J. Math. Soc. Japan
Vol. 64, No. 4 (2012) pp. 1249–1295
doi: 10.2969/jmsj/06441249

Dominated splitting of differentiable dynamics with C^1-topological weak-star property

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(Received Dec. 10, 2010)
(Revised Mar. 31, 2011)

Abstract. We study weak hyperbolicity of a differentiable dynamical system which is robustly free of non-hyperbolic periodic orbits of Markus type. Let $\boldsymbol{S}$ be a C^1-class vector field on a closed manifold M^n, which is free of any singularities. It is of C^1-*weak-star* in case there exists a C^1-neighborhood $\mathscr{U}$ of $\boldsymbol{S}$ such that for any $\boldsymbol{X} \in \mathscr{U}$, if P is a common periodic orbit of $\boldsymbol{X}$ and $\boldsymbol{S}$ with $\boldsymbol{S}_{\restriction P} = \boldsymbol{X}_{\restriction P}$, then P is hyperbolic with respect to $\boldsymbol{X}$. We show, in the framework of Liao theory, that $\boldsymbol{S}$ possesses the C^1-weak-star property if and only if it has a natural and nonuniformly hyperbolic dominated splitting on the set of periodic points $\mathrm{Per}(\boldsymbol{S})$, for the case $n = 3$.

1. Introduction.

Since the pioneer work of Peixoto [**37**] and specially the stability conjecture of Palis and Smale [**36**], it has been an important problem in differentiable dynamical systems to seek sufficient and necessary conditions for robustness of certain dynamics "P", where "P" is usually topological, such as the topological structure of orbits, shadowing, expansiveness etc., and where "P" might be global or local. Generally speaking, existence of such a robustness of "P" needs certain "uniformity" of the associated linear tangent map of the differentiable dynamical system itself.

The structural stability of orbits is the strongest robust dynamics, which implies uniform hyperbolicity on the non-wandering set, see, e.g., [**24**], [**29**], [**30**], [**42**], and [**17**]. Another important result of this type could be the star conjecture of Liao [**25**] for flows and Mañé [**29**] for diffeomorphisms, verified, respectively, by Liao [**24**] for 3-dimensional nonsingular flows and Mañé [**29**] for 2-dimensional diffeomorphisms and by Aoki [**2**] and Hayashi [**16**] for general diffeomorphisms and by Gan and Wen [**14**] for general nonsingular vector fields, which states that a C^1-diffeomorphism or a nonsingular C^1-vector field that is robustly free of any

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2010 *Mathematics Subject Classification.* Primary 37C10; Secondary 37D30, 37D05, 37C27, 34D30.

Key Words and Phrases. dominated splitting, weak-star property, Liao theory.

The author was partially supported by National Natural Science Foundation of China (Grant No. 11071112) and PAPD of Jiangsu Higher Education Institutions.

non-hyperbolic periodic orbits (also called the C^1*-star property*), satisfies Axiom A and the no cycle condition. The star conjecture is partially localized by the author in the recent work [**10**].

To attack the stability conjecture of Palis and Smale, Pliss [**38**], Liao [**22**], [**24**] and Mañé [**27**], [**29**] were led independently to the important notion "dominated splitting" of the tangent bundle into two invariant subbundles: one of them is definitely more contracted (or less expanded) than the other, after a uniform number of iterates. In the present paper, we show, using Liao theory, that there are the "nonuniform hyperbolicity" and "natural dominated splitting" of the periodic point set of a nonsingular differential system if and only if it is C^1-robustly free of the so-called non-hyperbolic periodic orbits of Markus type in the 3-dimensional case.

In this introductory section, we first introduce the basic concept C^1-weak-star property considered here and formulate our basic result proved in this paper.

1.1. C^1-weak-star property.

Let $\boldsymbol{S}\colon M^n \to TM^n$ be a C^1-class vector field defined on a closed manifold M^n of $\dim M^n = n \geq 2$, with a smooth Riemann structure $\langle \cdot, \cdot \rangle_x$, for $x \in M^n$. It gives naturally rise to flows on the state space M^n and its tangent bundle TM^n

$$(\boldsymbol{S}^t)_{t\in\mathbb{R}}\colon M^n \to M^n; \quad x \mapsto t.x$$

and

$$(D\boldsymbol{S}^t)_{t\in\mathbb{R}}\colon TM^n \to TM^n; \quad (x, v) \mapsto (t.x, D_x\boldsymbol{S}^t(v))$$

where $D_x\boldsymbol{S}^t\colon T_xM^n \mapsto T_{t.x}M^n$ is the derivative of $\boldsymbol{S}^t\colon M^n \to M^n$ at the base point $x \in M^n$ for any $t \in \mathbb{R}$.

As usual, an invariant set Λ of $(\boldsymbol{S}^t)_{t\in\mathbb{R}}$ (or $\boldsymbol{S}$-invariant set Λ for short) is said to be *hyperbolic* if the tangent bundle over Λ admits a $(D\boldsymbol{S}^t)_{t\in\mathbb{R}}$-invariant continuous splitting

$$T_\Lambda M^n = E^s \oplus \{\boldsymbol{S}_{\restriction\Lambda}\} \oplus E^u \qquad \text{where } \{\boldsymbol{S}_{\restriction\Lambda}\} = \operatorname{span}(\boldsymbol{S}_{\restriction\Lambda})$$

and constants $\lambda < 0, C > 0$ such that for any $t > 0$,

$$\|D\boldsymbol{S}^t(u)\| \leq C\|u\| \exp(\lambda t)\ \forall u \in E^s \quad \text{and} \quad \|D\boldsymbol{S}^{-t}(u)\| \leq C\|u\| \exp(\lambda t)\ \forall u \in E^u.$$

An $\boldsymbol{S}$-invariant set Λ is called *nonuniformly hyperbolic* if the tangent bundle over Λ admits a $(D\boldsymbol{S}^t)_{t\in\mathbb{R}}$-invariant (not necessarily continuous) splitting

$$T_\Lambda M^n = E^s \oplus \{\boldsymbol{S}_{\restriction\Lambda}\} \oplus E^u$$

and constants $\eta > 0, \tau > 0$ such that

$$\liminf_{L\to+\infty} \frac{1}{\tau L} \sum_{\ell=0}^{L-1} \log \|D\boldsymbol{S}^\tau \restriction E^s((\ell\tau).x)\| \le -\eta$$

and

$$\limsup_{L\to+\infty} \frac{1}{\tau L} \sum_{\ell=0}^{L-1} \log \|D\boldsymbol{S}^\tau \restriction E^u((\ell\tau).x)\|_{\mathrm{co}} \ge \eta$$

for all $x \in \Lambda$. Here and in the sequel $\| \bullet \|_{\mathrm{co}}$ means the minimal norm (also called co-norm in some literature) of the operator $\bullet$.

On the other hand, $\boldsymbol{S}$ can further induce the smooth linear skew-product (Poincaré) flow

$$\varPsi_{\boldsymbol{S}} \colon \mathbb{R} \times \boldsymbol{S}^\perp \to \boldsymbol{S}^\perp; \quad (t, (x, v)) \mapsto \big(t.x, \varPsi^t_{\boldsymbol{S},x}(v)\big)$$

on the $\boldsymbol{S}$-transversal tangent bundle of M^n

$$\boldsymbol{S}^\perp = \bigsqcup\nolimits_{x\in M^n} \boldsymbol{S}^\perp_x \qquad \text{where } \boldsymbol{S}^\perp_x = \big\{v \in T_x M^n \mid \langle \boldsymbol{S}(x), v\rangle_x = 0\big\}\ \forall x \in M^n.$$

We notice that every fiber $\boldsymbol{S}^\perp_x$ is of co-dimension 1 whenever $\boldsymbol{S}(x) \ne \mathbf{0}$, and if no confusion, we often simply write $\varPsi^t_{\boldsymbol{S},x}(v) = \varPsi^t_{\boldsymbol{S}}(v)$ for any $(x, v) \in \boldsymbol{S}^\perp$. Let

$$\mathrm{D}_-(x, \boldsymbol{S}) = \Big\{v \in \boldsymbol{S}^\perp_x \mid \lim_{t\to+\infty} \big\|\varPsi^t_{\boldsymbol{S}}(v)\big\| = 0\Big\}$$

and

$$\mathrm{D}_+(x, \boldsymbol{S}) = \Big\{v \in \boldsymbol{S}^\perp_x \mid \lim_{t\to-\infty} \big\|\varPsi^t_{\boldsymbol{S}}(v)\big\| = 0\Big\};$$

write

$$\mathrm{Ind}_{\boldsymbol{S}}(x) = \dim \mathrm{D}_-(x, \boldsymbol{S}).$$

Clearly, $\mathrm{D}_-(\cdot, \boldsymbol{S})$ and $\mathrm{D}_+(\cdot, \boldsymbol{S})$ both are $\varPsi_{\boldsymbol{S}}$-invariant subspaces of $\boldsymbol{S}^\perp$. As usual, $\mathrm{Ind}_{\boldsymbol{S}}(x)$ is still called the “index” of $\boldsymbol{S}$ at the base point $x \in M^n$.

A periodic point p of $\boldsymbol{S}$ with period τ is said to be "hyperbolic" if and only if $\varPsi_{\boldsymbol{S}}^{\tau}\colon \boldsymbol{S}_p^{\perp} \to \boldsymbol{S}_p^{\perp}$ does not have any eigenvalues of absolute value one; equivalently

$$\boldsymbol{S}_p^{\perp} = \mathrm{D}_-(p, \boldsymbol{S}) \oplus \mathrm{D}_+(p, \boldsymbol{S}).$$

Notice here that the hyperbolicity includes the contracting and expanding cases in the present paper.

From here on, $\mathrm{Per}(\boldsymbol{S})$ stands for the set of all periodic points of $\boldsymbol{S}$ and $\mathscr{O}_{\mathrm{per}}(\boldsymbol{S})$ denotes the set of all periodic orbits of $\boldsymbol{S}$. By $\mathscr{O}_{\mathrm{per}}(\boldsymbol{S} \restriction \varLambda)$ we mean the set of all periodic orbits which are contained in the subset $\varLambda$.

Let $\boldsymbol{S}$ be a C^{∞}-vector field of M^n. According to L. Markus [**31**] and [**35**, Lemma 2.5 of Chapter 3], for arbitrary $P \in \mathscr{O}_{\mathrm{per}}(\boldsymbol{S})$ and for any C^1-neighborhood $\mathscr{U}$ of $\boldsymbol{S}$, one can find some $\boldsymbol{X} \in \mathscr{U}$ such that $P \in \mathscr{O}_{\mathrm{per}}(\boldsymbol{X})$ is hyperbolic for $\boldsymbol{X}$ with $\boldsymbol{S}_{\restriction P} = \boldsymbol{X}_{\restriction P}$. This motivates us to study the following basic property described by Definition 1.1 below.

Let $\mathfrak{X}^1(M^n)$ denote the set of all C^1-class vector fields defined on M^n endowed with the usual C^1-norm.

Definition 1.1. Let $\boldsymbol{S} \in \mathfrak{X}^1(M^n)$ and let $\varLambda$ be a nonempty closed, not necessarily invariant, subset of M^n. $\boldsymbol{S}$ is said to possess the C^1*-weak-star property* on $\varLambda$, provided that there exists a C^1-neighborhood $\mathscr{U}$ of $\boldsymbol{S}$ in $\mathfrak{X}^1(M^n)$ such that, for any $\boldsymbol{X} \in \mathscr{U}$, every $P \in \mathscr{O}_{\mathrm{per}}(\boldsymbol{S} \restriction \varLambda) \cap \mathscr{O}_{\mathrm{per}}(\boldsymbol{X})$ with $\boldsymbol{S}_{\restriction P} = \boldsymbol{X}_{\restriction P}$, is hyperbolic with respect to $\boldsymbol{X}$.

Then, that $\boldsymbol{S}$ possesses the C^1-weak-star property means that $\boldsymbol{S}$ is robustly free of any non-hyperbolic *Markus-type* periodic orbits.

It should be noted here that, in general, for $P \in \mathscr{O}_{\mathrm{per}}(\boldsymbol{S}) \cap \mathscr{O}_{\mathrm{per}}(\boldsymbol{X})$, its prime periods under $\boldsymbol{S}$ and $\boldsymbol{X}$ might be different; it is enough to take $\boldsymbol{X} = (1+\ell^{-1})\boldsymbol{S}$ with ℓ large enough. However, there is no such difference in the case of diffeomorphisms.

Recall that $\boldsymbol{S}$ is called C^1*-star* in case there is a C^1-neighborhood $\mathscr{V}$ of $\boldsymbol{S}$ such that all periodic orbits of each $\boldsymbol{X} \in \mathscr{V}$ are hyperbolic for $\boldsymbol{X}$. We notice that even for the special case $\varLambda = M^n$, the C^1-weak-star property is conceptually weaker than the global C^1-star property, as is shown by the following simple example.

Example 1.2. Let $f\colon \mathbb{S}^2 \to \mathbb{S}^2$ be a C^1-diffeomorphism of the 2-dimensional unit sphere $\mathbb{S}^2$ with the South Pole S and the North Pole N, which has sources $\{\mathrm{N}, \mathrm{A}, \mathrm{B}\}$ and a saddle S such that $\Omega(f) = \{\mathrm{N}, \mathrm{S}, \mathrm{A}, \mathrm{B}\}$. Let $W^s(\mathrm{S}) = W^u(\mathrm{S})$ and $\{\mathrm{S}\} \cup W^s(\mathrm{S})$ be homeomorphic to the shape "∞" containing $\{\mathrm{A}, \mathrm{B}\}$. Then, the suspension $\mathscr{S}(f)$ of f on $\mathbb{S}^2 \times [0,1]$ obeys the C^1-weak-star property, but it is not a C^1-star system from [**14**] because there are two cycles at the periodic point $(\mathrm{S}, 0) \in \mathbb{S}^2 \times [0,1]$ of $\mathscr{S}(f)$.

So, the C^1-weak-star property introduced here is a generalization of the important C^1-star property that has been well studied by many literature; see, e.g., [**12**], [**22**], [**24**], [**29**], [**30**], [**2**], [**16**], [**14**] and the references therein.

1.2. Dominated and nonuniformly hyperbolic splitting.

In this paper, we push forward the methods of Liao developed in [**22**] to obtain the following basic theorem on the existence of a natural and uniform dominated splitting which possesses an additional nonuniform hyperbolicity, under the local C^1-weak-star condition introduced above.

Theorem A. *Let* $\boldsymbol{S} \in \mathfrak{X}^1(M^n)$ *have an invariant closed set* Λ *which contains no singularities of* $\boldsymbol{S}$*. If* $\boldsymbol{S}$ *possesses the* C^1*-weak-star property on* Λ*, then one can find numbers* $\eta > 0$ *and* $T > 0$ *satisfying the following two properties.*

(1) *"Natural dominated splitting": for any* $P \in \mathscr{O}_{\mathrm{per}}(\boldsymbol{S} \restriction \Lambda)$

$$\left\|\Psi^t_{\boldsymbol{S}\restriction \mathrm{D}_-(p,\boldsymbol{S})}\right\| \leq \exp(-\eta t) \qquad \text{if } \mathrm{Ind}_{\boldsymbol{S}}(P) = n-1, \tag{1.1a}$$

$$\left\|\Psi^t_{\boldsymbol{S}\restriction \mathrm{D}_+(p,\boldsymbol{S})}\right\|_{\mathrm{co}} \geq \exp(\eta t) \qquad \text{if } \mathrm{Ind}_{\boldsymbol{S}}(P) = 0, \tag{1.1b}$$

and

$$\frac{\|\Psi^t_{\boldsymbol{S}\restriction \mathrm{D}_-(p,\boldsymbol{S})}\|}{\|\Psi^t_{\boldsymbol{S}\restriction \mathrm{D}_+(p,\boldsymbol{S})}\|_{\mathrm{co}}} \leq \exp(-2\eta t) \quad \text{if } 1 \leq \mathrm{Ind}_{\boldsymbol{S}}(P) \leq n-2, \tag{1.1c}$$

for any $t \geq T$ *and all* $p \in P$.

(2) *"Nonuniform hyperbolicity": if* $P \in \mathscr{O}_{\mathrm{per}}(\boldsymbol{S} \restriction \Lambda)$ *has prime period* T_P *and if*

$$0 = t_0 < t_1 < \cdots < t_\ell = mT_P \quad \text{where } m \in \mathbb{N}$$

is a subdivision of the interval $[0, mT_P]$ *having* $t_k - t_{k-1} \geq T$ *for* $k = 1, \ldots, \ell$*, then*

$$\frac{1}{mT_P}\sum_{k=1}^{\ell} \log \left\|\Psi_{\boldsymbol{S}}^{t_k - t_{k-1}}{}_{\restriction \mathrm{D}_-(\boldsymbol{S}^{t_{k-1}}(p),\boldsymbol{S})}\right\| \leq -\eta$$

and

$$\frac{1}{mT_P}\sum_{k=1}^{\ell} \log \left\|\Psi_{\boldsymbol{S}}^{t_k - t_{k-1}}{}_{\restriction \mathrm{D}_+(\boldsymbol{S}^{t_{k-1}}(p),\boldsymbol{S})}\right\|_{\mathrm{co}} \geq \eta$$

for any $p \in P$.

Here $\boldsymbol{S}_p^{\perp} = \mathrm{D}_-(p, \boldsymbol{S}) \oplus \mathrm{D}_+(p, \boldsymbol{S})$ *is the natural hyperbolic splitting of the hyperbolic periodic trajectory* P *of* $\boldsymbol{S}$.

For any nonempty subset Θ of M^n, by $\mathrm{Cl}(\Theta)$ we denote the closure of Θ relative to M^n. Then the natural dominated splitting over $\mathscr{O}_{\mathrm{per}}(\boldsymbol{S} \restriction \Lambda)$ may be extended over $\mathrm{Cl}(\mathscr{O}_{\mathrm{per}}(\boldsymbol{S} \restriction \Lambda))$.

We notice here that in the case where $\Lambda = M^n$ and in addition $\boldsymbol{S}$ possesses the C^1-star property that is stronger than ours from Example 1.2, the above statements have been established independently by Liao [**22**] and Mañé [**29**, Proposition II.1]. Since the C^1-star condition implies the C^1-weak-star property, Theorem A is an extension of their theorems. In dynamical systems, some interesting basic dynamics sets like nonwandering set, homoclinic class, chain recurrent class and periodic orbit, usually are lower dimensional than the ambient manifold M^n, and so very sensitive to small perturbations. In view of this, our Theorem A above is more or less of interest.

This kind of "nonuniform hyperbolicity" presented by statement (2) of Theorem A above plays an important part for proving uniform hyperbolicity from robust dynamics in many important situations, for instance, in [**24**] and [**30**]. It is also very important in our subsequent applications [**11**]. In addition, (1.1a) and (1.1b) in statement (1) of Theorem A imply that there is at most a finite number of periodic orbits of indices 0 or $n-1$ for $\boldsymbol{S}$ having the C^1-weak-star property. This is an important property of the C^1-weak-star systems; this is because it removes the Newhouse phenomenon (the existence of infinitely many periodic attractors or periodic repellers [**33**]).

