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The intersection of two real forms in Hermitian symmetric spaces of compact type

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Abstract. We show that the intersections of two real forms, certain totally geodesic Lagrangian submanifolds, in Hermitian symmetric spaces of compact type are antipodal sets. The intersection number of two real forms is invariant under the replacement of the two real forms by congruent ones. If two real forms are congruent, then their intersection is a great antipodal set of them. It implies that any real form in Hermitian symmetric spaces of compact type is a globally tight Lagrangian submanifold. Moreover we describe the intersection of two real forms in the irreducible Hermitian symmetric spaces of compact type.

1. Introduction.

Let $\bar{M}$ be a Hermitian symmetric space. A submanifold M is called a *real form* of $\bar{M}$, if there exists an involutive anti-holomorphic isometry σ of $\bar{M}$ satisfying

$$M = \{x \in \bar{M} \mid \sigma(x) = x\}.$$

Any real form M is a totally geodesic Lagrangian submanifold of $\bar{M}$, which follows from Leung [**7**] or Lemma 1.1 in Takeuchi [**13**]. Leung [**7**] and Takeuchi [**13**] classified real forms of Hermitian symmetric spaces of compact type.

A subset S in a Riemannian symmetric space M is called an *antipodal set*, if the geodesic symmetry s_x fixes every point of S for every point x of S. The *2-number* $\#_2 M$ of M is the supremum of the cardinalities of antipodal sets of M. We call an antipodal set in M *great* if its cardinality attains $\#_2 M$. These were introduced by Chen and Nagano [**3**]. Takeuchi [**14**] proved that if M is a symmetric R-space, then

$$\#_2 M = \dim H_*(M, \boldsymbol{Z}_2), \tag{1.1}$$

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where $H_*(M, \boldsymbol{Z}_2)$ denotes the homology group of M with coefficient $\boldsymbol{Z}_2$. A compact Riemannian symmetric space is called a *symmetric R-space* if it is an orbit of the linear isotropy action of a Riemannian symmetric pair of semisimple type. We note that any real form of Hermitian symmetric spaces of compact type is a symmetric R-space, which is shown in [**13**].

THEOREM 1.1. *Let M be a Hermitian symmetric space of compact type. If two real forms L_1 and L_2 of M intersect transversally, then $L_1 \cap L_2$ is an antipodal set of L_1 and L_2.*

For a connected Riemannian manifold M we denote by $I_0(M)$ the identity component of the group of all isometries on M. We say that two submanifolds in a Hermitian symmetric space of compact type M are *congruent*, if one is transformed to another by an element of $I_0(M)$. Each element of $I_0(M)$ is a holomorphic isometry.

THEOREM 1.2. *Let M be a Hermitian symmetric space of compact type and let L_1, L_2, L_1', L_2' be real forms of M. We assume that L_1, L_1' are congruent and that L_2, L_2' are congruent. If L_1, L_2 intersect transversally and if L_1', L_2' intersect transversally, then $\#(L_1 \cap L_2) = \#(L_1' \cap L_2')$.*

THEOREM 1.3. *Let M be a Hermitian symmetric space of compact type and let L_1 and L_2 be real forms of M which are congruent to each other and intersect transversally. Then $L_1 \cap L_2$ is a great antipodal set of L_1 and L_2. That is, $\#(L_1 \cap L_2) = \#_2 L_1 = \#_2 L_2$.*

THEOREM 1.4. *Let M be an irreducible Hermitian symmetric space of compact type and let L_1 and L_2 be two real forms of M which intersect transversally.*

(1) *If $M = G_{2m}^{\boldsymbol{C}}(\boldsymbol{C}^{4m})$ $(m \geq 2)$, L_1 is congruent to $G_m^{\boldsymbol{H}}(\boldsymbol{H}^{2m})$ and L_2 is congruent to $U(2m)$, then*

$$\#(L_1 \cap L_2) = 2^m < \binom{2m}{m} = \#_2 L_1 < 2^{2m} = \#_2 L_2.$$

(2) *Otherwise, $L_1 \cap L_2$ is a great antipodal set of one of L_i's whose 2-number is less than or equal to another and we have*

$$\#(L_1 \cap L_2) = \min\{\#_2 L_1, \#_2 L_2\}.$$

REMARK 1.5. In the complex projective space $\boldsymbol{C}P^n$, any real form is congruent to the real projective space $\boldsymbol{R}P^n$ naturally embedded in $\boldsymbol{C}P^n$. Howard essentially showed the following statement in [**5**, pp. 26–27]. If two real forms L_1 and L_2 of $\boldsymbol{C}P^n$ intersect transversally, then there exists a unitary basis $u_1, \dots, u_{n+1}$ of $\boldsymbol{C}^{n+1}$ satisfying

$$L_1 \cap L_2 = \{\boldsymbol{C}u_1, \dots, \boldsymbol{C}u_{n+1}\}.$$

In particular $L_1 \cap L_2$ is a great antipodal set of L_1 and L_2, because $\#_2 \boldsymbol{R}P^n = n+1$. Thus Theorems stated above are generalizations of this statement. In this case $L_1 \cap L_2$ is also a great antipodal set of $\boldsymbol{C}P^n$, because $\#_2 \boldsymbol{C}P^n = n+1$.

Oh [**10**] introduced the notion of global tightness of Lagrangian submanifolds in a Hermitian symmetric space. We call a Lagrangian submanifold L of a Hermitian symmetric space M *globally tight*, if L satisfies

$$\#(L \cap g \cdot L) = \dim H_*(L, \boldsymbol{Z}_2)$$

for any $g \in I_0(M)$ with property that L intersects transversally with $g \cdot L$. We obtain the following corollary from (1.1) and Theorem 1.3.

COROLLARY 1.6. *Any real form of a Hermitian symmetric space of compact type is a globally tight Lagrangian submanifold.*

REMARK 1.7. We denote by $Q_n(\boldsymbol{C})$ the complex hyperquadric of complex dimension n, which is holomorphically isometric to the real oriented Grassmann manifold $\tilde{G}_2^{\boldsymbol{R}}(\boldsymbol{R}^{n+2})$. We regard $\tilde{G}_2^{\boldsymbol{R}}(\boldsymbol{R}^{n+2})$ as a submanifold in $\wedge^2 \boldsymbol{R}^{n+2}$ in a natural way and define a real form $S^{k,n-k}$ of $\tilde{G}_2^{\boldsymbol{R}}(\boldsymbol{R}^{n+2})$ by

$$S^{k,n-k} = S^k(\boldsymbol{R}e_1 + \cdots + \boldsymbol{R}e_{k+1}) \wedge S^{n-k}(\boldsymbol{R}e_{k+2} + \cdots + \boldsymbol{R}e_{n+2}),$$

where $S^m(V)$ is the unit sphere of a real Euclidean space V of dimension $m+1$. $Q_1(\boldsymbol{C}) = \boldsymbol{C}P^1 = S^2$ and its real form is a great circle, so its global tightness is well known. $Q_2(\boldsymbol{C}) = \boldsymbol{C}P^1 \times \boldsymbol{C}P^1 = S^2 \times S^2$ and its real forms $S^{0,2}$ and $S^{1,1}$ are globally tight, which Iriyeh and Sakai [**6**] proved in a different way. Recently they also proved that $S^{0,n}$ and $S^{1,n-1}$ are globally tight in $Q_n(\boldsymbol{C})$. After that the second author showed in [**15**] that the intersection of two real forms in $Q_n(\boldsymbol{C})$ is an antipodal set whose cardinality attains the smaller one of the 2-numbers of the two real forms. It is a corollary of the result that any real form in $Q_n(\boldsymbol{C})$ is a globally tight Lagrangian submanifold. The results in the present paper are generalizations of the results obtained in [**15**].

The organization of this paper is as follows. In Section 2 we briefly review some fundamental results on compact Riemannian symmetric spaces we need later. We prepare some properties of maximal tori of compact Riemannian symmetric spaces in Section 3. In Section 4 using properties of maximal tori obtained in Sections 2 and 3 we prove Theorem 1.1. The notion of polars of compact Riemannian symmetric spaces plays an essential role in the proofs of Theorems 1.2, 1.3 and 1.4. A relation between real forms and polars stated in Lemma 4.2 makes it possible to prove Theorem 1.3 by induction on polars. In Section 5 we prove Theorem 1.4 in each case of two real forms in irreducible Hermitian symmetric spaces of compact type using their classifications. In Section 6 we show some explicit descriptions of the intersections of two real forms in the complex Grassmann manifolds.

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2. Preliminaries.

We briefly review some fundamental results on compact Riemannian symmetric spaces in this section. After that we recall a result of Takeuchi [**12**] on maximal tori and a result of Sakai [**11**] on cut loci.

Let (G, K) be a compact symmetric pair with respect to an involutive automorphism θ of G. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K respectively. The involutive automorphism of $\mathfrak{g}$ induced from θ is also denoted by θ. Take an inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ which is invariant under θ and the adjoint representation of G. This inner product induces a Riemannian metric on the homogeneous manifold $M = G/K$. With respect to this metric M is a compact Riemannian symmetric space and any compact Riemannian symmetric space is obtained in this way. We have

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta(X) = X\}.$$

Set

$$\mathfrak{m} = \{X \in \mathfrak{g} \mid \theta(X) = -X\},$$

then we have a canonical orthogonal direct sum decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}.$$

We identify the tangent space of M at the origin o with $\mathfrak{m}$ and denote by $\mathrm{Exp} : \mathfrak{m} \to M$ the exponential map. Take a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{m}$. It is known that $A = \mathrm{Exp}\,\mathfrak{a}$ is a maximal totally geodesic flat submanifold of M, which is called a *maximal torus*. For $\lambda \in \mathfrak{a}$ we define root spaces

$$\mathfrak{m}_\lambda = \{X \in \mathfrak{m} \mid [H,[H,X]] = -\langle \lambda, H\rangle^2 X \quad (H \in \mathfrak{a})\},$$
$$\mathfrak{k}_\lambda = \{X \in \mathfrak{k} \mid [H,[H,X]] = -\langle \lambda, H\rangle^2 X \quad (H \in \mathfrak{a})\}$$

and define the root system R of $(\mathfrak{g}, \mathfrak{k})$ by

$$R = \{\lambda \in \mathfrak{a} - \{0\} \mid \mathfrak{m}_\lambda \neq \{0\}\}.$$

We take a fundamental system Π of R and denote by R_+ the set of positive roots with respect to Π. We have orthogonal direct sum decompositions ([**4**]):

$$\mathfrak{k} = \mathfrak{k}_0 + \sum_{\lambda \in R_+} \mathfrak{k}_\lambda, \quad \mathfrak{m} = \mathfrak{a} + \sum_{\lambda \in R_+} \mathfrak{m}_\lambda,$$

where $\mathfrak{k}_0 = \{X \in \mathfrak{k} \mid [X, \mathfrak{a}] = 0\}$. We denote by δ_i $(1 \leq i \leq s)$ the highest root of each irreducible factor of R and set

$$R^\# = \{\delta_i \mid 1 \leq i \leq s\}, \quad \Pi^\# = \Pi \cup R^\#.$$

For $\Delta \subset \Pi^\#$ we define

$$S^\Delta = \big\{H \in \mathfrak{a} \mid \langle \alpha, H\rangle > 0 \ (\alpha \in \Delta \cap \Pi),\ \langle \beta, H\rangle = 0 \ (\beta \in \Pi - \Delta),$$
$$\langle \delta_i, H\rangle < \pi \ (\delta_i \in \Delta \cap R^\#),\ \langle \delta_j, H\rangle = \pi \ (\delta_j \in R^\# - \Delta)\big\}.$$

Let $S = S^{\Pi^\#}$. We have

$$M = \bigcup_{k \in K} k \operatorname{Exp}(\bar{S}), \qquad \bar{S} = \bigcup_{\Delta \subset \Pi^\#} S^\Delta,$$

where $\bar{S}$ is the closure of S. We denote by $\bar{W}$ the affine Weyl group which is the semidirect product of the Weyl group of (G, K) and the lattice

$$\Gamma(A) = \{H \in \mathfrak{a} \mid \operatorname{Exp} H = o\}$$

of the maximal torus A. $\bar{W}$ naturally acts on $\mathfrak{a}$. We set

$$\bar{W}_S = \{\tau \in \bar{W} \mid \tau S = S\}.$$

Lemma 2.1 (Takeuchi [**12**, Lemma 1.7]). *If $k \operatorname{Exp} H_1 = \operatorname{Exp} H_2$ holds for $\Delta_1,\ \Delta_2 \subset \Pi^{\#}$, $H_1 \in S^{\Delta_1}$, $H_2 \in S^{\Delta_2}$ and $k \in K$, then there exists $\tau \in \bar{W}_S$ satisfying*

(1) $\tau S^{\Delta_1} = S^{\Delta_2}$,
(2) *for any* $H \in S^{\Delta_1}$, $k \operatorname{Exp} H = \operatorname{Exp} \tau H$,
(3) $\tau H_1 = H_2$.

In particular, $k \operatorname{Exp} S^{\Delta_1} = \operatorname{Exp} S^{\Delta_2}$.

