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C^1 subharmonicity of harmonic spans for certain discontinuously moving Riemann surfaces

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Abstract. We showed in [**3**] and [**4**] the variation formulas for Schiffer spans and harmonic spans of the moving domain $D(t)$ in $\mathbb{C}_z$ with parameter $t \in B = \{t \in \mathbb{C}_t : |t| < \rho\}$, respectively, such that each $\partial D(t)$ consists of a finite number of C^ω contours $C_j(t)$ $(j = 1 \ldots, \nu)$ in $\mathbb{C}_z$ and each $C_j(t)$ varies C^ωsmoothly with $t \in B$. This implied that, if the total space $\mathcal{D} = \bigcup_{t \in B}(t, D(t))$ is pseudoconvex in $B \times \mathbb{C}_z$, then the Schiffer span is logarithmically subharmonic and the harmonic span is subharmonic on B, respectively, so that we showed those applications. In this paper, we give the indispensable condition for generalizing these results to Stein manifolds. Precisely, we study the general variation under pseudoconvexity, i.e., the variation of domains $\mathcal{D} : t \in B \to D(t)$ is pseudoconvex in $B \times \mathbb{C}_z$ but neither each $\partial D(t)$ is smooth nor the variation is smooth for $t \in B$.

1. Introduction.

Let D be a domain in $\mathbb{C}_z$ bounded by C^ω smooth contours $C_1, \ldots, C_\nu$. We assume D contains two points $0, 1$. Then there exists the special univalent functions $P(z)$ and $Q(z)$, called the circular slit and the radial slit mappings on D, i.e.,

$$P(z) = \begin{cases} \dfrac{1}{z} + a_0 + a_1 z + \cdots & \text{at } z = 0, \\ A_1(z-1) + A_2(z-1)^2 + \cdots & \text{at } z = 1 \end{cases}$$

such that $|P(z)| = r_j$ (constant) on C_j $(j = 1, \ldots, \nu)$;

$$Q(z) = \begin{cases} \dfrac{1}{z} + b_0 + b_1 z + \cdots & \text{at } z = 0, \\ B_1(z-1) + B_2(z-1)^2 + \cdots & \text{at } z = 1 \end{cases}$$

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such that $\arg Q(z) = \theta_j$ (constant) on C_j $(j = 1, \dots, \nu)$.

We set $p(z) = \log|P(z)|$ (resp. $q(z) = \log|Q(z)|$) which is called the L_1- (resp. L_0-) principal function for $(D, 0, 1)$, and set $\alpha = \log|A_1|$ (resp. $\beta = \log|B_1|$) which is called the L_1-(resp. L_0-) constant for $(D, 0, 1)$:

$$p(z) = \begin{cases} \log \dfrac{1}{|z|} + h_0(z) & \text{at } z = 0, \text{ where } h_0(0) = 0, \\ \log|z-1| + \alpha + h_1(z) & \text{at } z = 1, \text{ where } h_1(1) = 0, \end{cases}$$

$$p(z) = c_j(\text{constant}) \text{ on } C_j \text{ and } \int_{C_j} \frac{\partial p}{\partial n_z} ds_z = 0 \ (j = 1, \dots, \nu);$$

$$q(z) = \begin{cases} \log \dfrac{1}{|z|} + \tilde{h}_0(z) & \text{at } z = 0, \text{ where } \tilde{h}_0(0) = 0, \\ \log|z-1| + \beta + \tilde{h}_1(z) & \text{at } z = 1, \text{ where } \tilde{h}_1(1) = 0, \end{cases}$$

$$\frac{\partial q}{\partial n_z} = 0 \text{ on } C_j \quad (j = 1, \dots, \nu).$$

We set $s = \alpha - \beta$, which is called the harmonic span for $(D, 0, 1)$ (see [**6**]).

Let $B = \{t \in \mathbb{C}_t : |t| < \rho\}$. We consider a variation of domains:

$$\mathcal{D} : t \in B \to D(t) \subset \mathbb{C}_z.$$

In this paper, we identify the variation $\mathcal{D}$ with the subset $\bigcup_{t \in B}(t, D(t))$ of $B \times \mathbb{C}_z$, and write

$$\mathcal{D} = \bigcup_{t \in B}(t, D(t)) \quad \text{and} \quad \partial\mathcal{D} = \bigcup_{t \in B}(t, \partial D(t)) \quad (\subset B \times \mathbb{C}_z).$$

Moreover, for a subset B_0 of B, we write $\mathcal{D}|_{B_0} : t \in B_0 \to D(t)$ and set $\mathcal{D}|_{B_0} = \bigcup_{t \in B_0}(t, D(t))$ and $\partial\mathcal{D}|_{B_0} = \bigcup_{t \in B_0}(t, \partial D(t))$. When each $D(t)$, $t \in B$ is a domain bounded by C^ω smooth contours $C_j(t)$ $(j = 1, \dots, \nu)$ in $\mathbb{C}_z$ and each $C_j(t)$ varies C^ω smoothly with parameter $t \in B$, we call the total space $\mathcal{D}$ is a *smooth variation.*

We assume that $\mathcal{D} \supset B \times \{0, 1\}$. Then each $D(t), t \in B$ carries the L_1-principal function $p(t, z)$, the L_1-constant $\alpha(t)$ and the harmonic span $s(t)$ for $(D(t), 0, 1)$. In this paper we shall use the following form in [**2**, Lemma 3] and the result in [**4**, Theorem 4.1]:

LEMMA 1.1 ([**2**]).

$$\frac{\partial \alpha(t)}{\partial t} = \frac{1}{\pi} \int_{\partial D(t)} k_1(t,z) \left| \frac{\partial p(t,z)}{\partial z} \right|^2 ds_z, \quad t \in B, \tag{1.1}$$

where $k_1(t,z) = (\partial\varphi/\partial t)/|\partial\varphi/\partial z|$ on $\partial\mathcal{D}$ does not depend on the choice of defining functions $\varphi(t,z)$ of $\partial\mathcal{D}$.

Here, for a point $p \in \partial\mathcal{D}$ and a neighborhood U of p, defining function $\varphi(t,z)$ for $U \cap \partial\mathcal{D}$ at p means a C^2 function φ in U satisfying $U \cap D = \{(t,z) \in U \mid \varphi(t,z) < 0\}$, $U \cap \partial D = \{(t,z) \in U \mid \varphi(t,z) = 0\}$ and $\partial\varphi/\partial z(t,z) \neq 0$ on $U \cap \partial\mathcal{D}$.

THEOREM 1.2 ([**4**]). *If the total space $\mathcal{D}$ is a pseudoconvex domain in $B \times \mathbb{C}_z$, then $s(t)$ is a C^ω subharmonic function on B.*

To apply these results to Stein manifolds, we need to study the general (non-smooth) variation $\mathcal{D}$ under pseudoconvexity, namely, the variation of domains $\mathcal{D} : t \in B \to D(t) \subset \mathbb{C}_z$ is pseudoconvex in $B \times \mathbb{C}_z$ such that neither $\partial D(t)$, $t \in B$ is always C^ω smooth nor the variation $t \in B \to \partial D(t)$ is C^ω smooth with $t \in B$.

Let $B = \{t \in \mathbb{C}_t : |t| < \rho\}$ and $\widetilde{\mathcal{D}}$ be a domain in $B \times \mathbb{C}_z$ such that $\widetilde{\mathcal{D}} \supset B \times \gamma$ where γ is an arc connecting 0 and 1 in $\mathbb{C}_z$. We set $\widetilde{D}(t) = \widetilde{\mathcal{D}} \cap (\{t\} \times \mathbb{C}_z)$ for $t \in B$, called the fiber of $\widetilde{\mathcal{D}}$ over t, so that $\widetilde{\mathcal{D}} = \bigcup_{t\in B}(t, \widetilde{D}(t))$. Let $\varphi(t,z)$ be a C^ω *strictly plurisubharmonic function* on $\widetilde{\mathcal{D}}$ and let

$$\hat{\mathcal{D}} = \{(t,z) \in \widetilde{\mathcal{D}} : \varphi(t,z) < 0\},$$
$$\hat{D}(t) = \{z \in \widetilde{D}(t) : \varphi(t,z) < 0\} \text{ for every } t \in B.$$

We set $\partial\hat{\mathcal{D}} = \bigcup_{t\in B}(t, \partial\hat{D}(t))$. Assume that $\hat{\mathcal{D}}$ is connected in $\widetilde{\mathcal{D}}$, $\hat{\mathcal{D}} \supset B \times \gamma$, and $\hat{D}(t) \Subset \widetilde{D}(t)$ for $t \in B$. Then $\hat{\mathcal{D}} = \bigcup_{t\in B}(t, \hat{D}(t))$ is pseudoconvex in $B \times \mathbb{C}_z$. We denote by $D(t)$, $t \in B$ the connected component of $\hat{D}(t)$ containing γ. Note that $\hat{D}(t) \neq D(t)$ in general. We set

$$\mathcal{D} = \bigcup_{t\in B}(t, D(t)) \quad \text{and} \quad \partial\mathcal{D} = \bigcup_{t\in B}(t, \partial D(t)).$$

