators with non-smooth kernels on irregular domains, Rev. Mat. Iberoamericana, **15** (1999), 233–265.

[14] X. T. Duong, E. M. Ouhabaz and A. Sikora, Plancherel-type estimates and sharp spectral multipliers, J. Funct. Anal., **196** (2002), 443–485.

[15] X. T. Duong, A. Sikora and L. Yan, Weighted norm inequalities, Gaussian bounds and sharp spectral multipliers, J. Funct. Anal., **260** (2011), 1106–1131.

[16] G. B. Folland and E. M. Stein, Hardy spaces on Homogeneous Groups, Math. Notes, **28**, Princeton University Press, 1982.

[17] C. Guillarmou, A. Hassell and A. Sikora, Restriction and spectral multiplier theorems on asymptotically conic manifolds, Anal. PDE, to appear.

[18] L. Hörmander, Estimates for translation invariant operators in L^p spaces, Acta Math., **104** (1960), 93–140.

[19] A. Hulanicki and E. M. Stein, Marcinkiewicz multiplier theorem for stratified groups, unpublished manuscript.

[20] S. Igari, A note on the Littlewood-Paley function $g^*(f)$, Tôhoku Math. J. (2), **18** (1966),

232–235.

[21] S. Igari and S. Kuratsubo, A sufficient condition for L^p-multipliers, Pacific J. Math., **38** (1971), 85–88.

[22] S. Lee, K. M. Rogers and A. Seeger, Improved bounds for Stein's square functions, preprint arXiv:1012.2159.

[23] E. M. Ouhabaz, Analysis of Heat Equations on Domains, London Math. Soc. Monogr., **31**, Princeton University Press, 2005.

[24] M. Reed and B. Simon, Methods of Modern Mathematical Physics. I, Academic Press, 1980.

[25] D. W. Robinson, Elliptic Operators and Lie Groups, Oxford Math. Monogr., The Clarendon Press, Oxford University Press, New York, 1991.

[26] A. Sikora, On the $L^2 \to L^\infty$ norms of spectral multipliers of "quasi-homogeneous" operators on homogeneous groups, Trans. Amer. Math. Soc., **351** (1999), 3743–3755.

[27] B. Simon, Maximal and minimal Schrödinger forms, J. Operator Theory, **1** (1979), 37–47.

[28] E. M. Stein, Localization and summability of multiple Fourier series, Acta Math., **100** (1958), 93–147.

[29] E. M. Stein, Harmonic Analysis: Real Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton Math. Ser., **43**, Princeton University Press, Princeton, NJ, 1993.

[30] E. M. Stein and G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces, Princeton Math. Ser., **32**, Princeton University Press, Princeton, NJ, 1971.

[31] T. Tao, A sharp bilinear restrictions estimate for paraboloids, Geom. Funct. Anal., **13** (2003), 1359–1384.

[32] N. Varopoulos, L. Saloff-Coste and T. Coulhon, Analysis and Geometry on Groups, Cambridge Tracts in Math., **100**, Cambridge University Press, Cambridge, 1992.

[33] K. Yosida, Functional Analysis (Fifth edition), Grundlehren Math. Wiss., **123**, Springer-Verlag, Berlin, 1978.

[34] A. Zygmund, Trigonometric Series. 2nd ed., **2**, Cambridge University Press, New York, 1959.

Peng Chen
Department of Mathematics
Sun Yat-sen (Zhongshan) University
Guangzhou 510275, P.R. China
E-mail: achenpeng1981@163.com

Xuan Thinh Duong
Department of Mathematics
Macquarie University
NSW 2109, Australia
E-mail: xuan.duong@mq.edu.au

Lixin Yan
Department of Mathematics
Sun Yat-sen (Zhongshan) University
Guangzhou 510275, P.R. China
E-mail: mcsylx@mail.sysu.edu.cn