For a compact Riemannian manifold X and $p \in X$, we denote by $C_p(X)$ and $\tilde{C}_p(X)$ the cut locus and the tangential cut locus of X with respect to p respectively.

Theorem 2.2 (Sakai [**11**]). *For a maximal torus A through the origin o of a compact Riemannian symmetric space $M = G/K$ we have*

$$\tilde{C}_o(A) = \mathfrak{a} \cap \tilde{C}_o(M), \qquad \tilde{C}_o(M) = \bigcup_{k \in K} \operatorname{Ad}(k) \tilde{C}_o(A),$$

$$C_o(A) = A \cap C_o(M), \qquad C_o(M) = \bigcup_{k \in K} k C_o(A).$$

3. Maximal tori.

We describe a maximal torus of the fixed point set of an involutive isometry of a symmetric R-space by the use of a canonical coordinate defined in Definition 3.2. We also show a property of the intersection of two maximal tori of a compact Riemannian symmetric space. For a Riemannian manifold X and an isometry ϕ of X we denote by $F(\phi, X)$ the fixed point set of ϕ. It is known that each connected component of $F(\phi, X)$ is a totally geodesic submanifold of X.

Lemma 3.1. *Let M be a compact Riemannian symmetric space. We assume that τ is an involutive isometry of M satisfying $\tau(o) = o$. We take the connected component M_1 of $F(\tau, M)$ through o and a maximal torus A_1 of M_1 through o. For a maximal torus A of M including A_1, we have $\tau(A) = A$.*

Proof. We denote by $\mathfrak{m}_1$, $\mathfrak{a}_1$ and $\mathfrak{a}$ the tangent spaces of M_1, A_1 and A at o respectively. We consider the space

$$\mathfrak{b}_1 = \{X \in \mathfrak{m}_1 \mid \langle X, \mathfrak{a}_1 \rangle = \{0\}\}.$$

If $\mathfrak{a}_1 = \mathfrak{m}_1$, we have $\mathfrak{b}_1 = \{0\} \subset \mathfrak{a}^{\perp}$. We shall show that $\mathfrak{b}_1 \subset \mathfrak{a}^{\perp}$ even in the case where $\mathfrak{a}_1 \neq \mathfrak{m}_1$. In this case $\mathfrak{b}_1$ is described by the root spaces of M_1 as follows:

$$\mathfrak{b}_1 = \sum_{\lambda \in (R_1)_+} (\mathfrak{m}_1)_\lambda.$$

For $\lambda \in R_1$ we can take $H_1 \in \mathfrak{a}_1$ satisfying $\langle \lambda, H_1 \rangle \neq 0$. For any $X \in (\mathfrak{m}_1)_\lambda$ we have $(\operatorname{ad} H_1)^2 X = -\langle \lambda, H_1 \rangle^2 X$, so

$$X = -\frac{1}{\langle \lambda, H_1 \rangle^2} (\operatorname{ad} H_1)^2 X.$$

Since $\mathfrak{a}_1 \subset \mathfrak{a}$ are abelian, for any $H_2 \in \mathfrak{a}$

$$\begin{aligned} \langle H_2, X \rangle &= -\frac{1}{\langle \lambda, H_1 \rangle^2} \langle H_2, (\operatorname{ad} H_1)^2 X \rangle \\ &= -\frac{1}{\langle \lambda, H_1 \rangle^2} \langle (\operatorname{ad} H_1)^2 H_2, X \rangle = 0. \end{aligned}$$

Hence we have $X \in \mathfrak{a}^\perp$ and $(\mathfrak{m}_1)_\lambda \subset \mathfrak{a}^\perp$. Therefore we obtain $\mathfrak{b}_1 \subset \mathfrak{a}^\perp$.

By the definition of M_1

$$\mathfrak{m}_1 = \{X \in \mathfrak{m} \mid d\tau_o(X) = X\}.$$

Since τ is an involutive isometry,

$$\mathfrak{m}_1^\perp = \{X \in \mathfrak{m} \mid d\tau_o(X) = -X\}.$$

We have showed $\mathfrak{m}_1 = \mathfrak{a}_1 + \mathfrak{b}_1$, $\mathfrak{a}_1 \subset \mathfrak{a}$, and $\mathfrak{b}_1 \subset \mathfrak{a}^\perp$, thus

$$\mathfrak{a} = \mathfrak{a}_1 + \mathfrak{a} \cap \mathfrak{m}_1^\perp.$$

For any $X \in \mathfrak{a}$ we can express $X = X_1 + X_2$ $(X_1 \in \mathfrak{a}_1,\ X_2 \in \mathfrak{a} \cap \mathfrak{m}_1^\perp)$ and obtain

$$d\tau_o(X) = d\tau_o(X_1 + X_2) = X_1 - X_2 \in \mathfrak{a}.$$

Therefore $\tau(A) = A$ holds. □

Definition 3.2. A compact Riemannian symmetric space M is said to be *cubic*, if its maximal torus A has an orthonormal basis of the lattice $\Gamma(A)$ for a suitable invariant metric. For a maximal torus A of a cubic compact Riemannian symmetric space we call a coordinate $x_1, \dots, x_r$ of $\mathfrak{a}$ satisfying

$$\Gamma(A) = \{(x_1, \dots, x_r) \mid x_i \in \pi \boldsymbol{Z}\}$$

a *canonical coordinate* of A.

A compact Riemannian symmetric space is cubic if and only if it is a symmetric R-space by Sätze 5 and 6 in Loos [**8**].

PROPOSITION 3.3. *Let M_2 be a cubic compact Riemannian symmetric space and τ be an involutive isometry of M_2 fixing the origin o. Let M_1 be the connected component of $F(\tau, M_2)$ through o. We take maximal tori A_i of M_i through o satisfying $A_1 \subset A_2$. There exists a canonical coordinate $x_1, \dots, x_r$ of A_2 satisfying*

$$d\tau_o(x_1, \dots, x_r) = (x_2, x_1, \dots, x_{2p}, x_{2p-1}, x_{2p+1}, \dots, x_q, -x_{q+1}, \dots, -x_r),$$
$$\mathfrak{a}_1 = \{(x_1, \dots, x_r) \mid x_1 = x_2, \dots, x_{2p-1} = x_{2p}, x_{q+1} = \cdots = x_r = 0\}.$$

PROOF. We have $\tau(A_2) = A_2$ by Lemma 3.1. We assume that the rank of M_2 is equal to r. Since M_2 is cubic, A_2 is isometric to the Riemann product $S^1_1 \times \cdots \times S^1_r$ of r copies of S^1. Each component x_i of a canonical coordinate of A_2 is a canonical coordinate of S^1_i. The image of S^1_i under τ is the same S^1_i or another S^1_j. If the image of S^1_i is S^1_j $(j \neq i)$, we change the order of the coordinate such that the image of S^1_{2i-1} is S^1_{2i} for $1 \leq i \leq p$. If the image of S^1_i is S^1_i itself, τ on S^1_i is the identity or reverses the orientation of S^1_i. Thus $d\tau_o$ maps $x_i \mapsto \pm x_i$. We change the order of the coordinate such that $d\tau_o$ maps $x_i \mapsto x_i$ for $2p+1 \leq i \leq q$ and that $d\tau_o$ maps $x_j \mapsto -x_j$ for $q+1 \leq j \leq r$. By the change of the coordinate we have

$$d\tau_o(x_1, \dots, x_r) = (x_2, x_1, \dots, x_{2p}, x_{2p-1}, x_{2p+1}, \dots, x_q, -x_{q+1}, \dots, -x_r).$$

Since $\mathfrak{a}_1$ is the 1-eigenspace of $d\tau_o$, we have

$$\mathfrak{a}_1 = \{(x_1, \dots, x_r) \mid x_1 = x_2, \dots, x_{2p-1} = x_{2p}, x_{q+1} = \cdots = x_r = 0\}. \qquad \square$$

Since a Hermitian symmetric space of compact type is cubic, we can apply Proposition 3.3 to it.

PROPOSITION 3.4. *Under the assumption of Proposition 3.3, if M_2 is a Hermitian symmetric space of compact type and if M_1 is a real form of M_2, then*

$$\mathfrak{a}_1 = \{(x_1, \ldots, x_{2p}, x_{2p+1}, \ldots, x_r) \mid x_{2i-1} = x_{2i}\ (1 \leq i \leq p)\}.$$

REMARK 3.5. Any real form M_1 of an irreducible Hermitian symmetric space of compact type M_2 is of maximal rank or satisfies $\mathrm{rank}(M_1) = \mathrm{rank}(M_2)/2$ and

$$\mathfrak{a}_1 = \{(x_1, \ldots, x_{2p}) \mid x_{2i-1} = x_{2i}\ (1 \leq i \leq p)\}.$$

PROOF. Let τ be an involutive anti-holomorphic isometry of M_2 which determines M_1 as its fixed point set. Lemma 3.1 implies $\tau(A_2) = A_2$. The maximal torus A_2 has a complexification $A_2^{\boldsymbol{C}} = \boldsymbol{C}P^1 \times \cdots \times \boldsymbol{C}P^1$ in M_2, that is, A_2 is a real form of $A_2^{\boldsymbol{C}}$. Each factor $\boldsymbol{C}P^1$ in $A_2^{\boldsymbol{C}}$ is holomorphically isometric to each other. $d\tau_o$ leaves $T_o(A_2^{\boldsymbol{C}})$ invariant, so we have $\tau(A_2^{\boldsymbol{C}}) = A_2^{\boldsymbol{C}}$. The image of each factor $\boldsymbol{C}P^1$ of $A_2^{\boldsymbol{C}}$ under τ is (1) itself or (2) another $\boldsymbol{C}P^1$. In the case of (1) τ induces an involutive anti-holomorphic isometry of $\boldsymbol{C}P^1$ and its fixed point set is a great circle. In the case of (2) τ induces an involutive anti-holomorphic isometry of $\boldsymbol{C}P^1 \times \boldsymbol{C}P^1 \cong Q_2(\boldsymbol{C})$ and its fixed point set is congruent to $S^{0,2}$ in $Q_2(\boldsymbol{C})$. Thus by a suitable change of the order of the coordinate of $\mathfrak{a}_2$ we have

$$\mathfrak{a}_1 = \{(x_1, \ldots, x_{2p}, x_{2p+1}, \ldots, x_r) \mid x_{2i-1} = x_{2i}\ (1 \leq i \leq p)\}. \qquad \square$$

LEMMA 3.6. *Let A_1, A_2 be two maximal tori of a compact Riemannian symmetric space through the origin o. We define the root system from A_2 and determine $S \subset \mathfrak{a}_2$. If $A_1 \cap A_2 \cap \mathrm{Exp}\, S^\Delta \neq \emptyset$ for a subset $\Delta \subset \Pi^\#$, then $\mathrm{Exp}\, S^\Delta \subset A_1 \cap A_2$.*

PROOF. We take a point $\mathrm{Exp}\, H_2$ ($H_2 \in S^\Delta$) in $A_1 \cap A_2 \cap \mathrm{Exp}\, S^\Delta$. Since this point also belongs to A_1, we can express it as $\mathrm{Exp}\, H_2 = \mathrm{Exp}\, X_1$ ($X_1 \in \mathfrak{a}_1$). Moreover we can take X_1 with $\|X_1\| = \|H_2\|$ by Theorem 2.2. Since all maximal tori are conjugate, there exists $k \in K$ satisfying $\mathrm{Ad}(k^{-1})\mathfrak{a}_1 = \mathfrak{a}_2$ and $\mathrm{Ad}(k^{-1})X_1 \in \bar{S}$. We set $H_1 = \mathrm{Ad}(k^{-1})X_1 \in \bar{S}$. We have $k\,\mathrm{Exp}\, H_1 = \mathrm{Exp}\, X_1 = \mathrm{Exp}\, H_2$. We take $H_1 \in S^{\Delta_1}$ satisfying $\Delta_1 \subset \Pi^\#$. Lemma 2.1 implies $k\,\mathrm{Exp}\, S^{\Delta_1} = \mathrm{Exp}\, S^\Delta \subset A_1 \cap A_2$. $\square$

4. The intersection of two real forms.

In this section we prove Theorem 1.1, using properties of maximal tori obtained in Sections 2 and 3. We recall the notion of polars of compact Riemannian symmetric spaces and show a relation between real forms and polars stated in Lemma 4.2. According to this, we can prove Theorem 1.3 by induction on polars.

PROOF OF THEOREM 1.1. The holomorphic sectional curvature of M is positive, so $L_1 \cap L_2 \neq \emptyset$ by Lemma 3.1 in [**15**]. If $L_1 \cap L_2$ consists of one point, there is noting to prove. So we assume that $\#(L_1 \cap L_2) \geq 2$ and take any two points of $L_1 \cap L_2$. We regard the one point as the origin o and denote by p another point. It is sufficient to prove that o and p are antipodal.

We take maximal tori A_i of L_i containing o, p and maximal tori A_i' of M containing A_i. We denote by $\mathfrak{a}_2'$ the maximal abelian subspace corresponding to A_2' and take $H_2 \in \mathfrak{a}_2'$ satisfying $p = \operatorname{Exp} H_2$. We take a fundamental system Π such that $H_2 \in \bar{S}$, where $S = S^{\Pi^\#}$. Lemma 3.6 implies $p \in \operatorname{Exp} S^\Delta \subset A_1' \cap A_2'$.