It also is interesting to note that under the hypothesis of Theorem A, every periodic points of $\boldsymbol{S}$ lying in Λ have only nonzero transversal Lyapunov exponents and such exponents are uniformly bounded away from zero. This generalizes a classical result of Franks [**12**, Theorem 1].

For the 3-dimensional case, we can obtain the following sufficient and necessary condition for C^1-weak-star property.

Theorem B. *Let* $\boldsymbol{S} \in \mathfrak{X}^1(M^3)$ *have an invariant closed set* Λ *which does not contain any singularities of* $\boldsymbol{S}$. *Then,* $\boldsymbol{S}$ *obeys the* C^1*-weak-star property on* Λ *if and only if each* $P \in \mathscr{O}_{\mathrm{per}}(\boldsymbol{S} \restriction \Lambda)$ *is hyperbolic and moreover, one can find numbers* $\eta > 0, T > 0$ *satisfying the properties* (1) *and* (2) *described in Theorem A.*

1.3. Weak-star property induced from other robust dynamics.

Let $\mathrm{Diff}^1(M^n)$ denote the set of all C^1-class diffeomorphisms of M^n endowed with the usual C^1-topology. We can deduce the local C^1-weak-star property of

$f \in \mathrm{Diff}^1(M^n)$ from some other C^1-robust dynamics which are considered recently in many papers. Propositions 1.4 and 1.6 below highlight somewhat our basic dominated and nonuniformly hyperbolic splitting theorem stated before.

Recall that a point $x \in M^n$ is said to be a *chain recurrent point* of f if for any $\delta > 0$ there can be found a δ-pseudo-orbit $\{x_i\}_{i=0}^{n_\delta}$ of f for some $n_\delta \geq 1$ such that $x_0 = x = x_{n_\delta}$. Let CR_f denote the set of all chain recurrent points of f. We define an equivalence relation $\leftrightsquigarrow$ on CR_f by $x \leftrightsquigarrow y$ if and only if for any $\delta > 0$ there exists a δ-pseudo-orbit $\{x_i\}_{i=0}^{n_\delta}$ of f for some $n_\delta \geq 1$ such that $x_0 = x$ and $x_{n_\delta} = y$. The equivalence classes $\mathrm{CR}_f(x)$ are called the *chain recurrent classes* of f, see, e.g., [**5**], [**3**], [**1**], [**40**], and [**43**]. Clearly, each chain recurrent class of f is a compact invariant set of f.

Definition 1.3. For any $x \in \mathrm{Per}(f)$, $\mathrm{CR}_f(x)$ is said to be C^1-*sustainedly shadowable* for f if there exists a C^1-neighborhood $\mathscr{U}$ of f such that for any $g \in \mathscr{U}$ and any $p \in \mathrm{Per}(f \restriction \mathrm{CR}_f(x)) \cap \mathrm{Per}(g)$, g has the shadowing property on $\mathrm{CR}_g(p)$. Similarly, the C^1-sustainedly shadowable property can be defined on the nonwandering set $\Omega(f)$ of f.

Notice here that our C^1-sustainedly shadowing is different from the C^1-stably shadowable condition considered in [**40**] and [**43**] where one requires g has the shadowing property on every $\mathrm{CR}_g(x_g)$. Precisely speaking, let x be a hyperbolic periodic point of f; then there exists a C^1-neighborhood $\mathscr{U}$ of f such that for any $g \in \mathscr{U}$, x has a continuation x_g near x; so, x_g lies in a unique chain component $\mathrm{CR}_g(x_g)$ of g. The C^1-stably shadowable condition in [**40**] and [**43**] requires that g has the shadowing property on $\mathrm{CR}_g(x_g)$. However, there f and g need not share a common periodic orbit in $\mathrm{CR}_f(x) \cap \mathrm{CR}_g(x_g)$. In addition, here we do not assume previously that x is hyperbolic.

However, the C^1-sustainedly shadowable property implies the C^1-weak-star property on a chain recurrent class from a discussion of Franks type, see Proposition 1.4 below, which implies that our Theorem A is a kind of extensions of Sakai [**40**, Theorems A and B]. Similar result can be obtained for the C^1-sustainedly shadowable property on the nonwandering set of a diffeomorphism.

Proposition 1.4. *Let $f \in \mathrm{Diff}^1(M^n)$ be arbitrarily given. Then, the following two statements hold.*

(1) *If for $x \in \mathrm{Per}(f)$, $\mathrm{CR}_f(x)$ is C^1-sustainedly shadowable, then it satisfies the C^1-weak-star property for f.*
(2) *If f is C^1-sustainedly shadowable on the nonwandering set $\Omega(f)$, then it possesses the C^1-weak-star property on $\Omega(f)$ and moreover it is hyperbolic on the closure* $\mathrm{Cl}(\mathrm{Per}(f))$.

Proof. The existence of the C^1-weak-star property follows from a Franks-type argument [**12**]. The hyperbolicity of $\mathrm{Cl}(\mathrm{Per}(f))$ in statement (2) of Proposition 1.4 follows from Theorem A and a Liao-wise sifting-shadowing combination as done in [**11**]. This proves Proposition 1.4. $\square$

A special case of $n = 2$ of Proposition 1.4 is a generalization of [**44**, Theorem A] for the continuous-time dynamics. Similar to the C^1-stably shadowable condition, C^1-stably expansive condition was considered in a number of papers, for example, in [**27**], [**28**], [**34**], [**32**], [**41**]. Now, we introduce a slightly different one.

Definition 1.5. f is said to be C^1-*sustainedly expansive* on the nonwandering set $\Omega(f)$ if there exists a C^1-neighborhood $\mathscr{U}$ of f such that each $g \in \mathscr{U}$ is expansive on $\Omega(g)$ if $\mathrm{Per}(f) \cap \mathrm{Per}(g) \neq \varnothing$. For any point $p \in \mathrm{Per}(f)$, one can define similarly the C^1-sustainedly expansive property on $\mathrm{CR}_f(p)$.

In [**27**] and [**32**], it is proved that if f is C^1-*stably expansive* on M^n, i.e., there is a C^1-neighborhood $\mathscr{V}$ of f such that every $g \in \mathscr{V}$ is expansive on M^n, then f is quasi-Anosov or equivalently satisfies both Axiom A and quasi-transversality condition. If f is C^1-robustly expansive on $\Omega(f)$, then it follows from [**2**] and [**16**] that f is Axiom A. Clearly our notion of C^1-sustainable expansiveness is weaker than the C^1-robustly expansive. Using an argument of Franks type, however, we can easily show that if f is C^1-sustainedly expansive on $\Omega(f)$, then f satisfies the C^1-weak-star property on $\Omega(f)$, see Proposition 1.6 below. Similarly, if f is C^1-sustainedly expansive on $\mathrm{CR}_f(p)$, then f obeys the C^1-weak-star property on $\mathrm{CR}_f(p)$.

Proposition 1.6. *Let $f \in \mathrm{Diff}^1(M^n)$ be arbitrarily given. Then, the following two statements hold.*

(1) *If f is C^1-sustainedly expansive on $\Omega(f)$, then it satisfies the C^1-weak-star property on $\Omega(f)$.*
(2) *If for $x \in \mathrm{Per}(f)$, f is C^1-sustainedly expansive on $\mathrm{CR}_f(x)$, then it obeys the C^1-weak-star property on $\mathrm{CR}_f(x)$.*

The statements of Propositions 1.4 and 1.6 will play a role of the starting point for one to further study the hyperbolicity of the C^1-sustainedly shadowable and expansive dynamical classes. We shall not discuss these applications here.

1.4. A simple version of of Theorem A.

Let $\mathfrak{X}^1(\mathbb{R}^n)$, where $n \geq 2$, be the space of all C^1-class vector fields on the n-dimensional Euclidean space $\mathbb{R}^n$ endowed with the uniform C^1-topology induced by the usual C^1-norm

$$\|\boldsymbol{X}-\boldsymbol{Y}\|_1=\sup\nolimits_{x\in\mathbb{R}^n}\{\|\boldsymbol{X}(x)-\boldsymbol{Y}(x)\|+\|\boldsymbol{X}'(x)-\boldsymbol{Y}'(x)\|\}$$

for all $\boldsymbol{X},\boldsymbol{Y}\in\mathfrak{X}^1(\mathbb{R}^n)$.

To simplify the symbols we will mainly prove the following Theorem A′ instead of Theorem A.

THEOREM A′. *Let* $\boldsymbol{S}\colon\mathbb{R}^n\to\mathbb{R}^n$ *be any nonsingular* C^1*-class vector field on* $\mathbb{R}^n$. *Assume* Λ *is an invariant, nonempty, and compact set of* $\boldsymbol{S}$. *If* $\boldsymbol{S}$ *possesses the* C^1*-weak-star property on* Λ, *then there are constants* $\eta>0, T>0$ *such that the following two statements hold.*

(1) $\boldsymbol{S}$ *has a natural and uniform* (η,T)*-dominated splitting on* $\mathrm{Per}(\boldsymbol{S}\restriction\Lambda)$, *i.e., for any* $t\geq T$ *and any* $x\in\mathrm{Per}(\boldsymbol{S}\restriction\Lambda)$

$$\begin{aligned}\left\|\varPsi^t_{\boldsymbol{S},x}\right\|&\leq\exp(-\eta t)\qquad \textit{if } \mathrm{Ind}_{\boldsymbol{S}}(x)=n-1,\\ \left\|\varPsi^t_{\boldsymbol{S},x}\right\|_{\mathrm{co}}&\geq\exp(\eta t)\qquad \textit{if } \mathrm{Ind}_{\boldsymbol{S}}(x)=0,\end{aligned}$$

and

$$\frac{\|\varPsi^t_{\boldsymbol{S}\restriction\mathrm{D}_-(x,\boldsymbol{S})}\|}{\|\varPsi^t_{\boldsymbol{S}\restriction\mathrm{D}_+(x,\boldsymbol{S})}\|_{\mathrm{co}}}\leq\exp(-2\eta t)\quad \textit{if } 1\leq\mathrm{Ind}_{\boldsymbol{S}}(x)\leq n-2.$$

(2) *Moreover, if* $x\in\mathrm{Per}(\boldsymbol{S}\restriction\Lambda)$ *has the prime period* $T_x\geq T$, *and*

$$0=t_0<t_1<\cdots<t_\ell=T_x$$

is a subdivision of $[0,T_x]$ *satisfying* $t_k-t_{k-1}\geq T$ *for* $k=1,\dots,\ell$, *then*

$$\frac{1}{T_x}\sum_{k=1}^{\ell}\log\left\|\varPsi_{\boldsymbol{S}}^{t_k-t_{k-1}}{}_{\restriction\mathrm{D}_-(\boldsymbol{S}^{t_{k-1}}(x),\boldsymbol{S})}\right\|\leq-\eta$$

and

$$\frac{1}{T_x}\sum_{k=1}^{\ell}\log\left\|\varPsi_{\boldsymbol{S}}^{t_k-t_{k-1}}{}_{\restriction\mathrm{D}_+(\boldsymbol{S}^{t_{k-1}}(x),\boldsymbol{S})}\right\|_{\mathrm{co}}\geq\eta.$$

Although this theorem is formally simpler than Theorem A, its proof contains all the key points of that. If we take the exponential projection $\exp\colon TM^n\to M^n$, one can translate its proof to a proof of Theorem A.

1.5. Applications.

If a nonsingular C^1-class differential system is robustly free of any nonhyperbolic Markus-type periodic orbits then based on Theorem A above, it is hyperbolic on the closure of all its periodic points under some other interesting conditions like shadowing and approximation conditions. Our principal results of applications of Theorem A will be the following Theorem C.

THEOREM C ([**11**]). *Let $\boldsymbol{S} \in \mathfrak{X}^1(M^n)$ be nonsingular, which possesses the C^1-weak-star property.*

(1) *If $\boldsymbol{S}$ has the shadowing property on the closure* $\mathrm{Cl}(\mathrm{Per}(\boldsymbol{S}))$, *then it is hyperbolic on* $\mathrm{Cl}(\mathrm{Per}(\boldsymbol{S}))$.
(2) *If every $(\boldsymbol{S}^t)_{t\in\mathbb{R}}$-minimal subset of* $\mathrm{Cl}(\mathrm{Per}(\boldsymbol{S}))$ *can be approached arbitrarily by periodic orbits of $\boldsymbol{S}$ in the sense of Hausdorff topology, then* $\mathrm{Cl}(\mathrm{Per}(\boldsymbol{S}))$ *is hyperbolic in the cases of $n = 3$ and 4.*

From [**6**], we see that the approximation condition assumed in statement (2) of Theorem C is C^1-generic. So, we can easily obtain the following C^1-generic result.

PROPOSITION 1.7. *If, for a C^1-generic $f \in \mathrm{Diff}^1(M^n)$ where $n = 2$ or 3, it possesses the C^1-weak-star property, then it satisfies Axiom A.*

PROOF. From the known C^1 closing lemma [**39**] or [**23**], it follows that for a C^1-generic $f \in \mathrm{Diff}^1(M^n)$, it satisfies $\Omega(f) = \mathrm{Cl}(\mathrm{Per}(f))$. Moreover, it follows from [**6**] that for a C^1-generic $f \in \mathrm{Diff}^1(M^n)$, its minimal sets can be approximated in the Hausdorff topology by its periodic orbits. So, the statement comes from Theorem C. □

Since the dynamics of Axiom A is not C^1-generic, Proposition 1.7 implies that the C^1-weak-star dynamics is also not C^1-generic in $\mathfrak{X}^1(M^n)$ for $n \geq 3$. Hence, it follows, from Propositions 1.4 and 1.6, that the C^1-sustainedly shadowable (resp. expansive) dynamics is not C^1-generic in $\mathrm{Diff}^1(M^n)$ too.

Let $\mathrm{Diff}_{\mathrm{loc}}^{1+\alpha}(M^n)$ be the set of all locally diffeomorphic transformations of class $\mathrm{C}^{1+\alpha}$, where $0 < \alpha \leq 1$. Anatole Katok has asked if $f \in \mathrm{Diff}_{\mathrm{loc}}^{1+\alpha}(M^n)$, which is Hölder conjugated to a C^1-expanding map (or a C^1-class Anosov diffeomorphism) is an expanding (or Anosov) one. On Katok's problem, there is an affirmative solution for the expanding case; see [**18**] and [**9**]. But recently Andrey Gogolev [**15**] presents a counterexample on $\mathbb{T}^2$ for the diffeomorphism case.

As a straightforward application of Theorem C, we can easily obtain the following result.

PROPOSITION 1.8. *Let $f \in \mathrm{Diff}^1(M^n)$ possess the C^1-weak-star property. If f is topologically conjugated to an Axiom A diffeomorphism, then f satisfies Axiom A itself.*

PROOF. Since f is topologically conjugated to an Axiom A diffeomorphism, f has the shadowing property on the nonwandering set $\Omega(f)$ of f and $\mathrm{Cl}(\mathrm{Per}(f)) = \Omega(f)$. Then the statement follows from Theorem C. □

Thus, under the same hypothesis of Proposition 1.8, if f is topologically conjugated to a transitive Anosov diffeomorphism, then f is Anosov itself.

1.6. Outline.

This paper is organized as follows. To prove Theorem A, we will introduce the theory of standard systems of differential equations in Section 2. In Section 3, we will introduce two technical linear perturbation lemmas that are due to S.-T. Liao. We will prove Theorem A′ and then Theorem A in Section 4, using Liao approaches. Finally, based on Theorem A we will prove Theorem B in Section 5.

2. Liao standard systems of differential equations.

In [**20**] and [**21**], Professor Shantao Liao established his theory of standard systems of differential equations for C^1-differential dynamical systems on closed manifolds. Since we will prove Theorem A′ in Liao's framework which is less known, we need to recall some basic results of Liao theory. Here our standard system is different from Liao's used in [**22**]. Moreover our treatments are much simpler than Liao's. In addition, we shall introduce the notion of "level vector field" following Liao [**22**].

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, where $n \geq 2$. We identify its tangent bundle $T\mathbb{R}^n$ with $\mathbb{R}^n \times \mathbb{R}^n$.

Throughout this section, let $\boldsymbol{S}\colon \mathbb{R}^n \to T\mathbb{R}^n$ be any fixed nonsingular C^1-class differential systems on $\mathbb{R}^n$ which has an invariant compact nonempty set $\Lambda \subset \mathbb{R}^n$.

2.1. Variational equations.

First $\boldsymbol{S}$ naturally generates the C^1-flow $(\boldsymbol{S}^t)_{t\in\mathbb{R}}\colon \mathbb{R}^n \to \mathbb{R}^n$; $\boldsymbol{S}^t(x) = t.x$ on the state space $\mathbb{R}^n$ as before. It further induces, on the tangent bundle $\mathbb{R}^n \times \mathbb{R}^n$ of $\mathbb{R}^n$, the smooth linear skew-product flow

$$(D\boldsymbol{S}^t)_{t\in\mathbb{R}}\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n; \quad (x, v) \mapsto (t.x, D_x\boldsymbol{S}^t(v))$$

where $D_x\boldsymbol{S}^t\colon T_x\mathbb{R}^n = \mathbb{R}^n \to T_{t.x}\mathbb{R}^n = \mathbb{R}^n; v \mapsto (\partial \boldsymbol{S}^t/\partial x)v$, corresponding to the variational equations

$$\begin{cases} \dot{x} = \boldsymbol{S}(x); \\ \dot{v} = \boldsymbol{S}'(x)v, \end{cases} \qquad ((x,v) \in \mathbb{R}^n \times \mathbb{R}^n)$$

on the extended (x,v)-phase-space $\mathbb{R}^n \times \mathbb{R}^n$, i.e., $(d/dt)D_x\boldsymbol{S}^t = \boldsymbol{S}'(t.x)D_x\boldsymbol{S}^t$. We define the natural $\boldsymbol{S}$-transversal vector bundle over Λ as follows:

$$\boldsymbol{S}_\Lambda^\perp = \bigsqcup\nolimits_{x\in\Lambda} \boldsymbol{S}_x^\perp \qquad \text{where the fiber at } x \text{ is } \boldsymbol{S}_x^\perp = \{v \in \mathbb{R}^n \mid \langle \boldsymbol{S}(x), v\rangle = 0\}.$$

Then, $\boldsymbol{S}$ further gives rise to the natural (smooth) linear skew-product flow on $\boldsymbol{S}_\Lambda^\perp$:

$$\Psi_{\boldsymbol{S}} \colon \mathbb{R} \times \boldsymbol{S}_\Lambda^\perp \to \boldsymbol{S}_\Lambda^\perp; \quad (t,(x,v)) \mapsto (t.x, \Psi_{\boldsymbol{S},x}^t(v))$$

where

$$\Psi_{\boldsymbol{S},x}^t(v) = D_x\boldsymbol{S}^t(v) - \left\langle D_x\boldsymbol{S}^t(v), \frac{\boldsymbol{S}(t.x)}{\|\boldsymbol{S}(t.x)\|}\right\rangle \frac{\boldsymbol{S}(t.x)}{\|\boldsymbol{S}(t.x)\|}.$$

In other words, $\Psi_{\boldsymbol{S},x}^t(v)$ is the component of $D_x\boldsymbol{S}^t(v)$ orthogonal to $\boldsymbol{S}(t.x)$.

