The classification of tilting modules over Harada algebras

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Abstract. In the 1980s, Harada introduced a class of algebras now called Harada algebras. The aim of this paper is to study Harada algebras in representation theoretical point of view. The paper concludes the following two results. The first is the classification of modules over left Harada algebras whose projective dimension is at most one. The second is the classification of tilting modules over left Harada algebras, which is done by giving a bijection between tilting modules over Harada algebras and tilting modules over direct products of upper triangular matrix algebras over a field.

1. Main results.

In the 1980s, Harada introduced a new class of algebras now called Harada algebras, which give a common generalization of QF-algebras and Nakayama algebras. We define left Harada algebras in terms of the structure of its projective modules since the results included in this paper will be shown by using only the conditions in Definition 1.2.

DEFINITION 1.1. Let T be a finitely generated module over a finite dimensional algebra R and $T \simeq \bigoplus_{i=1}^{n} T_i$ an indecomposable decomposition of T. Then T is called *basic* if T_i and T_j are not isomorphic for any $i \neq j$. The algebra R is called *basic* if R_R is basic.

DEFINITION 1.2. A basic finite dimensional algebra R is called a *left Harada algebra*[1] if a complete set of orthogonal primitive idempotents $\mathrm{Pi}(R)$ of R can be arranged such that $\mathrm{Pi}(R) = \{e_{ij}\}_{i=1, j=1}^{m, n_i}$ where

(1) $e_{i1}R$ is an injective R-module for any $i = 1, \ldots, m$,
(2) $e_{ij}R \simeq e_{i,j-1}J(R)$ for any $i = 1, \ldots, m$, $j = 2, \ldots, n_i$.

Here $J(R)$ is the Jacobson radial of R.

[1]In [**5**], [**8**], there conditions are shown to be equivalent to the original definition given in [**5**], which is stated in terms of left modules

Many authors have studied Harada algebras from ring structural and module structural viewpoints by [**6**], [**8**], [**9**], [**10**] etc. And those results were applied to study of the structure of QF-algebras and Nakayama algebras by [**6**], [**7**].

In this paper, we study left Harada algebras from modern representation theoretic viewpoints. The paper includes two main results which are presented as follows.

Throughout this paper, an algebra means a finite dimensional associative algebra over an algebraically closed field K. We always deal with finitely generated right modules over algebras. We denote by $J(M)$ the Jacobson radical of a module M, by $J^k(M)$ the k-th Jacobson radical of M and by $S(M)$ the socle of M.

Let R be a left Harada algebra as in Definition 1.2. We put

$$P_{ij} := J^{j-1}(e_{i1}R) \simeq e_{ij}R \qquad (1 \le i \le m,\ 1 \le j \le n_i) \qquad (1.1)$$

for simplicity. By the above conditions (1) and (2), we have a chain

$$P_{i1} \supset P_{i2} \supset \cdots \supset P_{in_i}$$

of indecomposable projective R-modules.

The first main result is the classification of R-modules whose projective dimension is equal to one. It is obvious that $\text{proj.dim}\,(P_{ik}/P_{il}) = 1$. We will prove that the converse holds in the following sense in Section 2.

THEOREM 1.3.   *A complete set of isomorphism classes of indecomposable R-modules whose projective dimension is equal to one is given as follows.*

$$\{P_{ik}/P_{il} \mid 1 \le i \le m,\ 1 \le k < l \le n_i\}.$$

Thus there are only finitely many indecomposable R-modules whose projective dimension are at most one.

The other main result is the classification of basic tilting R-modules. We recall the definition of tilting modules.

DEFINITION 1.4.   Let S be an algebra. An S-module T is called a *partial tilting module* if it satisfies the following conditions.

(1) $\text{proj.dim}\,T \le 1$.
(2) $\text{Ext}^1_S(T, T) = 0$.

A partial tilting S-module T is called a *tilting module* if it satisfies the following condition.

(3) There exists an exact sequence

$$0 \longrightarrow S_S \longrightarrow T_0 \longrightarrow T_1 \longrightarrow 0$$

where T_0, $T_1 \in \operatorname{add} T$.

From viewpoint of Morita theory, it is enough to consider basic tilting modules. We denote by $\operatorname{tilt}(S)$ the set of isomorphism classes of basic tilting S-modules.

The classification of tilting R-modules is done by reducing it to that of tilting modules over upper triangular matrix algebras which are typical example of left Harada algebras. In Section 3, we construct an ideal I of R such that $\overline{R} := R/I$ is isomorphic to $\mathrm{T}_{n_1}(K) \times \mathrm{T}_{n_2}(K) \times \cdots \times \mathrm{T}_{n_m}(K)$. Here $\mathrm{T}_n(K)$ is an $n \times n$ upper triangular matrix algebra over K. Then there is a natural functor $- \otimes_R \overline{R} : \operatorname{mod} R \longrightarrow \operatorname{mod} \overline{R}$. It will be shown that this functor gives a bijection from the set of isomorphism classes of R-modules whose projective dimension one to the set of isomorphism classes of $\overline{R}$-modules, and preserves vanishing of first extensions in Section 4. Thus the functor induces a bijection from the set of isomorphism classes of partial tilting R-modules to the set of isomorphism classes of partial tilting $\overline{R}$-modules. These facts imply the following result.

THEOREM 1.5. *There exists a bijection*

$$\operatorname{tilt}(R) \longrightarrow \operatorname{tilt}(\mathrm{T}_{n_1}(K)) \times \operatorname{tilt}(\mathrm{T}_{n_2}(K)) \times \cdots \times \operatorname{tilt}(\mathrm{T}_{n_m}(K)).$$

Moreover, we will include a combinatorial description of $\operatorname{tilt}(\mathrm{T}_n(K))$ which should be known for experts by using non-crossing partitions of a regular $(n + 2)$-polygon in Section 5. Combining this description with Theorem 1.5, we can classify tilting modules over a given left Harada algebra. Consequently we have the following result.

COROLLARY 1.6. *The number of basic tilting R-modules is equal to*

$$\prod_{i=1}^{m} \frac{1}{n_i + 1} \binom{2n_i}{n_i}.$$

2. Proof of Theorem 1.3.

In this section, we show Theorem 1.3. To prove it, we give key lemmas, that is, the properties of homomorphisms between indecomposable projective R-modules.

Let R be a left Harada algebra as in Definition 1.2. We use the notation (1.1).