We show that $p \in \operatorname{Exp} S^\Delta \subset A_1 \cap A_2$ by the use of Proposition 3.4.

We represent $\mathfrak{a}_2'$ by

$$\mathfrak{a}_2' = \{(x_1, \ldots, x_r)\}$$

with respect to a canonical coordinate of A_2. Proposition 3.4 implies that there exists an involutive permutation λ of $\{1, \ldots, r\}$ satisfying

$$\mathfrak{a}_2 = \{(x_1, \ldots, x_r) \mid x_i = x_{\lambda(i)} \ (1 \leq i \leq r)\}.$$

Since each irreducible factor of the root system of M is of type C or BC, we have

$$S^\Delta = \{(x_1, \ldots, x_r) \mid \pi/2 \geq x_1 = \cdots = x_{i_1} > x_{i_1+1} = \cdots = x_r \geq 0\}.$$

Moreover

$$S^\Delta \cap \{(x_1, \ldots, x_r) \mid x_i = x_{\lambda(i)} \ (1 \leq i \leq r)\} \neq \emptyset.$$

A point H_2 in S^Δ satisfies the equation $x_i = x_{\lambda(i)}$ for all i, so every point in S^Δ satisfies $x_i = x_{\lambda(i)}$ and we have

$$S^\Delta \subset \{(x_1, \ldots, x_r) \mid x_i = x_{\lambda(i)} \ (1 \leq i \leq r)\}.$$

We get a similar relation of inclusion with respect to $\mathfrak{a}_1, \mathfrak{a}_1'$, so we have $p \in \operatorname{Exp} S^\Delta \subset A_1 \cap A_2$.

If $\dim S^\Delta > 0$, the relation $p \in \operatorname{Exp} S^\Delta \subset A_1 \cap A_2$ contradicts the assumption that L_1 and L_2 intersect transversally. Therefore $\dim S^\Delta = 0$ and S^Δ is a vertex of $\bar{S}$. Since M is cubic, p is an antipodal point of o. There exist shortest geodesics joining o and p in L_1 and L_2, so o and p are antipodal in L_1 and L_2. $\square$

Definition 4.1. Let M be a compact connected Riemannian symmetric space and $p \in M$. We decompose the fixed point set $F(s_p, M)$ of the geodesic symmetry s_p at the origin p to the disjoint union of its connected components:

$$F(s_p, M) = \bigcup_{j=0}^{r} M_j^+,$$

where $M_0^+ = \{p\}$. We call each connected component M_j^+ a *polar* of M with respect to p.

Any polar is a totally geodesic submanifold. Chen-Nagano [**1**] introduced the notion polar. Nagano [**9**] determined the polars of each irreducible compact Riemannian symmetric space. Each polar of a Hermitian symmetric space of compact type is also a Hermitian symmetric space of compact type.

Lemma 4.2. *Let M be a Hermitian symmetric space of compact type and L be a real form of M through o. If a polar M^+ satisfies $L \cap M^+ \neq \emptyset$, then $L \cap M^+$ is a real form of M^+.*

Proof. We denote by τ the involutive anti-holomorphic isometry determining L. Since $o \in L$, we have $\tau(o) = o$ and $d\tau_o = 1_{T_oL} - 1_{T_o^\perp L}$. Thus $d(\tau \circ s_o)_o = d(s_o \circ \tau)_o$ and $\tau \circ s_o = s_o \circ \tau$. We have $s_o(\tau(x)) = \tau(s_o(x)) = \tau(x)$ for any $x \in F(s_o, M)$, hence we obtain $\tau(F(s_o, M)) = F(s_o, M)$.

We can take $p \in L \cap M^+$, because of the assumption that $L \cap M^+ \neq \emptyset$. Since $\tau(p) = p$, we have $\tau(M^+) = M^+$. Therefore τ induces an involutive anti-holomorphic isometry of M^+ and $L \cap M^+$ is a real form of M^+. $\square$

The following lemma shows that some properties of the intersection of two real forms of a Hermitian symmetric space of compact type can be reduced to those of the intersection of two real forms of each polar.

Lemma 4.3. *Let M be a Hermitian symmetric space of compact type, and denote by*

$$F(s_o, M) = \bigcup_{j=0}^{r} M_j^+$$

the polars of M with respect to the origin o.

(1) *If L is a real form of M through o, then the polars of L with respect to o is described by*

$$F(s_o, L) = \bigcup_{j=0}^{r} L \cap M_j^+,$$

and the following equality holds.

$$\#_2 L = \sum_{j=0}^{r} \#_2\big(L \cap M_j^+\big).$$

(2) *If L_1, L_2 are real forms of M through o, then we have*

$$L_1 \cap L_2 = \bigcup_{j=0}^{r} \big\{(L_1 \cap M_j^+) \cap (L_2 \cap M_j^+)\big\},$$

$$\#(L_1 \cap L_2) = \sum_{j=0}^{r} \#\big\{(L_1 \cap M_j^+) \cap (L_2 \cap M_j^+)\big\}.$$

Proof. (1) Since L is a totally geodesic submanifold through o, we have

$$F(s_o, L) = L \cap F(s_o, M) = \bigcup_{j=0}^{r} L \cap M_j^+.$$

Lemma 4.2 implies that $L \cap M_j^+$ is a real form of M_j^+ if $L \cap M_j^+$ is not empty. Any real form of a Hermitian symmetric space of compact type is connected by Theorem 3.8 in Leung [**7**], so $L \cap M_j^+$ is connected, hence it is a polar of L. L is a symmetric R-space by a result of Takeuchi [**13**], and the following equality holds by results of [**14**].

$$\#_2 L = \sum_{j=0}^{r} \#_2\big(L \cap M_j^+\big).$$

(2) $L_1 \cap L_2$ is antipodal in L_1 and L_2 by Theorem 1.1. $L_1 \cap L_2$ is also antipodal in M, so we have $L_1 \cap L_2 \subset F(s_o, M)$. Hence we get

$$L_1 \cap L_2 = \bigcup_{j=0}^{r} \big\{(L_1 \cap M_j^+) \cap (L_2 \cap M_j^+)\big\},$$

$$\#(L_1 \cap L_2) = \sum_{j=0}^{r} \#\big\{(L_1 \cap M_j^+) \cap (L_2 \cap M_j^+)\big\}. \qquad \square$$

Proof of Theorem 1.2. There exist $\phi_0, \phi_1 \in I_0(M)$ which satisfy $L_1' = \phi_0 L_1$ and $L_2' = \phi_1 L_2$. Since L_1' and L_2' intersect transversally, $\phi_0^{-1} L_1' = L_1$ and $\phi_0^{-1} L_2' = \phi_0^{-1}\phi_1 L_2$ intersect transversally, too. We set $g = \phi_0^{-1}\phi_1 \in I_0(M)$. L_1 and gL_2 intersect transversally and $\#(L_1' \cap L_2') = \#(L_1 \cap gL_2)$ holds. Thus the theorem reduces to the following statement.

(A) Assume that real forms L_1, L_2 in M and $g \in I_0(M)$ satisfy that L_1, L_2 intersect transversally and L_1, gL_2 intersect transversally, too. Then $\#(L_1 \cap L_2) = \#(L_1 \cap gL_2)$.

We can take $o \in L_1 \cap L_2$ and $p \in L_1 \cap gL_2$ by Lemma 3.1 in [**15**]. L_1 is an orbit of a subgroup of $I_0(M)$, so there exists $\phi_2 \in I_0(M)$ which satisfies $\phi_2 L_1 = L_1$ and $\phi_2(p) = o$. Then $\phi_2 L_1 = L_1$ and $\phi_2 g L_2$ intersect transversally. Since $o, \phi_2 g(o) \in \phi_2 g L_2$, there exists $\phi_3 \in I_0(M)$ which satisfies $\phi_3\phi_2 g L_2 = \phi_2 g L_2$ and $\phi_3\phi_2 g(o) = o$. We denote

$$K = \{\phi \in I_0(M) \mid \phi(o) = o\}.$$

We have $\phi_3\phi_2 g \in K$ and set $k = \phi_3\phi_2 g$. L_1 and $kL_2 = \phi_3\phi_2 g L_2 = \phi_2 g L_2$ intersect transversally. Since

$$\#(L_1 \cap gL_2) = \#(\phi_2(L_1 \cap gL_2)) = \#(L_1 \cap kL_2),$$

the statement (A) reduces to the following statement.

(B) Let L_1, L_2 be real forms in M which intersect transversally. If we take $o \in L_1 \cap L_2$ and $k \in I_0(M)$ which satisfies $k(o) = o$ and that L_1, kL_2 intersect transversally, then $\#(L_1 \cap L_2) = \#(L_1 \cap kL_2)$.

Now we prove $\#(L_1 \cap L_2) = \#(L_1 \cap kL_2)$. According to (2) of Lemma 4.3, we have

$$L_1 \cap L_2 = \bigcup_{j=0}^{r} \big\{(L_1 \cap M_j^+) \cap (L_2 \cap M_j^+)\big\},$$

$$L_1 \cap kL_2 = \bigcup_{j=0}^{r} \big\{(L_1 \cap M_j^+) \cap (kL_2 \cap M_j^+)\big\}$$

and

$$\#(L_1 \cap L_2) = \sum_{j=0}^{r} \#\big\{(L_1 \cap M_j^+) \cap (L_2 \cap M_j^+)\big\},$$

$$\#(L_1 \cap kL_2) = \sum_{j=0}^{r} \#\big\{(L_1 \cap M_j^+) \cap (kL_2 \cap M_j^+)\big\}.$$

The subsets $L_2 \cap M_j^+$ and $kL_2 \cap M_j^+$ are congruent in M_j^+ for each j. So $L_2 \cap M_j^+$ and $kL_2 \cap M_j^+$ are simultaneously empty, the same point or congruent real forms in M_j^+ for each j. In the first case, we have

$$\#\big\{(L_1 \cap M_j^+) \cap (L_2 \cap M_j^+)\big\} = 0 = \#\big\{(L_1 \cap M_j^+) \cap (kL_2 \cap M_j^+)\big\}.$$

In the second case, we have

$$\#\big\{(L_1 \cap M_j^+) \cap (L_2 \cap M_j^+)\big\} = 1 = \#\big\{(L_1 \cap M_j^+) \cap (kL_2 \cap M_j^+)\big\}.$$

In the third case, $L_2 \cap M_j^+$ and $kL_2 \cap M_j^+$ intersect transversally in M_j^+. The action of k on M_j^+ is contained in $I_0(M_j^+)$, so $L_2 \cap M_j^+$ and $kL_2 \cap M_j^+ = k(L_2 \cap M_j^+)$ are congruent real forms in M_j^+. We can apply the argument above to them in M_j^+. We take $o_j \in (L_1 \cap M_j^+) \cap (L_2 \cap M_j^+)$. By the argument above we can take $k_j \in I_0(M_j^+)$ which satisfies $k_j(o_j) = o_j$, $o_j \in (L_1 \cap M_j^+) \cap k_j(L_2 \cap M_j^+)$ and that $(L_1 \cap M_j^+) \cap (L_2 \cap M_j^+)$ and $(L_1 \cap M_j^+) \cap k_j(L_2 \cap M_j^+)$ are congruent in M_j^+. In particular we have

$$\#\big\{(L_1 \cap M_j^+) \cap (L_2 \cap M_j^+)\big\} = \#\big\{(L_1 \cap M_j^+) \cap k_j(L_2 \cap M_j^+)\big\}.$$

We denote by

$$F\big(s_{o_j}, M_j^+\big) = \bigcup_{k=0}^{r_j} M_{jk}^+$$

the polars of M_j^+ with respect to o_j. According to (2) of Lemma 4.3, we have

$$\big(L_1 \cap M_j^+\big) \cap \big(L_2 \cap M_j^+\big) = \bigcup_{l=0}^{r_j} \{((L_1 \cap M_j^+) \cap M_{jl}^+) \cap ((L_2 \cap M_j^+) \cap M_{jl}^+)\},$$

$$\big(L_1 \cap M_j^+\big) \cap k_j\big(L_2 \cap M_j^+\big) = \bigcup_{l=0}^{r_j} \{((L_1 \cap M_j^+) \cap M_{jl}^+) \cap (k_j(L_2 \cap M_j^+) \cap M_{jl}^+)\}$$

and

$$\#\big((L_1 \cap M_j^+) \cap (L_2 \cap M_j^+)\big) = \sum_{l=0}^{r_j} \#\big\{((L_1 \cap M_j^+) \cap M_{jl}^+) \cap ((L_2 \cap M_j^+) \cap M_{jl}^+)\big\},$$

$$\#\big(L_1 \cap M_j^+) \cap k_j(L_2 \cap M_j^+\big) = \sum_{l=0}^{r_j} \#\big\{((L_1 \cap M_j^+) \cap M_{jl}^+) \cap (k_j(L_2 \cap M_j^+) \cap M_{jl}^+)\big\}.$$

The subsets $(L_2 \cap M_j^+) \cap M_{jl}^+$ and $k_j(L_2 \cap M_j^+) \cap M_{jl}^+$ are congruent in M_{jl}^+ for each l. So $(L_2 \cap M_j^+) \cap M_{jl}^+$ and $k_j(L_2 \cap M_j^+) \cap M_{jl}^+$ are simultaneously empty, the same point or congruent real forms in M_{jl}^+ for each l. In the first and the second cases, we have

$$\begin{aligned} &\#\big\{((L_1 \cap M_j^+) \cap M_{jl}^+) \cap ((L_2 \cap M_j^+) \cap M_{jl}^+)\big\} \\ &\quad = \#\big\{((L_1 \cap M_j^+) \cap M_{jl}^+) \cap (k_j(L_2 \cap M_j^+) \cap M_{jl}^+)\big\}. \end{aligned}$$

In the third case, $(L_1 \cap M_j^+) \cap M_{jl}^+$ and $k_j(L_2 \cap M_j^+) \cap M_{jl}^+$ intersect transversally in M_{jl}^+. If we repeat this argument finitely many times, we reach the first and the second cases, because $\dim M_{jl}^+ < \dim M_j^+$. Hence we obtain (B) and (A), so complete the proof of the theorem. □

Proof of Theorem 1.3. Because of Theorem 1.1, the intersection $L_1 \cap L_2$ is an antipodal set of L_1 and L_2. In order to prove the theorem we may show $\#(L_1 \cap L_2) = \#_2 L_1 = \#_2 L_2$.