Then, Λ is hyperbolic if and only if there exists a $\Psi_{\boldsymbol{S}}$-invariant continuous splitting

$$\boldsymbol{S}_\Lambda^\perp = \mathrm{E}^s \oplus \mathrm{E}^u$$

and two constants $\lambda < 0$, $K > 0$, such that

$$\left\|\Psi_{\boldsymbol{S}}^t(u)\right\| \leq K\|u\| \exp(\lambda t) \ \forall u \in \mathrm{E}^s \quad \text{and} \quad \left\|\Psi_{\boldsymbol{S}}^{-t}(u)\right\| \leq K\|u\| \exp(\lambda t) \ \forall u \in \mathrm{E}^u.$$

for any $t \geq 0$.

As usual in Liao theory, let

$$\mathscr{F}_{\boldsymbol{S},n-1}^{*\sharp}(\Lambda) = \bigsqcup\nolimits_{x\in\Lambda} \mathscr{F}_{\boldsymbol{S},n-1}^{*\sharp}(x)$$

be the bundle of $\boldsymbol{S}$-transversal orthonormal $(n-1)$-frames of the tangent bundle $\mathbb{R}^n \times \mathbb{R}^n$ over Λ, where the fiber over x is defined as

$$\mathscr{F}_{\boldsymbol{S},n-1}^{*\sharp}(x) = \left\{\gamma = [v_1, \dots, v_{n-1}] \mid v_i \in \boldsymbol{S}_x^\perp,\ \langle v_i, v_j\rangle = \delta_{ij} \text{ for } 1 \leq i,j \leq n-1\right\}$$

equipped with the naturally induced smooth structure. Here δ_{ij} is the Kronecker symbol. Then, $\boldsymbol{S}$ naturally generates a frame skew-product flow:

$$\big(\mathscr{S}_t^{*\sharp}\big)_{t\in\mathbb{R}}\colon \mathbb{R}\times\mathscr{F}_{\boldsymbol{S},n-1}^{*\sharp}(\Lambda)\to\mathscr{F}_{\boldsymbol{S},n-1}^{*\sharp}(\Lambda);\quad (x,\gamma)\mapsto\big(t.x,\mathscr{S}_{t,x}^{*\sharp}(\gamma)\big), \tag{2.1}$$

where the factor $\mathscr{S}_{t,x}^{*\sharp}\colon\mathscr{F}_{\boldsymbol{S},n-1}^{*\sharp}(x)\to\mathscr{F}_{\boldsymbol{S},n-1}^{*\sharp}(t.x)$ is defined by applying the standard Gram-Schmidt orthogonalization procedure; see [**19**] and [**7**], for example.

We now make a Perronwise triangularization for the variational equation of $\boldsymbol{S}$ under the moving frames $(\mathscr{S}_{t,x}^{*\sharp}(\gamma))_{t\in\mathbb{R}}$. For that, we simply write

$$\gamma_x(t)=\mathscr{S}_{t,x}^{*\sharp}(\gamma)\qquad\forall t\in\mathbb{R}\text{ and }(x,\gamma)\in\mathscr{F}_{\boldsymbol{S},n-1}^{*\sharp}(\Lambda).$$

Because $\boldsymbol{S}$ is of C^1-class, $\gamma_x(t)$ also is of C^1-class in t; that is to say, $\partial\gamma_x(t)/\partial t$ makes sense and is continuous with respect to $(t,(x,\gamma))$ under the natural smooth structure.

Given any $(n-1)$-frame (x,γ) of $\boldsymbol{S}$, define an orthogonal non-autonomous coordinate transformation as follows:

$$\gamma_{x,t}\colon\mathbb{R}^{n-1}\to\boldsymbol{S}_{t.x}^{\perp};\quad z\mapsto\gamma_x(t)z:=\sum_{i=1}^{n-1}z^i\mathrm{col}_i(\gamma_x(t))$$

where $z=(z^1,\dots,z^{n-1})^{\mathrm{T}}\in\mathbb{R}^{n-1}$ is regarded as an $(n-1)$-dimensional column vector and the $(n-1)$-frame $\gamma_x(t)\in\mathbb{R}^{n\times(n-1)}$ as an n-by-$(n-1)$ matrix with columns $\mathrm{col}_1(\gamma_x(t)),\dots,\mathrm{col}_{n-1}(\gamma_x(t))$ successively. Then, there can be defined a family of linear isomorphisms

$$(C_{x,\gamma}^*(t))_{t\in\mathbb{R}}\colon\mathbb{R}^{n-1}\to\mathbb{R}^{n-1}\quad\text{ such that }\quad\begin{tikzcd}\mathbb{R}^{n-1}\arrow[r,"C_{x,\gamma}^*(t)"]\arrow[d,"\gamma_{x,0}"'] & \mathbb{R}^{n-1}\arrow[d,"\gamma_{x,t}"]\\ \boldsymbol{S}_x^{\perp}\arrow[r,"\Psi_{\boldsymbol{S},x}^t"] & \boldsymbol{S}_{t.x}^{\perp}.\end{tikzcd} \tag{2.2}$$

We now think of $C_{x,\gamma}^*(t)$ as an $(n-1)$-by-$(n-1)$ matrix under the standard basis of $\mathbb{R}^{n-1}$. Clearly, the function $t\mapsto(d/dt)C_{x,\gamma}^*(t)$ makes sense since $\boldsymbol{S}$ is of C^1-class, and by (2.2) we have

$$C_{x,\gamma}^*(t_1+t_2)=C_{\mathscr{S}_{t_1}^{*\sharp}(x,\gamma)}^*(t_2)\circ C_{x,\gamma}^*(t_1)\qquad\forall t_1,t_2\in\mathbb{R}.$$

Put

$$R^*_{x,\gamma}(t) = \left\{\frac{d}{dt}C^*_{x,\gamma}(t)\right\} \circ C^*_{x,\gamma}(t)^{-1} \qquad \forall (x,\gamma) \in \mathscr{F}^{*\sharp}_{\boldsymbol{S},n-1}(\varLambda)$$

which is continuous with respect to $(t,(x,\gamma))$. Given $(x,\gamma) \in \mathscr{F}^{*\sharp}_{\boldsymbol{S},n-1}(\varLambda)$ associated to $\boldsymbol{S}$, the linear differential equation

$$\dot{z} = R^*_{x,\gamma}(t)z, \qquad (t,z) \in \mathbb{R} \times \mathbb{R}^{n-1} \tag{$R^*_{x,\gamma}$}$$

is called the *Liao linearized system* of $\boldsymbol{S}$ under the moving frame $(\gamma_x(t))_{t\in\mathbb{R}}$.

Lemma 2.1 ([**19**], [**8**]). *The Liao linearized systems of $\boldsymbol{S}$ possess the following three properties.*

(1) *Uniform boundedness: $R^*_{x,\gamma}(t)$ is continuous in $(t,(x,\gamma)) \in \mathbb{R} \times \mathscr{F}^{*\sharp}_{\boldsymbol{S},n-1}(\varLambda)$ with*

$$\eta_\varLambda := \sup\left\{\sum_{i,j} |R^{*ij}_{x,\gamma}(t)|; t \in \mathbb{R},\ (x,\gamma) \in \mathscr{F}^{*\sharp}_{\boldsymbol{S},n-1}(\varLambda)\right\} < \infty.$$

(2) *Upper triangularity: $R^*_{x,\gamma}(t)$ is upper-triangular.*

(3) *Geometrical interpretation: Let $v = \gamma_{x,0}(\bar{z})$ for any $\bar{z} \in \mathbb{R}^{n-1}$ and any frame $(x,\gamma) \in \mathscr{F}^{*\sharp}_{\boldsymbol{S},n-1}(\varLambda)$. If $z(t) = z(t,\bar{z})$ is the solution of $(R^*_{x,\gamma})$ with $z(0) = \bar{z}$, then*

$$\varPsi^t_{\boldsymbol{S},x}(v) = \gamma_{x,t}(z(t)) \in \boldsymbol{S}^\perp_{t.x}.$$

Conversely, letting $z(t) = (z^1(t),\ldots,z^{n-1}(t))^{\mathrm{T}} \in \mathbb{R}^{n-1}$ be defined by

$$z^i(t) = \left\langle \varPsi^t_{\boldsymbol{S},x}(v), \mathrm{col}_i\gamma_x(t)\right\rangle \quad (i = 1,\ldots,n-1),$$

we have $\dot{z}(t) = R^*_{x,\gamma}(t)z(t)$ and $z(0) = \bar{z}$. Particularly, $C^*_{x,\gamma}(t)$ is the fundamental matrix of solution of $(R^*_{x,\gamma})$.

Statement (3) of Lemma 2.1 shows that $(R^*_{x,\gamma})$ is essentially the variational equations of the differential system $\boldsymbol{S}$ along the orbit $\boldsymbol{S}^{\mathbb{R}}(x)$.

2.2. Standard systems.

For $c > 0$, let $\mathbb{R}^{n-1}_c = \{z \in \mathbb{R}^{n-1}; \|z\| < c\}$. Let

$$\varSigma_{\boldsymbol{S}(t.x)} = \boldsymbol{S}^t(x) + \boldsymbol{S}^\perp_{t.x} \qquad \text{where } t.x = \boldsymbol{S}^t(x) \text{ as before}$$

be the cross-section of $\boldsymbol{S}$ at the base point $t.x$ for all $t \in \mathbb{R}$ and all $x \in \mathbb{R}^n$, viewed as a hyperplane in $\mathbb{R}^n$.

Given any $(x,\gamma) \in \mathscr{F}_{\boldsymbol{S},n-1}^{*\sharp}$, we consider the C^1-mapping

$$\mathfrak{S}_{x,\gamma}^{*}\colon \mathbb{R}\times\mathbb{R}^{n-1} \to \mathbb{R}^n$$

defined by

$$\mathfrak{S}_{x,\gamma}^{*}(t,z) = t.x + \gamma_{x,t}(z) \qquad \forall (t,z) \in \mathbb{R}\times\mathbb{R}^{n-1}.$$

It is known from [**8**] that there is a constant $\mathfrak{c} > 0$, which might rely on $\boldsymbol{S}$ but is independent of the choice of $(x,\gamma) \in \mathscr{F}_{\boldsymbol{S},n-1}^{*\sharp}(\Lambda)$, such that $\mathfrak{S}_{x,\gamma}^{*}$ is locally diffeomorphic from $\mathbb{R}\times\mathbb{R}_{\mathfrak{c}}^{n-1}$ into $\mathbb{R}^n$. In fact, there is some $\epsilon > 0$ so that for any $x \in \Lambda$, $\mathfrak{S}_{x,\gamma}^{*}$ is C^1-diffeomorphic from $(-\epsilon,\epsilon)\times\mathbb{R}_{2\mathfrak{c}}^{n-1}$ into $\mathbb{R}^n$.

Given any $(x,\gamma) \in \mathscr{F}_{\boldsymbol{S},n-1}^{*\sharp}(\Lambda)$ and any other $\boldsymbol{X} \in \mathfrak{X}^1(\mathbb{R}^n)$ nearby $\boldsymbol{S}$, we define a C^0-vector field on $\mathbb{R}\times\mathbb{R}_{\mathfrak{c}}^{n-1}$

$$\widehat{\boldsymbol{X}}_{x,\gamma}\colon \mathbb{R}\times\mathbb{R}_{\mathfrak{c}}^{n-1} \to \mathbb{R}\times\mathbb{R}^{n-1}$$

in this way:

$$\big(D_{(t,z)}\mathfrak{S}_{x,\gamma}^{*}\big)\widehat{\boldsymbol{X}}_{x,\gamma}(t,z) = \boldsymbol{X}(\mathfrak{S}_{x,\gamma}^{*}(t,z)) \qquad \forall (t,z) \in \mathbb{R}\times\mathbb{R}_{\mathfrak{c}}^{n-1}.$$

Since $\mathfrak{S}_{x,\gamma}^{*}$ is locally C^1-diffeomorphic associated to $\boldsymbol{S}$, $\widehat{\boldsymbol{X}}_{x,\gamma}(t,z)$ is well defined on $\mathbb{R}\times\mathbb{R}_{\mathfrak{c}}^{n-1}$. Particularly, for the special case $\boldsymbol{X} = \boldsymbol{S}$, we have

$$\widehat{\boldsymbol{S}}_{x,\gamma}(t,\mathbf{0}) = (1,\mathbf{0}) \in \mathbb{R}\times\mathbb{R}^{n-1}.$$

On the (t,z)-phase-space $\mathbb{R}\times\mathbb{R}_{\mathfrak{c}}^{n-1}$, we now consider the autonomous differential system

$$\frac{d}{d\mathfrak{t}}\begin{pmatrix} t\\ z\end{pmatrix} = \widehat{\boldsymbol{X}}_{x,\gamma}(t,z), \qquad (t,z) \in \mathbb{R}\times\mathbb{R}_{\mathfrak{c}}^{n-1} \text{ and } \mathfrak{t} \in \mathbb{R}.$$

Write

$$\widehat{\boldsymbol{X}}_{x,\gamma}(t,z) = \big(\widehat{\boldsymbol{X}}_{x,\gamma}^{0}(t,z),\dots,\widehat{\boldsymbol{X}}_{x,\gamma}^{n-1}(t,z)\big) \in \mathbb{R}\times\mathbb{R}^{n-1}.$$

Clearly, it follows from $\mathfrak{S}^*_{x,\gamma}(t,z)=\mathfrak{S}^*_{\mathscr{S}^{*\sharp}_t(x,\gamma)}(0,z)$ for any $(t,z)\in\mathbb{R}\times\mathbb{R}^{n-1}$ that

$$\widehat{\boldsymbol{X}}_{x,\gamma}(t,z)=\widehat{\boldsymbol{X}}_{\mathscr{S}^{*\sharp}_t(x,\gamma)}(0,z)\qquad\forall(t,z)\in\mathbb{R}\times\mathbb{R}^{n-1}_{\mathfrak{c}}.\tag{2.3}$$

Although $\mathfrak{S}^*_{x,\gamma}$ is only C^1-class, we can obtain more about the regularity of $\widehat{\boldsymbol{X}}_{x,\gamma}(t,z)$ with respect to the variable $z\in\mathbb{R}^{n-1}_{\mathfrak{c}}$.

First, by replacing $\mathfrak{c}$ by a more small positive constant if necessary, we can obtain the following lemma.

LEMMA 2.2. *There exists a* C^1*-neighborhood* $\mathscr{N}_{\boldsymbol{S}}$ *of* $\boldsymbol{S}$ *in* $\mathfrak{X}^1(\mathbb{R}^n)$ *such that for any* $\boldsymbol{X}\in\mathscr{N}_{\boldsymbol{S}}$*,*

$$\frac{1}{2}\le\widehat{\boldsymbol{X}}^0_{x,\gamma}(t,z)\le\frac{4}{2}$$

for any $(t,z,(x,\gamma))\in\mathbb{R}\times\mathbb{R}^{n-1}_{\mathfrak{c}}\times\mathscr{F}^{*\sharp}_{\boldsymbol{S},n-1}(\Lambda)$*.*

Thus, the following definition makes sense.

DEFINITION 2.3. Given any $(\boldsymbol{X},(x,\gamma))\in\mathscr{N}_{\boldsymbol{S}}\times\mathscr{F}^{*\sharp}_{\boldsymbol{S},n-1}(\Lambda)$, set

$$\boldsymbol{X}^*_{x,\gamma}(t,z)=\left(\frac{\widehat{\boldsymbol{X}}^1_{x,\gamma}(t,z)}{\widehat{\boldsymbol{X}}^0_{x,\gamma}(t,z)},\dots,\frac{\widehat{\boldsymbol{X}}^{n-1}_{x,\gamma}(t,z)}{\widehat{\boldsymbol{X}}^0_{x,\gamma}(t,z)}\right)\in\mathbb{R}^{n-1}\qquad\forall(t,z)\in\mathbb{R}\times\mathbb{R}^{n-1}_{\mathfrak{c}}.$$

The non-autonomous differential equations

$$\frac{dz}{dt}=\boldsymbol{X}^*_{x,\gamma}(t,z),\qquad(t,z)\in\mathbb{R}\times\mathbb{R}^{n-1}_{\mathfrak{c}}\tag{$\boldsymbol{X}^*_{x,\gamma}$}$$

is referred to as the *standard system of* $\boldsymbol{X}$ *associated to* $(\boldsymbol{S},(x,\gamma))$.

From (2.3), it follows easily that

$$\boldsymbol{X}^*_{x,\gamma}(t+t',z)=\boldsymbol{X}^*_{\mathscr{S}^{*\sharp}_t(x,\gamma)}(t',z)\qquad\forall t,t'\in\mathbb{R}\text{ and }z\in\mathbb{R}^{n-1}_{\mathfrak{c}}.$$

Note here that $\mathscr{S}^{*\sharp}_t$ is defined by $\boldsymbol{S}$ as in (2.1), not by $\boldsymbol{X}$. From now on, we rewrite $(\boldsymbol{X}^*_{x,\gamma})$ as a quasi-linear differential equations

$$\dot{z}=R^*_{x,\gamma}(t)z+\boldsymbol{X}^*_{\mathrm{rem}(x,\gamma)}(t,z),\qquad(t,z)\in\mathbb{R}\times\mathbb{R}^{n-1}_{\mathfrak{c}}\tag{$\boldsymbol{X}^*_{x,\gamma}$}$$

where

$$\boldsymbol{X}^*_{\mathrm{rem}(x,\gamma)}(t,z)=\boldsymbol{X}^*_{x,\gamma}(t,z)-R^*_{x,\gamma}(t)z$$

such that

$$\boldsymbol{X}^*_{\mathrm{rem}(x,\gamma)}(t+t',z)=\boldsymbol{X}^*_{\mathrm{rem}(\mathscr{S}_t^{*\sharp}(x,\gamma))}(t',z),\qquad \forall t,t'\in\mathbb{R}.$$

We notice also that $R^*_{x,\gamma}(t)$ is associated to $\boldsymbol{S}$, not to $\boldsymbol{X}$.

The following theorem is basic in the theory of standard systems of differential equations.

THEOREM 2.4 ([**20**]). *Let* $\mathscr{N}_{\boldsymbol{S}}$ *be given by Lemma* 2.2. *Then for any system* $\boldsymbol{X}\in\mathscr{N}_{\boldsymbol{S}}$, *the following statements hold*:

(1) $\boldsymbol{X}^*_{\mathrm{rem}(x,\gamma)}(t,z)$ *and* $\partial\boldsymbol{X}^*_{\mathrm{rem}(x,\gamma)}(t,z)/\partial z$ *both are continuous with respect to* $(t,z,(x,\gamma))$ *in* $\mathbb{R}\times\mathbb{R}^{n-1}_{\mathfrak{c}}\times\mathscr{F}^{*\sharp}_{\boldsymbol{S},n-1}(\Lambda)$.