Lemma 2.1.   *If a submodule of P_{i1} is not contained in $J(P_{in_i})$, then it is P_{ij} for some $1 \leq j \leq n_i$.*

Proof.   It follows from Definition 1.2 (2).   □

Lemma 2.2.   *Let $f : P_{ij} \longrightarrow P_{kl}$ be a homomorphism. Then the following assertions hold.*

(1) *f is monic if and only if $i = k$, $j \geq l$ and $\operatorname{Im} f = P_{kj}$.*
(2) *f is not monic if and only if $\operatorname{Im} f \subset J(P_{kn_k})$.*
(3) *Assume $i = k$ and $j < l$, we have $\operatorname{Im} f \subset J(P_{kn_k})$*
(4) *Assume $i \neq k$, we have $\operatorname{Im} f \subset J(P_{kn_k})$.*

Proof.   (1) We assume that f is monic. Then $i = k$ since $S(P_{ij}) \simeq S(P_{kl})$ and these are simple. By $\operatorname{length}(P_{ij}) \leq \operatorname{length}(P_{kl})$, we have $j \geq l$. By Lemma 2.1, the only submodule of P_{kl} whose length is equal to $\operatorname{length}(P_{ij})$ is P_{kj}. Thus we have $\operatorname{Im} f = P_{kj}$. The converse is obvious.

(2) We assume that $\operatorname{Im} f \not\subseteq J(P_{kn_k})$. By Lemma 2.1, there exists $0 \leq r \leq n_k - l$ such that $\operatorname{Im} f = P_{k,r+l}$. Therefore f is monic since f can be seen as an epimorphism between indecomposable projective R-modules. The converse follows from (1).

(3) Since $\operatorname{length}(P_{ij}) > \operatorname{length}(P_{il})$, there exists no monomorphism from P_{ij} to P_{il}. Therefore the assertion follows from (2).

(4) Since $i \neq k$, $S(P_{ij})$ and $S(P_{kl})$ are not isomorphic. Hence there exists no monomorphism from P_{ij} to P_{kl}. Therefore the assertion follows from (2).   □

Lemma 2.3.   *Let $f : P_{ij} \longrightarrow P_{il}$ be any monomorphism with $j \geq l$. Then the following assertions hold.*

(1) *For any homomorphism $g : P_{ij} \longrightarrow P_{il'}$ with $l \geq l'$, there exists a homomorphism $h : P_{il} \longrightarrow P_{il'}$ such that $g = hf$.*
(2) *For any homomorphism $g : P_{ij} \longrightarrow P_{st}$ which is not monic, there exists a homomorphism $h : P_{il} \longrightarrow P_{st}$ such that $g = hf$.*
(3) *For any homomorphism $g : P_{il'} \longrightarrow P_{il}$ with $l' \geq j$, there exists a homomorphism $h : P_{il'} \longrightarrow P_{ij}$ such that $g = fh$.*
(4) *For any homomorphism $g : P_{st} \longrightarrow P_{il}$ which is not monic, there exists a homomorphism $h : P_{st} \longrightarrow P_{ij}$ such that $g = fh$.*

Proof.   (1) Let $u : P_{il'} \longrightarrow P_{i1}$ be the inclusion map. Since P_{i1} is injective, there exists a homomorphism $h : P_{il} \longrightarrow P_{i1}$ such that $ug = hf$.

$$
\begin{array}{ccccc}
0 & \longrightarrow & P_{ij} & \overset{f}{\longrightarrow} & P_{il} \\
 & & \downarrow{\scriptstyle g} & & \\
 & & P_{il'} & \overset{h}{\cdot\cdot\cdot} & \\
 & & \downarrow{\scriptstyle u}\ \nearrow{\scriptstyle v} & & \\
 & & P_{i1} & &
\end{array}
$$

Since $l \geq l'$, we have $\operatorname{Im} h \subset P_{il'}$. We can see h as $h : P_{il} \longrightarrow P_{il'}$.

(2) Let $u : P_{st} \longrightarrow P_{s1}$ be the inclusion map. Since P_{s1} is injective, there exists a homomorphism $h : P_{il} \longrightarrow P_{s1}$ such that $ug = hf$.

$$
\begin{array}{ccccc}
0 & \longrightarrow & P_{ij} & \overset{f}{\longrightarrow} & P_{il} \\
 & & \downarrow{\scriptstyle g} & & \\
 & & P_{st} & \overset{h}{\cdot\cdot\cdot} & \\
 & & \downarrow{\scriptstyle u}\ \nearrow{\scriptstyle v} & & \\
 & & P_{s1} & &
\end{array}
$$

If h is monic, then $ug = hf$ is monic, hence g is monic. This is a contradiction. Therefore h is not monic. By Lemma 2.2 (2), we have $\operatorname{Im} h \subset J(P_{sn_s}) \subset P_{st}$. We can see h as $h : P_{il} \longrightarrow P_{st}$.

(3) By Lemma 2.2 (1), (3), we have $\operatorname{Im} f = P_{ij}$ and $\operatorname{Im} g \subset P_{il'}$. Since $l' \geq j$, we have $\operatorname{Im} g \subset \operatorname{Im} f$. Since $P_{il'}$ is projective, there exists a homomorphism $h : P_{il'} \longrightarrow P_{ij}$ such that $g = fh$.

$$
\begin{array}{ccc}
 & & P_{il'} \\
 & {\scriptstyle h}\swarrow & \downarrow{\scriptstyle g} \\
P_{ij} & \overset{f}{\longrightarrow} P_{ij} & \longrightarrow 0
\end{array}
$$

(4) By Lemma 2.2 (1), (2), we have $\operatorname{Im} g \subset J(P_{in_i}) \subset P_{ij} = \operatorname{Im} f$. The assertion follows by the same argument as in the proof of (3). $\qquad\square$

The following result gives Theorem 1.3.

Lemma 2.4. *Let Q_i and Q'_j be indecomposable projective R-modules and*

$$f : Q := Q_1 \oplus \cdots \oplus Q_k \longrightarrow Q' := Q'_1 \oplus \cdots \oplus Q'_l$$

a monomorphism. Then there exists automorphisms $\varphi \in \mathrm{Aut}_R(Q)$, $\psi \in \mathrm{Aut}_R(Q')$ such that

$$\psi f \varphi^{-1} = \begin{pmatrix} f_1 & & 0 \\ & \ddots & \\ 0 & & f_k \\ \hline & 0 & \\ & \vdots & \\ & 0 & \end{pmatrix} : Q_1 \oplus \cdots \oplus Q_k \longrightarrow Q'_1 \oplus \cdots \oplus Q'_l.$$

Proof. We proceed by induction on k. First we consider the case $k = 1$. Then Q is an indecomposable projective R-module. We write f as

$$f : Q \xrightarrow{\begin{pmatrix} f_1 \\ \vdots \\ f_l \end{pmatrix}} Q', \quad f_i : Q \longrightarrow Q_i \quad (1 \le i \le l).$$