We can suppose that $o \in L_1 \cap L_2$ without loss of generality. According to (2) of Lemma 4.3, we have

$$L_1 \cap L_2 = \bigcup_{j=0}^{r} \big\{(L_1 \cap M_j^+) \cap (L_2 \cap M_j^+)\big\},$$

$$\#(L_1 \cap L_2) = \sum_{j=0}^{r} \#\big\{(L_1 \cap M_j^+) \cap (L_2 \cap M_j^+)\big\}.$$

According to (1) of Lemma 4.3, the polars of L_i are described by

$$F(s_o, L_i) = \bigcup_{j=0}^{r} L_i \cap M_j^+$$

and the following equality for $i = 1, 2$ holds.

$$\#_2 L_i = \sum_{j=0}^{r} \#_2\big(L_i \cap M_j^+\big).$$

Since L_1 and L_2 are congruent, we have $\#_2 L_1 = \#_2 L_2$. The subsets $L_1 \cap M_j^+$ and $L_2 \cap M_j^+$ are congruent in M_j^+ for each j. So $L_1 \cap M_j^+$ and $L_2 \cap M_j^+$ are simultaneously empty, the same point or congruent real forms in M_j^+ for each j. In the first and the second cases, we have

$$\#\big\{(L_1 \cap M_j^+) \cap (L_2 \cap M_j^+)\big\} = \#\big(L_i \cap M_j^+\big) = \#_2\big(L_i \cap M_j^+\big)$$

for $i = 1, 2$. In the third case, $L_1 \cap M_j^+$ and $L_2 \cap M_j^+$ intersect transversally in M_j^+. We can apply the argument above to them in M_j^+. We take $o_j \in (L_1 \cap M_j^+) \cap (L_2 \cap M_j^+)$ and denote by

$$F\big(s_{o_j}, M_j^+\big) = \bigcup_{k=0}^{r_j} M_{jk}^+$$

the polars of M_j^+ with respect to o_j. We have

$$\begin{aligned}&\#\big\{(L_1 \cap M_j^+) \cap (L_2 \cap M_j^+)\big\}\\ &\quad = \sum_{k=0}^{r_j} \#\big\{((L_1 \cap M_j^+) \cap M_{jk}^+) \cap ((L_2 \cap M_j^+) \cap M_{jk}^+)\big\}.\end{aligned}$$

The subsets $(L_1 \cap M_j^+) \cap M_{jk}^+$ and $(L_2 \cap M_j^+) \cap M_{jk}^+$ are simultaneously empty, the same point or congruent real forms in M_{jk}^+ for each k. In the first and the second cases, we have

$$\begin{aligned}&\#\big\{((L_1 \cap M_j^+) \cap M_{jk}^+) \cap ((L_2 \cap M_j^+) \cap M_{jk}^+)\big\}\\ &\quad = \#\big((L_i \cap M_j^+) \cap M_{jk}^+\big) = \#_2\big((L_i \cap M_j^+) \cap M_{jk}^+\big)\end{aligned}$$

for $i = 1, 2$. In the third case, $(L_1 \cap M_j^+) \cap M_{jk}^+$ and $(L_2 \cap M_j^+) \cap M_{jk}^+$ intersect transversally in M_{jk}^+. We can apply the argument above to them in M_{jk}^+. If we repeat this argument finitely many times, we reach the first and the second cases. Thus we obtain

$$\#(L_1 \cap L_2) = \#_2 L_1 = \#_2 L_2. \qquad \square$$

At the last of this section we show an example of real forms.

EXAMPLE 4.4. Let $M = (\boldsymbol{C}P^1)^4$ and $\tau_1, \tau_2 : \boldsymbol{C}P^1 \to \boldsymbol{C}P^1$ be involutive anti-holomorphic isometries of $\boldsymbol{C}P^1$. τ_1, τ_2 are conjugate under holomorphic isometries of $\boldsymbol{C}P^1$. We assume that the real forms determined by τ_1 and τ_2 intersect transversally. We define L_1, L_2 by

$$L_1 = \{(x, y, \tau_1(x), \tau_1(y)) \mid x, y \in \boldsymbol{C}P^1\},$$
$$L_2 = \{(x, \tau_2(x), y, \tau_2(y)) \mid x, y \in \boldsymbol{C}P^1\}.$$

Lemma 5.8 implies that L_1 and L_2 are real forms in M. Moreover L_1 and L_2 are transformed to each other by a holomorphic isometry of M. Let $\{o, \bar{o}\}$ be the intersection of the real forms determined by τ_1 and τ_2 and denote $o^4 = (o, o, o, o) \in M$. We have

$$L_1 \cap F(s_{o^4}, M) = \{(o, o, o, o), (o, \bar{o}, o, \bar{o}), (\bar{o}, o, \bar{o}, o), (\bar{o}, \bar{o}, \bar{o}, \bar{o})\},$$
$$L_2 \cap F(s_{o^4}, M) = \{(o, o, o, o), (o, o, \bar{o}, \bar{o}), (\bar{o}, \bar{o}, o, o), (\bar{o}, \bar{o}, \bar{o}, \bar{o})\}.$$

It implies

$$L_1 \cap L_2 = \{(o, o, o, o), (\bar{o}, \bar{o}, \bar{o}, \bar{o})\}.$$

$\{(o, o, o, o), (o, \bar{o}, o, \bar{o}), (\bar{o}, o, \bar{o}, o), (\bar{o}, \bar{o}, \bar{o}, \bar{o})\}$ is a great antipodal set of L_1 and $\{(o, o, o, o), (o, o, \bar{o}, \bar{o}), (\bar{o}, \bar{o}, o, o), (\bar{o}, \bar{o}, \bar{o}, \bar{o})\}$ is a great antipodal set of L_2. Therefore we obtain

$$\#(L_1 \cap L_2) = 2 < 4 = \#_2 L_1 = \#_2 L_2.$$

5. Irreducible Hermitian symmetric spaces of compact type.

We treat the intersection of two real forms which are not congruent in irreducible Hermitian symmetric spaces of compact type in this section.

PROOF OF THEOREM 1.4. We apply Lemma 4.3 to two real forms which are not congruent in each irreducible Hermitian symmetric space of compact type and compare their intersection number with their 2-numbers. The list of irreducible Hermitian symmetric spaces of compact type and their real forms which we have to show the statements of the theorem is, according to the results of Leung [**7**] or Takeuchi [**13**], as follows:

M	L_1	L_2
$Q_n(\boldsymbol{C})$	$S^{k,n-k}$	$S^{l,n-l}$
$G_{2q}^{\boldsymbol{C}}(\boldsymbol{C}^{2m+2q})$	$G_q^{\boldsymbol{H}}(\boldsymbol{H}^{m+q})$	$G_{2q}^{\boldsymbol{R}}(\boldsymbol{R}^{2m+2q})$
$G_n^{\boldsymbol{C}}(\boldsymbol{C}^{2n})$	$U(n)$	$G_n^{\boldsymbol{R}}(\boldsymbol{R}^{2n})$
$G_{2m}^{\boldsymbol{C}}(\boldsymbol{C}^{4m})$	$G_m^{\boldsymbol{H}}(\boldsymbol{H}^{2m})$	$U(2m)$
$Sp(2m)/U(2m)$	$Sp(m)$	$U(2m)/O(2m)$
$SO(4m)/U(2m)$	$U(2m)/Sp(m)$	$SO(2m)$
$E_6/T\cdot Spin(10)$	$F_4/Spin(9)$	$G_2^{\boldsymbol{H}}(\boldsymbol{H}^4)/\boldsymbol{Z}_2$
$E_7/T\cdot E_6$	$T\cdot(E_6/F_4)$	$(SU(8)/Sp(4))/\boldsymbol{Z}_2$

In the case of the complex hyperquadric $Q_n(\boldsymbol{C})$, the statement of (2) was already obtained in [**15**]. The other cases are showed in Theorems 5.1–5.7. $\square$

Theorem 5.1. *If real forms L_1 congruent to $G_q^{\boldsymbol{H}}(\boldsymbol{H}^{m+q})$ and L_2 congruent to $G_{2q}^{\boldsymbol{R}}(\boldsymbol{R}^{2m+2q})$ in $G_{2q}^{\boldsymbol{C}}(\boldsymbol{C}^{2m+2q})$ intersect transversally, then their intersection $L_1\cap L_2$ is a great antipodal set of L_1 and*

$$\#(L_1\cap L_2)=\#_2L_1=\binom{m+q}{q}\le\binom{2m+2q}{2q}=\#_2L_2.$$

Theorem 5.2. *If real forms L_1 congruent to $U(n)$ and L_2 congruent to $G_n^{\boldsymbol{R}}(\boldsymbol{R}^{2n})$ in $G_n^{\boldsymbol{C}}(\boldsymbol{C}^{2n})$ intersect transversally, then their intersection $L_1\cap L_2$ is a great antipodal set of L_1 and*

$$\#(L_1\cap L_2)=\#_2L_1=2^n\le\binom{2n}{n}=\#_2L_2.$$

Theorem 5.3. *If real forms L_1 congruent to $G_m^{\boldsymbol{H}}(\boldsymbol{H}^{2m})$ and L_2 congruent to $U(2m)$ in $G_{2m}^{\boldsymbol{C}}(\boldsymbol{C}^{4m})$ intersect transversally, then*

$$\#(L_1\cap L_2)=2^m,\qquad \min\{\#_2L_1,\#_2L_2\}=\binom{2m}{m}.$$

If $m=1$, then $\#(L_1\cap L_2)=\min\{\#_2L_1,\#_2L_2\}$ holds. If $m\ge 2$, then $\#(L_1\cap L_2)<\min\{\#_2L_1,\#_2L_2\}$ holds.

Theorem 5.4. *If real forms L_1 congruent to $Sp(m)$ and L_2 congruent to $U(2m)/O(2m)$ in $Sp(2m)/U(2m)$ intersect transversally, then their intersection $L_1 \cap L_2$ is a great antipodal set of L_1 and*

$$\#(L_1 \cap L_2) = \#_2 L_1 = 2^m \leq 2^{2m} = \#_2 L_2.$$

Theorem 5.5. *If real forms L_1 congruent to $U(2m)/Sp(m)$ and L_2 congruent to $SO(2m)$ in $SO(4m)/U(2m)$ intersect transversally, then their intersection $L_1 \cap L_2$ is a great antipodal set of L_1 and*

$$\#(L_1 \cap L_2) = \#_2 L_1 = 2^m \leq 2^{2m-1} = \#_2 L_2.$$

Theorem 5.6. *If real forms L_1 congruent to $F_4/Spin(9)$ and L_2 congruent to $G_2^{\boldsymbol{H}}(\boldsymbol{H}^4)/\boldsymbol{Z}_2$ in $E_6/T \cdot Spin(10)$ intersect transversally, then their intersection $L_1 \cap L_2$ is a great antipodal set of L_1 and*

$$\#(L_1 \cap L_2) = \#_2 L_1 = 3 < 27 = \#_2 L_2.$$

Theorem 5.7. *If real forms L_1 congruent to $T \cdot (E_6/F_4)$ and L_2 congruent to $(SU(8)/Sp(4))/\boldsymbol{Z}_2$ in $E_7/T \cdot E_6$ intersect transversally, then their intersection $L_1 \cap L_2$ is a great antipodal set of L_1 and*

$$\#(L_1 \cap L_2) = \#_2 L_1 = 8 < 56 = \#_2 L_2.$$

Polars of irreducible Hermitian symmetric spaces of compact type are not irreducible in general. In order to treat their real forms we prepare the following Lemma 5.8 and Proposition 5.9.

Lemma 5.8. *Let M be a Hermitian symmetric space of compact type and $\tau : M \to M$ be an involutive anti-holomorphic isometry. The transformation defined by $(x, y) \mapsto (\tau(y), \tau(x))$ is an involutive anti-holomorphic isometry of $M \times M$. Its fixed point set is given by*

$$D_\tau(M) = \{(x, \tau(x)) \mid x \in M\}.$$

The conclusions of Lemma 5.8 directly follow from the assumptions, so we omit its proof.