We thus have two variations of domains in $B \times \mathbb{C}_z$:

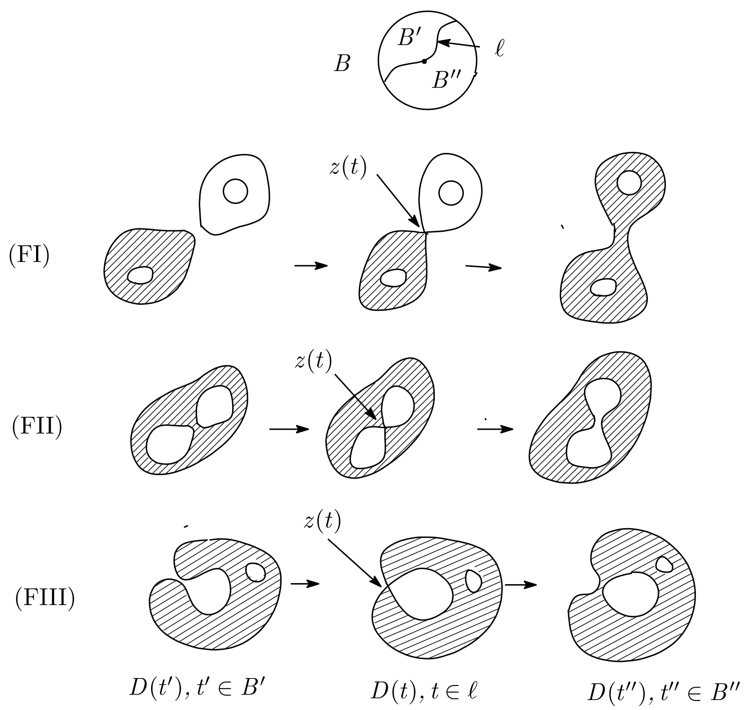
$$\hat{\mathcal{D}} : t \in B \to \hat{D}(t), \quad \mathcal{D} : t \in B \to D(t).$$

The total space $\mathcal{D}$ as well as $\hat{\mathcal{D}}$ is pseudoconvex in $B \times \mathbb{C}_z$. In general, the variation $\mathcal{D}$ is discontinuous. For [**3**] and [**4**], in this paper we treat $\mathcal{D}$ with the following cases: there exists a C^ω simple arc ℓ which divides B into two domains B' and B'', i.e., $B = B' \cup \ell \cup B''$ such that, for any $t \in B' \cup B''$ the boundary $\partial D(t)$ of the domain $D(t)$ in $\widetilde{D}(t)$ is C^ω smooth, and for any $t \in \ell$ the boundary $\partial D(t)$ of $D(t)$ has only one singular point $z(t)$, namely, $\varphi(t, z(t)) = (\partial\varphi/\partial z)(t, z(t)) = 0$, which is of the case (c1) or (c2) mentioned below, precisely, if $z(t_0)$ for some $t_0 \in \ell$ is of case (c1) (resp. (c2)), then $z(t)$ for each $t \in \ell$ is of the same case (c1) (resp. (c2)). Let $C(t)$ denote the connected component of $\partial\hat{D}(t)$ passing through $z(t)$. Then

(c1) $C(t)$ consists of two closed curves $C_1(t)$ and $C_1^*(t)$, and one of them, say $C_1(t)$, is one of boundary components of $D(t)$, so that $[C_1^*(t) \setminus \{z(t)\}] \cap \partial D(t) = \emptyset$ (we say that the variation $\mathcal{D}$ is of case (c1));

(c2) $C(t)$ is one of the boundary components of $D(t)$ (we say that the variation $\mathcal{D}$ is of case (c2)).

We find two distinct points of $\partial D(t)$ over $z(t)$ in case (c2). For example, if the shadowed part below is $D(t)$, then the singular point $z(t)$ is of case (c1) for (FI), and of case (c2) for (FII) and (FIII).



In the general case, using the strictly plurisubharmonicity of $\varphi(t,z)$, we find in [**4**] an increasing sequence $a_n \nearrow 0$ $(n \to \infty)$ such that, if we construct $\mathcal{D}_{a_n}$ from $\widehat{\mathcal{D}}_{a_n} := \{\varphi < a_n\}$ in $\widetilde{\mathcal{D}}$ like as $\mathcal{D}$ in $\widehat{\mathcal{D}} = \{\varphi(t,z) < 0\}$ in $\widetilde{\mathcal{D}}$, then there exists a finite number of C^ω arcs $\{\ell_k\}_{k=1,\dots,\nu}$ (depending on a_n) in B such that, if we put $\{t_1,\dots,t_q\}$ the common points of ℓ_i and ℓ_j or ℓ_i itself (for any $i,j = 1,\dots,\nu$), and if we set $\ell'_k = \ell_k \setminus \{t_1,\dots,t_q\}$ $(k=1,\dots,\nu)$, then at each point $t^0 \in \ell'_k$ we find a small disk B^0 of center t^0 such that $\mathcal{D}_{a_n}$ over B^0 with $\ell'_k \cap B^0$ is of the case (c1) or (c2) like $\mathcal{D}$ (over B) with ℓ (in B) in the above figures.

Each $D(t)$, $t \in B$ carries the L_1-principal function $p(t,z)$ and L_1-constant $\alpha(t)$ for $(D(t),0,1)$, similarly the L_0-principal function $q(t,z)$ and the L_0-constant $\beta(t)$, and then the harmonic span $s(t)(=\alpha(t)-\beta(t))$ for $(D(t),0,1)$.

Theorem 1.3. *Under the above situation,*

(i) *the harmonic span $s(t)$ for $(D(t),0,1)$ for the variation $\mathcal{D}$ of case* (c1) *is C^1 subharmonic on B (we give a concrete example such that $s(t)$ is C^1 subharmonic on B not of class C^2);*

(ii) *there exists a counterexample for the variation $\mathcal{D}$ of case* (c2) *such that $s(t)$ is neither of class C^1 nor subharmonic on B.*

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2. C^1 subharmonicity of harmonic spans $s(t)$ for the variation $\mathcal{D}$ of case (c1).

Under the above notation, we shall state more precisely the situation of case (c1) for the proof of Theorem 1.3 (i). Let $\mathcal{D}$ be a variation of case (c1). Then there exists a smooth arc l in B which separates B into two domains B' and B'' with the following three conditions:

(I) $\hat{D}(t)$, $t \in B'$ consists of two disjoint domains $D(t)$ and $D^*(t)$. Here, $\partial D(t) = \sum_{j=1}^m C_j(t)$ and $\partial D^*(t) = C_1^*(t) + \sum_{j=m+1}^n C_j(t)$ are C^ω smooth contours. $C_1(t)$ and $C_1^*(t)$ are the outer boundary components of $D(t)$ and $D^*(t)$, respectively.

(II) $\hat{D}(t)$, $t \in l$ consists of two disjoint domains $D(t)$ and $D^*(t)$ such that the outer boundary components $C_1(t)$ and $C_1^*(t)$ of $D(t)$ and $D^*(t)$ are piecewise C^ω smooth contours with only one corner, respectively, whose corner is the common one $z(t)$, i.e., $C_1(t) \cap C_1^*(t) = \{z(t)\}$; and the other boundary components C_j $(j = 2,\dots,m,m+1,\dots,n)$ of them are C^ω smooth contours;

(III) $\hat{D}(t)$, $t \in B''$ is connected, i.e., $\hat{D}(t) = D(t)$. Here, $\partial D(t) = \sum_{j=1}^n C_j(t)$

are C^ω smooth contours, and $C_1(t)$ is the outer boundary component of $D(t)$.

Lemma 2.1. *Under the above situation, if the variation $\mathcal{D} : t \in B \to D(t)$ is of case* (c1)*, then $\alpha(t)$ and $\beta(t)$ are C^1 subharmonic and C^1 superharmonic on B, respectively, and hence $s(t)$ is C^1 subharmonic on B.*

Proof. Since the proof for $\beta(t)$ is similar to that for $\alpha(t)$, we shall prove that $\alpha(t)$ *is C^1 subharmonic on B*. Since $\mathcal{D}|_{B'} : t \in B' \to D(t)$ and $\mathcal{D}|_{B''} : t \in B'' \to D(t)$ are smooth variations, we see that the L_1-principal function $p(t,z)$ for $(D(t), 0, 1)$ is of class C^ω for $(t,z) \in (\mathcal{D}|_{B'} \cup \mathcal{D}|_{B''}) \setminus [(B' \cup B'') \times \{0,1\}]$, and $\alpha(t)$ is subharmonic of class C^ω on $B' \cup B''$.