Since $S(Q)$ is simple, there exists an monomorphism in $\{f_1, \ldots, f_l\}$. So we can assume that $f_1, \ldots, f_r$ are monic and $f_{r+1}, \ldots, f_l$ are not monic. We assume that $\mathrm{length}(Q'_1) \le \mathrm{length}(Q'_i)$ for $2 \le i \le r$. Then for any $2 \le j \le r$ there exists a homomorphism $h_j : Q'_1 \longrightarrow Q'_j$ such that $f_j = h_j f_1$ by Lemma 2.3 (1). Moreover, for any $r + 1 \le j \le n$ there exists a homomorphism $h_j : Q'_1 \longrightarrow Q'_j$ such that $f_j = h_j f_1$ by Lemma 2.3 (2). Let

$$\psi = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \hline -h_2 & 1 & & 0 \\ \vdots & & \ddots & \\ -h_l & 0 & & 1 \end{pmatrix} \in \mathrm{Aut}_R(Q'_1 \oplus \cdots \oplus Q'_l).$$

Then we have

$$\psi f = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \hline -h_2 & 1 & & 0 \\ \vdots & & \ddots & \\ -h_l & 0 & & 1 \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_l \end{pmatrix} = \begin{pmatrix} f_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Next we assume that $k \geq 2$ and that the assertion holds for $k-1$. We assume that $\mathrm{length}(Q_k) \leq \mathrm{length}(Q_i)$ for $1 \leq i \leq k-1$. By applying the induction hypotheses to $f|_{Q_1 \oplus \cdots \oplus Q_{k-1}}$, we can assume that

$$f|_{Q_1 \oplus \cdots \oplus Q_{k-1}} : Q_1 \oplus \cdots \oplus Q_{k-1} \xrightarrow{\begin{pmatrix} f_1 & & 0 \\ & \ddots & \\ 0 & & f_{k-1} \\ \hline & 0 & \\ & \vdots & \\ & 0 & \end{pmatrix}} Q', \quad f_i : Q_i \longrightarrow Q'_i$$

$$(1 \leq i \leq k-1).$$

Therefore we can write f as

$$f : Q \xrightarrow{\begin{pmatrix} f_1 & & 0 & \vline & g_1 \\ & \ddots & & \vline & \vdots \\ 0 & & f_{k-1} & \vline & g_{k-1} \\ \hline 0 & \cdots & 0 & \vline & g_k \\ \vdots & & \vdots & \vline & \vdots \\ 0 & \cdots & 0 & \vline & g_l \end{pmatrix}} Q', \quad g_i : Q_k \longrightarrow Q'_i \quad (1 \leq i \leq l).$$

By Lemma 2.3 (3), (4), and the assumption on Q_k, for any $1 \leq i \leq k-1$ there exists a homomorphism $h_i : Q_k \longrightarrow Q_i$ such that $g_i = f_i h_i$. Let

$$\varphi = \begin{pmatrix} 1 & & 0 & \vline & h_1 \\ & \ddots & & \vline & \vdots \\ 0 & & 1 & \vline & h_{k-1} \\ \hline & 0 & & \vline & 1 \end{pmatrix} \in \mathrm{Aut}(Q_1 \oplus \cdots \oplus Q_k).$$

Then we have

$$\begin{pmatrix} f_1 & & 0 & \vline & \\ & \ddots & & \vline & 0 \\ 0 & & f_{k-1} & \vline & \\ \hline 0 & \cdots & 0 & \vline & g_k \\ \vdots & & \vdots & \vline & \vdots \\ 0 & \cdots & 0 & \vline & g_l \end{pmatrix} \varphi = \begin{pmatrix} f_1 & & 0 & \vline & g_1 \\ & \ddots & & \vline & \vdots \\ 0 & & f_{k-1} & \vline & g_{k-1} \\ \hline 0 & \cdots & 0 & \vline & g_k \\ \vdots & & \vdots & \vline & \vdots \\ 0 & \cdots & 0 & \vline & g_l \end{pmatrix} = f.$$

By applying the same argument as in the case $k = 1$ to

$$\begin{pmatrix} g_k \\ \vdots \\ g_l \end{pmatrix},$$

the assertion follows.   □

Now we can prove Theorem 1.3.

PROOF OF THEOREM 1.3.   The projective dimension of P_{ik}/P_{il} is obviously equal to 1. Let X be an indecomposable R-module whose projective dimension is equal to one. Then there exists an exact sequence

$$0 \longrightarrow Q \longrightarrow Q' \longrightarrow X \longrightarrow 0$$

such that Q and Q' are projective R-modules. By Lemma 2.4 and since X is an indecomposable R-module, Q and Q' must be indecomposable R-modules. By Lemma 2.2 (1), X is isomorphic to one of P_{ik}/P_{il}.   □

3.   Triangular factor algebras of Harada algebras.

In this section, we keep the notations from the previous section. We define a special factor algebra $\overline{R} = R/I$ of R which is isomorphic to a direct product $\mathrm{T}_{n_1}(K) \times \cdots \times \mathrm{T}_{n_m}(K)$ of upper triangular matrix algebras over K. The algebra $\overline{R}$ contains important information about R which is seen in Lemma 4.3 and Proposition 4.4.

We start by defining an ideal I of R. We put

$$e_{ij}R \supset I_{ij} := J^{n_i - j + 1}(e_{ij}R) \quad (1 \le i \le m,\ 1 \le j \le n_i),$$

and

$$R \supset I := \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n_i} I_{ij}.$$

Then we have the following result.

LEMMA 3.1.   *I is an ideal of R.*

PROOF.   Clearly I is a right ideal of R. We show that I is a left ideal of R. It

is enough to show that $rx \in I_{kl} = J^{n_k-l+1}(e_{kl}R)$ for any $x \in I_{ij} = J^{n_i-j+1}(e_{ij}R)$ and any $r \in e_{kl}R$. We consider the homomorphism

$$\varphi_r : I_{ij} \ni x \longmapsto rx \in e_{kl}R$$

of right R-modules. Since I_{ij} is indecomposable and non-projective, we have $\mathrm{Im}\,\varphi_r \subset J^{n_k-l+1}(e_{kl}R) = I_{kl}$. Therefore I is a left ideal of R. □

By Lemma 3.1, we can consider the factor algebra

$$\overline{R} := R/I.$$

For an element $a \in R$, we put $\bar{a} := a + I \in \overline{R}$.

We put

$$e_i := e_{i1} + e_{i2} + \cdots + e_{in_i}$$

for $1 \leq i \leq m$.

Now we show the following description of $\overline{R}$.