(2) *Given any* $(x,\gamma)\in\mathscr{F}^{*\sharp}_{\boldsymbol{S},n-1}(\Lambda)$ *and* $t'<t''$, *if* $z^*(t)=z^*_{\boldsymbol{X},x,\gamma}(t;t_0,z)$ *where* $t,t_0\in(t',t'')$, *is the solution of* $(\boldsymbol{X}^*_{x,\gamma})$ *with* $z^*(t_0)=z$, *then*

$$\boldsymbol{X}^{\mathbb{t}(t;t_0)}(\mathfrak{S}^*_{x,\gamma}(t_0,z))=\mathfrak{S}^*_{x,\gamma}(t,z^*(t))\in\Sigma_{\boldsymbol{S}(t.x)}$$

where

$$\mathbb{t}(t;t_0)=\int_{t_0}^{t}\frac{1}{\widehat{\boldsymbol{X}}^0_{x,\gamma}(\tau,z^*(\tau))}\,d\tau.$$

(3) $\boldsymbol{S}^*_{x,\gamma}(t,z)$ *is of* C^1*-class in* $z\in\mathbb{R}^{n-1}_{\mathfrak{c}}$ *with* $\boldsymbol{S}^*_{x,\gamma}(t,\boldsymbol{0})=\boldsymbol{0}$ *for all* t, *such that*

$$\frac{\partial\boldsymbol{S}^*_{x,\gamma}(t,z)}{\partial z}\to\frac{\partial\boldsymbol{S}^*_{x,\gamma}(t,\boldsymbol{0})}{\partial z}=R^*_{x,\gamma}(t)\quad \textit{as } z\to\boldsymbol{0}$$

uniformly for $(t,(x,\gamma))\in\mathbb{R}\times\mathscr{F}^{*\sharp}_{\boldsymbol{S},n-1}(\Lambda)$.

REMARK 2.5. Statement (2) of Theorem 2.4 shows that if x is a periodic point of $\boldsymbol{S}$ with period τ_x, then

$$\boldsymbol{S}^{\mathbb{t}(\tau_x;0)}\colon\Sigma_{\boldsymbol{S}(x)}\to\Sigma_{\boldsymbol{S}(x)}$$

is well defined near x, which is just the classical Poincaré map of $\boldsymbol{S}$ at x, and

$C^*_{x,\gamma}(\tau_x)\colon \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ is its linear approximation. However, certain care must be taken in interpreting the characteristic multipliers of x, since the coordinate system of $T_x\Sigma_{\boldsymbol{S}(x)}$ is γ at instant $t = 0$ and $\gamma_x(\tau_x)$ at instant $t = \tau_x$.

2.3. Level vector fields.

As mentioned before, there is a constant $\epsilon > 0$, which is independent of the choices of $(x, \gamma) \in \mathscr{F}^{*\sharp}_{\boldsymbol{S},n-1}(\Lambda)$, such that $\mathfrak{S}^*_{x,\gamma}\colon [0, \epsilon] \times \mathbb{R}^{n-1}_{\mathfrak{c}} \to \mathbb{R}^n$ is C^1-diffeomorphic. If $T > 0$ and $0 < \zeta < \mathfrak{c}$ are such that

$$\mathfrak{S}^*_{x,\gamma}\colon [0, T] \times \mathbb{R}^{n-1}_{\zeta} \to \mathbb{R}^n$$

is C^1-diffeomorphic into $\mathbb{R}^n$, then the cylinder $[0, T] \times \mathbb{R}^{n-1}_{\zeta}$ is said to be *admissible for* $(\boldsymbol{S}, x)$. Notice here that the admissibility is independent of the choice of the frames $\gamma \in \mathscr{F}^{*\sharp}_{\boldsymbol{S},n-1}(x)$.

We introduce a necessary notion following Liao [**22**].

Definition 2.6. Let $[0, T] \times \mathbb{R}^{n-1}_{\zeta}$ be admissible for $(\boldsymbol{S}, x)$ where $x \in \Lambda$, and

$$\boldsymbol{Z}(t, z) = (0, \boldsymbol{Z}^1, \ldots, \boldsymbol{Z}^{n-1})$$

a C^1-vector field on $\mathbb{R}^n$ such that $\boldsymbol{Z}_{\restriction \mathbb{R}^n - [0,T] \times \mathbb{R}^{n-1}_{\zeta}} \equiv \boldsymbol{0}$. Such a $\boldsymbol{Z}$ is called a *level vector field* on the cylinder $[0, T] \times \mathbb{R}^{n-1}_{\zeta}$. For any $\gamma \in \mathscr{F}^{*\sharp}_{\boldsymbol{S},n-1}(x)$, define naturally a C^0-vector field, write $\Pi^*_{x,\gamma}(\boldsymbol{Z})$, on $\mathbb{R}^n$ as follows:

$$\Pi^*_{x,\gamma}(\boldsymbol{Z})(w) = \begin{cases} \big(D\mathfrak{S}^*_{x,\gamma}\big)\boldsymbol{Z}(t, z) & \text{if } w = \mathfrak{S}^*_{x,\gamma}(t, z),\ 0 \le t \le T,\ \|z\| \le \zeta \\ \boldsymbol{0} & \text{if } w \in \mathbb{R}^n - \mathfrak{S}^*_{x,\gamma}\big([0, T] \times \mathbb{R}^{n-1}_{\zeta}\big). \end{cases}$$

The following facts are useful for proving Theorem A, which correspond to [**22**, Propositions 3.1, 3.3 and 3.4].

Theorem 2.7. *Let $\boldsymbol{Z}(t, z) = \big(0, \widehat{\boldsymbol{S}}^0_{x,\gamma}(t, z)\boldsymbol{Z}^*(t, z)\big) \in \mathbb{R}^n$ be a level vector field on an admissible cylinder $[0, T] \times \mathbb{R}^{n-1}_{\zeta}$ for $(\boldsymbol{S}, x)$, where $(x, \gamma) \in \mathscr{F}^{*\sharp}_{\boldsymbol{S},n-1}(\Lambda)$. Then, the following three statements hold*:

(1) $\Pi^*_{x,\gamma}(\boldsymbol{Z})$ *is of* C^1*-class.*
(2) *There are numbers $\xi_* > 0, \zeta_* > 0$ such that for any $(x, \gamma) \in \mathscr{F}^{*\sharp}_{\boldsymbol{S},n-1}(\Lambda)$ and any $T > 0$*

$$\big\|\Pi^*_{x,\gamma}(\boldsymbol{Z})\big\|_1 \le \xi_* \|\boldsymbol{Z}\|_1$$

if $[0,T]\times\mathbb{R}^{n-1}_{\zeta}, \zeta\le\zeta_*$, *is an admissible cylinder of* $(\boldsymbol{S},x)$ *and* $\boldsymbol{Z}(t,z)$ *is a level vector field on it.*

(3) *Let* $\boldsymbol{X}=\boldsymbol{S}+\Pi^*_{x,\gamma}(\boldsymbol{Z})$. *For any* $\varepsilon>0$, *there is a* $\delta=\delta(\boldsymbol{S},\varepsilon)>0$ *such that*

$$\widehat{\boldsymbol{X}}_{x,\gamma}(t,z)=\widehat{\boldsymbol{S}}_{x,\gamma}(t,z)+\boldsymbol{Z}(t,z)$$

and

$$\boldsymbol{X}^*_{x,\gamma}(t,z)=\boldsymbol{S}^*_{x,\gamma}(t,z)+\boldsymbol{Z}^*(t,z) \quad \text{and} \quad \|\Pi^*_{x,\gamma}(\boldsymbol{Z})\|_1<\varepsilon$$

if $[0,T]\times\mathbb{R}^{n-1}_{\zeta}$ *with* $\zeta\le\delta$ *is an admissible cylinder of* $(\boldsymbol{S},x)$ *and* $\boldsymbol{Z}(t,z)$ *is a level vector field on it satisfying* $\|\boldsymbol{Z}\|_1\le\delta$.

PROOF. Let $C(s;(t,z))\in[0,T]\times\mathbb{R}^{n-1}_{\zeta}$ be an integral curve of the level vector field $\boldsymbol{Z}$ in $\mathbb{R}^n$ with $C(0)=(t,z)$, for $s\in(-\tau,\tau)$ for some $\tau>0$ sufficiently small. Clearly,

$$C(s;(t,z))=(t,C^1(s;(t,z)),\dots,C^{n-1}(s;(t,z)))$$

lies in $\{t\}\times\mathbb{R}^{n-1}_{\mathfrak{c}}$ for all $s\in(-\tau,\tau)$. Then for $w=\mathfrak{S}^*_{x,\gamma}(t,z)$, we have

$$\begin{aligned}\mathscr{C}(s;w)&:=\mathfrak{S}^*_{x,\gamma}(C(s;(t,z)))\\&=t.x+\gamma_{x,t}(C^1(s;(t,z)),\dots,C^{n-1}(s;(t,z)))\in\Sigma_{\boldsymbol{S}(t.x)},\end{aligned}$$

is an integral curve of the field $\Pi^*_{x,\gamma}(\boldsymbol{Z})$ in $\mathbb{R}^n$ with $\mathscr{C}(0;w)=w$. Thus,

$$\begin{aligned}\Pi^*_{x,\gamma}(\boldsymbol{Z})(w)=\left.\frac{d\mathscr{C}(s;w)}{ds}\right|_{s=0}&=\gamma_x(t)\big(\boldsymbol{Z}^1(t,z),\dots,\boldsymbol{Z}^{n-1}(t,z)\big)^{\mathrm{T}}\\&=\gamma_x(t)\big(\boldsymbol{Z}^1({\mathfrak{S}^*_{x,\gamma}}^{-1}(w)),\dots,\boldsymbol{Z}^{n-1}({\mathfrak{S}^*_{x,\gamma}}^{-1}(w))\big)^{\mathrm{T}}\end{aligned}\tag{2.4}$$

is of C^1-class with respect to w by the definition of admissible cylinder. This proves statement (1) of Theorem 2.7.

Next, we are going to prove statement (2) of Theorem 2.7. By (2.4) we have

$$\left\|\Pi^*_{x,\gamma}(\boldsymbol{Z})\right\|_0\le\|\boldsymbol{Z}\|_0$$

for any $(x,\gamma)\in\mathscr{F}^{*\sharp}_{\boldsymbol{S},n-1}(\mathit{\Lambda})$ and for any admissible cylinder $[0,T]\times\mathbb{R}^{n-1}_{\zeta}$ of $(\boldsymbol{S},x)$. Note that

$$w = w(t,z) = \mathfrak{S}^*_{x,\gamma}(t,z) = t.x + \gamma_{x,t}(z) \qquad \forall (t,z) \in [0,T] \times \mathbb{R}^{n-1}_\zeta.$$

From (2.4) again, it follows that

$$\left\| \frac{\partial \Pi^*_{x,\gamma}(\boldsymbol{Z})(w)}{\partial w} \right\| \le \left\| \frac{\partial \{\gamma_x(t)(\boldsymbol{Z}^1(t,z),\ldots,\boldsymbol{Z}^{n-1}(t,z))^{\mathrm{T}}\}}{\partial (t,z)} \right\| \left\| \frac{\partial {\mathfrak{S}^*_{x,\gamma}}^{-1}(w)}{\partial w} \right\|.$$

However,

$$\begin{aligned}
&\left\| \frac{\partial \{\gamma_x(t)(\boldsymbol{Z}^1(t,z),\ldots,\boldsymbol{Z}^{n-1}(t,z))^{\mathrm{T}}\}}{\partial t} \right\| \\
&\quad \le \left\| \frac{d\gamma_x(t)}{dt} \big(\boldsymbol{Z}^1(t,z),\ldots,\boldsymbol{Z}^{n-1}(t,z)\big)^{\mathrm{T}} \right\| + \left\| \gamma_x(t) \left(\frac{\partial \boldsymbol{Z}^1(t,z)}{\partial t}, \ldots, \frac{\partial \boldsymbol{Z}^{n-1}(t,z)}{\partial t} \right)^{\mathrm{T}} \right\| \\
&\quad \le \left\| \frac{d\gamma_x(t)}{dt} \right\| \|\boldsymbol{Z}\|_0 + \left\| \frac{\partial \boldsymbol{Z}(t,z)}{\partial t} \right\|
\end{aligned}$$

and

$$\begin{aligned}
\left\| \frac{\partial \{\gamma_x(t)(\boldsymbol{Z}^1(t,z),\ldots,\boldsymbol{Z}^{n-1}(t,z))^{\mathrm{T}}\}}{\partial z} \right\| &= \left\| \gamma_x(t) \left[\frac{\partial \boldsymbol{Z}^i(t,z)}{\partial z^j} \right]_{(n-1)\times(n-1)} \right\| \\
&\le \left\| \frac{\partial \boldsymbol{Z}(t,z)}{\partial z} \right\|.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\left\| \Pi^*_{x,\gamma}(\boldsymbol{Z}) \right\|_1 &\le \|\boldsymbol{Z}\|_0 + \sup_w \left\| \frac{\partial \Pi^*_{x,\gamma}(\boldsymbol{Z})(w)}{\partial w} \right\| \\
&\le \|\boldsymbol{Z}\|_0 + \sup_w \left\| \frac{\partial {\mathfrak{S}^*_{x,\gamma}}^{-1}(w)}{\partial w} \right\| \left(\left\| \frac{d\gamma_x(t)}{dt} \right\| \|\boldsymbol{Z}\|_0 + \left\| \frac{\partial \boldsymbol{Z}(t,z)}{\partial (t,z)} \right\| \right) \\
&\le \sup_w \left(1 + \left\| \frac{\partial {\mathfrak{S}^*_{x,\gamma}}^{-1}(w)}{\partial w} \right\| \right) \left(1 + \left\| \frac{d\gamma_x(t)}{dt} \right\| \right) \|\boldsymbol{Z}\|_1
\end{aligned}$$

and

$$\frac{\partial \mathfrak{S}^*_{x,\gamma}(t,z)}{\partial (t,z)} = \left[\boldsymbol{S}(t.x) + \frac{d}{dt}\gamma_{x,t}(z),\ \gamma_x(t) \right]_{n\times n}$$

where $w = w(t, z)$.

On the other hand, we see that $(d/dt)\gamma_{x,t}$ is jointly continuous with respect to $(t,(x,\gamma))$ and uniformly continuous with respect to $(t,(x,\gamma)) \in [0,T]\times \mathscr{F}^{*\sharp}_{\boldsymbol{S},n-1}(\Lambda)$. So $\|(\partial {\mathfrak{S}^*_{x,\gamma}}^{-1}(w))/\partial w\|$ is continuous with respect to $(t,(x,\gamma))$.

Thus, by the group property of $(\mathscr{S}^{*\sharp}_t)_{t\in\mathbb{R}}$ and the compactness of $\mathscr{F}^{*\sharp}_{\boldsymbol{S},n-1}(\Lambda)$, there exist constants $\xi_* > 0, \zeta_* > 0$ such that for any $(x,\gamma) \in \mathscr{F}^{*\sharp}_{\boldsymbol{S},n-1}(\Lambda)$ and any $T > 0$

$$\left\|\Pi^*_{x,\gamma}(\boldsymbol{Z})\right\|_1 \le \xi_* \|\boldsymbol{Z}\|_1$$

whenever Z is a level vector field on an admissible cylinder $[0,T]\times\mathbb{R}^{n-1}_\zeta$ for $(\boldsymbol{S},x)$ and for $\zeta \le \zeta_*$. This proves statement (2) of Theorem 2.7.

Statement (3) of Theorem 2.7 comes immediately from Definition 2.3 and statement (2) of Theorem 2.7 proved above. The proof of Theorem 2.7 is therefore completed. □

Theorem 2.7 will play the role of the Franks lemma [**12**]. This kind of special small perturbation $\boldsymbol{X}$ of $\boldsymbol{S}$ defined by Theorem 2.7 is very important in the proof of Theorem A′.

2.4. Level vector fields based on periodic orbits.

We now assume that $p \in \Lambda$ is a periodic point of the differential system $\boldsymbol{S}$ with prime period T_p. For any $\gamma \in \mathscr{F}^{*\sharp}_{\boldsymbol{S},n-1}(p)$, $\gamma_p(T_p) = \mathscr{S}^{*\sharp}_{T_p,p}(\gamma)$ belongs to $\mathscr{F}^{*\sharp}_{\boldsymbol{S},n-1}(p)$ as well. Let

$$\langle\gamma,\gamma_p(T_p)\rangle = (\langle \mathrm{col}_i\gamma, \mathrm{col}_j\gamma_p(T_p)\rangle)_{(n-1)\times(n-1)} \tag{2.5}$$

be the correlation $(n-1)$-by-$(n-1)$ matrix of γ and $\gamma_p(T_p)$. Then, we have

$$\gamma_p(T_p) = \big(\mathrm{col}_1\gamma_p(T_p),\dots,\mathrm{col}_{n-1}\gamma_p(T_p)\big) = \gamma\langle\gamma,\gamma_p(T_p)\rangle.$$

We will use the following results in the proof of Theorem A′.