Proposition 5.9. (1) *Let M_1, M_2 be Hermitian symmetric spaces of compact type, L_1, L_1' two real forms of M_1, and L_2, L_2' two real forms of M_2.*

Then $L_1 \times L_2$ *and* $L'_1 \times L'_2$ *are real forms of* $M_1 \times M_2$ *and we have* $(L_1 \times L_2) \cap (L'_1 \times L'_2) = (L_1 \cap L'_1) \times (L_2 \cap L'_2)$. *If* L_1, L'_1 *intersect transversally and if* L_2, L'_2 *intersect transversally, then* $L_1 \times L_2$ *and* $L'_1 \times L'_2$ *intersect transversally and we have* $\#\{(L_1 \times L_2) \cap (L'_1 \times L'_2)\} = \#(L_1 \cap L'_1)\#(L_2 \cap L'_2)$.

(2) *Let* L_1, L_2 *be real forms of a Hermitian symmetric space* M *of compact type and* $\tau : M \to M$ *an involutive anti-holomorphic isometry. We have*

$$(L_1 \times L_2) \cap D_\tau(M) = \{(x, \tau(x)) \mid x \in L_1 \cap \tau^{-1}(L_2)\}.$$

Two real forms $L_1 \times L_2$ *and* $D_\tau(M)$ *of* $M \times M$ *intersect transversally, if and only if* L_1 *and* $\tau^{-1}(L_2)$ *intersect transversally. In this case, we have*

$$\#\{(L_1 \times L_2) \cap D_\tau(M)\} = \#\{L_1 \cap \tau^{-1}(L_2)\}.$$

Here $\tau^{-1}(L_2)$ *and* L_2 *are congruent.*

(3) *If* M *is a Hermitian symmetric space of compact type and if* $\tau_1, \tau_2 : M \to M$ *are involutive anti-holomorphic isometries which are conjugate with respect to holomorphic isometries, then* $D_{\tau_1}(M)$ *and* $D_{\tau_2}(M)$ *are congruent. Moreover if* $D_{\tau_1}(M)$ *and* $D_{\tau_2}(M)$ *transversally intersect, then* $\#(D_{\tau_1}(M) \cap D_{\tau_2}(M)) = \#_2 M$.

Proof. (1) The conclusions of (1) directly follow from the assumptions.

(2) The intersection of $L_1 \times L_2$ and $D_\tau(M)$ is given by

$$(L_1 \times L_2) \cap D_\tau(M) = \{(x, \tau(x)) \mid x \in L_1 \cap \tau^{-1}(L_2)\}.$$

This equality implies that $L_1 \times L_2$ and $D_\tau(M)$ intersect transversally in $M \times M$ if and only if L_1 and $\tau^{-1}(L_2)$ intersect transversally in M. In this case, we have

$$\#\{(L_1 \times L_2) \cap D_\tau(M)\} = \#\{L_1 \cap \tau^{-1}(L_2)\}.$$

Let τ_i be the involutive anti-holomorphic isometry determining L_i. We have $\tau^{-1}(L_2) = \tau^{-1} \circ \tau_2(L_2)$ and $\tau^{-1} \circ \tau_2$ is a holomorphic isometry of M. Hence $\tau^{-1}(L_2)$ and L_2 are congruent.

(3) By the assumption there exists an holomorphic isometry g satisfying $\tau_2 = g\tau_1 g^{-1}$. Any point in $D_{\tau_i}(M)$ is equal to $(x, \tau_i(x))$ for $x \in M$, thus

$$(x, \tau_2(x)) = (g \times g)(g^{-1}(x), \tau_1 g^{-1}(x)).$$

This implies $D_{\tau_2}(M) = (g \times g)D_{\tau_1}(M)$ and $D_{\tau_1}(M)$ and $D_{\tau_2}(M)$ are congruent. Moreover if $D_{\tau_1}(M)$ and $D_{\tau_2}(M)$ transversally intersect, then according to Theorem 1.3

$$\#(D_{\tau_1}(M) \cap D_{\tau_2}(M)) = \#_2(D_{\tau_1}(M)) = \#_2 M. \qquad \square$$

PROOF OF THEOREM 5.1. We prove the theorem by induction on q, m. If $q = m = 1$, then $G_2^{\boldsymbol{C}}(\boldsymbol{C}^4)$ is holomorphically isometric to the complex hyperquadric $Q_3(\boldsymbol{C})$, and real forms $G_1^{\boldsymbol{H}}(\boldsymbol{H}^2)$ and $G_2^{\boldsymbol{R}}(\boldsymbol{R}^4)$ in $G_2^{\boldsymbol{C}}(\boldsymbol{C}^4)$ are respectively congruent to $S^{0,4}$ and $S^{2,2}$ in $Q_3(\boldsymbol{C})$. According to the result in [**15**], $L_1 \cap L_2$ is a great antipodal set of L_1 and

$$\#(L_1 \cap L_2) = \#_2 L_1 = 2 < 6 = \#_2 L_2.$$

Next we consider the case of general q, m. By [**9**, (3.12)] the polars of $G_{2q}^{\boldsymbol{C}}(\boldsymbol{C}^{2m+2q})$ are given by

$$M_j^+ = G_j^{\boldsymbol{C}}(\boldsymbol{C}^{2q}) \times G_{2q-j}^{\boldsymbol{C}}(\boldsymbol{C}^{2m}) \qquad (0 \leq j \leq 2q)$$

and the polars of $G_q^{\boldsymbol{H}}(\boldsymbol{H}^{m+q})$ and of $G_{2q}^{\boldsymbol{R}}(\boldsymbol{R}^{2m+2q})$ are given by

$$G_k^{\boldsymbol{H}}(\boldsymbol{H}^q) \times G_{q-k}^{\boldsymbol{H}}(\boldsymbol{H}^m) \qquad (0 \leq k \leq q),$$

and

$$G_k^{\boldsymbol{R}}(\boldsymbol{R}^{2q}) \times G_{2q-k}^{\boldsymbol{R}}(\boldsymbol{R}^{2m}) \qquad (0 \leq k \leq 2q)$$

respectively. By Lemma 4.2 for $0 \leq j \leq 2q$ we have

$$G_q^{\boldsymbol{H}}(\boldsymbol{H}^{m+q}) \cap M_j^+ = \begin{cases} \emptyset & (j : \text{odd}) \\ G_k^{\boldsymbol{H}}(\boldsymbol{H}^q) \times G_{q-k}^{\boldsymbol{H}}(\boldsymbol{H}^m) & (j = 2k) \end{cases}$$

$$G_{2q}^{\boldsymbol{R}}(\boldsymbol{R}^{2m+2q}) \cap M_j^+ = G_j^{\boldsymbol{R}}(\boldsymbol{R}^{2q}) \times G_{2q-j}^{\boldsymbol{R}}(\boldsymbol{R}^{2m}).$$

L_1 and L_2 are congruent to $G_q^{\boldsymbol{H}}(\boldsymbol{H}^{m+q})$ and $G_{2q}^{\boldsymbol{R}}(\boldsymbol{R}^{2m+2q})$ by the action of the isotropy subgroup K of $G_{2q}^{\boldsymbol{C}}(\boldsymbol{C}^{2m+2q})$ at the origin respectively. We note that K is connected. Each of M_j^+ is invariant under the action of K, the intersections $L_1 \cap M_j^+$ and $L_2 \cap M_j^+$ are congruent to $G_q^{\boldsymbol{H}}(\boldsymbol{H}^{m+q}) \cap M_j^+$ and $G_{2q}^{\boldsymbol{R}}(\boldsymbol{R}^{2m+2q}) \cap M_j^+$ in M_j^+ respectively. By the assumption of induction we have

$$\#\{(L_1 \cap M_{2k}^+) \cap (L_2 \cap M_{2k}^+)\} = \binom{q}{k}\binom{m}{q-k}.$$

Lemma 4.3 implies

$$\#(L_1 \cap L_2) = \sum_{k=0}^{q} \#\{(L_1 \cap M_{2k}^+) \cap (L_2 \cap M_{2k}^+)\} = \sum_{k=0}^{q} \binom{q}{k}\binom{m}{q-k}.$$

This is equal to the coefficient of x^q when we expand $(1+x)^q(1+x)^m$, thus

$$\#(L_1 \cap L_2) = \binom{m+q}{q} = \#_2 L_1 < \binom{2m+2q}{2q} = \#_2 L_2. \qquad \square$$

Proof of Theorem 5.2. By [**9**, (3.12)] the polars of $G_n^{\boldsymbol{C}}(\boldsymbol{C}^{2n})$ are given by

$$M_j^+ = G_j^{\boldsymbol{C}}(\boldsymbol{C}^n) \times G_{n-j}^{\boldsymbol{C}}(\boldsymbol{C}^n) \qquad (0 \leq j \leq n)$$

and the polars of $G_n^{\boldsymbol{R}}(\boldsymbol{R}^{2n})$ are given by

$$G_k^{\boldsymbol{R}}(\boldsymbol{R}^n) \times G_{n-k}^{\boldsymbol{R}}(\boldsymbol{R}^n) \qquad (0 \leq k \leq n).$$

By [**9**, (3.3)] the polars of $U(n)$ are given by

$$G_k^{\boldsymbol{C}}(\boldsymbol{C}^n) \qquad (0 \leq k \leq n).$$

We note that $G_j^{\boldsymbol{C}}(\boldsymbol{C}^n)$ and $G_{n-j}^{\boldsymbol{C}}(\boldsymbol{C}^n)$ are holomorphically isometric. By Lemma 4.2 for $0 \leq j \leq n$ we have

$$U(n) \cap M_j^+ = D_{\tau_j}\big(G_j^{\boldsymbol{C}}(\boldsymbol{C}^n)\big)$$
$$G_n^{\boldsymbol{R}}(\boldsymbol{R}^{2n}) \cap M_j^+ = G_j^{\boldsymbol{R}}(\boldsymbol{R}^n) \times G_{n-j}^{\boldsymbol{R}}(\boldsymbol{R}^n).$$

Since L_1 and L_2 are congruent to $U(n)$ and $G_n^{\boldsymbol{R}}(\boldsymbol{R}^{2n})$ in $G_n^{\boldsymbol{C}}(\boldsymbol{C}^{2n})$ respectively, $L_1 \cap M_j^+$ and $L_2 \cap M_j^+$ are congruent to $U(n) \cap M_j^+$ and $G_n^{\boldsymbol{R}}(\boldsymbol{R}^{2n}) \cap M_j^+$ in M_j^+ respectively. Lemma 5.8 and Theorem 1.3 imply

$$\begin{aligned}
&\#\{(L_1 \cap M_j^+) \cap (L_2 \cap M_j^+)\} \\
&\quad = \#\{(G_j^{\boldsymbol{R}}(\boldsymbol{R}^n) \times G_{n-j}^{\boldsymbol{R}}(\boldsymbol{R}^n)) \cap D_{\tau_j}(G_j^{\boldsymbol{C}}(\boldsymbol{C}^n))\} = \binom{n}{j}.
\end{aligned}$$

Lemma 4.3 implies

$$\#(L_1 \cap L_2) = \sum_{j=0}^{n} \#\{(L_1 \cap M_j^+) \cap (L_2 \cap M_j^+)\} = \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

On the other hand

$$\#_2 U(n) = 2^n \leq \binom{2n}{n} = \#_2 G_n^{\boldsymbol{R}}(\boldsymbol{R}^{2n}),$$

so we have

$$\min\{\#_2 L_1, \#_2 L_2\} = 2^n. \qquad \square$$

Proof of Theorem 5.3. By [**9**, (3.12)] the polars of $G_{2m}^{\boldsymbol{C}}(\boldsymbol{C}^{4m})$ are given by

$$M_j^+ = G_j^{\boldsymbol{C}}(\boldsymbol{C}^{2m}) \times G_{2m-j}^{\boldsymbol{C}}(\boldsymbol{C}^{2m}) \qquad (0 \leq j \leq 2m).$$

We note that $G_j^{\boldsymbol{C}}(\boldsymbol{C}^{2m})$ and $G_{2m-j}^{\boldsymbol{C}}(\boldsymbol{C}^{2m})$ are holomorphically isometric. By [**9**, (3.3)] the polars of $U(2m)$ are given by

$$G_k^{\boldsymbol{C}}(\boldsymbol{C}^{2m}) \qquad (0 \leq k \leq 2m)$$

and by [**9**, (3.12)] the polars of $G_m^{\boldsymbol{H}}(\boldsymbol{H}^{2m})$ are given by

$$G_k^{\boldsymbol{H}}(\boldsymbol{H}^m) \times G_{m-k}^{\boldsymbol{H}}(\boldsymbol{H}^m) \qquad (0 \leq k \leq m).$$