PROPOSITION 3.2. *Under the hypotheses above, the following assertions hold.*

(1) $\{\overline{e_i} \mid 1 \leq i \leq m\}$ *is a set of orthogonal central idempotents of* $\overline{R}$ *and there exists a K-algebra isomorphism*

$$\overline{R}\overline{e_j} \simeq \mathrm{T}_{n_i}(K).$$

(2) *There exists a K-algebra isomorphism*

$$\overline{R} \simeq \mathrm{T}_{n_1}(K) \times \mathrm{T}_{n_2}(K) \times \cdots \times \mathrm{T}_{n_m}(K).$$

To prove the above proposition, we describe all indecomposable projective $\overline{R}$-modules as factor modules of indecomposable projective R-modules. Since $I \subset J(R)$, we have that

$$\{\overline{e_{ij}} \mid 1 \leq i \leq m,\ 1 \leq j \leq n_i\}$$

is a complete set of orthogonal primitive idempotents of $\overline{R}$. For $1 \leq i \leq m$, $1 \leq j \leq n_i$, there exists an isomorphism

$$\overline{P}_{i,j} := \overline{e_{ij}R} = e_{ij}R/J^{n_i-j+1}(e_{ij}R) \simeq P_{ij}/J^{n_i-j+1}(P_{ij}) = P_{ij}/J(P_{in_i}).$$

By Definition 1.2 (b),

$$0 \subset P_{in_i}/J(P_{in_i}) \subset P_{i,n_i-1}/J(P_{in_i})$$
$$\subset \cdots \subset P_{i,j+1}/J(P_{in_i}) \subset P_{ij}/J(P_{in_i}) = \overline{P}_{i,j} \tag{3.1}$$

is a unique composition series of $\overline{P}_{i,j}$ as an $\overline{R}$-module. Therefore any indecomposable projective $\overline{R}$-module is serial and its composition factors are not isomorphic to each other.

From the above argument, we can prove Proposition 3.2.

PROOF OF PROPOSITION 3.2.   (1) We calculate $\operatorname{Hom}_{\overline{R}}(\overline{P}_{i,j}, \overline{P}_{k,l})$. If $i \neq k$, $\overline{P}_{i,j}$ and $\overline{P}_{k,l}$ have no common composition factors. So we have

$$\operatorname{Hom}_{\overline{R}}(\overline{P}_{i,j}, \overline{P}_{k,l}) = 0.$$

If $i = k$, we can easily see that

$$\operatorname{Hom}_{\overline{R}}(\overline{P}_{i,j}, \overline{P}_{i,l}) \simeq \begin{cases} K & (j \geq l) \\ 0 & (j < l) \end{cases}$$

by composition series (3.1).

Thus we have the following isomorphisms as K-vector space.

$$\overline{e_i}\,\overline{R}\,\overline{e_j} \simeq \operatorname{Hom}_{\overline{R}}(\overline{e_j R}, \overline{e_i R})$$

$$\simeq \begin{pmatrix} \operatorname{Hom}_{\overline{R}}(\overline{P}_{j,1}, \overline{P}_{i,1}) & \operatorname{Hom}_{\overline{R}}(\overline{P}_{j,2}, \overline{P}_{i,1}) & \cdots & \operatorname{Hom}_{\overline{R}}(\overline{P}_{j,n_j}, \overline{P}_{i,1}) \\ \operatorname{Hom}_{\overline{R}}(\overline{P}_{j,1}, \overline{P}_{i,2}) & \operatorname{Hom}_{\overline{R}}(\overline{P}_{j,2}, \overline{P}_{i,2}) & \cdots & \operatorname{Hom}_{\overline{R}}(\overline{P}_{j,n_j}, \overline{P}_{i,2}) \\ \vdots & \vdots & \ddots & \vdots \\ \operatorname{Hom}_{\overline{R}}(\overline{P}_{j,1}, \overline{P}_{i,n_i}) & \operatorname{Hom}_{\overline{R}}(\overline{P}_{j,2}, \overline{P}_{i,n_i}) & \cdots & \operatorname{Hom}_{\overline{R}}(\overline{P}_{j,n_j}, \overline{P}_{i,n_i}) \end{pmatrix}$$

$$\simeq \begin{cases} \begin{pmatrix} K & K & \cdots & K \\ & K & \cdots & K \\ & & \ddots & \vdots \\ 0 & & & K \end{pmatrix} & (i = j) \\ 0 & (i \neq j). \end{cases}$$

It is easily seen that the above isomorphism gives a K-algebra isomorphism when $i = j$.

(2) By (1), we have the following K-algebra isomorphism.

$$\overline{R} \simeq \begin{pmatrix} \overline{e_1}\overline{R}\overline{e_1} & \overline{e_1}\overline{R}\overline{e_2} & \cdots & \overline{e_1}\overline{R}\overline{e_m} \\ \overline{e_2}\overline{R}\overline{e_1} & \overline{e_2}\overline{R}\overline{e_2} & \cdots & \overline{e_2}\overline{R}\overline{e_m} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{e_m}\overline{R}\overline{e_1} & \overline{e_m}\overline{R}\overline{e_2} & \cdots & \overline{e_m}\overline{R}\overline{e_m} \end{pmatrix}$$

$$\simeq \begin{pmatrix} \mathrm{T}_{n_1}(K) & & & 0 \\ & \mathrm{T}_{n_2}(K) & & \\ & & \ddots & \\ 0 & & & \mathrm{T}_{n_m}(K) \end{pmatrix}. \qquad \square$$

Next we give a description of indecomposable $\overline{R}$-modules. By Proposition 3.2 and the unique composition series (3.1) of $P_{ij}/J(P_{in_i})$, a complete set of isomorphism classes of indecomposable nonprojective R-modules is given by

$$\{P_{ik}/P_{il} \mid 1 \le i \le m, \ 1 \le k < l \le n_i\}.$$

We put

$$P_{i,k,l} := P_{ik}/P_{il}$$

for $1 \le i \le m$, $1 \le k < l \le n_i$ for simplicity.

We have the following diagram for any $1 \le i \le m$ by our definitions.

$$\begin{array}{ccccccccccc}
\overline{P}_{i,1} & \longrightarrow & P_{i,1,n_i} & \longrightarrow & P_{i,1,n_i-1} & \longrightarrow & \cdots & \longrightarrow & P_{i,1,3} & \longrightarrow & P_{i,1,2} \\
\cup & & \cup & & \cup & & & & \cup & & \\
\overline{P}_{i,2} & \longrightarrow & P_{i,2,n_i} & \longrightarrow & P_{i,2,n_i-1} & \longrightarrow & \cdots & \longrightarrow & P_{i,2,3} & & \\
\cup & & \cup & & \cup & & & & & & \\
\vdots & & \vdots & & \vdots & & & & & & \\
\cup & & \cup & & \cup & & & & & & \\
\overline{P}_{i,n_i-2} & \longrightarrow & P_{i,n_i-2,n_i} & \longrightarrow & P_{i,n_i-2,n_i-1} & & & & & & \\
\cup & & \cup & & & & & & & & \\
\overline{P}_{i,n_i-1} & \longrightarrow & P_{i,n_i-1,n_i} & & & & & & & & \\
\cup & & & & & & & & & & \\
\overline{P}_{i,n_i} & & & & & & & & & &
\end{array}$$

In the above diagram, arrows from left to right mean natural epimorphisms. We remark that the above diagram is the AR-quiver of $\mathrm{mod}(\overline{Re_i})$ (see Section 5).