It thus suffices to prove that $\alpha(t)$ *is of class C^1 on B*. Fix any $t_0 \in l$ and $B_0 = \{|t - t_0| < r_0\} \Subset B$. We set $B_0' = B_0 \cap B'$ and $B_0'' = B_0 \cap B''$, so that $B_0 = B_0' \cup B_0'' \cup l|_{B_0}$. By the C^1 smoothness of $\alpha(t)$ on B and the C^ω subharmonicity of $\alpha(t)$ on $B' \cup B''$, we have by Green's formula

$$\begin{aligned}\int_{\partial B_0} \frac{\partial \alpha}{\partial n_z} ds_z &= \int_{\partial B_0'} \frac{\partial \alpha}{\partial n_z} ds_z + \int_{\partial B_0''} \frac{\partial \alpha}{\partial n_z} ds_z \\ &= \iint_{B_0'} \Delta \alpha dxdy + \iint_{B_0''} \Delta \alpha dxdy \geq 0. \end{aligned} \tag{2.1}$$

From C^1 smoothness of $\alpha(t)$ on B,

$$\int_{\partial B_0} \frac{\partial \alpha}{\partial n_z} ds_z = \int_0^{2\pi} \frac{\partial \alpha}{\partial r}(t_0 + r_0 e^{i\theta}) r_0 d\theta = r_0 \left[\frac{d}{dr} \int_0^{2\pi} \alpha(t_0 + re^{i\theta}) d\theta \right]_{r=r_0}.$$

If we set $S(r) := \int_0^{2\pi} \alpha(t_0 + re^{i\theta}) d\theta$ for $0 \leq r < \rho$, then we see from (2.1) that $dS/dr(r_0) \geq 0$, so that $S(r)$ is an increasing function on $(0, \rho)$. Thus, $S(r_0) \geq \lim_{r \to 0} S(r) = 2\pi\alpha(t_0)$, and hence $1/2\pi \int_0^{2\pi} \alpha(t_0 + re^{i\theta}) d\theta \geq \alpha(t_0)$. Thus, $\alpha(t)$ is subharmonic on B_0.

Let $t_0 \in l$ be fixed. It is enough to show that $\alpha(t)$ *is of class C^1 at the point* t_0. Precisely speaking, it suffices to prove that

(A) $\alpha(t)$ is continuous on B;
(B) $\{\partial \alpha(t)/\partial t\}_{t \in B' \cup B''}$ is a Cauchy sequence at the point t_0. Precisely, for given $\epsilon > 0$, there exists a disk $B_\epsilon = \{|t - t_0| < \rho_\epsilon\}$ in B such that

$$\left| \frac{\partial \alpha}{\partial t}(t') - \frac{\partial \alpha}{\partial t}(t'') \right| < \epsilon \quad \text{for every } t',\ t'' \in B_\epsilon' \cup B_\epsilon'', \tag{2.2}$$

where $B_\epsilon \setminus l = B'_\epsilon \cup B''_\epsilon$ such that $B'_\epsilon \subset B'$ and $B''_\epsilon \subset B''$.

Proof of (A). It suffices to show that $p(t,z) \to p(t_0,z)$ as $t \in B' \cup B'' \to t_0$ locally uniformly in $\mathcal{D}$. We consider the circular slit mapping $P(t,z)$ on each $D(t)$, $t \in B$ such that (i) $\log|P(t,z)| = p(t,z)$ on $D(t)$, (ii) $P(t,z) - 1/z$ is regular at $z=0$, (iii) $P(t,1)=0$. Since $P(t,z), t \in B$ is univalent on $D(t)$ with condition (ii), it follows from Koebe's distortion theorem that $\{P(t,z)\}_{t\in B'\cup B''}$ is a normal family in the following sense: for any given sequence $\{P(t_k,z)\}_k$ where $t_k \in B'\cup B''$ such that $t_k \to t_0$ as $k \to \infty$, there exists a subsequence $\{P(t_{k_j},z)\}_j$ which locally uniformly converges a certain univalent function $F(z)$ on $D(t_0)$. Then $F(z)$ is a circular slit mapping on $D(t_0)$ since each $P(t_{k_j},z)$ is a circular slit mapping on $D(t)$, $t_{k_j} \in B' \cup B''$. We remark that the number of the slits might change but it is a finite number (at most n as in (I) and (III)). Moreover, $F(z) - 1/z$ is regular at $z=0$ and $F(1)=0$ by the conditions (ii) and (iii). Consequently, $F(z) = P(t_0,z)$ on $D(t_0)$, independent of the choice of subsequences. It follows that $P(t,z) \to P(t_0,z)$ as $t \in B' \cup B'' \to t_0$ locally uniformly in $\mathcal{D}$. Thus, $p(t,z)$ is continuous for $(t,z) \in \mathcal{D}$, and hence $\alpha(t)$ is continuous on B. □

Proof of (B). We divide the proof of (B) into 4 short steps.

[1st step] We shall show that *there exist a disk $B_1 = \{|t-t_0| < \rho_1\}$ in B and a constant $A>0$ such that*

$$\left|\frac{\partial p(t,z)}{\partial z}\right| \le A\left|\frac{\partial \varphi(t,z)}{\partial z}\right|, \quad z \in C_1(t),\ t \in B'_1 \cup B''_1.$$

Here, $B_1 \setminus l = B'_1 \cup B''_1$ such that $B'_1 \subset B'$ and $B''_1 \subset B''$.

Let γ be an arc connecting poles 0 and 1. Let L_0 be a closed curve in $D(t_0)$ which bounds a subdomain V_0 of $D(t_0)$ and $V_0 \supset \gamma$. We draw closed curves L_j $(j=2,\dots,m,m+1,\dots,n)$ in $\hat{D}(t_0)$ close to $C_j(t_0)$ such that L_j $(j=0,2,\dots,n)$ are mutually disjoint in $\hat{D}(t_0)$. We can take a small disk $B_1 = \{|t-t_0| < \rho_1\}$ in B such that $\hat{D}(t) \supset L_j$ $(j=0,2,\dots,n)$ for every $t \in B_1$. Since $L_j \subset D(t)$ $(j=2,\dots,m)$ for $t \in B'_1$ and $L_j \subset D(t)$ $(j=2,\dots,n)$ for $t \in B''$, we consider a subdomain $E(t)$ of $D(t)$ for $t \in B'_1 \cup B''_1$ (where $B_1 \setminus l = B'_1 \cup B''_1$ such that $B'_1 \subset B'$ and $B''_1 \subset B''$) such that

$$\partial E(t) = \begin{cases} C_1(t) + L_0 + \sum_{j=2}^m L_j & \text{if } t \in B'_1; \\ C_1(t) + L_0 + \sum_{j=2}^n L_j & \text{if } t \in B''_1. \end{cases}$$

We set $p(t,z) = c_1(t)$ constant on $C_1(t)$ for $t \in B_1$, and

$$
\begin{aligned}
K &= \sup\{\varphi(t,z) : (t,z) \in B_1 \times [L_0 \cup L_2 \cup \cdots \cup L_n]\}, \\
m(t) &= \begin{cases} \sup\{|p(t,z) - c_1(t)| : z \in L_0 \cup L_2 \cup \cdots \cup L_m\} & \text{if } t \in B_1'; \\ \sup\{|p(t,z) - c_1(t)| : z \in L_0 \cup L_2 \cup \cdots \cup L_m \cdots \cup L_n\} & \text{if } t \in B_1'', \end{cases} \\
M &= \sup\{m(t) : t \in B_1' \cup B_1''\}.
\end{aligned}
$$

Then we have $-\infty < K < 0$ and $0 < M < \infty$. We set $A = -M/K > 0$. Let $t \in B_1' \cup B_1''$ be fixed. We consider the following function $\phi(t,z)$ on $E(t)$ defined by

$$\phi(t,z) = A\varphi(t,z) + |p(t,z) - c_1(t)|, \quad z \in E(t).$$

Since $\varphi(t,z)$ is subharmonic and $p(t,z)$ is harmonic on $E(t)$, $\phi(t,z)$ is subharmonic on $E(t)$. Since $\phi(t,z) = 0$ on $C_1(t)$ and $\phi(t,z) \le 0$ on $\partial E(t) \setminus C_1(t)$, it follows from the maximum principle that $\phi(t,z) \le 0$ on $E(t)$, namely,

$$A\varphi(t,z) \le -|p(t,z) - c_1(t)|, \quad z \in E(t) \cup \partial E(t).$$

Since $\varphi(t,z) = |p(t,z) - c_1(t)| = 0$ on $C_1(t)$ and $\varphi(t,z) < 0$ on $E(t)$, it follows that

$$A \frac{\partial \varphi(t,z)}{\partial n_z} \ge \left| \frac{\partial (p(t,z) - c_1(t))}{\partial n_z} \right|, \quad z \in C_1(t),$$

where n_z is the unit outer normal vector of $C_1(t)$ at z. This is identical with $A|\partial\varphi(t,z)/\partial z| \ge |\partial p(t,z)/\partial z|$ for $z \in C_1(t)$, which proves the 1st step.