By Lemma 4.2 for $0 \leq j \leq 2m$

$$\begin{aligned}
G_m^{\boldsymbol{H}}(\boldsymbol{H}^{2m}) \cap M_j^+ &= \begin{cases} \emptyset & (j : \text{odd}) \\ G_k^{\boldsymbol{H}}(\boldsymbol{H}^m) \times G_{m-k}^{\boldsymbol{H}}(\boldsymbol{H}^m) & (j = 2k) \end{cases} \\
U(2m) \cap M_j^+ &= D_{\tau_j}(G_j^{\boldsymbol{C}}(\boldsymbol{C}^{2m})).
\end{aligned}$$

Since L_1 and L_2 are congruent to $G^{\boldsymbol{H}}_m(\boldsymbol{H}^{2m})$ and $U(2m)$ in $G^{\boldsymbol{C}}_{2m}(\boldsymbol{C}^{4m})$ respectively, $L_1 \cap M^+_j$ and $L_2 \cap M^+_j$ are congruent to $G^{\boldsymbol{H}}_m(\boldsymbol{H}^{2m}) \cap M^+_j$ and $U(2m) \cap M^+_j$ in M^+_j respectively. Lemma 5.8 and Theorem 1.3 imply

$$\begin{aligned}
&\#\big\{(L_1 \cap M^+_{2k}) \cap (L_2 \cap M^+_{2k})\big\} \\
&\quad = \#\big\{(G^{\boldsymbol{H}}_k(\boldsymbol{H}^m) \times G^{\boldsymbol{H}}_{m-k}(\boldsymbol{H}^m)) \cap D_{\tau_{2k}}(G^{\boldsymbol{C}}_{2k}(\boldsymbol{C}^{2m}))\big\} = \binom{m}{k}.
\end{aligned}$$

Lemma 4.3 implies

$$\#(L_1 \cap L_2) = \sum_{k=0}^{m} \#\big\{(L_1 \cap M^+_{2k}) \cap (L_2 \cap M^+_{2k})\big\} = \sum_{k=0}^{m} \binom{m}{k} = 2^m.$$

On the other hand

$$\#_2 G^{\boldsymbol{H}}_m(\boldsymbol{H}^{2m}) = \binom{2m}{m} \leq 2^{2m} = \#_2 U(2m),$$

so we have

$$\min\{\#_2 L_1, \#_2 L_2\} = \binom{2m}{m}. \qquad \square$$

PROOF OF THEOREM 5.4. By [**9**, (3.21)] the polars of $Sp(2m)/U(2m)$ are given by

$$M^+_j = G^{\boldsymbol{C}}_j(\boldsymbol{C}^{2m}) \qquad (0 \leq j \leq 2m).$$

By [**9**, (3.10)] the polars of $Sp(m)$ are given by

$$G^{\boldsymbol{H}}_k(\boldsymbol{H}^m) \qquad (0 \leq k \leq m)$$

and by [**9**, (3.17)] the polars of $U(2m)/O(2m)$ are given by

$$G^{\boldsymbol{R}}_k(\boldsymbol{R}^{2m}) \qquad (0 \leq k \leq 2m).$$

By Lemma 4.2 for $0 \leq j \leq 2m$

$$Sp(m) \cap M_j^+ = \begin{cases} \emptyset & (j : \text{odd}) \\ G_k^{\boldsymbol{H}}(\boldsymbol{H}^m) & (j = 2k) \end{cases}$$

$$(U(2m)/O(2m)) \cap M_j^+ = G_j^{\boldsymbol{R}}(\boldsymbol{R}^{2m}).$$

Since L_1 and L_2 are congruent to $Sp(m)$ and $U(2m)/O(2m)$ in $Sp(2m)/U(2m)$ respectively, $L_1 \cap M_j^+$ and $L_2 \cap M_j^+$ are congruent to $Sp(m) \cap M_j^+$ and $(U(2m)/O(2m)) \cap M_j^+$ in M_j^+ respectively. Thus by Theorem 5.1 we have

$$\#\{(L_1 \cap M_{2k}^+) \cap (L_2 \cap M_{2k}^+)\} = \binom{m}{k}.$$

Lemma 4.3 implies

$$\#(L_1 \cap L_2) = \sum_{k=0}^{m} \#\{(L_1 \cap M_{2k}^+) \cap (L_2 \cap M_{2k}^+)\} = \sum_{k=0}^{m} \binom{m}{k} = 2^m.$$

On the other hand

$$\#_2 Sp(m) = 2^m \leq 2^{2m} = \#_2(U(2m)/O(2m)),$$

so we have

$$\min\{\#_2 L_1, \#_2 L_2\} = 2^m. \qquad \square$$

Proof of Theorem 5.5. By [**9**, (3.20)] the polars of $SO(4m)/U(2m)$ are given by

$$M_j^+ = G_{2j}^{\boldsymbol{C}}(\boldsymbol{C}^{2m}) \qquad (0 \leq j \leq m).$$

By [**9**, (3.19)] the polars of $U(2m)/Sp(m)$ are given by

$$G_k^{\boldsymbol{H}}(\boldsymbol{H}^m) \qquad (0 \leq k \leq m)$$

and by [**9**, (3.6)] the polars of $SO(2m)$ are given by

$$G_{2k}^{\boldsymbol{R}}(\boldsymbol{R}^{2m}) \qquad (0 \leq k \leq m).$$

By Lemma 4.2 for $0 \leq j \leq m$

$$
\begin{aligned}
(U(2m)/Sp(m)) \cap M_j^+ &= G_j^{\boldsymbol{H}}(\boldsymbol{H}^m)\\
SO(2m) \cap M_j^+ &= G_{2j}^{\boldsymbol{R}}(\boldsymbol{R}^{2m}).
\end{aligned}
$$

Since L_1 and L_2 are congruent to $U(2m)/Sp(m)$ and $SO(2m)$ in $SO(4m)/U(2m)$ respectively, $L_1 \cap M_j^+$ and $L_2 \cap M_j^+$ are congruent to $(U(2m)/Sp(m)) \cap M_j^+$ and $SO(2m) \cap M_j^+$ in M_j^+ respectively. Thus by Theorem 5.1 we have

$$\#\{(L_1 \cap M_j^+) \cap (L_2 \cap M_j^+)\} = \binom{m}{j}.$$

Lemma 4.3 implies

$$\#(L_1 \cap L_2) = \sum_{j=0}^{m} \#\{(L_1 \cap M_j^+) \cap (L_2 \cap M_j^+)\} = \sum_{j=0}^{m} \binom{m}{j} = 2^m.$$

On the other hand

$$\#_2(U(2m)/Sp(m)) = 2^m \le 2^{2m-1} = \#_2 SO(2m),$$

so we have

$$\min\{\#_2 L_1, \#_2 L_2\} = 2^m. \qquad \square$$

Before the proofs of Theorem 5.6 and Theorem 5.7 we make some preparation for treating the cases of exceptional type. Let $M = M_1 \times M_2$ be a Riemann product of compact symmetric spaces M_1 and M_2. Since the geodesic symmetry s_o at $o = (o_1, o_2) \in M_1 \times M_2$ is defined by $s_o(x) = (s_{o_1}(x_1), s_{o_2}(x_2))$ for $(x_1, x_2) \in M_1 \times M_2$, we have

$$F(s_o, M_1 \times M_2) = F(s_{o_1}, M_1) \times F(s_{o_2}, M_2). \tag{5.2}$$

We assume that a discrete subgroup $\boldsymbol{Z}_\mu$ of the isometry group of a compact symmetric space M acts freely on M and the quotient space $M/\boldsymbol{Z}_\mu$ is a symmetric space. Let $\pi : M \to M/\boldsymbol{Z}_\mu$ be the projection. Then we have $s_{\pi(x)}(\pi(y)) = \pi(s_x(y))$ for $x, y \in M$. If $\mu = 2^k n$ where n is an odd number, π is a composition of k double covering maps and a n-fold covering map. Hence, it is enough to consider the cases where μ is odd and where $\mu = 2$ for knowing the polars of $M/\boldsymbol{Z}_\mu$.

DEFINITION 5.10. Let M be a compact connected Riemannian symmetric space and $p \in M$. If $\bar{p} \in M$ is an isolated point in $F(s_p, M) - \{p\}$, $\bar{p}$ is called a *pole* with respect to p. When there is a pole $\bar{p}$ with respect to p in M, the set of midpoints of the geodesic segments from p to $\bar{p}$ is called the *centrosome* with respect to p and $\bar{p}$ and denoted by $C(p, \bar{p})$. Each connected component of a centrosome is called a *centriole*.

LEMMA 5.11. *Let M be a compact connected Riemannian symmetric space. Assume that a discrete subgroup $\boldsymbol{Z}_2$ of the isometry group of M acts freely on M and the quotient space $M/\boldsymbol{Z}_2$ is a symmetric space. Let $\pi : M \to M/\boldsymbol{Z}_2$ be the projection. Then, if $\pi(x) = \pi(y)$ for $x, y \in M$, either $x = y$ or y is a pole with respect to x.*

PROOF. We show that if $\pi(x) = \pi(y)$ and $x \neq y$, y is a pole with respect to x. The geodesic symmetry s_x preserves $\pi^{-1}(\pi(x)) = \{x, y\}$ and fixes x, so s_x fixes y. We can take a neighborhood U of $\pi(x)$ so that each connected component $\tilde{U}_i$ $(i = 1, 2)$ of $\pi^{-1}(U)$ is homeomorphic to U under π with $x \in \tilde{U}_1$ and $y \in \tilde{U}_2$ and that $\pi(x)$ is the only fixed point of $s_{\pi(x)}$ on U. Then y is the only fixed point of s_x on $\tilde{U}_2$. In fact, if $s_x(y') = y'$ for $y' \neq y$, then $\pi(y') \in U$ and $\pi(y') \neq \pi(x)$ and $s_{\pi(x)}(\pi(y')) = \pi(s_x(y')) = \pi(y')$. This contradicts that $\pi(x)$ is the only fixed point of $s_{\pi(x)}$ on U. Hence y is a pole with respect to x. □

LEMMA 5.12. *Let M be a compact connected Riemannian symmetric space. Assume that a discrete subgroup $\boldsymbol{Z}_\mu$ of the isometry group of M acts freely on M and the quotient space $M/\boldsymbol{Z}_\mu$ is a symmetric space. Let $\pi : M \to M/\boldsymbol{Z}_\mu$ be the projection and let $[x]$ denote $\pi(x)$ for $x \in M$.*

(1) *If $\mu = 2$, for every polar $(M/\boldsymbol{Z}_2)^+$ in $M/\boldsymbol{Z}_2$ with respect to $[o]$ there exists either a polar M^+ in M with respect to o or a centriole C in M with respect to o and $\bar{o}$ satisfying $\pi^{-1}([o]) = \{o, \bar{o}\}$ which π maps onto $(M/\boldsymbol{Z}_2)^+$.*
(2) *If μ is odd, for every polar $(M/\boldsymbol{Z}_\mu)^+$ in $M/\boldsymbol{Z}_\mu$ with respect to $[o]$ there exists a polar M^+ in M with respect to o such that $\pi(M^+) = (M/\boldsymbol{Z}_\mu)^+$. Moreover, the restriction of π to M^+ is an isomorphism.*

PROOF. For $x \in M$, $[x] \in F(s_{[o]}, M/\boldsymbol{Z}_\mu)$ if and only if $[s_o(x)] = [x]$.

(1) If $[x] \in F(s_{[o]}, M/\boldsymbol{Z}_2)$, we have either $s_x(x) = x$ or $s_o(x) = \bar{x}$ by Lemma 5.11, where $\bar{x}$ is a pole with respect to x and $\pi^{-1}([x]) = \{x, \bar{x}\}$. Let $(M/\boldsymbol{Z}_2)^+$ be a polar through $[x]$ in $M/\boldsymbol{Z}_2$. When $s_o(x) = x$, if we take a polar M^+ through x in M, we have $\pi(M^+) = (M/\boldsymbol{Z}_2)^+$. When $s_o(x) = \bar{x}$, we have

$$s_o \circ s_x = s_o \circ s_x \circ s_o \circ s_o = s_{s_o(x)} \circ s_o = s_{\bar{x}} \circ s_o = s_x \circ s_o$$

because $\bar{x}$ is a pole with respect to x, which is equivalent to $s_{\bar{x}} = s_x$. Hence x

belongs to the centrosome $C(o, \bar{o})$ by [**3**, Proposition 3.4]. If we take a centriole C through x in M, we have $\pi(C) = (M/\boldsymbol{Z}_2)^+$.

(2) In the proof of [**3**, Proposition 3.1], it is shown that if $s_{[o]}([x]) = ([x])$, then s_o fixes the one point x' in $\pi^{-1}([x])$. If we take a polar M^+ through x', we have $\pi(M^+) = (M/\boldsymbol{Z}_\mu)^+$ for a polar $(M/\boldsymbol{Z}_\mu)^+$ through $[x]$ in $M/\boldsymbol{Z}_\mu$. If $\pi(x) = \pi(y)$ for $x, y \in M^+$, then $y \in \pi^{-1}([x])$ and $x = y$ follows what is stated above. Hence the restriction of π to M^+ is injective, so it is an isomorphism. □

DEFINITION 5.13. Let M be a compact connected Riemannian symmetric space and $p \in M$. For any $x \in F(s_p, M)$ we denote by $M^+_{(x)}$ the connected component of $F(s_p, M)$ through x. We call the connected component of $F(s_x \circ s_p, M)$ through x the *meridian* to $M^+_{(x)}$ at x and denote it by $M^-_{(x)}$.