4. Proof of Theorem 1.5.

The aim of this section is to prove Theorem 1.5. We keep the notations from the previous section.

Now we give the correspondence of Theorem 1.5 precisely. A key role is played by the functor

$$F := - \otimes_R \overline{R} : \mathrm{mod}\, R \longrightarrow \mathrm{mod}\, \overline{R}.$$

THEOREM 4.1.   *Under the hypotheses above, we have a bijection*

$$F : \mathrm{tilt}(R) \ni T \longmapsto F(T) \in \mathrm{tilt}(\overline{R}).$$

By Proposition 3.2 and Theorem 4.1, we have the following corollary immediately.

COROLLARY 4.2.   *Under the hypotheses above, we have a bijection*

$$\mathrm{tilt}(R) \ni T \longmapsto (F(T)\overline{e_1}, \ldots, F(T)\overline{e_m}) \in \mathrm{tilt}(\overline{Re_1}) \times \cdots \times \mathrm{tilt}(\overline{Re_m}).$$

Hence we have Theorem 1.5.

In the rest of this section, we prove Theorem 4.1. Let $\mathscr{P}$ be the category of R-modules whose projective dimension is at most one. We define the full subcategories $\mathscr{P}_i$ of $\mathscr{P}$ for $1 \le i \le m$ by

$$\mathscr{P}_i := \mathrm{add}\{P_{ij},\ P_{i,k,l} \mid 1 \le j \le n_i,\ 1 \le k < l \le n_i\}.$$

By Theorem 1.3, we have

$$\mathscr{P} = \mathrm{add}(\mathscr{P}_1 \cup \mathscr{P}_2 \cup \cdots \cup \mathscr{P}_m).$$

To prove Theorem 4.1, we show that the restriction on F to $\mathscr{P}$ gives a bijection from the isomorphism classes of R-modules which are in $\mathscr{P}$ to the isomorphism classes of $\overline{R}$-modules which preserves vanishing property of first extensions (Lemma 4.3 and Proposition 4.4).

LEMMA 4.3.   *Under the hypotheses above, the following assertions hold.*

(1) *The restriction on F to $\mathscr{P}$ induces a bijection between the isomorphism classes of R-modules which lie in $\mathscr{P}$ and the isomorphism classes of $\overline{R}$-modules.*

(2) *The restriction on F to $\mathscr{P}_i$ induces a bijection between the isomorphism classes of R-modules which lie in $\mathscr{P}_i$ and the isomorphism classes of $\overline{Re_i}$-modules.*

PROOF. We calculate $F(M)$ for an indecomposable R-module M which lies in $\mathscr{P}$. We have isomorphisms

$$F(P_{ij}) = P_{ij} \otimes_R \overline{R} \simeq P_{ij}/(P_{ij}I) = P_{ij}/J^{n_i}(P_{i1}) = \overline{P}_{ij},$$

$$F(P_{i,k,l}) = P_{i,k,l} \otimes_R \overline{R} \simeq P_{i,k,l}/(P_{i,k,l}I) = P_{i,k,l}.$$

for $1 \le i \le m$, $1 \le j \le n_i$ and $1 \le k < l \le n_i$. The assertion follows. $\square$

PROPOSITION 4.4. *For any $X, Y \in \mathscr{P}$, $\mathrm{Ext}^1_R(X, Y) = 0$ if and only if $\mathrm{Ext}^1_{\overline{R}}(F(X), F(Y)) = 0$.*

We divide a proof of the above result into a few steps. First we show that $\mathrm{Ext}^1_R(X, Y)$ vanishes for any $X \in \mathscr{P}_i$ and $Y \in \mathscr{P}_u$ if $i \ne u$.

LEMMA 4.5. *If $i \ne u$, then $\mathrm{Ext}^1_R(X, Y) = 0$ for any $X \in \mathscr{P}_i$ and $Y \in \mathscr{P}_u$.*

PROOF. It is obvious that $\mathrm{Ext}^1_R(P_{ij}, \mathscr{P}_u) = 0$. We show $\mathrm{Ext}^1_R(P_{i,k,l}, \mathscr{P}_u) = 0$ for $1 \le k < l \le n_i$.

First we show $\mathrm{Ext}^1_R(P_{i,k,l}, P_{uv}) = 0$ for $1 \le v \le n_u$. We take a projective resolution

$$0 \longrightarrow P_{il} \xrightarrow{\ f\ } P_{ik} \longrightarrow P_{i,k,l} \longrightarrow 0 \tag{4.1}$$

of $P_{i,k,l}$ in $\mathrm{mod}\, R$. By applying $\mathrm{Hom}_R(-, P_{uv})$ to the above exact sequence, we have an exact sequence

$$\mathrm{Hom}_R(P_{ik}, P_{uv}) \xrightarrow{\mathrm{Hom}(f, P_{uv})} \mathrm{Hom}_R(P_{il}, P_{uv}) \longrightarrow \mathrm{Ext}^1_R(P_{i,k,l}, P_{uv}) \longrightarrow 0.$$

By the assumption $i \ne u$, there is no monomorphism from P_{il} to P_{uv} since the simple socles $S(P_{il})$ and $S(P_{uv})$ are not isomorphic. By Lemma 2.3 (2), $\mathrm{Hom}(f, P_{uv})$ is an epimorphism. Therefore we have $\mathrm{Ext}^1_R(P_{i,k,l}, P_{uv}) = 0$.

Next we show $\mathrm{Ext}^1_R(P_{i,k,l}, P_{u,s,t}) = 0$ for $1 \le s < t \le n_u$. By applying $\mathrm{Hom}_R(-, P_{u,s,t})$ to (4.1), we have an exact sequence

$$\mathrm{Hom}_R(P_{ik}, P_{u,s,t}) \longrightarrow \mathrm{Hom}_R(P_{il}, P_{u,s,t}) \longrightarrow \mathrm{Ext}^1_R(P_{i,k,l}, P_{u,s,t}) \longrightarrow 0.$$

By the assumption $i \neq u$, $P_{il}/J(P_{il})$ does not appear in composition factors of $P_{u,s,t}$. Therefore we have $\mathrm{Hom}_R(P_{il}, P_{u,s,t}) = 0$, and so $\mathrm{Ext}^1_R(P_{i,k,l}, P_{u,s,t}) = 0$. $\square$

Next we show that $\mathrm{Ext}^1_R(X, Y) = 0$ if and only if $\mathrm{Ext}^1_R(F(X), F(Y)) = 0$ for any $X, Y \in \mathscr{P}_i$. We need the following lemma.