[2nd step] We set $N = \sup\{|\partial\varphi(t,z)/\partial t|, |\partial\varphi(t,z)/\partial z| : (t,z) \in \mathcal{D}|_{B_1}\} < \infty$, where $\mathcal{D}|_{B_1} = \bigcup_{t \in B_1}(t, D(t)) \subset B_1 \times \mathbb{C}_z$. To prove (2.2), let $\epsilon > 0$ be given. Then we shall show that *there exists a small disk* $V = \{|z - z(t_0)| < r\}$ *in* $\mathbb{C}_z$ *such that*

$$\left| \int_{C_1(t) \cap V} k_1(t,z) \left| \frac{\partial p(t,z)}{\partial z} \right|^2 ds_z \right| < \frac{\epsilon}{2}, \quad t \in B_1' \cup B_1''.$$

Here, $C_1(t) \cap V$ *for some* t *may be empty.*

Let $V = \{|z - z(t_0)| < r\}$ be a disk in $\mathbb{C}_z$ such that

$$\int_{C_1(t) \cap V} ds_z < \frac{\epsilon}{2A^2N^2}, \quad t \in B_1.$$

Note that $\varphi(t,z)$ is a defining function of $\partial\mathcal{D}$. From Lemma 1.1 and the 1st step, we have for $t \in B_1' \cup B_1''$,

$$|k_1(t,z)|\left|\frac{\partial p(t,z)}{\partial z}\right|^2 = \left|\frac{\frac{\partial \varphi}{\partial t}}{\frac{\partial \varphi}{\partial z}}\right|\left|\frac{\partial p}{\partial z}\right|^2 \le \left|\frac{\frac{\partial \varphi}{\partial t}}{\frac{\partial \varphi}{\partial z}}\right| A^2 \left|\frac{\partial \varphi}{\partial z}\right|^2$$

$$= \left|\frac{\partial \varphi}{\partial t}\right| A^2 \left|\frac{\partial \varphi}{\partial z}\right| \le A^2 N^2 \quad \text{on } C_1(t).$$

We thus have that, for $t \in B_1' \cup B_1''$,

$$\left|\int_{C_1(t)\cap V} k_1(t,z)\left|\frac{\partial p(t,z)}{\partial z}\right|^2 ds_z\right| \le A^2N^2 \int_{C_1(t)\cap V} ds_z < \frac{\epsilon}{2},$$

which is desired.

[3rd step] We take $B_2 = \{|t-t_0| < \rho_2\}$ in B_1 such that each $D(t)\cap V$, $t \in B_2$ is a non-empty simply connected domain in V. We set $\hat{\mathcal{G}} = \mathcal{D}|_{B_2} \setminus (B_2 \times V) = \bigcup_{t\in B_2}(t, D(t)\setminus V) \subset B_2 \times \mathbb{C}_z$, where $\mathcal{D}|_{B_2} : t \in B_2 \to D(t)$, and $B_2 \setminus l = B_2' \cup B_2''$ such that $B_2' \subset B_1'$ and $B_2'' \subset B_1''$, so that $\hat{\mathcal{G}}$ consists of two disjoint domains $\mathcal{G}$ and $\mathcal{G}^*$ such that

$$\mathcal{G} = \bigcup_{t\in B_2}(t, G(t)), \quad \mathcal{G}^* = \bigcup_{t\in B_2''}(t, G^*(t)).$$

Thus, both $G(t)$, $t \in B_2$ and $G^*(t)$, $t \in B_2''$ are piecewise C^ω smooth domains with two corners on ∂V, and both variations $\mathcal{G} : t \in B_2 \to G(t)$ and $\mathcal{G}^* : t \in B_2'' \to G^*(t)$ are C^ω smooth with $t \in B_2$ and B_2'', respectively. Under the above situation, we shall prove that *there exists a disk $B_3 = \{|t-t_0| < \rho_3\}$ in B_2 such that, for every $t', t'' \in B_3' \cup B_3''$,*

$$\left|\int_{(\partial D(t'))\setminus V} k_1(t',z)\left|\frac{\partial p(t',z)}{\partial z}\right|^2 ds_z - \int_{(\partial D(t''))\setminus V} k_1(t'',z)\left|\frac{\partial p(t'',z)}{\partial z}\right|^2 ds_z\right| < \frac{\epsilon}{2}.$$

Here, $B_3 \setminus l = B_3' \cup B_3''$ such that $B_3' \subset B_2'$ and $B_3'' \subset B_2''$.

Let W be a thin tubular neighborhood of $\partial G(t_0)(= \partial[D(t_0)\setminus V])$ in $\mathbb{C}_z$. We can take a disk $B_3 = \{|t-t_0| < \rho_3\}$ in B_2 such that each $p(t,z)$, $t \in B_3$ is harmonically extended to W. It follows from (A) that $p(t,z) \to p(t_0,z)$ as $t \in B_3 \to t_0$ uniformly in W, so that $\partial p(t,z)/\partial z \to \partial p(t_0,z)/\partial z$ as $t \in B_3 \to t_0$ uniformly in W. Since $k_1(t,z) = (\partial\varphi(t,z)/\partial t)/|\partial\varphi(t,z)/\partial z|$ is of class C^ω for $(t,z) \in W$ and $\partial G(t) \to \partial G(t_0)$ as $t \in B_3 \to t_0$ uniformly, we have

$$\lim_{t\in B_3\to t_0}\int_{\partial G(t)\setminus\partial V} k_1(t,z)\left|\frac{\partial p(t,z)}{\partial z}\right|^2 ds_z = \int_{\partial G(t_0)\setminus\partial V} k_1(t_0,z)\left|\frac{\partial p(t_0,z)}{\partial z}\right|^2 ds_z. \tag{2.3}$$

Similarly, we find a neighborhood W^* of $\partial G^*(t_0)$ ($= \partial[D^*(t_0)\setminus V]$ where $D^*(t_0)$ is defined in (II)) in $\mathbb{C}_z$ such that each $p(t,z)$, $t\in B_3''$ is harmonically extended to W^*. On the other hand, on any given compact set $K \Subset D^*(t_0)$, $p(t,z)$ for $t\in B''$ uniformly converges to a certain harmonic function $\hat{p}(z)$ on $D^*(t_0)$ such that the boundary values are $\hat{p}(z)=c_1(t_0)$ on $C_1^*(t_0)$ and $\hat{p}(z)$ is a certain constant $\hat{c}_j$ on each $C_j(t_0)$ ($j=m+1,\dots,n$). Since $\int_{C_j(t)} *dp(t,z)=0$ for $t\in B_2''$, we have $\int_{C_j(t)} *d\hat{p}(z)=0$. This implies that $\hat{p}(z)\equiv c_1(t_0)$ on $D^*(t_0)$. It follows that $p(t,z)\to c_1(t_0)$ as $t\in B_3''\to t_0$ uniformly on W^*, so that $\partial p(t,z)/\partial z\to 0$ as $t\in B_3''\to t_0$ on W^*. By the same argument as above, we have

$$\lim_{t\in B_3''\to t_0}\int_{\partial G^*(t)\setminus\partial V} k_1(t,z)\left|\frac{\partial p(t,z)}{\partial z}\right|^2 ds_z = 0. \tag{2.4}$$

Since $(\partial D(t))\setminus V = \partial G(t)\setminus\partial V$ for $t\in B_3'$ and $(\partial D(t))\setminus V = [\partial G(t)\setminus\partial V]\cup[\partial G^*(t)\setminus\partial V]$ for $t\in B_3''$, we obtain from (2.3) and (2.4) that

$$\lim_{t\in B_3'\cup B_3''\to t_0}\int_{(\partial D(t))\setminus V} k_1(t,z)\left|\frac{\partial p(t,z)}{\partial z}\right|^2 ds_z = \int_{(\partial D(t_0))\setminus V} k_1(t_0,z)\left|\frac{\partial p(t_0,z)}{\partial z}\right|^2 ds_z.$$

This immediately implies the 3rd step.