LEMMA 5.14 ([**2**, Theorem 2.9]). *Let N be a compact connected Riemannian symmetric space and let M be a totally geodesic submanifold of N. Let $M^+_{(x)}$ be a polar of M through x with respect to $o \in M$. Then, there is a polar $N^+_{(x)}$ of N with respect to o such that $M^+_{(x)} = N^+_{(x)} \cap M$. Moreover, $M^+_{(x)}$ (resp. $M^-_{(x)}$) is a totally geodesic submanifold of $N^+_{(x)}$ (resp. $N^-_{(x)}$).*

PROOF OF THEOREM 5.6. The polars of $E_6/T \cdot Spin(10)$ with respect to the origin o are given by

$$M^+_0 = \{o\}, \qquad M^+_1 = Q_8(\boldsymbol{C}), \qquad M^+_2 = SO(10)/U(5)$$

by [**9**, (4.9)]. On the other hand, the polars of $F_4/Spin(9)$ with respect to the origin o are $\{o\}$ and S^8 by [**9**, (4.9)] and the polars of $G^{\boldsymbol{H}}_2(\boldsymbol{H}^4)/\boldsymbol{Z}_2$ with respect to the origin o are $\{o\}$, $S^{4,4}$ and $SO(5)$ by [**9**, (3.13)]. Then by Lemma 5.14 we have

$$\begin{aligned}
(F_4/Spin(9)) \cap M^+_0 &= \{o\} \\
(F_4/Spin(9)) \cap M^+_1 &= S^8 \\
(F_4/Spin(9)) \cap M^+_2 &= \emptyset,
\end{aligned}$$

and

$$\begin{aligned}
\left(G^{\boldsymbol{H}}_2(\boldsymbol{H}^4)/\boldsymbol{Z}_2\right) \cap M^+_0 &= \{o\} \\
\left(G^{\boldsymbol{H}}_2(\boldsymbol{H}^4)/\boldsymbol{Z}_2\right) \cap M^+_1 &= S^{4,4} \\
\left(G^{\boldsymbol{H}}_2(\boldsymbol{H}^4)/\boldsymbol{Z}_2\right) \cap M^+_2 &= SO(5).
\end{aligned}$$

Since L_1 and L_2 are congruent to $F_4/Spin(9)$ and $G_2^{\boldsymbol{H}}(\boldsymbol{H}^4)/\boldsymbol{Z}_2$ in $E_6/T\cdot Spin(10)$ respectively, $L_1 \cap M_j^+$ and $L_2 \cap M_j^+$ are congruent to $(F_4/Spin(9)) \cap M_j^+$ and $(G_2^{\boldsymbol{H}}(\boldsymbol{H}^4)/\boldsymbol{Z}_2) \cap M_j^+$ in M_j^+ respectively. Thus by Theorem 1 in [**15**] we have

$$\#\big\{(L_1 \cap M_1^+) \cap (L_2 \cap M_1^+)\big\} = 2.$$

Lemma 4.3 implies

$$\#(L_1 \cap L_2) = 1 + 2 = 3.$$

On the other hand

$$\#_2(F_4/Spin(9)) = 3 < 27 = \#_2\big(G_2^{\boldsymbol{H}}(\boldsymbol{H}^4)/\boldsymbol{Z}_2\big),$$

so we have

$$\min\{\#_2 L_1, \#_2 L_2\} = 3. \qquad \square$$

Proof of Theorem 5.7. By [**9**, (4.8)] the polars of $E_7/T\cdot E_6$ with respect to the origin o are given by

$$M_0^+ = \{o\}, \qquad M_1^+ \cong M_2^+ = E_6/T\cdot Spin(10), \qquad M_3^+ = \{\bar{o}\},$$

where $\bar{o}$ is the pole of o. On the other hand, the polars of $T\cdot(E_6/F_4)$ with respect to the origin o are $\{o, \bar{o}\}$ and two copies of $F_4/Spin(9)$ by (5.2) and Lemma 5.12, here we note that $T\cdot(E_6/F_4) = (T\times(E_6/F_4))/\boldsymbol{Z}_3$. And the polars of $(SU(8)/Sp(4))/\boldsymbol{Z}_2$ with respect to the origin o are $\{o, \bar{o}\}$ and two copies of $G_2^{\boldsymbol{H}}(\boldsymbol{H}^4)/\boldsymbol{Z}_2$ by Lemma 5.12. Then we have

$$\begin{aligned}
&(T\cdot(E_6/F_4)) \cap M_0^+ = \{o\}\\
&(T\cdot(E_6/F_4)) \cap M_i^+ = F_4/Spin(9) \qquad (i = 1, 2)\\
&(T\cdot(E_6/F_4)) \cap M_3^+ = \{\bar{o}\}.
\end{aligned}$$

We also have

$$\begin{aligned}
&((SU(8)/Sp(4))/\boldsymbol{Z}_2) \cap M_0^+ = \{o\}\\
&((SU(8)/Sp(4))/\boldsymbol{Z}_2) \cap M_i^+ = G_2^{\boldsymbol{H}}(\boldsymbol{H}^4)/\boldsymbol{Z}_2 \qquad (i = 1, 2)\\
&((SU(8)/Sp(4))/\boldsymbol{Z}_2) \cap M_3^+ = \{\bar{o}\}.
\end{aligned}$$

Since L_1 and L_2 are congruent to $T \cdot (E_6/F_4)$ and $(SU(8)/Sp(4))/\boldsymbol{Z}_2$ in $E_7/T \cdot E_6$ respectively, $L_1 \cap M_j^+$ and $L_2 \cap M_j^+$ are congruent to $(T \cdot (E_6/F_4)) \cap M_j^+$ and $((SU(8)/Sp(4))/\boldsymbol{Z}_2) \cap M_j^+$ in M_j^+ respectively. Thus by Theorem 5.6 we have

$$\#\{(L_1 \cap M_i^+) \cap (L_2 \cap M_i^+)\} = 3 \qquad (i = 1, 2).$$

Lemma 4.3 implies

$$\#(L_1 \cap L_2) = 1 + 3 + 3 + 1 = 8.$$

On the other hand

$$\#_2(T \cdot (E_6/F_4)) = 8 < 56 = \#_2((SU(8)/Sp(4))/\boldsymbol{Z}_2),$$

so we have

$$\min\{\#_2 L_1, \#_2 L_2\} = 8. \qquad \square$$

6. Explicit descriptions of the intersections of two real forms.

In this section we explicitly describe the intersections of two real forms congruent to real Grassmann manifolds or quaternionic Grassmann manifolds in complex Grassmann manifolds.

In order to describe explicitly real forms congruent to $G_r^{\boldsymbol{R}}(\boldsymbol{R}^{n+r})$ in $G_r^{\boldsymbol{C}}(\boldsymbol{C}^{n+r})$, we give an explicit description of any Lagrangian subspace in the complex Euclidean space.

Lemma 6.1. *For any Lagrangian subspace V in $\boldsymbol{C}^n$ there exist an orthonormal basis $v_1, \dots, v_n$ of $\boldsymbol{R}^n$ and $\theta_1, \dots, \theta_n \in \boldsymbol{R}$ satisfying*

$$V = \left\langle e^{\sqrt{-1}\theta_1} v_1, \dots, e^{\sqrt{-1}\theta_n} v_n \right\rangle_{\boldsymbol{R}}.$$

Proof. The Lagrangian subspaces in $\boldsymbol{C}^n$ are naturally corresponding to the elements in $U(n)/O(n)$. We denote by $e_1, \dots, e_n$ the standard unitary basis of $\boldsymbol{C}^n$.

$$\left\{\left\langle e^{\sqrt{-1}\theta_1} e_1, \dots, e^{\sqrt{-1}\theta_n} e_n \right\rangle_{\boldsymbol{R}} \mid \theta_j \in \boldsymbol{R}\right\}$$

is a maximal torus of $U(n)/O(n)$. For any Lagrangian subspace V in $\boldsymbol{C}^n$ there exist $g \in O(n)$ and $\theta_j \in \boldsymbol{R}$ satisfying

$$V = g\big\langle e^{\sqrt{-1}\theta_1}e_1, \ldots, e^{\sqrt{-1}\theta_n}e_n\big\rangle_{\boldsymbol{R}} = \big\langle e^{\sqrt{-1}\theta_1}ge_1, \ldots, e^{\sqrt{-1}\theta_n}ge_n\big\rangle_{\boldsymbol{R}}.$$

We put $v_j = ge_j$. Then $v_1, \ldots, v_n$ is an orthonormal basis of $\boldsymbol{R}^n$ and

$$V = \big\langle e^{\sqrt{-1}\theta_1}v_1, \ldots, e^{\sqrt{-1}\theta_n}v_n\big\rangle_{\boldsymbol{R}}. \qquad \square$$

THEOREM 6.2. *Let L_1, L_2 be two real forms congruent to $G_r^{\boldsymbol{R}}(\boldsymbol{R}^{n+r})$ in $G_r^{\boldsymbol{C}}(\boldsymbol{C}^{n+r})$. We assume that L_1, L_2 intersect transversally. There exists a unitary basis $u_1, \ldots, u_{n+r}$ of $\boldsymbol{C}^{n+r}$ satisfying*

$$L_1 \cap L_2 = \{\langle u_{i_1}, \ldots, u_{i_r}\rangle_{\boldsymbol{C}} \mid 1 \leq i_1 < \cdots < i_r \leq n+r\}.$$

PROOF. We first suppose that $L_1 = G_r^{\boldsymbol{R}}(\boldsymbol{R}^{n+r})$. L_2 is congruent to $G_r^{\boldsymbol{R}}(\boldsymbol{R}^{n+r})$, so there exists $g \in U(n+r)$ satisfying

$$L_2 = gG_r^{\boldsymbol{R}}(\boldsymbol{R}^{n+r}) = G_r^{\boldsymbol{R}}(g\boldsymbol{R}^{n+r}).$$

$g\boldsymbol{R}^{n+r}$ is a Lagrangian subspace in $\boldsymbol{C}^{n+r}$, thus by Lemma 6.1 there exist an orthonormal basis $v_1, \ldots, v_{n+r}$ of $\boldsymbol{R}^{n+r}$ and $\theta_1, \ldots, \theta_{n+r} \in \boldsymbol{R}$ satisfying

$$g\boldsymbol{R}^{n+r} = \big\langle e^{\sqrt{-1}\theta_1}v_1, \ldots, e^{\sqrt{-1}\theta_{n+r}}v_{n+r}\big\rangle_{\boldsymbol{R}}.$$

Hence we have

$$L_2 = G_r^{\boldsymbol{R}}\big(\big\langle e^{\sqrt{-1}\theta_1}v_1, \ldots, e^{\sqrt{-1}\theta_{n+r}}v_{n+r}\big\rangle_{\boldsymbol{R}}\big).$$

For $1 \leq i_1 < \cdots < i_r \leq n+r$

$$\big\langle e^{\sqrt{-1}\theta_1}v_{i_1}, \ldots, e^{\sqrt{-1}\theta_1}v_{i_r}\big\rangle_{\boldsymbol{C}} = \langle v_{i_1}, \ldots, v_{i_r}\rangle_{\boldsymbol{C}}$$

and we obtain

$$L_1 \cap L_2 \supset \{\langle v_{i_1}, \ldots, v_{i_r}\rangle_{\boldsymbol{C}} \mid 1 \leq i_1 < \cdots < i_r \leq n+r\}.$$

Since L_1, L_2 intersect transversally, by Theorem 1.1 $L_1 \cap L_2$ is a great antipodal set of L_1 and L_2. Therefore we have

$$\binom{n+r}{r} \leq \#(L_1 \cap L_2) = \#_2 L_1 = \#_2 L_2 = \binom{n+r}{r}$$

and

$$L_1 \cap L_2 = \{\langle v_{i_1}, \dots, v_{i_r} \rangle_{\boldsymbol{C}} \mid 1 \le i_1 < \cdots < i_r \le n+r\}.$$

We suppose that $L_1 = G_r^{\boldsymbol{R}}(\boldsymbol{R}^{n+r})$, so $v_1, \dots, v_{n+r}$ is an orthonormal basis of $\boldsymbol{R}^{n+r}$. In a general case there exists a unitary basis $u_1, \dots, u_{n+r}$ of $\boldsymbol{C}^{n+r}$ satisfying

$$L_1 \cap L_2 = \{\langle u_{i_1}, \dots, u_{i_r} \rangle_{\boldsymbol{C}} \mid 1 \le i_1 < \cdots < i_r \le n+r\}. \qquad \square$$

If we regard $\boldsymbol{C}^{2m+2q}$ as a quaternionic vector space of quaternionic dimension $m+q$, then we can regard quaternionic subspaces of quaternionic dimension q as complex subspaces of complex dimension $2q$. This induces an embedding of $G_q^{\boldsymbol{H}}(\boldsymbol{H}^{m+q})$ in $G_{2q}^{\boldsymbol{C}}(\boldsymbol{C}^{2m+2q})$.