LEMMA 4.6. *For any $1 \leq i \leq m$, $1 \leq j \leq n_i$ and $1 \leq k < l \leq n_i$, the natural epimorphism $\varphi : P_{ij} \longrightarrow \overline{P}_{i,j}$ induces an isomorphism*

$$\mathrm{Hom}(\varphi, P_{i,k,l}) : \mathrm{Hom}_{\overline{R}}(\overline{P}_{i,j}, P_{i,k,l}) = \mathrm{Hom}_R(\overline{P}_{i,j}, P_{i,k,l}) \longrightarrow \mathrm{Hom}_R(P_{ij}, P_{i,k,l}).$$

PROOF. It is obvious that $\mathrm{Hom}_R(\overline{P}_{i,j}, P_{i,k,l}) = \mathrm{Hom}_{\overline{R}}(\overline{P}_{i,j}, P_{i,k,l})$ holds. We show that $\mathrm{Hom}(\varphi, P_{i,k,l})$ is an isomorphism.

Since φ is epic, we have that $\mathrm{Hom}(\varphi, P_{i,k,l})$ is monic. Since any $f \in \mathrm{Hom}_R(P_{ij}, P_{i,k,l})$ satisfies $\mathrm{Ker}\, f \supset P_{ij}I = \mathrm{Ker}\, \varphi$, we have that f factors through φ. Thus $\mathrm{Hom}(\varphi, P_{i,k,l})$ is an isomorphism. $\square$

PROPOSITION 4.7. *For fixed $1 \leq i \leq m$, the following assertions hold.*

(1) *For $1 \leq j \leq n_i$, $\mathrm{Ext}^1_R(P_{ij}, X) = 0$ for any $X \in \mathscr{P}$.*
(2) *For $1 \leq j \leq n_i$, $\mathrm{Ext}^1_{\overline{R}}(\overline{P}_{i,j}, X) = 0$ for any $X \in \mathrm{mod}\,\overline{R}$.*
(3) *For $1 \leq k < l \leq n_i$ and $1 \leq s < t \leq n_i$, there exists a K-vector space isomorphism*

$$\mathrm{Ext}^1_R(P_{i,k,l}, P_{i,s,t}) \simeq \mathrm{Ext}^1_{\overline{R}}(P_{i,k,l}, P_{i,s,t}).$$

(4) *For $1 \leq k < l \leq n_i$ and $1 \leq j \leq n_i$, $\mathrm{Ext}^1_R(P_{i,k,l}, P_{ij}) = 0$ if and only if $\mathrm{Ext}^1_{\overline{R}}(P_{i,k,l}, \overline{P}_{i,j}) = 0$.*

PROOF. (1) (2) Obvious.
(3) We have a natural projective resolution

$$0 \longrightarrow P_{il} \xrightarrow{\ f\ } P_{ik} \longrightarrow P_{i,k,l} \longrightarrow 0 \tag{4.2}$$

of $P_{i,k,l}$ in $\mathrm{mod}\,R$ and a natural projective resolution

$$0 \longrightarrow \overline{P}_{i,l} \xrightarrow{\ f'\ } \overline{P}_{i,k} \longrightarrow P_{i,k,l} \longrightarrow 0 \tag{4.3}$$

of $P_{i,k,l}$ in $\mathrm{mod}\,\overline{R}$. For natural epimorphisms $\varphi : P_{ik} \longrightarrow \overline{P}_{i,k}$ and $\varphi' : P_{il} \longrightarrow \overline{P}_{i,l}$, we have the following commutative diagram.

$$0 \longrightarrow P_{il} \xrightarrow{\ f\ } P_{ik} \longrightarrow P_{i,k,l} \longrightarrow 0$$
$$\Big\downarrow{\varphi'} \qquad \Big\downarrow{\varphi}$$
$$0 \longrightarrow \overline{P}_{i,l} \xrightarrow{\ f'\ } \overline{P}_{i,k} \longrightarrow P_{i,k,l} \longrightarrow 0$$

By applying $\mathrm{Hom}_R(-, P_{i,s,t})$ to the upper row and applying $\mathrm{Hom}_{\overline{R}}(-, P_{i,s,t})$ to the lower row, we have the following commutative diagram.

$$\mathrm{Hom}_{\overline{R}}(\overline{P}_{i,k}, P_{i,s,t}) \longrightarrow \mathrm{Hom}_{\overline{R}}(\overline{P}_{i,l}, P_{i,s,t}) \longrightarrow \mathrm{Ext}^1_{\overline{R}}(P_{i,k,l}, P_{i,s,t}) \longrightarrow 0$$
$$\Big\| \qquad\qquad \Big\|$$
$$\mathrm{Hom}_R(\overline{P}_{i,k}, P_{i,s,t}) \qquad\qquad \mathrm{Hom}_R(\overline{P}_{i,l}, P_{i,s,t})$$
$${\scriptstyle \mathrm{Hom}(\varphi, P_{i,s,t})}\Big\downarrow \qquad\qquad {\scriptstyle \mathrm{Hom}(\varphi', P_{i,s,t})}\Big\downarrow$$
$$\mathrm{Hom}_R(P_{ik}, P_{i,s,t}) \longrightarrow \mathrm{Hom}_R(P_{il}, P_{i,s,t}) \longrightarrow \mathrm{Ext}^1_R(P_{i,k,l}, P_{i,s,t}) \longrightarrow 0$$

By Lemma 4.6, $\mathrm{Hom}(\varphi, P_{i,s,t})$ and $\mathrm{Hom}(\varphi', P_{i,s,t})$ are isomorphisms. Consequently we have an isomorphism $\mathrm{Ext}^1_R(P_{i,k,l}, P_{i,s,t}) \simeq \mathrm{Ext}^1_{\overline{R}}(P_{i,k,l}, P_{i,s,t})$.

(4) By applying $\mathrm{Hom}_R(-, P_{ij})$ to (4.2), we have an exact sequence

$$\mathrm{Hom}_R(P_{ik}, P_{ij}) \xrightarrow{\mathrm{Hom}(f, P_{ij})} \mathrm{Hom}_R(P_{il}, P_{ij}) \longrightarrow \mathrm{Ext}^1_R(P_{i,k,l}, P_{ij}) \longrightarrow 0.$$

It can be seen that $\mathrm{Ext}^1_R(P_{i,k,l}, P_{ij}) = 0$ if and only if $\mathrm{Hom}(f, P_{ij})$ is an epimorphism.