[4th step] (2.2) *holds for* $B_\epsilon = B_3$.

Since $C_1(t)\cap V = (\partial D(t))\cap V$ for $t\in B_3$, the 2nd and the 3rd steps together with (1.1) imply that, for $t',t''\in B_3'\cup B_3''$,

$$\begin{aligned}
&\left|\frac{\partial\alpha}{\partial t}(t') - \frac{\partial\alpha}{\partial t}(t'')\right| \\
&\quad\le \frac{1}{\pi}\left(\int_{(\partial D(t'))\setminus V} k_1(t',z)\left|\frac{\partial p(t',z)}{\partial z}\right|^2 ds_z - \int_{(\partial D(t''))\setminus V} k_1(t'',z)\left|\frac{\partial p(t'',z)}{\partial z}\right|^2 ds_z\right) \\
&\qquad + \frac{1}{\pi}\left|\int_{(\partial D(t'))\cap V} k_1(t',z)\left|\frac{\partial p(t',z)}{\partial z}\right|^2 ds_z\right| \\
&\qquad + \frac{1}{\pi}\left|\int_{(\partial D(t''))\cap V} k_1(t'',z)\left|\frac{\partial p(t'',z)}{\partial z}\right|^2 ds_z\right|
\end{aligned}$$

$$< \frac{1}{\pi}\left(\frac{\epsilon}{2} + \epsilon\right) < \epsilon,$$

which is desired. $\square$

Therefore, we see from (A) and (B) that $\alpha(t)$ is subharmonic on B. Similarly, $\beta(t)$ is superharmonic on B, and then $s(t)$ is subharmonic on B. $\square$

We shall give an example of the variation $\mathcal{D}$ of case (c1) such that $s(t)$ is C^1 subharmonic on B but not of class C^2.

EXAMPLE. Let $B = \{t \in \mathbb{C} : |t-1| < \rho\}$ where $0 < \rho < 1/4$. For each $t \in B$, we consider the following domain in $\mathbb{C}_z$:

$$\hat{D}(t) = \{z \in \mathbb{C}_z : |z-1||z+1| < |t|\},$$

so that each $\hat{D}(t) = \hat{D}(|t|)$. We set

$$B' = \{z \in B : |t| < 1\}, \quad l = \{z \in B : |t| = 1\}, \quad B'' = \{z \in B : |t| > 1\}.$$

Let $D(t)$ denote the connected component of $\hat{D}(t)$ containing two points $\{1, \sqrt{3/2}\}$. Then we have the variation of domains $\mathcal{D} : t \in B \to D(t)$ in $B \times \mathbb{C}_z$, and $\mathcal{D} = \bigcup_{t \in B}(t, D(t))$ is pseudoconvex in $B \times \mathbb{C}_z$ of case (c1), namely, (i) $\hat{D}(t), \in B'$ consists of two disjoint domains $D(t)$ and $D^*(t)$ in $\mathbb{C}_z$ bounded by two C^ω smooth closed curves $C_1(t) = \partial D(t)$ in $\{\Re z > 0\}$ and $C_1^*(t) = \partial D^*(t)$ in $\{\Re z < 0\}$; (ii) $\hat{D}(t)$, $t \in l$ consists of two disjoint domains $D(t)$ and $D^*(t)$ in $\mathbb{C}_z$ bounded by the lemniscate $\{|z+1||z-1| = 1\}$ passing through 0. The boundary components $C_1(t) = \partial D(t)$ in $\{\Re z \geq 0\}$ and $C_1^*(t) = \partial D^*(t)$ in $\{\Re z \leq 0\}$ are piecewise C^ω smooth contours with only one corner, respectively, whose corner is the common singular point $z(t) = 0$, i.e., $C_1(t) \cap C_1^*(t) = \{0\}$; (iii) $\hat{D}(t)$, $t \in B''$ is a simply connected domain in $\mathbb{C}_z$, i.e., $\hat{D}(t) = D(t)$, bounded by a C^ω smooth closed curve $C_1(t) = \partial D(t)$.

For constructing the exact form of the harmonic span $s(t)$ for $\mathcal{D}$, we recall the following in [**2**, p. 129] which is based on the example of Theorem 1.4 in [**1**, Section 5].

PROPOSITION 2.2. (1) *The harmonic span is invariant under the holomorphic mappings, precisely, let R be a domain in $\mathbb{C}_z$ and let $a, b \in R$, $a \neq b$. Let $w = f(z)$ be any holomorphic mapping from R onto R_1 and set $a_1 = f(a)$ and $b_1 = f(b)$. Then the harmonic span s for (R, a, b) is equal to the harmonic span s_1 for (R_1, a_1, b_1).*

(2) *Let* $R=\{|z|<r\}$ *be a disk in* $\mathbb{C}_z$ *and let* $\xi\in R$, $\xi\neq 0$. *Then the harmonic span* s *for* $(R,0,\xi)$ *is* $s=\log\{1/(1-(|\xi|/r)^2)\}$.

To give the exact form of $s(t)$ for $t\in B$, we divide two cases: (i) $t\in B'\cup\ell$ and (ii) $t\in B''$. Consider the mapping $w=f(z):=z^2-1$. It maps the z-plane to the two-sheeted Riemann surface R_1 over the w-plane with the branch point $w=-1$.

In case (i), $w=f(z)$ maps $D(t)$ to the disk $\{|w|<|t|\ (\leq 1)\}$, one of two univalent sheets of R_1 over $\{|w|<|t|\}$, and two points $\{1,\sqrt{3/2}\}$ to $\{0,1/2\}$. It follows from Proposition 2.2 that $s(t)=\log\{1/(1-(1/2|t|)^2)\}$.

In case (ii), we further consider

$$w_1=\frac{w}{t},\quad w_2=\frac{w_1+\frac{1}{t}}{1+\frac{1}{t}w_1},\quad w_3=\sqrt{w_2},\quad w_4=\frac{w_3-\sqrt{\frac{1}{t}}}{1+\frac{1}{\sqrt{t}}w_3}.$$

Then the composite function $W=F(z)=w_4\circ w_3\circ w_2\circ w_2\circ w_1\circ f(z)$ on $D(t)$ conformally maps the domain $D(t)$ to $\{|W|\leq 1\}$, and two points $\{1,\sqrt{3/2}\}$ to $\{0,\xi(t)\}$. Here $\xi(t)=(\sqrt{3}\sqrt{2|t|^2+1}-2|t|+1)/2\sqrt{t}(|t|+1)$. It follows from Proposition 2.2 that

$$s(t)=\log\frac{1}{1-\left(\left|\frac{\sqrt{3}\sqrt{2|t|^2+1}-2|t|+1}{2\sqrt{t}(|t|+1)}\right|\right)^2}.$$

By the calculation we see that $s(t)$ is C^ω subharmonic on $B'\cup B''$. For the C^1 smoothness but not C^2 smoothness of $s(t)$ on B, it is enough to check the following:

$$h(t):=\begin{cases}\frac{\sqrt{3}\sqrt{2|t|^2+1}-2|t|+1}{\sqrt{|t|}(|t|+1)}, & t\in B''\\ \frac{1}{|t|}, & t\in B'\cup l\end{cases}$$

is of class C^1 *but not of class* C^2 *on* B. Indeed, for $t=1\in l$, we have $h(1)=1$ and $\lim_{t\to 1-0}h'(t)=\lim_{t\to 1+0}h'(t)=h'(1)=-1$, but $\lim_{t\to 1-0}h''(t)=2\neq 25/12=\lim_{t\to 1+0}h''(t)$.

3. Counterexample for the variation $\mathcal{D}$ of case (c2).

We shall give a counterexample of the variation $\mathcal{D}$ of case (c2) such that $\alpha(t)$ (resp. $\beta(t)$) is neither of class C^1 on B nor subharmonic (resp. superharmonic) on B, and $s(t)$ is neither of class C^1 on B nor subharmonic on B. Since the proof is similar to other cases, we give a counterexample for $\alpha(t)$ of a variation $\mathcal{D}$ of case (c2) (like the figure of type (FII)).