Theorem 6.3. *Let L_1, L_2 be two real forms congruent to $G_q^{\boldsymbol{H}}(\boldsymbol{H}^{m+q})$ in $G_{2q}^{\boldsymbol{C}}(\boldsymbol{C}^{2m+2q})$. We assume that L_1, L_2 intersect transversally. There exists a unitary basis $v_1, \dots, v_{2m+2q}$ of $\boldsymbol{C}^{2m+2q}$ satisfying*

$$L_1 \cap L_2 = \{\langle v_{2i_1-1}, v_{2i_1}, \dots, v_{2i_q-1}, v_{2i_q} \rangle_{\boldsymbol{C}} \mid 1 \le i_1 < \cdots < i_q \le m+q\}.$$

Proof. We first consider the case of $q=1$ and prove the statement by induction on m. L_1, L_2 are two real forms congruent to $G_1^{\boldsymbol{H}}(\boldsymbol{H}^{m+1})$ in $G_2^{\boldsymbol{C}}(\boldsymbol{C}^{2m+2})$. In the case of $m=1$, $G_2^{\boldsymbol{C}}(\boldsymbol{C}^{2+2})$ is holomorphically isometric to the complex hyperquadric $Q_4(\boldsymbol{C})$ and L_1, L_2 in $G_2^{\boldsymbol{C}}(\boldsymbol{C}^4)$ are congruent to $S^{0,4}$ in $Q_4(\boldsymbol{C})$. The statement of $S^{0,4}$ in $Q_4(\boldsymbol{C})$ was already showed in [**15**]. There exists a unitary basis $v_1, \dots, v_4$ of $\boldsymbol{C}^4$ satisfying that $\langle v_1, v_2 \rangle_{\boldsymbol{C}}$ and $\langle v_3, v_4 \rangle_{\boldsymbol{C}}$ are quaternionic subspaces of quaternionic dimension 1 in $\boldsymbol{C}^4 = \boldsymbol{H}^2$ and

$$L_1 \cap L_2 = \{\langle v_1, v_2 \rangle_{\boldsymbol{C}}, \langle v_3, v_4 \rangle_{\boldsymbol{C}}\}.$$

We next consider the case of $m \ge 2$. By Lemma 3.1 in [**15**] we have $L_1 \cap L_2 \ne \emptyset$. We denote by $u_1, u_2, e_1, \dots, e_{2m}$ the standard unitary basis of $\boldsymbol{C}^{2m+2}$. We can suppose that $o = \langle u_1, u_2 \rangle_{\boldsymbol{C}} \in L_1 \cap L_2$. The polars of $G_2^{\boldsymbol{C}}(\boldsymbol{C}^{2m+2})$ with respect to o are given by

$$\{o\}, \quad G_1^{\boldsymbol{C}}(\langle e_1, \dots, e_{2m} \rangle_{\boldsymbol{C}}) \times G_1^{\boldsymbol{C}}(\langle u_1, u_2 \rangle_{\boldsymbol{C}}), \quad G_2^{\boldsymbol{C}}(\langle e_1, \dots, e_{2m} \rangle_{\boldsymbol{C}}).$$

We have

$$G_1^{\boldsymbol{H}}(\boldsymbol{H}^{m+1}) \cap \left(G_1^{\boldsymbol{C}}(\boldsymbol{C}^{2m}) \times G_1^{\boldsymbol{C}}(\boldsymbol{C}^2)\right) = \emptyset,$$

thus for $j = 1, 2$

$$L_j \cap \left(G_1^{\boldsymbol{C}}(\boldsymbol{C}^{2m}) \times G_1^{\boldsymbol{C}}(\boldsymbol{C}^2)\right) = \emptyset.$$

Moreover

$$G_1^{\boldsymbol{H}}(\boldsymbol{H}^{m+1}) \cap G_2^{\boldsymbol{C}}(\boldsymbol{C}^{2m}) = G_1^{\boldsymbol{H}}(\boldsymbol{H}^m),$$

thus $L_j \cap G_2^{\boldsymbol{C}}(\boldsymbol{C}^{2m})$ are congruent to $G_1^{\boldsymbol{H}}(\boldsymbol{H}^m)$ in $G_2^{\boldsymbol{C}}(\boldsymbol{C}^{2m})$. By the assumption of induction $L_1 \cap L_2 \cap G_2^{\boldsymbol{C}}(\boldsymbol{C}^{2m})$ is congruent to

$$\{\langle e_1, e_2\rangle_{\boldsymbol{C}}, \dots, \langle e_{2m-1}, e_{2m}\rangle_{\boldsymbol{C}}\}.$$

Hence we obtain

$$L_1 \cap L_2 \supset \{o, \langle v_1, v_2\rangle_{\boldsymbol{C}}, \dots, \langle v_{2m-1}, v_{2m}\rangle_{\boldsymbol{C}}\}.$$

Since L_1, L_2 intersect transversally, by Theorem 1.3 $L_1 \cap L_2$ is a great antipodal set of L_1 and L_2. Therefore we have

$$m + 1 \leq \#(L_1 \cap L_2) = \#_2 L_1 = \#_2 L_2 = m + 1$$

and

$$L_1 \cap L_2 = \{o, \langle v_1, v_2\rangle_{\boldsymbol{C}}, \dots, \langle v_{2m-1}, v_{2m}\rangle_{\boldsymbol{C}}\},$$

which complete the proof in the case of $q = 1$.

We consider the case of $q \geq 2$. By Lemma 3.1 in [**15**] we have $L_1 \cap L_2 \neq \emptyset$. We denote by $u_1, \dots, u_{2q}, e_1, \dots, e_{2m}$ the standard unitary basis of $\boldsymbol{C}^{2m+2q}$. We can suppose that $o = \langle u_1, \dots, u_{2q}\rangle_{\boldsymbol{C}} \in L_1 \cap L_2$. The polars of $G_{2q}^{\boldsymbol{C}}(\boldsymbol{C}^{2m+2q})$ are given by

$$G_i^{\boldsymbol{C}}(\langle e_1, \dots, e_{2m}\rangle_{\boldsymbol{C}}) \times G_{2q-i}^{\boldsymbol{C}}(\langle u_1, \dots, u_{2q}\rangle_{\boldsymbol{C}}) \quad (0 \leq i \leq 2q).$$

We have

$$G_q^{\boldsymbol{H}}(\boldsymbol{H}^{m+q}) \cap \left(G_1^{\boldsymbol{C}}(\boldsymbol{C}^{2m}) \times G_{2q-1}^{\boldsymbol{C}}(\boldsymbol{C}^{2q})\right) = \emptyset,$$

thus for $j = 1, 2$

$$L_j \cap \big(G_1^{\boldsymbol{C}}(\boldsymbol{C}^{2m}) \times G_{2q-1}^{\boldsymbol{C}}(\boldsymbol{C}^{2q})\big) = \emptyset.$$

Moreover

$$G_q^{\boldsymbol{H}}(\boldsymbol{H}^{m+q}) \cap \big(G_2^{\boldsymbol{C}}(\boldsymbol{C}^{2m}) \times G_{2q-2}^{\boldsymbol{C}}(\boldsymbol{C}^{2q})\big) = G_1^{\boldsymbol{H}}(\boldsymbol{H}^m) \times G_{q-1}^{\boldsymbol{H}}(\boldsymbol{H}^q),$$

thus $L_j \cap (G_2^{\boldsymbol{C}}(\boldsymbol{C}^{2m}) \times G_{2q-2}^{\boldsymbol{C}}(\boldsymbol{C}^{2q}))$ are congruent to $G_1^{\boldsymbol{H}}(\boldsymbol{H}^m) \times G_{q-1}^{\boldsymbol{H}}(\boldsymbol{H}^q)$ in $G_2^{\boldsymbol{C}}(\boldsymbol{C}^{2m}) \times G_{2q-2}^{\boldsymbol{C}}(\boldsymbol{C}^{2q})$. By the result in the case of $q = 1$ there exists a unitary basis $v_1, \ldots, v_{2m}$ of $\boldsymbol{C}^{2m}$ satisfying that

$$\langle v_1, v_2\rangle_{\boldsymbol{C}}, \ldots, \langle v_{2m-1}, v_{2m}\rangle_{\boldsymbol{C}}$$

are quaternionic subspaces of quaternionic dimension 1 in $\boldsymbol{C}^{2m} = \boldsymbol{H}^m$, there exists a unitary basis $w_1, \ldots, w_{2q}$ of $\boldsymbol{C}^{2q}$ satisfying

$$\langle w_1, w_2\rangle_{\boldsymbol{C}}, \ldots, \langle w_{2q-1}, w_{2q}\rangle_{\boldsymbol{C}}$$

are quaternionic subspaces of quaternionic dimension 1 in $\boldsymbol{C}^{2q} = \boldsymbol{H}^q$, and they satisfy

$$\begin{aligned}
&L_1 \cap L_2 \cap \big(G_2^{\boldsymbol{C}}(\boldsymbol{C}^{2m}) \times G_{2q-2}^{\boldsymbol{C}}(\boldsymbol{C}^{2q})\big) \\
&\quad = \{\langle v_1, v_2\rangle_{\boldsymbol{C}}, \ldots, \langle v_{2m-1}, v_{2m}\rangle_{\boldsymbol{C}}\} \\
&\qquad \times \{\langle w_3, \ldots, w_{2q}\rangle_{\boldsymbol{C}}, \langle w_1, w_2, \hat{w}_3, \hat{w}_4, \ldots, w_{2q}\rangle_{\boldsymbol{C}}, \ldots, \langle w_1, \ldots, w_{2q-2}\rangle_{\boldsymbol{C}}\}.
\end{aligned}$$

L_1, L_2 are congruent to $G_q^{\boldsymbol{H}}(\boldsymbol{H}^{m+q})$, hence we have

$$\begin{aligned}
&L_1 \cap L_2 \cap \big(G_4^{\boldsymbol{C}}(\boldsymbol{C}^{2m}) \times G_{2q-4}^{\boldsymbol{C}}(\boldsymbol{C}^{2q})\big) \\
&\quad \supset \{\langle v_{2i-1}, v_{2i}, v_{2j-1}, v_{2j}\rangle_{\boldsymbol{C}} \mid 1 \leq i < j \leq m\} \\
&\qquad \times \{\langle \ldots, \hat{w}_{2k-1}, \hat{w}_{2k}, \ldots, \hat{w}_{2l-1}, \hat{w}_{2l}, \ldots\rangle_{\boldsymbol{C}} \mid 1 \leq k < l \leq q\}.
\end{aligned}$$

Similar relations of inclusion holds for

$$L_1 \cap L_2 \cap \big(G_{2i}^{\boldsymbol{C}}(\boldsymbol{C}^{2m}) \times G_{2q-2i}^{\boldsymbol{C}}(\boldsymbol{C}^{2q})\big) \qquad (1 \leq i \leq q).$$

Therefore $L_1 \cap L_2$ contains the direct sums of any q subspaces of

$$\langle v_1, v_2\rangle_{\boldsymbol{C}}, \ldots, \langle v_{2m-1}, v_{2m}\rangle_{\boldsymbol{C}}, \langle w_1, w_2\rangle_{\boldsymbol{C}}, \ldots, \langle w_{2q-1}, w_{2q}\rangle_{\boldsymbol{C}}.$$

The unitary basis $v_1, \dots, v_{2m}, w_1, \dots, w_{2q}$ is renamed $v_1, \dots, v_{2m+2q}$. Then we obtain

$$L_1 \cap L_2 \supset \{\langle v_{2i_1-1}, v_{2i_1}, \dots, v_{2i_q-1}, v_{2i_q}\rangle_{\boldsymbol{C}} \mid 1 \leq i_1 < \cdots < i_q \leq m+q\}.$$

Since L_1, L_2 intersect transversally, by Theorem 1.3 $L_1 \cap L_2$ is a great antipodal set of L_1 and L_2. Therefore we have

$$\binom{m+q}{q} \leq \#(L_1 \cap L_2) = \#_2 L_1 = \#_2 L_2 = \binom{m+q}{q}$$

and

$$L_1 \cap L_2 = \{\langle v_{2i_1-1}, v_{2i_1}, \dots, v_{2i_q-1}, v_{2i_q}\rangle_{\boldsymbol{C}} \mid 1 \leq i_1 < \cdots < i_q \leq m+q\}. \qquad \square$$

THEOREM 6.4. *Let L_1 be a real forms congruent to $G_q^{\boldsymbol{H}}(\boldsymbol{H}^{m+q})$ and L_2 a real form congruent to $G_{2q}^{\boldsymbol{R}}(\boldsymbol{R}^{2m+2q})$ in $G_{2q}^{\boldsymbol{C}}(\boldsymbol{C}^{2m+2q})$. We assume that L_1, L_2 intersect transversally. There exists a unitary basis $v_1, \dots, v_{2m+2q}$ of $\boldsymbol{C}^{2m+2q}$ satisfying*

$$L_1 \cap L_2 = \{\langle v_{2i_1-1}, v_{2i_1}, \dots, v_{2i_q-1}, v_{2i_q}\rangle_{\boldsymbol{C}} \mid 1 \leq i_1 < \cdots < i_q \leq m+q\}.$$

The proof of the theorem is similar to those of Theorems 6.2 and 6.3, so we omit it.

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