Proposition 2.8. *Given any periodic point $p \in \mathrm{Per}(\boldsymbol{S}\restriction\Lambda)$ with the prime period T_p and any $\varepsilon > 0$, let $\boldsymbol{Z} = (0,\widetilde{\boldsymbol{Z}})$ be a level vector field on an admissible cylinder $[0,T]\times\mathbb{R}^{n-1}_\zeta$ of $(\boldsymbol{S},p)$ such that $\boldsymbol{Z}(t,\mathbf{0}) = \mathbf{0}$ for all $t \in \mathbb{R}$ and $\|\boldsymbol{Z}\|_1 \le \delta$, where $\delta = \delta(\boldsymbol{S},\varepsilon)$ is as in statement (3) of Theorem 2.7. Let $\gamma \in \mathscr{F}^{*\sharp}_{\boldsymbol{S},n-1}(p)$ and $\boldsymbol{S}^{[0,T_p]}(p)\cap\mathfrak{S}^*_{p,\gamma}([0,T]\times\mathbb{R}^{n-1}_\zeta) = \boldsymbol{S}^{[0,T]}(p)$ and $T < T_p$. Then, $P := \boldsymbol{S}^{[0,T_p]}(p)$ is also a periodic orbit of $\boldsymbol{X} = \boldsymbol{S} + \Pi^*_{p,\gamma}(\boldsymbol{Z})$ with $\boldsymbol{S}_{\restriction P} = \boldsymbol{X}_{\restriction P}$. Moreover, if $\boldsymbol{Z}_\sharp(t)$ is the standard fundamental solution of matrix of the equation*

$$\frac{dz}{dt} = \left(R^*_{p,\gamma}(t) + \frac{\partial \widetilde{\boldsymbol{Z}}(t,z)}{\partial z}\bigg|_{z=\mathbf{0}} \right) z,$$

then the eigenvalues of the matrices $\langle \gamma, \gamma_p(T_p)\rangle \boldsymbol{Z}_\sharp(T_p)$ *and* $\boldsymbol{Z}_\sharp(T_p)\langle \gamma, \gamma_p(T_p)\rangle$ *are just those of* $\boldsymbol{X}$ *at* p*, respectively.*

Proof. Let $\boldsymbol{X} = \boldsymbol{S} + \Pi^*_{p,\gamma}(\boldsymbol{Z})$. Since $\boldsymbol{S}(t.p) = \boldsymbol{X}(t.p)$ for any $t \in \mathbb{R}$ where $t.p = \boldsymbol{S}^t(p)$, $\boldsymbol{S}^{[0,T_p]}(p)$ is still a periodic orbit of $\boldsymbol{X}$ with $\boldsymbol{S}^t(p) = \boldsymbol{X}^t(p)$ for all t. We write

$$\widetilde{\boldsymbol{X}}(t,z) = \widehat{\boldsymbol{S}}^0_{p,\gamma}(t,z)\boldsymbol{X}^*(t,z).$$

By a direct calculation, we easily have

$$\frac{\partial \widehat{\boldsymbol{S}}^0_{p,\gamma}(t,z)\boldsymbol{Z}^*(t,z)}{\partial z}\bigg|_{z=\mathbf{0}} = \frac{\partial \boldsymbol{Z}^*(t,z)}{\partial z}\bigg|_{z=\mathbf{0}}.$$

Thus, $\boldsymbol{Z}_\sharp(t)$ is also the standard fundamental matrix solution of the equation

$$\frac{dz}{dt} = \left(R^*_{p,\gamma}(t) + \frac{\partial \boldsymbol{Z}^*(t,z)}{\partial y}\bigg|_{z=\mathbf{0}} \right) z,$$

where $\boldsymbol{Z}^*(t,z)$ is as in statement (3) of Theorem 2.7.

To validate the second part of Proposition 2.8, we consider the Poincaré map of $\boldsymbol{X}$ at p

$$\psi^{T_p}_{\boldsymbol{X},p} \colon \Sigma_{\boldsymbol{S}(p)} \to \Sigma_{\boldsymbol{S}(T_p.p)} = \Sigma_{\boldsymbol{S}(p)}.$$

Notice that $\Sigma_{\boldsymbol{S}(p)} = \Sigma_{\boldsymbol{X}(p)}$ and $\boldsymbol{X}^*_{p,\gamma}(t,\mathbf{0}) = \mathbf{0}$ for all $t \in \mathbb{R}$ and so $z^*_{\boldsymbol{X},p,\gamma}(t) = \mathbf{0}$ is a solution of the standard system $(\boldsymbol{X}^*_{p,\gamma})$ of $\boldsymbol{X}$ associated to $(\boldsymbol{S},(p,\gamma))$. From statement (3) of Theorem 2.7 and Theorem 2.4, it follows that $D_p\psi^{T_p}_{\boldsymbol{X},p} = \boldsymbol{Z}_\sharp(T_p)$ if we let $\gamma, \gamma_p(T_p)$ serve as the bases of $\boldsymbol{S}^\perp_p$ and $\boldsymbol{S}^\perp_{T_p.p}$, respectively. So, if we let γ serve simultaneously as the basis of $\boldsymbol{S}^\perp_p$ and $\boldsymbol{S}^\perp_{T_p.p}$, then

$$D_p\psi^{T_p}_{\boldsymbol{X},p} = \langle \gamma, \gamma_p(T_p)\rangle \boldsymbol{Z}_\sharp(T_p).$$

This shows that the eigenvalues of $\langle \gamma, \gamma_p(T_p)\rangle \boldsymbol{Z}_\sharp(T_p)$ are just those of $\boldsymbol{X}$ at p.

Similarly, one can prove that the eigenvalues of $\boldsymbol{Z}_\sharp(T_p)\langle \gamma, \gamma_p(T_p)\rangle$ are just those of $\boldsymbol{X}$ at p. This proves Proposition 2.8. □

3. Two perturbation lemmas of Liao.

In this section, we will introduce two perturbation lemmas due to Liao [**22**]. Since they are unavailable for English readers, we now restate them without proof.

Let us consider a linear differential equations of order $\mathfrak{i}$

$$\frac{dy}{dt} = A(t)y \qquad (t, y) \in \mathbb{R} \times \mathbb{R}^{\mathfrak{i}}$$

where the $\mathfrak{i}$-by-$\mathfrak{i}$ coefficient matrix $A(t)$ is continuous in t such that

$$\sup_{t\in\mathbb{R}} \|A(t)\| \leq a_* < \infty.$$

By $y_A(t, y)$ we denote its solutions with $y_A(0, y) = y$ for any $y \in \mathbb{R}^{\mathfrak{i}}$.

Let $f_* \colon \mathbb{R} \to [0, 1]$ be a smooth bump function needed later such that

$$f_*|_{(-\infty,\,1/8]} \equiv 0, \quad f_*|_{[7/8,\,\infty)} \equiv 1, \quad f_*|_{[1/4,\,3/4]} \equiv 1/\sqrt{2}.$$

and

$$1 \leq b_* := \sup_{t\in\mathbb{R}} |f_*'(t)| < \infty.$$

Then, the following two lemmas are useful.

Lemma 3.1 ([**22**]). *For any* $\varrho \in (0, 1)$, $y_0, y_\sharp \in \mathbb{R}^{\mathfrak{i}}$ *with* $\|y_0\| = 1 = \|y_\sharp\|$ *and* $0 < T_1 < T < \infty$ *such that*

$$\frac{\|y_A(T, y_0)\|}{\|y_A(T, y_\sharp)\|} \leq \frac{\lambda_*^2}{8} \exp(\lambda \varrho T_1/32)$$

and

$$T \geq \max\left\{\frac{16 a_* T_1}{\varrho},\, 2\lambda T_1 + \frac{64}{\varrho} \log \frac{2}{\lambda_*},\, T_1 + 2\right\},$$

where

$$\lambda = \frac{\varrho}{4 b_* \exp(2a_*)}, \quad \lambda_* = \frac{\lambda}{2} \exp(-\varrho/2),$$

there is a linear perturbed equation

$$\frac{dy}{dt} = [A(t) + B_\sharp(t)]y \qquad (t, y) \in \mathbb{R} \times \mathbb{R}^{\mathfrak{i}} \tag{$\sharp$}$$

which satisfies that

(i) $B_\sharp(t)$ *is continuously differentiable in* t *such that* $B_\sharp(t)|_{(-\infty,\,0]\cup[T-1/8,\,\infty)} \equiv 0$ *and*

$$\sup_{t\in\mathbb{R}}\|B_\sharp(t)\| < \varrho;$$

(ii) *there exists a solution* $y_\sharp(t)$ *such that* $y_\sharp(0) = y_0$, $y_\sharp(T) = y_A(T, y_\sharp)$ *or* $-y_A(T, y_\sharp)$.

The second lemma is stated as follows:

Lemma 3.2 ([**22**]). *Let* $\varrho \in (0, 1)$ *and* $0 = t_0 < t_1 < \cdots < t_\ell = T < \infty$ *and* $y_0, y_\natural \in \mathbb{R}^{\mathfrak{i}}$ *such that* $\|y_0\| = 1 = \|y_A(T, y_\natural)\|$. *If*

$$\|y_A(t_{\ell-1}, y_\natural)\| = \inf_{y\in\mathbb{R}^{\mathfrak{i}},\|y_A(T,y)\|=1} \|y_A(t_{\ell-1}, y)\|$$

and

$$t_k - t_{k-1} \ge \max\left\{\frac{16a_*\bar{T}}{\varrho},\ 2\lambda\bar{T} + \frac{64}{\varrho}\log\frac{2}{\lambda_*},\ \bar{T} + 2\right\}, \quad k = 1, \dots, \ell,$$

where $\lambda = \varrho/(4b_* \exp(2a_*))$, $\lambda_* = (\lambda/2)\exp(-\varrho/2)$ *and where*

$$\bar{T} = \frac{32}{\lambda\varrho}\log\frac{32}{\lambda_*^2},$$

then there is a linear perturbed equation

$$\frac{dy}{dt} = [A(t) + B_\natural(t)]y \qquad (t, y) \in \mathbb{R} \times \mathbb{R}^{\mathfrak{i}} \tag{$\natural$}$$

which satisfies that

(i) $B_\natural(t)$ *is continuously differentiable in* t *such that* $B_\natural(t)|_{(-\infty,\,0]\cup[T-1/8,\,\infty)} \equiv 0$ *and*

$$\sup_{t\in\mathbb{R}}\|B_\natural(t)\| < \varrho;$$

(ii) *there exists a solution* $y_\natural(t)$ *such that* $y_\natural(0) = y_0$ *and*

$$\frac{y_\natural(T)}{\|y_\natural(T)\|} = y_A(T, y_\natural) \text{ or } -y_A(T, y_\natural)$$

and

$$\frac{\|y_\natural(t_k)\|}{\|y_\natural(t_{k-1})\|} = \sup_{y \in \mathbb{R}^{\mathrm{i}}, \|y_A(t_{k-1}, y)\|=1} \|y_A(t_k, y)\| \quad \text{for } k = 1, \dots, \ell.$$

Particularly,

$$\|y_\natural(T)\| = \prod_{k=1}^{\ell} \sup_{y \in \mathbb{R}^{\mathrm{i}}, \|y_A(t_{k-1}, y)\|=1} \|y_A(t_k, y)\|.$$

Let $\mathscr{A}_t \colon \mathbb{R}^{\mathrm{i}} \to \mathbb{R}^{\mathrm{i}}$; $y \mapsto y_A(t, y)$ for all $t \in \mathbb{R}$. Then

$$\inf_{y \in \mathbb{R}^{\mathrm{i}}, \|y_A(T, y)\|=1} \|y_A(t_{\ell-1}, y)\| = \|\mathscr{A}_{t_{\ell-1}} \circ \mathscr{A}_T^{-1}\|_{\mathrm{co}}$$

and

$$\sup_{y \in \mathbb{R}^{\mathrm{i}}, \|y_A(t_{k-1}, y)\|=1} \|y_A(t_k, y)\| = \left\|\mathscr{A}_{t_k} \circ \mathscr{A}_{t_{k-1}}^{-1}\right\|$$

for $k = 1, \dots, \ell$.

4. Existence of weak hyperbolicity.

This section will be devoted to proving Theorem A stated in Section 1.2. As described before, we need to prove mainly the simple version Theorem A$'$ stated in Section 1.4.

4.1. Proof of Theorem A$'$.

Throughout this subsection, let $\boldsymbol{S} \colon \mathbb{R}^n \to T\mathbb{R}^n$ be any given C^1-class vector field on $\mathbb{R}^n$ having no singularities. Let $\Lambda \subset \mathbb{R}^n$ be an invariant, nonempty, and compact set of the dynamical system $\boldsymbol{S}$. We simply write

$$\boldsymbol{S}^t(x) = t.x \quad \text{and} \quad \Psi^t_{\boldsymbol{S},x}(v) = \Psi^t(v) \qquad \forall t \in \mathbb{R} \text{ and } (x, v) \in \boldsymbol{S}^{\perp}_{\Lambda}.$$

For convenience, we now reformulate Theorem A$'$ as follows:

Theorem 4.1. *If $\boldsymbol{S}$ obeys the* C^1*-weak-star property on* Λ*; that is to say, there is a* C^1*-neighborhood* $\mathscr{U}^*$ *of* $\boldsymbol{S}$ *in* $\mathfrak{X}^1(\mathbb{R}^n)$ *such that, for any* $\boldsymbol{V} \in \mathscr{U}^*$*, every* P *in* $\mathscr{O}_{\mathrm{per}}(\boldsymbol{S} \restriction \Lambda) \cap \mathscr{O}_{\mathrm{per}}(\boldsymbol{V})$ *with* $\boldsymbol{S}_{\restriction P} = \boldsymbol{V}_{\restriction P}$*, is hyperbolic with respect to* $\boldsymbol{V}$*, then there are constants* $\eta > 0$ *and* $T > 0$*, for which the following two statements hold.*

(1) $\boldsymbol{S}$ *has a natural and uniform dominated splitting on* $\mathrm{Per}(\boldsymbol{S} \restriction \Lambda)$*; that is*

$$\begin{aligned} \|\Psi^t\| &\le \exp(-\eta t) && \text{if } \dim \mathrm{D}_-(x, \boldsymbol{S}) = n-1,\\ \|\Psi^t\|_{\mathrm{co}} &\ge \exp(\eta t) && \text{if } \dim \mathrm{D}_-(x, \boldsymbol{S}) = 0,\\ \frac{\|\Psi^t{}_{\restriction \mathrm{D}_-(x,\boldsymbol{S})}\|}{\|\Psi^t{}_{\restriction \mathrm{D}_+(x,\boldsymbol{S})}\|_{\mathrm{co}}} &\le \exp(-2\eta t) && \text{if } 1 \le \dim \mathrm{D}_-(x, \boldsymbol{S}) \le n-2, \end{aligned}$$

for any $t \ge T$ *and any* $x \in \mathrm{Per}(\boldsymbol{S} \restriction \Lambda)$*.*

(2) *If* $x \in \mathrm{Per}(\boldsymbol{S} \restriction \Lambda)$ *has the prime period* $T_x \ge T$ *and*

$$0 = t_0 < t_1 < \cdots < t_\ell = T_x, \quad \ell \ge 1$$

is a subdivision of $[0, T_x]$ *satisfying* $t_k - t_{k-1} \ge T$ *for* $k = 1, \ldots, \ell$*, then*

$$\frac{1}{T_x} \sum_{k=1}^{\ell} \log \left\| \Psi^{t_k - t_{k-1}}{}_{\restriction \mathrm{D}_-(t_{k-1}.x, \boldsymbol{S})} \right\| \le -\eta$$

and

$$\frac{1}{T_x} \sum_{k=1}^{\ell} \log \left\| \Psi^{t_k - t_{k-1}}{}_{\restriction \mathrm{D}_+(t_{k-1}.x, \boldsymbol{S})} \right\|_{\mathrm{co}} \ge \eta.$$

Here the subspaces $\mathrm{D}_-(x, \boldsymbol{S})$ and $\mathrm{D}_+(x, \boldsymbol{S})$ of $T_x\mathbb{R}^n$ both are defined as in Section 1.1.

Proof. For the clarity, we will divide the proof of Theorem 4.1 into several steps.

Step 1: Define a C^1-neighborhood $\mathscr{W}$ of $\boldsymbol{S}$ in $\mathfrak{X}^1(\mathbb{R}^n)$ as follows:

$$\mathscr{W} = \mathscr{U}^* \cap \mathscr{N}_{\boldsymbol{S}}$$

where $\mathscr{N}_{\boldsymbol{S}}$ is defined by Lemma 2.2. At first, let us choose two constants $\varepsilon > 0$ and $\delta > 0$ such that

(a) if $\boldsymbol{V} \in \mathfrak{X}^1(\mathbb{R}^n)$ satisfies $\|\boldsymbol{V} - \boldsymbol{S}\|_1 < \varepsilon$ then $\boldsymbol{V} \in \mathscr{W}$;
(b) $\delta = \delta(\boldsymbol{S}, \varepsilon)$ satisfies the requirement of statement (3) of Theorem 2.7 for the case where ε is given as in (a).

And put

$$\varrho = \frac{\min\{\delta,\ 1\}}{4(1+2b_*)} \qquad \text{with } 0 < \varrho < 1, \tag{4.1}$$

where $b_* \geq 1$ is defined as in Section 3 associated to the bump function $f_*(t)$.

Step 2: For any $P \in \mathscr{O}_{\mathrm{per}}(\boldsymbol{S} \upharpoonright \Lambda)$, from the hyperbolicity of $\boldsymbol{S}$ at P it follows that there exists a natural Ψ-invariant decomposition over P

$$\boldsymbol{S}_x^{\perp} = \mathrm{D}_-(x, \boldsymbol{S}) \oplus \mathrm{D}_+(x, \boldsymbol{S}), \quad \text{and simply write} \quad \mathfrak{i}_x = \mathrm{Ind}_{\boldsymbol{S}}(x) \qquad \forall x \in P.$$

Then, take an arbitrary $(n-1)$-frame $\gamma \in \mathscr{F}^{*\sharp}_{\boldsymbol{S},n-1}(x)$ at the base point $x \in P$ such that

$$\mathrm{col}_k\gamma \in \mathrm{D}_-(x, \boldsymbol{S}) \quad \text{for } k = 1, \ldots, \mathfrak{i}_x \quad \text{if } \mathfrak{i}_x \geq 1. \tag{4.2}$$

We will apply Liao's Lemmas 3.1 and 3.2 stated in Section 3 to the Liao linearized equations of $\boldsymbol{S}$ under the chosen $(n-1)$-frame (x, γ)

$$\frac{dz}{dt} = R^*_{x,\gamma}(t)z \tag{$R^*_{x,\gamma}$}$$

given as in Section 2.1. Write

$$R^*_{x,\gamma}(t) = \begin{bmatrix} Q_{x,\gamma}(t) & Q''_{x,\gamma}(t) \\ \mathbf{0}_{(n-1-\mathfrak{i}_x)\times\mathfrak{i}_x} & Q'_{x,\gamma}(t) \end{bmatrix}, \quad z = \begin{bmatrix} y \\ y' \end{bmatrix} \in \mathbb{R}^{\mathfrak{i}_x} \times \mathbb{R}^{n-1-\mathfrak{i}_x}$$

where $Q_{x,\gamma}(t)$ is a matrix of $\mathfrak{i}_x \times \mathfrak{i}_x$. By $z_{x,\gamma}(t, z)$ we denote the solution of $(R^*_{x,\gamma})$ with $z_{x,\gamma}(0, z) = z$ for any $z \in \mathbb{R}^{n-1}$. In the case $\mathfrak{i}_x \geq 1$, let $Y_{x,\gamma}(t)$ be the standard fundamental solution matrix of the following subsystem of $(R^*_{x,\gamma})$

$$\dot{y} = Q_{x,\gamma}(t)y, \qquad (t, y) \in \mathbb{R} \times \mathbb{R}^{\mathfrak{i}_x}.$$

From Lemma 2.1, it follows that $\|\Psi^t(\gamma_{x,0}(z))\| = \|z_{x,\gamma}(t, z)\|$ for any $t \in \mathbb{R}$ and any $z \in \mathbb{R}^{n-1}$. Thus,

$$\|\varPsi^t{}_{\restriction \mathrm{D}_-(t'.x,\boldsymbol{S})}\| = \sup_{y\in\mathbb{R}^{\mathfrak{i}_x},\|Y_{x,\gamma}(t')y\|=1} \|Y_{x,\gamma}(t'+t)y\| \qquad \forall t\geq 0 \text{ and } t'\in\mathbb{R}.$$

Step 3: We first prove the following claim:

Claim 1. *There exists a constant $T_\varLambda > 0$ such that, if $P\in\mathscr{O}_{\mathrm{per}}(\boldsymbol{S}\restriction\varLambda)$ having prime period $T_P\geq T_\varLambda$ and if $0=t_0<t_1<\cdots<t_\ell=T_P$ is a subdivision of $[0,T_P]$ with $t_k-t_{k-1}\geq T_\varLambda$ for $k=1,\dots,\ell$, then the inequalities*

$$\frac{1}{T_P}\sum_{k=1}^{\ell}\log\left\|\varPsi^{t_k-t_{k-1}}{}_{\restriction \mathrm{D}_-(t_{k-1}.x,\boldsymbol{S})}\right\|\leq -\varrho/4 \tag{4.3}$$

and

$$\frac{1}{T_P}\sum_{k=1}^{\ell}\log\left\|\varPsi^{t_k-t_{k-1}}{}_{\restriction \mathrm{D}_+(t_{k-1}.x,\boldsymbol{S})}\right\|_{\mathrm{co}}\geq \varrho/4 \tag{4.4}$$

for any $x\in P$, are satisfied.