Let $B = \{|t-1| < \rho\}$ where $0 < \rho < 1$. We consider the following Levi flat domain in $B \times \mathbb{P}_z$:

$$\mathcal{D} = \bigcup_{t \in B} (t, D(t)), \quad D(t) = \{z \in \mathbb{P}_z : |z-1||z+1| > |t|\},$$

so that each $D(t) \ni \infty$ and $D(t) = D(|t|)$. Let

$$B' = \{z \in B : |t| < 1\}, \quad l = \{z \in B : |t| = 1\}, \quad B'' = \{z \in B : |t| > 1\}.$$

$D(t)$, $t \in B'$ is a domain in $\mathbb{P}_z$ bounded by two smooth closed curves $C_1(t)$ in $\{\Re z < 0\}$ and $C_2(t)$ in $\{\Re z > 0\}$; $D(t)$, $t \in B''$ is a simply connected domain in $\mathbb{P}_z$ bounded by a smooth closed curve $C(t)$; and $D(t)$, $t \in l$ is a simply connected domain in $\mathbb{P}_z$ bounded by the lemniscate $C = \{|z-1||z+1| = 1\}$ passing through 0. We set $C \cap \{\Re z \le 0\} = C_1$ and $C \cap \{\Re z \ge 0\} = C_2$, so that C_1 and C_2 are piecewise C^ω smooth contours with one common corner $z = 0$, namely, $C_1 \cap C_2 = \{0\}$. We note that, for $t_0 \in l$, $\lim_{t \in B' \to t_0} C_1(t) = C_1$ and $\lim_{t \in B' \to t_0} C_2(t) = C_2$. The variation $\mathcal{D} : t \in B \to D(t)$ is of case (c2).

For $t \in B'$, we denote by $\omega_1(t,z)$ the harmonic measure for $(D(t), C_1(t))$, i.e., $\omega_1(t,z)$ is harmonic on $D(t)$ and continuous on $\overline{D(t)}$ such that $\omega_1(t,z) = 1$ (resp. 0) on $C_1(t)$ (resp. $C_2(t)$). For $t \in l$, we remark that $D(t) = D(1)$, and consider the harmonic measure $\omega_1(z)$ for $(D(1), C_1)$, i.e., $\omega_1(z)$ is harmonic on $D(1)$ and continuous on $\overline{D(1)} \setminus \{0\}$ such that $\omega_1(z) = 1$ (resp. 0) on $C_1 \setminus \{0\}$ (resp. $C_2 \setminus \{0\}$). We see that $\lim_{t \in B' \to 1} \omega_1(t,z) = \omega_1(z)$ uniformly on every compact set in $D(1)$.

We fix two point $a, b \in D(1)$, $a \ne b$ such that $\omega_1(a) \ne \omega_1(b)$. We set $K := |\omega_1(a) - \omega_1(b)| > 0$. If necessary, take a smaller disk B of center $t = 1$, then we may assume that

$$\frac{K}{2} \le |\omega_1(t,a) - \omega_1(t,b)| \le 2K \quad \text{for } t \in B'. \tag{3.1}$$

On each $D(t)$, $t \in B$, we consider L_1-principal function p for $(D(t), a, b)$, i.e.,

$$p(t,z) = \log \frac{1}{|z-a|} + h_a(t,z) \quad \text{at } z = a, \text{ where } h_a(t,a) = 0;$$

$$p(t,z) = \log |z-b| + \alpha(t) + h_b(t,z) \quad \text{at } z = b, \text{ where } h_b(t,b) = 0;$$

$$p(t,z) = \begin{cases} \text{const. } c(t) \text{ on } C(t) \text{ and } \int_{C(t)} (\partial p/\partial n_z) ds_z = 0 \quad \text{for } t \in B'' \cup l, \\ \text{const. } c_j(t) \text{ on } C_j(t) \text{ and } \int_{C_j(t)} (\partial p/\partial n_z) ds_z = 0 \ (j = 1,2) \quad \text{for } t \in B', \end{cases}$$

where $C(t) = C = C_1 \cup C_2$ for $t \in l$. The L_1-constant $\alpha(t)$ for $(D(t), a, b)$ is clearly of class C^ω on $B \setminus l = B' \cup B''$. Since $D(t) = D(|t|)$, we have $p(t, z) = p(|t|, z)$ and $\alpha(t) = \alpha(|t|)$ for $t \in B$.

We have proved in [**1**, Theorem 1.4.] that $\alpha(t)$ is C^ω subharmonic on $B \setminus l = B' \cup B''$. For $t_0 \in l$, we let $\partial/\partial n_{t_0}$ denote the outer normal derivative of l (whose direction is counter clockwise) at t_0. Then we have

Theorem 3.1. (1) *$\alpha(t)$ is continuous on B,*
(2) *$\alpha(t)$ is not of class C^1 on B, precisely,*

(i) $\dfrac{\partial \alpha(t)}{\partial t}$ *is continuous on $B'' \cup l$;*

(ii) $\displaystyle\lim_{h \nearrow 0} \frac{\alpha(t_0 + h n_{t_0}) - \alpha(t_0)}{h} = \infty$ *for every $t_0 \in l$,*

(3) *$\alpha(t)$ is not subharmonic at any $t_0 \in l$.*

To prove this theorem, we prepare the following lemmas.

We represent $\alpha(t)$, $t \in B$ by use of functions concerning the Green functions: For $t \in B$, we consider the Green function $g_a(t, z)$ (resp. $g_b(t, z)$) and the Robin constant $\lambda_a(t)$ (resp. $\lambda_b(t)$) for $(D(t), a)$ (resp. $(D(t), b)$), i.e.,

$$g_a(t, z) = \log \frac{1}{|z - a|} + \lambda_a(t) + \mathfrak{h}_a(t, z) \quad \text{at } z = a, \text{ where } \mathfrak{h}_a(t, a) = 0;$$

$$g_b(t, z) = \log \frac{1}{|z - b|} + \lambda_b(t) + \mathfrak{h}_b(t, z) \quad \text{at } z = b, \text{ where } \mathfrak{h}_b(t, b) = 0$$

with $g_a(t, z) = g_b(t, z) = 0$ on $\partial D(t)$.

For $t \in B'$, we have

$$\omega_1(t, z) = \frac{-1}{2\pi} \int_{C_1(t)} \frac{\partial g_z(t, \zeta)}{\partial n_\zeta} ds_\zeta \quad \text{for } z \in D(t), \tag{3.2}$$

and consider the Dirichlet integral $\|d\omega_1(t, \cdot)\|_{D(t)}$ of $\omega_1(t, z)$ over $D(t)$,

$$\|d\omega_1(t, \cdot)\|_{D(t)} := \iint_{D(t)} \left[\left(\frac{\partial \omega_1(t, z)}{\partial x} \right)^2 + \left(\frac{\partial \omega_1(t, z)}{\partial y} \right)^2 \right] dx dy.$$

Under these notations, we have

LEMMA 3.2.

$$\alpha(t) = \begin{cases} 2g_a(t,b) - (\lambda_a(t) + \lambda_b(t)) & \textit{for } t \in B'' \cup l; \\ 2g_a(t,b) - (\lambda_a(t) + \lambda_b(t)) & \\ \quad -2\pi\{\omega_1(t,a) - \omega_1(t,b)\}^2 \dfrac{1}{\|d\omega_1(t,\cdot)\|^2_{D(t)}} & \textit{for } t \in B'. \end{cases}$$

PROOF. We set, for $t \in B$,

$$G(t,z) := g_a(t,z) - g_b(t,z) \quad \text{on } D(t);$$
$$u(t,z) := p(t,z) - G(t,z) \quad \text{on } D(t),$$

so that $G(t,z) = 0$ on $\partial D(t)$ and $\int_{\partial D(t)} (\partial G(t,z)/\partial n_z) ds_z = 0$, and $u(t,z)$ is a harmonic function on $D(t)$ such that

$$u(t,a) = -\lambda_a(t) + g_b(t,a) = -\lambda_a(t) + g_a(t,b);$$
$$u(t,b) = \alpha(t) - g_a(t,b) + \lambda_b(t);$$
$$u(t,z) = \begin{cases} c(t) \text{ on } C(t) & \text{for } t \in B'' \cup l, \\ c_i(t) \text{ on } C_j(t) \ (j = 1,2) & \text{for } t \in B'. \end{cases}$$

We first show the case $t \in B'' \cup l$. Since $u(t,z) = c(t)$ on $\partial D(t)$, we see from the maximum principal for harmonic function that $u(t,z) \equiv c(t)$ in $D(t)$. In particular, at $z = a \in D(t)$, $c(t) = u(t,a) = -\lambda_a(t) + g_a(t,b)$. Let $\gamma_\epsilon(a)$ (resp. $\gamma_\epsilon(b)$) be the circle of center a (resp. b) with radius $0 < \epsilon \ll 1$. By Green's formula, we have

$$\int_{C(t)-\gamma_\epsilon(a)-\gamma_\epsilon(b)} u(t,z) \frac{\partial p(t,z)}{\partial n_z} ds_z = \int_{C(t)-\gamma_\epsilon(a)-\gamma_\epsilon(b)} p(t,z) \frac{\partial u(t,z)}{\partial n_z} ds_z. \quad (3.3)$$

Letting $\epsilon \to 0$, the left hand side of (3.3) is

$$c(t) \int_{C(t)} \frac{\partial p}{\partial n_z} ds_z + 2\pi(u(t,a) - u(t,b)) = 2\pi(u(t,a) - u(t,b))$$

by the property of L_1-principal function p, and the right hand side of (3.3) is

$$c(t) \int_{C(t)} \frac{\partial u}{\partial n_z} ds_z = c(t) \biggl(\int_{C(t)} \frac{\partial p}{\partial n_z} ds_z - \int_{C(t)} \frac{\partial G}{\partial n_z} ds_z \biggr) = 0.$$

Therefore, the assertion is shown.