We show that $\mathrm{Hom}(f, P_{ij})$ is an epimorphism if and only if $j \le k$ or $l < j$. First we assume that $j > k$ and $l \ge j$. By $l \ge j$, there exists a monomorphism from P_{il} to P_{ij}. But there are no monomorhisms from P_{ik} to P_{ij} by $j > k$. Since P_{ik} has simple socle, gf is not monic for any $g \in \mathrm{Hom}_R(P_{ik}, P_{ij})$. Thus $\mathrm{Hom}(f, P_{ij})$ is not an epimorphism. Next we assume $j \le k$. By Lemma 2.3 (1), $\mathrm{Hom}(f, P_{ij})$ is an epimorphism. Finally we assume $l < j$. Then by $\mathrm{length}(P_{il}) > \mathrm{length}(P_{ij})$, there are no monomorphisms from P_{il} to P_{ij}. By Lemma 2.3 (2), $\mathrm{Hom}(f, P_{ij})$ is an epimorphism.

On the other hand, by applying $\mathrm{Hom}_{\overline{R}}(-, \overline{P}_{i,j})$ to (4.3), we have an exact sequence

$$\mathrm{Hom}_{\overline{R}}(\overline{P}_{i,k}, \overline{P}_{i,j}) \xrightarrow{\mathrm{Hom}(f', \overline{P}_{i,j})} \mathrm{Hom}_{\overline{R}}(\overline{P}_{i,l}, \overline{P}_{i,j}) \longrightarrow \mathrm{Ext}^1_{\overline{R}}(P_{i,k,l}, \overline{P}_{i,j}) \longrightarrow 0.$$

It can be seen that $\mathrm{Ext}^1_{\overline{R}}(P_{i,k,l}, \overline{P}_{i,j}) = 0$ if and only if $\mathrm{Hom}(f', \overline{P}_{i,j})$ is an epimor-

phism. We can show that $\mathrm{Hom}(f', \overline{P}_{i,j})$ is an epimorphism if and only if $j \leq k$ or $l < j$ by the same argument as above.

Consequently $\mathrm{Ext}^1_R(P_{i,k,l}, P_{ij}) = 0$ if and only if $j \leq k$ or $l < j$ which is equivalent to $\mathrm{Ext}^1_{\overline{R}}(P_{i,k,l}, \overline{P}_{i,j}) = 0$. □

By Lemma 4.5 and Proposition 4.7, we have Proposition 4.4.

Finally to prove Theorem 4.1, we need the following well-known proposition which describes a very useful equivalent condition of tilting modules.

PROPOSITION 4.8 ([**2**]). *Let R be an algebra and T a partial tilting R-module. Then the following conditions are equivalent.*

(1) *T is a tilting module.*
(2) *The number of pairwise nonisomorphic indecomposable direct summand of T is equal to the number of pairwise nonisomorphic simple R-modules.*

Now we can prove Theorem 4.1.

PROOF OF THEOREM 4.1. Any basic tilting R-module lies in $\mathscr{P}$. By Lemma 4.3 and Proposition 4.4, F induces a bijection from the set of isomorphism classes of partial tilting R-modules to that of isomorphism classes of partial tilting $\overline{R}$-modules. Moreover by Proposition 4.8, F induces a bijection from $\mathrm{tilt}(R)$ to $\mathrm{tilt}(\overline{R})$ since the number of isomorphism classes of simple R-modules is equal to that of isomorphism classes of simple $\overline{R}$-modules. □

5. Combinatorial description of tilting $\mathrm{T}_n(K)$-modules.

In this section, we recall the well-known classification of basic tilting modules over the upper triangular matrix algebra $\mathrm{T}_n(K)$. It was done by constructing a bijection between $\mathrm{tilt}(\mathrm{T}_n(K))$ and the set of non-crossing partitions of the regular $(n + 2)$-polygon into triangles. By this classification and Theorem 4.2, we can classify basic tilting modules over left Harada algebras.
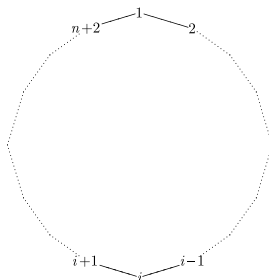
First we introduce coordinates in the AR-quiver of $\mathrm{T}_n(K)$ as follows.
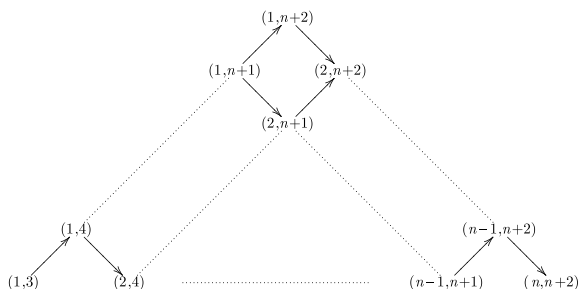
We remark that the vertex (i, j) corresponds the $T_n(K)$-module

$$M_{ij} = \begin{pmatrix} \overset{j-2}{0} \ ... \ 0 \ \overset{1}{K} \ ... \ K \end{pmatrix} \Big/ \begin{pmatrix} \overset{i}{0} \ ... \ 0 \ \overset{1}{K} \ ...K \end{pmatrix} = \begin{pmatrix} \overset{j-2}{0} \ ... \ 0 \ \overset{i}{K} \ ... \ K \ \overset{1}{0} \ ... \ 0 \end{pmatrix}.$$

Next we consider a regular $(n+2)$-polygon R_{n+2} whose vertices are numbered as follows.

We denote by $D(R_{n+2})$ the set of all diagonals of R_{n+2} except edges of R_{n+2}. We call a subset S of $D(R_{n+2})$ a *non-crossing partition* of R_{n+2} if S satisfies the following conditions.

(1) Any two distinct diagonals in S do not cross except at their endpoints.
(2) R_{n+2} is divided into triangles by diagonals in S.

We denote by $\mathscr{P}_{n+2}$ the set of an non-crossing partitions of R_{n+2}.

Now we construct the correspondence Φ from $\mathscr{P}_{n+2}$ to tilt$(T_n(K))$. We denote by (i, j) the diagonal between i and j for $i < j$. It is obvious that there exists a bijection

$$D(R_{n+2}) \ni (i, j) \longmapsto M_{ij} \in \{M_{ij} \mid (i, j) \neq (1, n+2)\}.$$

We take $S \in \mathscr{P}_{n+2}$. Since any non-crossing partition of R_{n+2} consists of $n - 1$ diagonals, S can be putted by

$$S = \{(i_1, j_1), (i_2, j_2), \ldots, (i_{n-1}, j_{n-1})\}.$$

Then we define

$$\Phi(S) := M_{1, n+2} \oplus \left(\bigoplus_{k=1}^{n-1} M_{i_k, j_k} \right).$$

This correspondence $\Phi : \mathscr{P}_{n+2} \longrightarrow \mathrm{tilt}(\mathrm{T}_n(K))$ is well-defined, and moreover the following hold.