Here ϱ is as in (4.1) and $T_\varLambda$ will be defined by (4.7b) below.

Proof. Let $P\in\mathscr{O}_{\mathrm{per}}(\boldsymbol{S}\restriction\varLambda)$ with the prime period T_P and a subdivision of $[0,T_P]$

$$0=t_0<t_1<\cdots<t_\ell=T_P,\quad \ell\geq 1$$

have been arbitrarily given. Let

$$\mathfrak{i}=\mathrm{Ind}_{\boldsymbol{S}}(P).$$

Since the case of $\mathfrak{i}=0$ can be handled by considering $-\boldsymbol{S}$ instead of $\boldsymbol{S}$, without loss of generality we may assume $1\leq\mathfrak{i}\leq n-1$. Let $x\in P$ be a periodic point with the prime period $T_x=T_P$.

First, pick up some $(n-1)$-frame $\gamma\in\mathscr{F}^{*\sharp}_{\boldsymbol{S},n-1}(x)$ at the base point x satisfying (4.2) and consider $(R^*_{x,\gamma})$ associated to $(\boldsymbol{S},(x,\gamma))$. To apply Lemma 3.2, we first take and then fix some $y_\natural\in\mathbb{R}^{\mathfrak{i}}$ such that

$$\|Y_{x,\gamma}(T_x)y_\natural\|=1\quad\text{and}\quad\|Y_{x,\gamma}(t_{\ell-1})y_\natural\|=\inf_{y\in\mathbb{R}^{\mathfrak{i}},\|Y_{x,\gamma}(T_x)y\|=1}\|Y_{x,\gamma}(t_{\ell-1})y\|.$$

Let $z_\natural=(Y_{x,\gamma}(T_x)y_\natural,\mathbf{0})^{\mathrm{T}}\in\mathbb{R}^{\mathfrak{i}}\times\mathbb{R}^{n-1-\mathfrak{i}}$, and set

$$z_0 = \langle \gamma, \gamma_x(T_x) \rangle z_\natural \in \mathbb{R}^{n-1},$$

where $\gamma_x(T_x) = \mathscr{S}^{*\sharp}_{T_x,x}(\gamma)$ and $\mathscr{S}^{*\sharp}_{T_x,x} \colon \mathscr{F}^{*\sharp}_{\boldsymbol{S},n-1}(x) \to \mathscr{F}^{*\sharp}_{\boldsymbol{S},n-1}(x)$ is defined as in (2.1). Condition (4.2) implies that $\mathrm{col}_k \gamma_x(T_x) \in \mathrm{D}_-(x, \boldsymbol{S})$ for $k = 1, \ldots, \mathfrak{i}$ and hence the correlation matrix of γ and $\gamma_x(T_x)$ has the form as follows:

$$\langle \gamma, \gamma_x(T_x) \rangle = \begin{bmatrix} C_{x,\gamma} & \mathbf{0} \\ \mathbf{0} & * \end{bmatrix} \tag{4.5}$$

where $C_{x,\gamma}$ is of $\mathfrak{i} \times \mathfrak{i}$ and $*$ is some matrix of $(n-1-\mathfrak{i}) \times (n-1-\mathfrak{i})$ given as in (2.5). So,

$$z_0 = (C_{x,\gamma} Y_{x,\gamma}(T_x) y_\natural, \mathbf{0}) \in \mathbb{R}^{\mathfrak{i}} \times \mathbb{R}^{n-1-\mathfrak{i}}.$$

Let

$$y_0 = C_{x,\gamma} Y_{x,\gamma}(T_x) y_\natural \in \mathbb{R}^{\mathfrak{i}}. \tag{4.6}$$

Since $\langle \gamma, \gamma_x(T_x) \rangle$ is an orthogonal matrix, we have $\|y_0\| = \|z_\natural\| = 1$.

Let the constants $\lambda, \lambda_*, \bar{T}$ and T_Λ be defined as

$$\lambda = \frac{\varrho}{4b_* \exp(2\eta_\Lambda)}, \quad \lambda_* = \frac{\lambda}{2} \exp(-\varrho/2),$$

and

$$\bar{T} = \frac{32}{\lambda\varrho} \log \frac{32}{\lambda_*^2}, \tag{4.7a}$$

$$T_\Lambda = \max\left\{ \frac{16\eta_\Lambda \bar{T}}{\varrho},\ 2\lambda\bar{T} + \frac{64}{\varrho} \log \frac{2}{\lambda_*},\ \bar{T} + 2 \right\}, \tag{4.7b}$$

where η_Λ is defined as in Lemma 2.1. It is easy to see that T_Λ is independent of the choice of the $(n-1)$-frame (x, γ).

Then, if

$$t_k - t_{k-1} \geq T_\Lambda \qquad \text{for } k = 1, \ldots, \ell, \tag{4.8}$$

then by applying Lemma 3.2 with $A(t) = Q_{x,\gamma}(t)$, $a_* = \eta_\Lambda$, and $T = T_x$, one can find a linear equation

$$\frac{dy}{dt} = [Q_{x,\gamma}(t) + B_{x,\gamma}(t)]\, y, \qquad (t, y) \in \mathbb{R} \times \mathbb{R}^{\mathfrak{i}}$$

such that $B_{x,\gamma}(t)$ is continuously differentiable in t,

$$B_{x,\gamma}(t)|_{(-\infty,\, 0] \cup [T_x - 1/8,\, +\infty)} = 0, \quad \text{and } \sup_{t\in\mathbb{R}} \|B_{x,\gamma}(t)\| < \varrho. \tag{4.9}$$

Observe that it has a solution $y_\natural(t)$ such that

$$y_\natural(0) = y_0 \quad \text{and} \quad y_\natural(T_x) = \|y_\natural(T_x)\| Y_{x,\gamma}(T_x) y_\natural \text{ or } -\|y_\natural(T_x)\| Y_{x,\gamma}(T_x) y_\natural, \tag{4.10}$$

and

$$\|y_\natural(T_x)\| = \prod_{k=1}^{\ell} \sup_{y\in\mathbb{R}^{\mathfrak{i}}, \|Y_{x,\gamma}(t_{k-1})y\|=1} \|Y_{x,\gamma}(t_k) y\|. \tag{4.11}$$

Next, we are going to prove (4.3) of Claim 1 under condition (4.8).

We will define a level vector field on $\mathbb{R}^n$ as in Section 2.3. First, we take a constant ξ_0 such that

$$\frac{\xi_0}{T_x} \int_0^{T_x} f_*(s/T_x) f_*(1 - s/T_x)\, ds = \frac{\varrho}{4} \quad \text{and} \quad 0 < \xi_0 < \varrho. \tag{4.12}$$

Put

$$T = T_x - \frac{1}{8}.$$

Since x has prime period $T_x > T$, we can take a constant $\zeta \in (0, \mathfrak{c})$ so that the cylinder $[0, T] \times \mathbb{R}^{n-1}_\zeta$ is admissible for $(\boldsymbol{S}, x)$ and

$$\boldsymbol{S}^{[0,T_x]}(x) \cap \mathfrak{S}^*_{x,\gamma}\big([0, T] \times \mathbb{R}^{n-1}_\zeta\big) = [0, T].x,$$

where $\mathfrak{S}^*_{x,\gamma}$ is the standard mapping defined by $(\boldsymbol{S}, (x, \gamma))$ as in Section 2.2. Particularly, we require

$$\zeta < \frac{\delta}{2[\max_{0\le t\le T_x} \|(d/dt) B_{x,\gamma}(t)\| + 2\varrho b_* T_x^{-1} + \varrho] + 1}. \tag{4.13}$$

For any $\xi \in [0, \xi_0]$, define a vector field on $\mathbb{R}^n$ as follows:

$$\boldsymbol{Z}_{\gamma,\xi}(t,z) = \big(0,\ \widetilde{\boldsymbol{Z}}_{\gamma,\xi}(t,z)\big) = \big(0, f_*(2(1-\|z\|/\zeta))B_{\gamma,\xi}(t)z\big)$$

for any $(t,z) \in \mathbb{R} \times \mathbb{R}^{n-1}$, where

$$B_{\gamma,\xi}(t) = \frac{\xi}{\xi_0}\begin{bmatrix} B_{x,\gamma}(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix}_{(n-1)\times(n-1)} + \xi f_*(t/T_x) f_*(1-t/T_x) I_{n-1}.$$

Here I_{n-1} is the identity matrix on $\mathbb{R}^{n-1}$. It is easy to see that $\boldsymbol{Z}_{\gamma,\xi}(t,z)$ is level and of C^1-class on the admissible cylinder $[0,T] \times \mathbb{R}^{n-1}_{\zeta}$ of $(\boldsymbol{S}, x)$, and

$$\widetilde{\boldsymbol{Z}}_{\gamma,\xi}(t,\mathbf{0}) = \mathbf{0}\ \forall t \in \mathbb{R} \quad \text{and} \quad \widetilde{\boldsymbol{Z}}_{\gamma,\xi}(t,z) = B_{\gamma,\xi}(t)z \quad \text{for } \|z\| \le \zeta/2.$$

Let

$$\boldsymbol{X}_{\gamma,\xi} = \boldsymbol{S} + \Pi^*_{x,\gamma}(\boldsymbol{Z}_{\gamma,\xi}).$$

Then $\boldsymbol{X}_{\gamma,\xi} \in \mathfrak{X}^1(\mathbb{R}^n)$ by statement (1) of Theorem 2.7. Moreover, we have $\|\Pi^*_{x,\gamma}(\boldsymbol{Z}_{\gamma,\xi})\|_1 < \varepsilon$ and so $\boldsymbol{X}_{\gamma,\xi} \in \mathscr{W}$ for any $\xi \in [0,\xi_0]$ by (a) in Step 1 before. Indeed, from

$$\frac{\partial \widetilde{\boldsymbol{Z}}_{\gamma,\xi}(t,z)}{\partial z} = \begin{cases} B_{\gamma,\xi}(t) & \text{if } \|z\| \le \zeta/2, \\ \mathbf{0} & \text{if } \|z\| \ge \zeta, \\ B_{\gamma,\xi}(t)\hat{f}_*(z) & \text{if } \zeta/2 \le \|z\| \le \zeta, \end{cases}$$

where $\hat{f}_*(z) = f_*(2(1-\|z\|/\zeta))I_{n-1} - 2f'_*(2(1-\|z\|/\zeta))(z/\zeta)(z^{\mathrm{T}}/\|z\|)$, we obtain by (4.1)

$$\sup_{(t,z)\in\mathbb{R}^n} \left\| \frac{\partial \widetilde{\boldsymbol{Z}}_{\gamma,\xi}(t,z)}{\partial z} \right\| < (\varrho + \xi_0)(1+2b_*) < \delta/2.$$

Similarly, we can obtain by (4.9)

$$\begin{aligned} \sup_{(t,z)\in\mathbb{R}^n} \left\| \frac{\partial \widetilde{\boldsymbol{Z}}_{\gamma,\xi}(t,z)}{\partial t} \right\| &\le \zeta \left\| \frac{d}{dt} B_{\gamma,\xi}(t) \right\| \\ &\le \zeta \bigg(\max_{0\le t\le T} \left\| \frac{d}{dt} B_{x,\gamma}(t) \right\| + \frac{2\xi_0 b_*}{T_x} \bigg). \end{aligned}$$

In addition,

$$\sup_{(t,z)\in\mathbb{R}^{n+1}} \|\boldsymbol{Z}_{\gamma,\xi}(t,z)\| \le \zeta\varrho.$$

Therefore, by (4.13) we have

$$\|\boldsymbol{Z}_{\gamma,\xi}\|_1 < \delta.$$

This implies, from statement (3) of Theorem 2.7 and (b) in Step 1, that $\|\Pi^*_{x,\gamma}(\boldsymbol{Z}_{\gamma,\xi})\|_1 < \varepsilon$ as desired.

Clearly, from Proposition 2.8, $\boldsymbol{S}^{[0,T_x]}(x)$ is also a periodic orbit of $\boldsymbol{X}_{\gamma,\xi}$ with $\boldsymbol{S}^t(x) = \boldsymbol{X}^t_{\gamma,\xi}(x)$ for all $0 \le t \le T_x$, since $\boldsymbol{Z}_{\gamma,\xi}(t,\mathbf{0}) = \mathbf{0}$ for all $t \in \mathbb{R}$. Let $N_{\gamma,\xi}(t)$ be the standard fundamental solution of matrix of the equations

$$\frac{dz}{dt} = \left[R^*_{x,\gamma}(t) + \frac{\partial \widetilde{\boldsymbol{Z}}_{\gamma,\xi}(t,z)}{\partial z}\bigg|_{z=\mathbf{0}}\right] z, \qquad (t,z) \in \mathbb{R}\times\mathbb{R}^{n-1}. \tag{4.14}$$

Then, from Proposition 2.8 it follows that $N_{\gamma,\xi}(T_x)\langle\gamma,\gamma_x(T_x)\rangle$ has the same eigenvalues as the periodic point x of the differential system $\boldsymbol{X}_{\gamma,\xi}$. From

$$\frac{\partial \widetilde{\boldsymbol{Z}}_{\gamma,\xi}(t,z)}{\partial z}\bigg|_{z=\mathbf{0}} = B_{\gamma,\xi}(t)$$

and statement (2) of Lemma 2.1, it is easily seen that $N_{\gamma,\xi}(t)$ is such that

$$N_{\gamma,\xi}(t) = \begin{bmatrix} L_{\gamma,\xi}(t) & * \\ \mathbf{0} & * \end{bmatrix} \qquad \text{where } L_{\gamma,\xi}(t) \text{ is a matrix of } \mathfrak{i}\times\mathfrak{i}.$$

So, from (4.5)

$$N_{\gamma,\xi}(T_x)\langle\gamma,\gamma_x(T_x)\rangle = \begin{bmatrix} L_{\gamma,\xi}(T_x)C_{x,\gamma} & * \\ \mathbf{0} & * \end{bmatrix}$$

Let $\rho_\gamma(\xi)$ be the spectral radius of the $\mathfrak{i}$-by-$\mathfrak{i}$ matrix $L_{\gamma,\xi}(T_x)C_{x,\gamma}$. Then, $\rho_\gamma(\xi)$ is continuous with respect to $\xi \in [0,\xi_0]$, since $L_{\gamma,\xi}(T_x)$ is continuous.

Now, on the contrary, assume that (4.3) is not true. Thus according to (4.11),

$$\|y_{\natural}(T_x)\| > \exp(-\varrho T_x/4).$$

For $\xi = \xi_0$, let

$$\hat{y}(t) = y_\natural(t)\exp\left\{\int_0^t \xi_0 f_*(s/T_x) f_*(1 - s/T_x)\, ds\right\}, \quad \hat{y}(0) = y_\natural(0).$$

Since $\hat{y}(t)$ is a solution of the equation

$$\frac{dy}{dt} = \big[Q_{x,\gamma}(t) + B_{x,\gamma}(t) + \xi_0 f_*(t/T_x) f_*(1 - t/T_x) I_{\mathrm{i}}\big] y,$$

$(\hat{y}(t), \mathbf{0})$ is a solution of (4.14) by the definition of $B_{\gamma,\xi}(t)$, and further

$$\hat{y}(T_x) = L_{\gamma,\xi}(T_x)\hat{y}(0).$$

It follows from (4.6) and (4.10) that

$$\begin{aligned}\hat{y}(T_x) &= L_{\gamma,\xi}(T_x) C_{x,\gamma} Y_{x,\gamma}(T_x) y_\natural \\ &= \begin{cases} \|y_\natural(T_x)\| \exp\left\{\displaystyle\int_0^{T_x} \xi_0 f_*(s/T_x) f_*(1 - s/T_x)\, ds\right\} Y_{x,\gamma}(T_x) y_\natural \\ \text{or} \\ -\|y_\natural(T_x)\| \exp\left\{\displaystyle\int_0^{T_x} \xi_0 f_*(s/T_x) f_*(1 - s/T_x)\, ds\right\} Y_{x,\gamma}(T_x) y_\natural \end{cases}\end{aligned}$$

and so $L_{\gamma,\xi}(T_x) C_{x,\gamma}$ has an eigenvalue

$$\|y_\natural(T_x)\| \exp\left\{\int_0^{T_x} \xi_0 f_*(s/T_x) f_*(1 - s/T_x)\, ds\right\}$$

or

$$-\|y_\natural(T_x)\| \exp\left\{\int_0^{T_x} \xi_0 f_*(s/T_x) f_*(1 - s/T_x)\, ds\right\}.$$

Thus, $\rho_\gamma(\xi_0) > 1$ by (4.12).

On the other hand, for the case where $\xi = 0$, we have $\widetilde{\boldsymbol{Z}}_{\gamma,0}(t, z) = \mathbf{0}$ for any $t \in \mathbb{R}$. So $L_{\gamma,0}(t) = Y_{x,\gamma}(t)$ and further $\rho_\gamma(0)$ is the spectral radius of the linear transformation

$$\Psi_x^{T_x}{}_{\restriction \mathrm{D}_-(x,\boldsymbol{S})} \colon \mathrm{D}_-(x, \boldsymbol{S}) \to \mathrm{D}_-(x, \boldsymbol{S}) \qquad (\text{where } \mathrm{D}_-(T_x.x, \boldsymbol{S}) = \mathrm{D}_-(x, \boldsymbol{S})).$$

Since x is a hyperbolic periodic point of $\boldsymbol{S}$ with $\mathrm{Ind}_{\boldsymbol{S}}(x) \geq 1$ and $\mathrm{D}_{-}(x, \boldsymbol{S})$ is just the stable subspace, we have $\rho_\gamma(0) < 1$. Therefore, by the continuity of $\rho_\gamma(\xi)$ with respect to ξ, there is some $\xi' \in [0, \xi_0]$ such that $\rho_\gamma(\xi') = 1$.