We next show the case $t \in B'$. By Green's formula, we have

$$\int_{C_1(t)+C_2(t)} \frac{\partial \omega_1(t,z)}{\partial n_z} ds_z = 0;$$

$$\int_{C_1(t)} \frac{\partial \omega_1(t,z)}{\partial n_z} ds_z = \|d\omega_1(t,\cdot)\|^2_{D(t)};$$

$$\int_{C_1(t)} \frac{\partial G(t,z)}{\partial n_z} ds_z = -2\pi(\omega_1(t,a) - \omega_1(t,b)) \qquad \text{by (3.2).} \tag{3.4}$$

We also have

$$\int_{C_1(t)+C_2(t)-\gamma_\epsilon(a)-\gamma_\epsilon(b)} \omega_1(t,z) \frac{\partial p(t,z)}{\partial n_z} ds_z = \int_{C_1(t)+C_2(t)-\gamma_\epsilon(a)-\gamma_\epsilon(b)} p(t,z) \frac{\partial \omega_1(t,z)}{\partial n_z} ds_z. \tag{3.5}$$

Letting $\epsilon \to 0$, the left hand side of (3.5) is

$$\int_{C_1(t)} \frac{\partial p}{\partial n_z} ds_z + 0 + 2\pi(\omega_1(t,a) - \omega_1(t,b)) = 2\pi(\omega_1(t,a) - \omega_1(t,b))$$

by the property of ω_1 and p, and the right hand side of (3.5) is

$$c_1(t) \int_{C_1(t)} \frac{\partial \omega_1(t,z)}{\partial n_z} ds_z + c_2(t) \int_{C_2(t)} \frac{\partial \omega_1(t,z)}{\partial n_z} ds_z = (c_1(t) - c_2(t)) \|d\omega_1(t,\cdot)\|^2_{D(t)}.$$

$$\therefore \quad c_1(t) - c_2(t) = 2\pi(\omega_1(t,a) - \omega_1(t,b)) \frac{1}{\|d\omega_1(t,\cdot))\|^2_{D(t)}}. \tag{3.6}$$

Similarly,

$$\int_{C_1(t)+C_2(t)-\gamma_\epsilon(a)-\gamma_\epsilon(b)} u(t,z) \frac{\partial p(t,z)}{\partial n_z} ds_z = \int_{C_1(t)+C_2(t)-\gamma_\epsilon(a)-\gamma_\epsilon(b)} p(t,z) \frac{\partial u(t,z)}{\partial n_z} ds_z.$$

Letting $\epsilon \to 0$, we see from the boundary behavior of $p(t,z)$ and (3.4) that

$$\begin{aligned}2\pi(u(t,a)-u(t,b)) &= c_1(t)\int_{C_1(t)}\frac{\partial(p-G)}{\partial n_z}ds_z + c_2(t)\int_{C_2(t)}\frac{\partial(p-G)}{\partial n_z}ds_z\\ &= -(c_1(t)-c_2(t))\int_{C_1(t)}\frac{\partial G(t,z)}{\partial n_z}ds_z\\ &= 2\pi(c_1(t)-c_2(t))(\omega_1(t,a)-\omega_1(t,b)),\end{aligned}$$

for which we substitute (3.6) to obtain

$$u(t,a)-u(t,b) = 2\pi(\omega_1(t,a)-\omega_1(t,b))^2\frac{1}{\|d\omega_1(t,\cdot))\|^2_{D(t)}}.$$

This implies the desired formula in case $t \in B'$. □

Remark 1. Let $t_0 \in l$. As shown above, we have $p(t_0,z) = c(t_0)+g_a(t_0,z)-g_b(t_0,z)$ on $D(t_0)$. We thus have

$$\int_{C_1}\frac{\partial p(t_0,z)}{\partial n_z}ds_z = \int_{C_1}\frac{\partial g_a(t_0,z)}{\partial n_z}ds_z - \int_{C_1}\frac{\partial g_b(t_0,z)}{\partial n_z}ds_z = 2\pi(\omega_1(a)-\omega_1(b)) \neq 0.$$

On the other hand, $\int_{C_1(t)}(\partial p(t,z)/\partial n_z)ds_z = 0$ for $t \in B'$; as $t \in B' \to t_0 \in l$, both $C_1(t) \to C_1$ and $p(t,z) \to p(t_0,z)$ uniformly converges in any compact set $K \Subset D(t)$. This difference is the key of our counterexample.

Lemma 3.3. *$\lambda_a(t)$ and $g_a(t,b)$ are of class C^1 on B.*

Proof. By Lemma 4.1 in [**7**], C^1-ness of $\lambda_a(t)$ on B was shown (cf [**5**]). Its proof is available for the result for $g_a(t,b)$ only replacing the following variation formula used in the proof:

$$\frac{\partial\lambda_a(t)}{\partial t} = -\frac{1}{\pi}\int_{\partial D(t)}k_1(t,z)\left|\frac{\partial g_a(t,z)}{\partial n_z}\right|^2 ds_z \quad \text{for } t \in B'\cup B'',$$

by the Hadamard variation formula:

$$\frac{\partial g_a(t,b)}{\partial t} = -\frac{1}{\pi}\int_{\partial D(t)}k_1(t,z)\left|\frac{\partial g_a(t,z)}{\partial n_z}\right|\left|\frac{\partial g_b(t,z)}{\partial n_z}\right|ds_z$$
$$\text{for } t \in B'\cup B''.$$ □

Remark 2. The Robin constant $\lambda_a(t)$ and the Green function $g_a(t,b)$ are of class C^1 on B for all variations $\mathcal{D}$ of cases (c1) and (c2). We showed that the L_1-constant $\alpha(t)$ is of class C^1 on B for the variation $\mathcal{D}$ of case (c1) in Section 2. In this section, we shall prove below that $\alpha(t)$ for the variation $\mathcal{D}$ of case (c2) is not always of class C^1 by using the C^1-ness of $\lambda_a(t)$ and $g_a(t,b)$.

Lemma 3.4.

$$\|d\omega_1(t,\cdot)\|^2_{D(t)} \le \frac{3\pi}{\sqrt[4]{1-|t|}} \quad \textit{for } t \in B'.$$

Proof. For $t \in B'$, we set $|t| = r$, so that $0 < 1-\rho < r < 1$, $D(t) = D(r)$, $\omega_1(t,z) = \omega_1(r,z)$, and

$$\{y=0\} \cap \partial D(r) = \{y=0\} \cap [C_1(r) \cup C_2(r)] = \{\pm\sqrt{1-r}, \pm\sqrt{1+r}\}.$$

We simply set $\epsilon = \sqrt{1-r}$, so that $0 < \epsilon < \sqrt{\rho}$. We consider the following two disks:

$$\Delta_1(\epsilon) = \{|z + (3+\epsilon)| \le 3\}, \quad \Delta_2(\epsilon) = \{|z - (3+\epsilon)| \le 3\}.$$

By simple consideration, we have $C_1(r) \subset \Delta_1(\epsilon)$ and $C_2(r) \subset \Delta_2(\epsilon)$. The boundaries $\widetilde{C}_1(\epsilon) = -\partial\Delta_1(\epsilon)$ and $\widetilde{C}_2(\epsilon) = -\partial\Delta_2(\epsilon)$ (circles with clockwise direction) touch $C_1(r)$ and $C_2(r)$ at the points $(-\epsilon, 0)$ and $(\epsilon, 0)$, respectively. We consider the following domain in $\mathbb{P}_z$:

$$E(\epsilon) = \mathbb{P}_z \setminus (\Delta_1(\epsilon) \cup \Delta_2(\epsilon)),$$

so that $\partial E(\epsilon) = \widetilde{C}_1(\epsilon) + \widetilde{C}_2(\epsilon)$ and $D(r) \supset E(\epsilon)$. We consider the harmonic measure $\Omega(\epsilon, z)$ for $(E(\epsilon), \widetilde{C}_1(\epsilon))$ such that $\Omega(\epsilon, z) = 1$ (resp. 0) on $\widetilde{C}_1(\epsilon)$ (resp. $\widetilde{C}_2(\epsilon)$). It follows from Dirichlet principle that

$$\|d\omega_1(r,\cdot)\|^2_{D(r)} \le \|d\Omega(\epsilon,\cdot)\|^2_{E(\epsilon)}.$$