THEOREM 5.1.   *The above correspondence* $\Phi : \mathscr{P}_{n+2} \longrightarrow \mathrm{tilt}(\mathrm{T}_n(K))$ *is a bijection.*

EXAMPLE 5.2.   We consider $n = 3$ case. We classify basic tilting $\mathrm{T}_3(K)$-modules by using Theorem 5.1. All of partitions of the regular pentagon into triangles are given as follows.

(1)

(2)

(3)

(4)

(5)
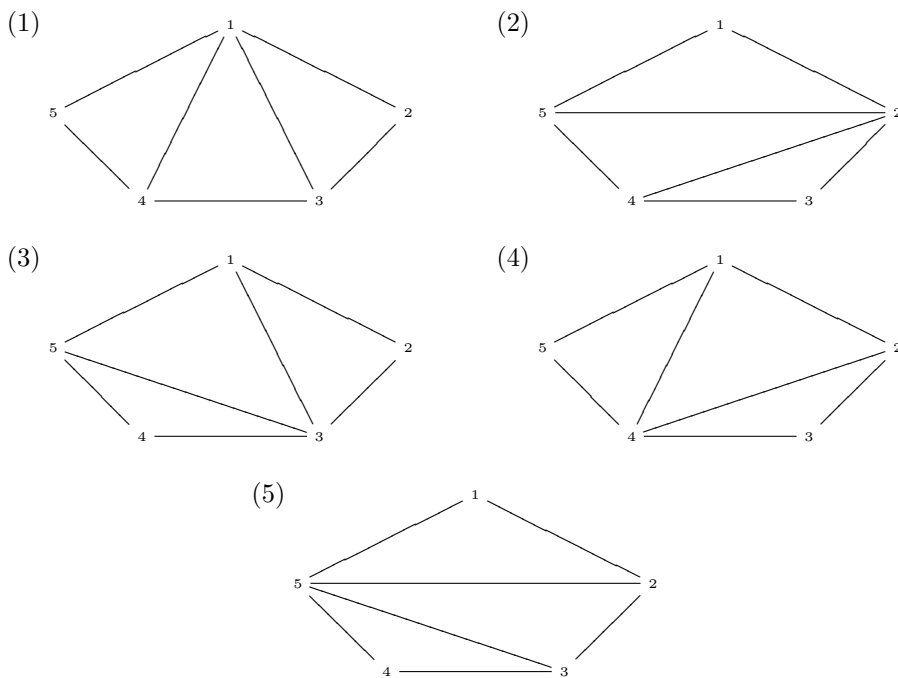
Therefore the number of basic tilting $\mathrm{T}_3(K)$-modules is equal to 5 and all of basic tilting $\mathrm{T}_3(K)$-modules are given as follows.

(1) $\big(\, K\ K\ K\,\big) \oplus \big(\, 0\ K\ K\,\big) \oplus \big(\, 0\ 0\ K\,\big)$,
(2) $\big(\, K\ K\ K\,\big) \oplus \big(\, K\ K\ 0\,\big) \oplus \big(\, 0\ K\ 0\,\big)$,
(3) $\big(\, K\ K\ K\,\big) \oplus \big(\, K\ 0\ 0\,\big) \oplus \big(\, 0\ 0\ K\,\big)$,
(4) $\big(\, K\ K\ K\,\big) \oplus \big(\, 0\ K\ K\,\big) \oplus \big(\, 0\ K\ 0\,\big)$,
(5) $\big(\, K\ K\ K\,\big) \oplus \big(\, K\ K\ 0\,\big) \oplus \big(\, K\ 0\ 0\,\big)$.

Now we show examples of classifications of tilting modules over left Harada algebras.

EXAMPLE 5.3. (1) Let R be a local quasi-Frobenius algebra. Then we consider block extension (cf. [**3**], [**10**])

$$R(n) = \begin{pmatrix} R & \cdots & R \\ & \ddots & \vdots \\ J(R) & & R \end{pmatrix}$$

for $n \in \mathbf{N}$ of R which is a subalgebra of $n \times n$ full matrix algebra over R. We can show that

(a) the first row is a injective module,
(b) the i-th row is the Jacobson radical of the $(i-1)$-th row for $2 \leq i \leq n$.

In particular $R(n)$ is a left Harada algebra with $m = 1$ and $n_1 = n$ in Definition 1.2.

By Corollary 4.2, we have a bijection $F : \mathrm{tilt}(R(n)) \longrightarrow \mathrm{tilt}(\mathrm{T}_n(K))$. We can obtain all basic tilting $R(n)$-modules from the definition of F and Theorem 5.1.
(2) Let R be a basic quasi-Frobenius algebra which has a complete set of orthogonal primitive idempotents $\{e, f\}$. Then we can represent R as follows.

$$R \simeq \begin{pmatrix} eRe & eRf \\ fRe & fRf \end{pmatrix}.$$

We put $Q := eRe$, $W := fRf$, $A := eRf$ and $B := fRe$. Now we consider the block extension (cf. [**3**], [**10**])

$$R(n_1, n_2) = \left(\begin{array}{ccc|ccc} Q & \cdots & Q & A & \cdots & A \\ & \ddots & \vdots & \vdots & & \vdots \\ J(Q) & & Q & A & \cdots & A \\ \hline B & \cdots & B & W & \cdots & W \\ \vdots & & \vdots & & \ddots & \vdots \\ B & \cdots & B & J(W) & & W \end{array} \right)$$

for $n_1, n_2 \in \mathbf{N}$ of R which is a subalgebra of $\mathrm{End}_R((eR)^{n_1} \oplus (fR)^{n_2})$. We can show that

(a) the first and $n_1 + 1$ row are injective modules,
(b) the i-th row is the Jacobson radical of the $(i-1)$-th row for $2 \leq i \leq n_1$ and $n_1 + 2 \leq i \leq n_1 + n_2$.

In particular $R(n_1, n_2)$ is a left Harada algebra with $m = 2$ in Definition 1.2.

By Corollary 4.2, we have a bijection $F : \mathrm{tilt}(R(n_1, n_2)) \longrightarrow \mathrm{tilt}(\mathrm{T}_{n_1}(K)) \times \mathrm{tilt}(\mathrm{T}_{n_2}(K))$. We can obtain all basic tilting $R(n_1, n_2)$-modules from the definition of F and Theorem 5.1.

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