This contradicts the fact that $\boldsymbol{X}_{\gamma,\xi'} \in \mathscr{W}$ which shares the same periodic orbit $P = \boldsymbol{S}^{[0,T_x]}(x)$ with $\boldsymbol{S}$. So, under condition (4.8), the inequality (4.3) holds. Considering $-\boldsymbol{S}$, we can similarly show the inequality (4.4).

This proves Claim 1. $\square$

Step 4: Let

$$\Upsilon = \log\left(\frac{2\exp(-2\eta_\Lambda T_\Lambda)}{\sqrt{25[6+\exp(-2\eta_\Lambda T_\Lambda)]^2 + 4\exp(-4\eta_\Lambda T_\Lambda)} - 5[6+\exp(-2\eta_\Lambda T_\Lambda)]}\right).$$

It is clear that $0 < \Upsilon < \infty$. Define two constants

$$T_* = \max\{2T_\Lambda, 4\Upsilon/\varrho\} \quad \text{and} \quad T_\star = 2T_\Lambda(1 + 2\eta_\Lambda T_\Lambda/\log 2),$$

where T_Λ as in (4.7b) and η_Λ as in statement (1) of Lemma 2.1.

Next, we will show

CLAIM 2. *If $P \in \mathscr{O}_{\mathrm{per}}(\boldsymbol{S} \restriction \Lambda)$ has the prime period $T_P \geq T_*$ satisfying* $\mathrm{ind}_{\boldsymbol{S}}(P) = \mathfrak{i}$, $1 \leq \mathfrak{i} \leq n-2$, *then*

$$\frac{1}{t}\log\frac{\|\Psi^t{}_{\restriction \mathrm{D}_+(x,\boldsymbol{S})}\|_{\mathrm{co}}}{\|\Psi^t{}_{\restriction \mathrm{D}_-(x,\boldsymbol{S})}\|} \geq \frac{1}{T_\star}\log 2 \qquad \forall t \geq T_\star$$

for any $x \in P$.

PROOF. Fix any $P \in \mathscr{O}_{\mathrm{per}}(\boldsymbol{S} \restriction \Lambda)$ with prime period $T_P \geq T_*$ and $1 \leq \mathfrak{i} \leq n-2$. To prove Claim 2, we first prove

$$\log\left\|\Psi^{T_\Lambda}{}_{\restriction \mathrm{D}_+(x,\boldsymbol{S})}\right\|_{\mathrm{co}} - \log\left\|\Psi^{T_\Lambda}{}_{\restriction \mathrm{D}_-(x,\boldsymbol{S})}\right\| \geq \log 2 \qquad \forall x \in P. \tag{4.15}$$

Hereafter, let $x \in P$ be given. We can take $u_- \in \mathrm{D}_-(x, \boldsymbol{S})$ and $u_+ \in \mathrm{D}_+(x, \boldsymbol{S})$ such that $\|u_-\| = 1 = \|u_+\|$ and

$$\log\left\|\Psi_x^{T_\Lambda}(u_-)\right\| = \log\left\|\Psi^{T_\Lambda}{}_{\restriction \mathrm{D}_-(x,\boldsymbol{S})}\right\|$$

and

$$\log\left\|\Psi_x^{T_\Lambda}(u_+)\right\| = \log\left\|\Psi^{T_\Lambda}{}_{\restriction \mathrm{D}_+(x,\boldsymbol{S})}\right\|_{\mathrm{co}}.$$

From the assumption $T_P \geq T_*$ and by (4.3) and (4.4) of Claim 1, we have

$$\max\left\{\|\Psi_x^{T_P}(u_-)\|,\ \|\Psi_x^{-T_P}(u_+)\|\right\} \leq \exp(-\varrho T_*/4) < 1. \tag{4.16}$$

Thus, there is some $\bar{u} \in \boldsymbol{S}_x^{\perp}$ and $\bar{r} > 0,\ r > 0$ such that

$$\bar{u} = \bar{r}u_- + r\frac{\Psi_x^{-T_P}(u_+)}{\|\Psi_x^{-T_P}(u_+)\|} \quad \text{and} \quad \|\bar{u}\| = 1 = \left\|\Psi_x^{T_P}(\bar{u})\right\|.$$

Let $u_0 = \Psi_x^{T_P}(\bar{u})$ and $r_0 = r/\|\Psi_x^{-T_P}(u_+)\|$. Then

$$\bar{u} = \bar{r}u_- + r_0\Psi_x^{-T_P}(u_+) \quad \text{and} \quad u_0 = r_0u_+ + \bar{r}\Psi_x^{T_P}(u_-). \tag{4.17}$$

Take a frame $\gamma \in \mathscr{F}^{*\sharp}_{\boldsymbol{S},n-1}(x)$ such that (4.2), and let $z_\gamma(t,z)$ be the solutions of $(R^*_{x,\gamma})$ associated to $(\boldsymbol{S},(x,\gamma))$ with $z_\gamma(0,z) = z$ for all $z \in \mathbb{R}^{n-1}$. By Lemma 2.1, it is easy to see that

$$\exp(-\eta_\Lambda t) \leq \left\|\Psi_x^t\right\| \leq \exp(\eta_\Lambda t) \quad \forall t > 0. \tag{4.18}$$

Take $\bar{z}, z_-, z_0, z_+ \in \mathbb{R}^n$ such that

$$\bar{u} = \gamma_{x,0}(\bar{z}), \quad u_- = \gamma_{x,0}(z_-), \quad u_0 = \gamma_{x,0}(z_0) \quad \text{and} \quad u_+ = \gamma_{x,0}(z_+).$$

Then $\|\bar{z}\| = \|z_-\| = \|z_0\| = \|z_+\| = 1$. We claim

$$\frac{\|z_\gamma(T_\Lambda, z_-)\|}{\|z_\gamma(T_\Lambda, \bar{z})\|} \leq \frac{4}{3}, \tag{4.19}$$

and

$$\frac{\|z_\gamma(T_\Lambda, z_0)\|}{\|z_\gamma(T_\Lambda, z_+)\|} \leq \frac{4}{3}. \tag{4.20}$$

In fact, from (4.17) we have

$$\bar{r}^2 + 2\bar{r}r_0\left\langle u_-, \Psi_x^{-T_P}(u_+)\right\rangle + r_0^2\left\|\Psi_x^{-T_P}(u_+)\right\| = 1$$

and

$$r_0^2 + 2\bar{r}r_0\big\langle u_+, \Psi_x^{T_P}(u_-)\big\rangle + \bar{r}^2\big\|\Psi_x^{T_P}(u_-)\big\| = 1.$$

This implies that

$$\begin{aligned} 0 &= \bar{r}^2\big(1 - \|\Psi_x^{T_P}(u_-)\|^2\big) + 2\bar{r}r_0\big(\langle u_-, \Psi_x^{-T_P}(u_+)\rangle - \langle u_+, \Psi_x^{T_P}(u_-)\rangle\big) \\ &\quad + r_0^2\big(\|\Psi_x^{-T_P}(u_+)\|^2 - 1\big). \end{aligned}$$

Let

$$\begin{aligned} a &= 1 - \big\|\Psi_x^{T_P}(u_-)\big\|^2, \\ b &= 2\big(\big\langle u_-, \Psi_x^{-T_P}(u_+)\big\rangle - \big\langle u_+, \Psi_x^{T_P}(u_-)\big\rangle\big) \\ c &= 1 - \big\|\Psi_x^{-T_P}(u_+)\big\|^2. \end{aligned}$$

Then, from (4.16)

$$\frac{\bar{r}}{r_0} = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a} \le \frac{4 + \sqrt{4^2 + 4}}{2a} < \frac{5}{a}$$

and similarly $r_0/\bar{r} \le 5/a$. Thus, we obtain

$$\max\left\{\frac{r_0}{\bar{r}}, \frac{\bar{r}}{r_0}\right\} < \frac{5}{1 - \exp(-\varrho T_*/2)}.$$

Moreover, by the definition of T_*

$$\begin{aligned} \max\left\{\frac{r_0}{\bar{r}}\big\|\Psi_x^{-T_P}(u_+)\big\|, \frac{\bar{r}}{r_0}\big\|\Psi_x^{T_P}(u_-)\big\|\right\} &< \frac{5\exp(-\varrho T_*/4)}{1 - \exp(-\varrho T_*/2)} \\ &< \frac{\exp(-2\eta_\Lambda T_\Lambda)}{6 + \exp(-2\eta_\Lambda T_\Lambda)}. \end{aligned}$$

By $\bar{u} = \bar{r}u_- + r_0\Psi_x^{-T_P}(u_+)$ and the triangle inequality, we have

$$\frac{1}{1 + r_0/\bar{r}\|\Psi_x^{-T_P}(u_+)\|} \le \bar{r} \le \frac{1}{1 - r_0/\bar{r}\|\Psi_x^{-T_P}(u_+)\|}$$

and further

$$|\bar{r}-1| \le \frac{r_0/\bar{r}\|\Psi_x^{-T_P}(u_+)\|}{1-r_0/\bar{r}\|\Psi_x^{-T_P}(u_+)\|}.$$

Hence

$$\begin{aligned}\|u_- - \bar{u}\| &\le \|u_- - \bar{r}u_-\| + \|\bar{r}u_- - \bar{u}\| \le |\bar{r}-1| + \bar{r}\frac{r_0}{\bar{r}}\left\|\Psi_x^{-T_P}(u_+)\right\| \\ &\le \frac{2}{1-r_0/\bar{r}\|\Psi_x^{-T_P}(u_+)\|}\frac{r_0}{\bar{r}}\left\|\Psi_x^{-T_P}(u_+)\right\| \\ &< \frac{1}{3}\exp(-2\eta_\Lambda T_\Lambda),\end{aligned}$$

and moreover

$$\|z_- - \bar{z}\| < \frac{1}{3}\exp(-2\eta_\Lambda T_\Lambda).$$

On the other hand, from $\|z_\gamma(T_\Lambda,\bar{z})\| \ge \exp(-\eta_\Lambda T_\Lambda)$ and

$$\left\|z_\gamma(T_\Lambda,z_-) - z_\gamma(T_\Lambda,\bar{z})\right\| = \|z_\gamma(T_\Lambda, z_- - \bar{z})\| \le \|z_- - \bar{z}\|\exp(\eta_\Lambda T_\Lambda)$$

it follows that

$$\frac{\|z_\gamma(T_\Lambda,z_-)\|}{\|z_\gamma(T_\Lambda,\bar{z})\|} \le 1 + \frac{\|z_- - \bar{z}\|\exp(\eta_\Lambda T_\Lambda)}{\|z_\gamma(T_\Lambda,\bar{z})\|} \le \frac{4}{3},$$

which shows (4.19).

Similarly, beginning with $u_0 = r_0u_+ + \bar{r}\Psi_x^{T_P}(u_-)$, we can prove the estimate (4.20).

To prove (4.15), on the contrary, we may assume by Lemma 2.1

$$\frac{\|z_\gamma(T_\Lambda,z_+)\|}{\|z_\gamma(T_\Lambda,z_-)\|} = \frac{\|\Psi_x^{T_\Lambda}(u_+)\|}{\|\Psi_x^{T_\Lambda}(u_-)\|} < 2.$$

Thus, from (4.19) and (4.20) we have

$$
\begin{aligned}
\frac{\|z_\gamma(T_\Lambda, z_0)\|}{\|z_\gamma(T_\Lambda, \bar z)\|} &= \frac{\|z_\gamma(T_\Lambda, z_0)\|}{\|z_\gamma(T_\Lambda, z_+)\|} \cdot \frac{\|z_\gamma(T_\Lambda, z_+)\|}{\|z_\gamma(T_\Lambda, z_-)\|} \cdot \frac{\|z_\gamma(T_\Lambda, z_-)\|}{\|z_\gamma(T_\Lambda, \bar z)\|} \\
&\le \frac{32}{9} < \frac{\lambda_*^2}{8} \exp(\lambda \varrho \bar T/32)
\end{aligned}
$$

where $\bar T$ is as in (4.7a). Applying Lemma 3.1 with $A(t) = R^*_{x,\gamma}(t)$, $T_1 = \bar T$ and $T = T_\Lambda$, we can obtain a linear equation

$$
\frac{d\tilde z}{dt} = \big[R^*_{x,\gamma}(t) + \widetilde B_\gamma(t)\big]\tilde z, \quad (t, \tilde z) \in \mathbb{R} \times \mathbb{R}^{n-1}
$$

such that $\widetilde B_\gamma(t)$ is continuously differentiable in t, $\widetilde B_\gamma(t)_{\restriction(-\infty,\,0]\cup[T_\Lambda - 1/8,\,\infty)} = \mathbf{0}$ and

$$
\sup\nolimits_{t\in\mathbb{R}} \big\|\widetilde B_\gamma(t)\big\| < \varrho.
$$

Observe that it has a solution $\tilde z(t)$ such that

$$
\tilde z(0) = z_0, \quad \tilde z(T_\Lambda) = z_\gamma(T_\Lambda, \bar z) \text{ or } -z_\gamma(T_\Lambda, \bar z)
$$

and so

$$
\tilde z(T_P) = z_\gamma(T_P, \bar z) \text{ or } -z_\gamma(T_P, \bar z). \tag{4.21}
$$

Now, as in the proof of Claim 1, we take a small $\zeta > 0$ so that the cylinder $[0, T_\Lambda - 1/8] \times \mathbb{R}^{n-1}_\zeta$ is admissible for $(\boldsymbol{S}, (x, \gamma))$, and with respect to the level field

$$
\boldsymbol{Z}_\gamma(t, z) = \big(0, \widetilde{\boldsymbol{Z}}_\gamma(t, z)\big) = \big(0, f_*(2(1 - \|z\|/\zeta))\widetilde B_\gamma(t) z\big)
$$

we have a C^1-vector field $\boldsymbol{X}_\gamma = \boldsymbol{S} + \Pi^*_{x,\gamma}(\boldsymbol{Z}_\gamma) \in \mathscr{W}$, which still has P as a periodic orbit of prime period T_P whose eigenvalues are just the eigenvalues of the matrix $\langle\gamma, \gamma_x(T_P)\rangle N_\gamma(T_P)$ from Proposition 2.8. Here $N_\gamma(t)$ satisfies

$$
\frac{dN_\gamma(t)}{dt} = \left(R^*_{x,\gamma}(t) + \left.\frac{\partial \widetilde{\boldsymbol{Z}}_\gamma(t, z)}{\partial z}\right|_{z=\mathbf{0}}\right) N_\gamma(t), \quad N_\gamma(0) = I_{n-1}.
$$

However, it is easily seen that $(\partial \widetilde{\boldsymbol{Z}}_\gamma(t, z)/\partial z)_{z=\mathbf{0}} = \widetilde B_\gamma(t)$ and from $\bar u = \gamma_{x,0}(\bar z)$ and Lemma 2.1 we have $\gamma_{x,T_P}(z_\gamma(T_P, \bar z)) = \Psi_x^{T_P}(\bar u) = u_0 = \gamma_{x,0}(z_0)$. Hence, by (4.21) we have

$$z_0 = \langle \gamma, \gamma_x(T_P) \rangle z_\gamma(T_P, \bar{z}) = \pm \langle \gamma, \gamma_x(T_P) \rangle \tilde{z}(T_P) = \pm \langle \gamma, \gamma_x(T_P) \rangle N_\gamma(T_P) z_0$$

which implies that x has an eigenvalue of absolute value one with respect to $\boldsymbol{X}_\gamma$. It is a contradiction since $\boldsymbol{X}_\gamma \in \mathscr{W} \subset \mathscr{U}^*$ and $P = \boldsymbol{X}_\gamma^{[0,T_P]}(x) \subset \Lambda$. This proves the inequality (4.15).

Next, we prove Claim 2 from (4.15). If $t \geq T_\star$ and $kT_\Lambda \leq t < (k+1)T_\Lambda$ for some $k \geq 2$ (noting that $T_\star \geq 2T_\Lambda$), then

$$\begin{aligned} &\frac{1}{t}\big[\log \|\Psi^t{}_{\restriction \mathrm{D}_+(x,\boldsymbol{S})}\|_{\mathrm{co}} - \log \|\Psi^t{}_{\restriction \mathrm{D}_-(x,\boldsymbol{S})}\|\big] \\ &\quad \geq \frac{1}{t}\left\{ \sum_{j=0}^{k-1} \big[\log \|\Psi^{T_\Lambda}{}_{\restriction \mathrm{D}_+(jT_\Lambda.x,\boldsymbol{S})}\|_{\mathrm{co}} - \log \|\Psi^{T_\Lambda}{}_{\restriction \mathrm{D}_-(jT_\Lambda.x,\boldsymbol{S})}\|\big] + (t-kT_\Lambda)\frac{\log 2}{T_\Lambda} \right\} \\ &\qquad - \frac{1}{t}\left\{ \big|\log \|\Psi^{t-kT_\Lambda}{}_{\restriction \mathrm{D}_+(kT_\Lambda.x,\boldsymbol{S})}\|_{\mathrm{co}}\big| \right. \\ &\qquad\qquad \left. + \big|\log \|\Psi^{t-kT_\Lambda}{}_{\restriction \mathrm{D}_-(kT_\Lambda.x,\boldsymbol{S})}\|\big| + (t-kT_\Lambda)\frac{\log 2}{T_\Lambda} \right\}. \end{aligned}$$

In addition, by (4.18)

$$\begin{aligned} &-\frac{1}{t}\left\{ \big|\log \|\Psi^{t-kT_\Lambda}{}_{\restriction \mathrm{D}_+(kT_\Lambda.x,\boldsymbol{S})}\|_{\mathrm{co}}\big| + \big|\log \|\Psi^{t-kT_\Lambda}{}_{\restriction \mathrm{D}_-(kT_\Lambda.x,\boldsymbol{S})}\|\big| + (t-kT_\Lambda)\frac{\log 2}{T_\Lambda} \right\} \\ &\quad \geq \frac{-(2\eta_\Lambda T_\Lambda + \log 2)}{t}. \end{aligned}$$

Thus, by (4.15)

$$\begin{aligned} \frac{1}{t}\log \frac{\|\Psi^t{}_{\restriction \mathrm{D}_+(x,\boldsymbol{S})}\|_{\mathrm{co}}}{\|\Psi^t{}_{\restriction \mathrm{D}_-(x,\boldsymbol{S})}\|} &\geq \left(\frac{1}{T_\Lambda} - \frac{1}{t}\right)\log 2 - \frac{2\eta_\Lambda T_\Lambda}{t} \\ &\geq \left(\frac{1}{T_\Lambda} - \frac{1}{T_\star}\right)\log 2 - \frac{2\eta_\Lambda T_\Lambda}{T_\star} \\ &= \frac{\log 2}{T_\star}. \end{aligned}$$

This proves Claim 2. □

Step 5: To complete the proof of Theorem 4.1, let

$$\Lambda_{0,n-1} = \{x \in \mathrm{Per}(\boldsymbol{S} \restriction \Lambda) \mid \mathrm{Ind}_{\boldsymbol{S}}(x) = 0 \text{ or } n-1\}.$$