To calculate the exact form of $\|d\Omega(\epsilon,\cdot)\|^2_{E(\epsilon)}$, we simply set $h(\epsilon,z) = 2(\Omega(\epsilon,z) - 1/2)$ on $E(\epsilon)$, which is a harmonic function on $E(\epsilon)$ such that

$$h(\epsilon,z) = \begin{cases} 1 & \text{on } \widetilde{C}_1(\epsilon); \\ -1 & \text{on } \widetilde{C}_2(\epsilon); \\ 0 & \text{on the } y\text{-axis.} \end{cases}$$

It follows from Schwarz reflection principle for the symmetric domain $E(\epsilon)$ with circle boundaries that $h(\epsilon, z)$ is harmonically extended to a certain two-punctured domain $\mathfrak{E}(\epsilon)$ of the form $\mathfrak{E}(\epsilon) = \mathbb{P}_z \setminus \{\eta, -\eta\}$, where $\eta = \eta(\epsilon)$ is some point depending on ϵ such that $\epsilon < \eta < 3+\epsilon$, $\lim_{z\to-\eta} h(\epsilon, z) = +\infty$, and $\lim_{z\to\eta} h(\epsilon, z) = -\infty$. By the symmetries of domain $\mathfrak{E}(\epsilon)$ and function $h(\epsilon, z)$ with respect to the y-axis, we have

$$h(\epsilon, z) = c \log \left| \frac{z-\eta}{z+\eta} \right| \quad \text{on } \mathfrak{E}(\epsilon),$$

where $c = c(\epsilon) > 0$ is a certain constant depending on ϵ. Since $h(\epsilon, \epsilon) = h(\epsilon, 6+\epsilon) = -1$, we have

$$\begin{aligned}
&\eta = \sqrt{6\epsilon + \epsilon^2}, \\
&c = 1/\log \left| \frac{\epsilon+\eta}{\epsilon-\eta} \right| = 1/\log \left(\frac{1 + (\sqrt{\epsilon}/\sqrt{6+\epsilon})}{1 - (\sqrt{\epsilon}/\sqrt{6+\epsilon})} \right) \leq \frac{3}{\sqrt{\epsilon}} \quad \text{for } 0 < \epsilon < 1.
\end{aligned}$$

On the other hand, we have by Cauchy formula

$$\begin{aligned}
\|dh(r, \cdot)\|^2_{E(\epsilon)} &= \int_{\widetilde{C}_1(\epsilon)} 1 \cdot \frac{\partial h(\epsilon, z)}{\partial n_z} ds_z + \int_{\widetilde{C}_2(\epsilon)} (-1) \cdot \frac{\partial h(\epsilon, z)}{\partial n_z} ds_z \\
&= \frac{2}{i} \int_{\widetilde{C}_1(\epsilon) - \widetilde{C}_2(\epsilon)} \frac{\partial h(\epsilon, z)}{\partial z} dz \\
&= \frac{c}{i} \int_{C_1(\epsilon) - C_2(\epsilon)} \left(\frac{1}{z-\eta} - \frac{1}{z+\eta} \right) dz = 4\pi c.
\end{aligned}$$

We conclude that, for $0 < \epsilon = \sqrt{1-r} < \sqrt{\rho}$,

$$\begin{aligned}
&\|d\omega_1(r, \cdot)\|^2_{D(r)} \leq \|d\Omega(\epsilon, \cdot)\|^2_{E(\epsilon)} = \frac{1}{4} \|dh(\epsilon, \cdot)\|^2_{E(\epsilon)} \\
&= \pi c \leq \frac{3\pi}{\sqrt{\epsilon}} = \frac{3\pi}{\sqrt[4]{1-r}}.
\end{aligned} \qquad \square$$

Proof of Theorem 3.1. For a fixed $t_0 \in l$, it follows from (3.2) and Lemma 3.4 that

$$\lim_{t\in B''\to t_0} (\omega_1(t, a) - \omega_1(t, b))^2 \frac{1}{\|d\omega_1(t, \cdot)\|^2_{D(t)}} \leq \lim_{t\in B''\to t_0} (2K)^2 \frac{\sqrt[4]{1-|t|}}{3\pi} = 0.$$

By Lemmas 3.2 and 3.3, we have $\lim_{t\in B''\to t_0}\alpha(t)=\alpha(t_0)$, so that (1) is proved.

The assertion (2)-(i) directly follows Lemmas 3.2 and 3.3. For (2)-(ii), since $\alpha(t)=\alpha(|t|)$ on $B=\{|t-1|<\rho\}$ and l is an arc of $\{|t|=1\}$ near $t=1$, it suffices to prove in case $t_0=1$ and $\lim_{r\nearrow 1}(\alpha(1)-\alpha(r))/(1-r)=+\infty$. Let the intervals $I=(1-\rho,1+\rho)$ and $I'=(1-\rho,1)$. For simplicity, we set $\Lambda(r)=2g_a(r,b)-(\lambda_a(r)+\lambda_b(r))$ on I; $\widetilde{K}(r)=2\pi(\omega_1(r,a)-\omega_1(r,b))^2$ on I'; and $H(r)=1/\|d\omega_1(r,\cdot)\|^2_{D(r)}$ on I', so that

$$\alpha(r)=\begin{cases}\Lambda(r) & \text{on } I\setminus I';\\ \Lambda(r)-\widetilde{K}(r)H(r) & \text{on } I'.\end{cases}$$

Note that $\alpha(1)=\Lambda(1)$ by $1\in I\setminus I'$ or equivalently $\lim_{r\nearrow 1}H(r)=0$ by continuity (1). Since $\Lambda(r)$ is of class C^1 on I by Lemma 3.3, $\widetilde{K}(r)\geq \pi K^2/2>0$ on I' by (3.1), and $H(r)\geq \sqrt[4]{1-r}/3\pi$ on I' by Lemma 3.4, it follows that

$$\begin{aligned}\lim_{r\nearrow 1}\frac{\alpha(1)-\alpha(r)}{1-r}&=\Lambda'(1)+\lim_{r\nearrow 1}\frac{\widetilde{K}(r)H(r)-\widetilde{K}(1)H(1)}{1-r}\\ &\geq \Lambda'(1)+\lim_{r\nearrow 1}\frac{\pi K^2/2\cdot\sqrt[4]{1-r}/3\pi}{1-r}\geq \Lambda'(1)+\frac{K^2}{6}\lim_{r\nearrow 1}\frac{1}{\sqrt[4]{(1-r)^3}}=+\infty,\end{aligned}$$

which proves (2)-(ii). Therefore, $\alpha(t)$ is not of class C^1 on B.

To prove (3) by contradiction, we assume that $\alpha(t)$ is subharmonic on B. We consider the conformal mapping $T:t\in B\to\tau=\log t\in\mathbb{C}_\tau$, where $\log 1=0$. We set $\Delta=T(B)$ and $\widetilde{\alpha}(\tau)=\alpha(t)$, where $\tau=\log t$. We see from assertion (1) that $\widetilde{\alpha}(\tau)$ is continuous and subharmonic on Δ. Since $\widetilde{\alpha}(\tau)=\widetilde{\alpha}(\tau_1)$ where $\tau_1=\Re\tau$, it follows from assertion (2) that there exists a small disk $\Delta_0=\{|\tau|<r_0\}$ in Δ and some constants $0<m<M<\infty$ such that

$$\widetilde{\alpha}(\tau)\leq\begin{cases}\widetilde{\alpha}(0)+m\tau_1 & \text{for } \tau\in\Delta_0''\cup\widetilde{l};\\ \widetilde{\alpha}(0)+M\tau_1 & \text{for } \tau\in\Delta_0',\end{cases}$$

where $\Delta_0'=\Delta_0\cap\{\tau_1<0\}$, $\widetilde{l}=\Delta_0\cap\{\tau_1=0\}$, $\Delta_0''=\Delta_0\cap\{\tau_1>0\}$. It follows that

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\widetilde{\alpha}(r_0e^{i\theta})d\theta&\leq\widetilde{\alpha}(0)+\frac{1}{2\pi}\int_0^{2\pi}(m+M)r_0\cos\theta\,d\theta\\ &=\widetilde{\alpha}(0)+\frac{1}{\pi}\int_0^{\pi/2}(m-M)r_0\cos\theta\,d\theta<\widetilde{\alpha}(0),\end{aligned}$$

which contradicts with the subharmonicity of $\widetilde{\alpha}(\tau)$ on Δ. □

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