Then, $\Lambda_{0,n-1}$ consists of at most finite number of "isolated" periodic orbits from [**24**, Theorem 3.1], whose proof relies only on Claim 1 proved before. Let

$$\Lambda_{T_*} = \{x \in \mathrm{Per}(\boldsymbol{S} \restriction \Lambda) \mid 1 \leq \mathrm{ind}_{\boldsymbol{S}}(x) \leq n-2 \text{ and } T_x \leq T_*\},$$

where T_x denotes the prime period of x under $\boldsymbol{S}$. Then, Λ_{T_*} also consists of at most finite number of periodic orbits of $\boldsymbol{S}$. Therefore there is some $\eta' > 0$ and $T' > 0$ such that for any $t \geq T'$ and any $x \in \Lambda_{0,n-1} \cup \Lambda_{T_*}$

$$\left\|\Psi_x^t\right\| \leq \exp(-\eta' t) \text{ if } \mathrm{Ind}_{\boldsymbol{S}}(x) = n-1, \quad \left\|\Psi_x^t\right\|_{\mathrm{co}} \geq \exp(\eta' t) \text{ if } \mathrm{Ind}_{\boldsymbol{S}}(x) = 0$$

and

$$\left.\begin{aligned} &\|\Psi^t{}_{\restriction \mathrm{D}_-(x,\boldsymbol{S})}\| \leq \exp(-\eta' t) \\ &\|\Psi^t{}_{\restriction \mathrm{D}_+(x,\boldsymbol{S})}\|_{\mathrm{co}} \geq \exp(\eta' t) \end{aligned}\right\} \quad \text{if } 1 \leq \mathrm{Ind}_{\boldsymbol{S}}(x) \leq n-2.$$

Then, from Claims 1 and 2 it follows immediately that the constants

$$\eta = \min\left\{\eta',\ \frac{\varrho}{4},\ \frac{\log 2}{2T_\star}\right\} \quad \text{and} \quad T = \max\{T',\ T_*,\ T_\star\}$$

satisfy the requirements of Theorem 4.1 at every $x \in \mathrm{Per}(\boldsymbol{S} \restriction \Lambda)$. Thus, the proof of Theorem 4.1 is complete. □

4.2. Proof of Theorem A.

Now, we turn to prove Theorem A based on Theorem A′. Let M^n be as before and let $\boldsymbol{S} \in \mathfrak{X}^1(M^n)$. We need a uniformity lemma stated as follows:

LEMMA 4.2. *Assume that $\boldsymbol{S}$ obeys the C^1-weak-star property on Λ and $\mathbb{T} > 0$. Then, there exists a C^1-neighborhood $\mathscr{U}_{\mathbb{T}}$ of $\boldsymbol{S}$ and two numbers $\theta > 0$ and $\Delta > 0$ such that, for any $\boldsymbol{V} \in \mathscr{U}_{\mathbb{T}}$ and any $P \in \mathscr{O}_{\mathrm{per}}(\boldsymbol{S} \restriction \Lambda) \cap \mathscr{O}_{\mathrm{per}}(\boldsymbol{V})$ with $\boldsymbol{S}_{\restriction P} = \boldsymbol{V}_{\restriction P}$ and period $T_P \leq \mathbb{T}$, we have*

$$\frac{1}{t} \log \left\|\Psi^t_{\boldsymbol{V}\restriction \mathrm{D}_-(x,\boldsymbol{V})}\right\| \leq -\theta \quad \textit{and} \quad \frac{1}{t} \log \left\|\Psi^t_{\boldsymbol{V}\restriction \mathrm{D}_+(x,\boldsymbol{V})}\right\|_{\mathrm{co}} \geq \theta$$

for any $t \geq \Delta$ and all $x \in P$.

PROOF. For any $\boldsymbol{V} \in \mathscr{U}$ where $\mathscr{U}$ is as in Definition 1.1, put

$$P_{\mathbb{T},\Lambda}(\boldsymbol{V}) = \bigcup \big\{P \in \mathscr{O}_{\mathrm{per}}(\boldsymbol{S} \restriction \Lambda) \cap \mathscr{O}_{\mathrm{per}}(\boldsymbol{V}) \mid \boldsymbol{S}_{\restriction P} = \boldsymbol{V}_{\restriction P} \text{ and } T_P \leq \mathbb{T}\big\}.$$

It is easy to check that $P_{\mathbb{T},\Lambda}(\boldsymbol{V})$ is hyperbolic with respect to $\boldsymbol{V}$, since it consists of at most finitely many periodic orbits of $\boldsymbol{V}$. So, there are $\theta_{\boldsymbol{S}} > 0$ and $T_{\boldsymbol{S}} \geq \mathbb{T}$ such that for any $y \in P_{\mathbb{T},\Lambda}(\boldsymbol{S})$

$$\left\|\Psi_{\boldsymbol{S}}^{T_{\boldsymbol{S}}}{}_{\restriction \mathrm{D}_-(y,\boldsymbol{S})}\right\| \leq \exp(-4\theta_{\boldsymbol{S}}T_{\boldsymbol{S}}) \quad \text{and} \quad \left\|\Psi_{\boldsymbol{S}}^{T_{\boldsymbol{S}}}{}_{\restriction \mathrm{D}_+(y,\boldsymbol{S})}\right\|_{\mathrm{co}} \geq \exp(4\theta_{\boldsymbol{S}}T_{\boldsymbol{S}}).$$

Then, by a standard argument (the Dependence of Solutions on Initial Conditions and Parameters of ODE) we can always take a C^1-neighborhood $\mathscr{U}_{\mathbb{T}}$ of $\boldsymbol{S}$ such that for any $\boldsymbol{V} \in \mathscr{U}_{\mathbb{T}}$ and any $y \in P_{\mathbb{T},\Lambda}(\boldsymbol{V})$

$$\left\|\Psi_{\boldsymbol{V}}^{T_{\boldsymbol{S}}}{}_{\restriction \mathrm{D}_-(y,\boldsymbol{V})}\right\| \leq \exp(-2\theta_{\boldsymbol{S}}T_{\boldsymbol{S}}) \quad \text{and} \quad \left\|\Psi_{\boldsymbol{V}}^{T_{\boldsymbol{S}}}{}_{\restriction \mathrm{D}_+(y,\boldsymbol{V})}\right\|_{\mathrm{co}} \geq \exp(2\theta_{\boldsymbol{S}}T_{\boldsymbol{S}}).$$

This, together with Lemma 2.1, completes the proof of Lemma 4.2. □

Now, we reformulate Theorem A as follows:

THEOREM 4.3. *If $\boldsymbol{S} \in \mathfrak{X}^1(M^n)$ obeys the* C^1*-weak-star property on some invariant closed set Λ containing no singularities, then one can find $\widetilde{\eta} > 0$, $\widetilde{T} > 0$ such that the following two statements hold.*

(1) *For any $P \in \mathscr{O}_{\mathrm{per}}(\boldsymbol{S} \restriction \Lambda)$,*

$$\begin{aligned}
\left\|\Psi^t_{\boldsymbol{S}\restriction \mathrm{D}_-(p,\boldsymbol{S})}\right\| &\leq \exp(-\widetilde{\eta}t) && \text{if } \dim \mathrm{D}_-(p,\boldsymbol{S}) = n-1,\\
\left\|\Psi^t_{\boldsymbol{S}\restriction \mathrm{D}(p,\boldsymbol{S})}\right\|_{\mathrm{co}} &\geq \exp(\widetilde{\eta}t) && \text{if } \dim \mathrm{D}_-(p,\boldsymbol{S}) = 0,\\
\frac{\|\Psi^t_{\boldsymbol{S}\restriction \mathrm{D}_-(p,\boldsymbol{S})}\|}{\|\Psi^t_{\boldsymbol{S}\restriction \mathrm{D}_+(p,\boldsymbol{S})}\|_{\mathrm{co}}} &\leq \exp(-2\widetilde{\eta}t) && \text{if } 1 \leq \dim \mathrm{D}_-(p,\boldsymbol{S}) \leq n-2.
\end{aligned}$$

for any $t \geq \widetilde{T}$ and all $p \in P$.

(2) *Moreover, if $P \in \mathscr{O}_{\mathrm{per}}(\boldsymbol{S} \restriction \Lambda)$ has prime period T_P and*

$$0 = t_0 < t_1 < \cdots < t_\ell = mT_P \quad \text{where } m \in \mathbb{N}$$

is a subdivision of $[0, mT_P]$ satisfying $t_k - t_{k-1} \geq \widetilde{T}$ for $k = 1, \ldots, \ell$, then

$$\frac{1}{mT_P}\sum_{k=1}^{\ell} \log \left\|\Psi_{\boldsymbol{S}}^{t_k - t_{k-1}}{}_{\restriction \mathrm{D}_-(\boldsymbol{S}^{t_{k-1}}(p),\boldsymbol{S})}\right\| \leq -\widetilde{\eta}$$

and

$$\frac{1}{mT_P}\sum_{k=1}^{\ell}\log\left\|\varPsi_{\boldsymbol{S}}^{t_k-t_{k-1}}{}_{\restriction \mathrm{D}_+(\boldsymbol{S}^{t_{k-1}}(p),\boldsymbol{S})}\right\|_{\mathrm{co}}\geq\widetilde{\eta}$$

for any $p\in P$.

(3) *There are at most finite number of periodic contracted* (*or expanded*) *orbits of $\boldsymbol{S}$ contained in $\varLambda$.*

Proof. Statements (1) and (2) of Theorem 4.3 follow immediately from Lemma 4.2 and Theorem 4.1. Statement (3) of Theorem 4.3 comes from statements (1) and (2) of Theorem 4.3 using an argument similar to that of [**24**, Theorem 3.1]. This thus completes the proof of Theorem 4.3. □

Remark 4.4. In fact, under the hypothesis of Theorem 4.3, one can find a C^1-neighborhood $\mathscr{U}$ of $\boldsymbol{S}$ which satisfies that statements (1) and (2) of Theorem 4.3 are still fulfilled when $\boldsymbol{S}$ is replaced by $\boldsymbol{X}$ and $P\in\mathscr{O}_{\mathrm{per}}(\boldsymbol{S}\restriction\varLambda)\cap\mathscr{O}_{\mathrm{per}}(\boldsymbol{X})$ with $\boldsymbol{S}_{\restriction P}=\boldsymbol{X}_{\restriction P}$, for any $\boldsymbol{X}\in\mathscr{U}$.

5. A criterion of weak-star condition.

In this section, we will devote our attention to proving the sufficiency part of Theorem B, which provides us with a criterion of C^1-weak-star property in the 3-dimensional case. For its proof, the main obstacle is that every ergodic measure in $\varLambda$ need not be approximated arbitrarily by periodic measures in $\varLambda$.

Theorem 5.1. *Let $\varLambda\subseteq M^3$ be an invariant compact set of a differential system $\boldsymbol{X}\in\mathfrak{X}^1(M^3)$ with $\boldsymbol{X}(x)\neq\boldsymbol{0}$ for $x\in\varLambda$. Assume there are constants $\eta>0$ and $T>0$, for which there hold the following three conditions*:

(1) *Each $P\in\mathscr{O}_{\mathrm{per}}(\boldsymbol{X}\restriction\varLambda)$ is hyperbolic, that is to say, $\boldsymbol{X}_p^{\perp}=\mathrm{D}_-(p,\boldsymbol{X})\oplus\mathrm{D}_+(p,\boldsymbol{X})$ for any $p\in P$.*

(2) *$\boldsymbol{X}$ possesses a natural (η,T)-dominated splitting*:

$$\frac{\|\varPsi_{\boldsymbol{X}\restriction\mathrm{D}_-(p,\boldsymbol{X})}^t\|}{\|\varPsi_{\boldsymbol{X}\restriction\mathrm{D}_+(p,\boldsymbol{X})}^t\|_{\mathrm{co}}}\leq\exp(-2\eta t)\quad\forall p\in P\qquad \textit{if}\ \mathrm{Ind}_{\boldsymbol{X}}(P)\neq 0,2$$

for any $t\geq T$ and all $P\in\mathscr{O}_{\mathrm{per}}(\boldsymbol{X}\restriction\varLambda)$.

(3) *If $P\in\mathscr{O}_{\mathrm{per}}(\boldsymbol{X}\restriction\varLambda)$ has prime period T_P and if*

$$0=t_0<t_1<\cdots<t_\ell=mT_P\quad \textit{where}\ m\in\mathbb{N}$$

is a subdivision of $[0,mT_P]$ satisfying $t_k-t_{k-1}\geq T$ for $k=1,\ldots,\ell$, then

$$\frac{1}{mT_P}\sum_{k=1}^{\ell}\log\left\|\Psi_{\boldsymbol{X}}^{t_k-t_{k-1}}{}_{\restriction \mathrm{D}_-(\boldsymbol{X}^{t_{k-1}}(p),\boldsymbol{X})}\right\| \leq -\eta \tag{5.1a}$$

and

$$\frac{1}{mT_P}\sum_{k=1}^{\ell}\log\left\|\Psi_{\boldsymbol{X}}^{t_k-t_{k-1}}{}_{\restriction \mathrm{D}_+(\boldsymbol{X}^{t_{k-1}}(p),\boldsymbol{X})}\right\|_{\mathrm{co}} \geq \eta \tag{5.1b}$$

for any $p \in P$.

Then $\boldsymbol{X}$ *possesses the* C^1*-weak-star property on* Λ.

Proof. We simply assume that $\boldsymbol{X} \in \mathfrak{X}^1(M^3)$ does not have any singularities, and let $\Lambda_{\mathfrak{i}} = \mathrm{Cl}(\{p \in \mathrm{Per}(\boldsymbol{X} \restriction \Lambda) \mid \mathrm{Ind}_{\boldsymbol{X}}(p) = \mathfrak{i}\})$ for $\mathfrak{i} = 0, 1, 2$. The conditions (5.1a) and (5.1b) imply that $\Lambda_0 \cup \Lambda_2$ consists of at most a finite numbers of periodic repellers and periodic attractors and so $\Lambda_0 \cup \Lambda_2$ is isolated from Λ_1. Therefore, to prove the statement of Theorem 5.1, it is sufficient to verify that $\boldsymbol{X}$ possesses the C^1-weak-star property on Λ_1.

Then from the persistence of dominated splitting, see, e.g., [**4**, Appendix B.1], there can be found a neighborhood U of Λ_1 in M^3 and a C^1-neighborhood $\mathscr{V}$ of $\boldsymbol{X}$ in $\mathfrak{X}^1(M^3)$ such that for every $\boldsymbol{Y} \in \mathscr{V}$, it has no singularities and possesses an $(\eta/2, T; 1)$-dominated splitting over the maximal $\boldsymbol{Y}$-invariant set $K_{\boldsymbol{Y}}$ in U

$$\boldsymbol{Y}_x^{\perp} = E(x,\boldsymbol{Y}) \oplus F(x,\boldsymbol{Y}) \qquad \text{with } \dim E(x,\boldsymbol{Y}) = 1\ \forall x \in K_{\boldsymbol{Y}}.$$

Note that $\Lambda_1 \subseteq K_{\boldsymbol{X}}$ and $E(\cdot)\colon \boldsymbol{Y} \mapsto E(\boldsymbol{Y})$ is continuous in the sense that

$$\|E(\boldsymbol{X}) - E(\boldsymbol{Y})\| := \sup_{x \in K_{\boldsymbol{X}} \cap K_{\boldsymbol{Y}}} \measuredangle(E(x,\boldsymbol{X}), E(x,\boldsymbol{Y})) \to 0 \quad \text{as } \boldsymbol{Y} \to \boldsymbol{X} \text{ in C}^1\text{-norm}$$

(analogous statement holds for $F(\cdot)\colon \boldsymbol{Y} \mapsto F(\boldsymbol{Y})$). Here $\measuredangle$ denotes the angle between the two subspaces $E(x,\boldsymbol{X}), E(x,\boldsymbol{Y})$ of $\boldsymbol{Y}_x^{\perp}$. Clearly, $\mathrm{D}_-(x,\boldsymbol{X}) = E(x,\boldsymbol{X})$ and $\mathrm{D}_+(x,\boldsymbol{X}) = F(x,\boldsymbol{X})$ for all $x \in \mathrm{Per}(\boldsymbol{X} \restriction \Lambda_1)$ with $\mathrm{Ind}_{\boldsymbol{X}}(x) = 1$.

For any $\boldsymbol{Y} \in \mathscr{V}$, we define the so-called Liao qualitative function

$$\omega^s(\cdot,\boldsymbol{Y})\colon K_{\boldsymbol{Y}} \to \mathbb{R}; \quad x \mapsto \left.\frac{d}{dt}\right|_{t=0} \left\|\Psi_{\boldsymbol{Y},x}^{t} \restriction E(x,\boldsymbol{Y})\right\|.$$

Similarly, $\omega^u(\cdot,\boldsymbol{Y})$ may be defined on $K_{\boldsymbol{Y}}$ based on $F(\boldsymbol{Y})$. From [**7**, Theorem 2.3], we see that $\omega^s(\cdot,\boldsymbol{Y})$ and $\omega^u(\cdot,\boldsymbol{Y})$ both are continuous functions for any $\boldsymbol{Y} \in \mathscr{V}$.

We now define another qualitative function

$$\omega^s(\cdot, \boldsymbol{X}|\boldsymbol{Y})\colon K_{\boldsymbol{X}} \to \mathbb{R}; \quad x \mapsto \begin{cases} \omega^s(x, \boldsymbol{Y}) & \text{if } x \in K_{\boldsymbol{X}} \cap K_{\boldsymbol{Y}}, \\ \omega^s(x, \boldsymbol{X}) & \text{if } x \in K_{\boldsymbol{X}} \setminus K_{\boldsymbol{Y}}. \end{cases}$$

Clearly, $\omega^s(\cdot, \boldsymbol{X}|\boldsymbol{Y})$ is Borel measurable, not necessarily continuous, for any system $\boldsymbol{Y} \in \mathscr{V}$.

From the calculations in the proof of [**7**, Theorem 2.3] and the continuity of $E(\cdot)$, there holds the following

Claim 3. *If $\boldsymbol{Y}_n \in \mathscr{V}$ converges to $\boldsymbol{X}$ in the sense of* C^1*-norm, then $\omega^s(x, \boldsymbol{X}|\boldsymbol{Y}_n)$ converges to $\omega^s(x, \boldsymbol{X})$ uniformly for $x \in K_{\boldsymbol{X}}$.*

Similar statement holds for $\omega^u(\cdot, \boldsymbol{X}|\boldsymbol{Y})$.

For any $P \in \mathscr{O}_{\mathrm{per}}(\boldsymbol{Y})$, where $\boldsymbol{Y} \in \mathscr{V}$, let $\mu_{P,\boldsymbol{Y}}$ be the unique ergodic probability measure of $\boldsymbol{Y}$ supported on P. Then from the Liao spectrum theorem, see [**26**], [**7**], [**8**] for example, we have the following

Claim 4. *Let $P \in \mathscr{O}_{\mathrm{per}}(\boldsymbol{Y} \restriction U)$ be arbitrarily given, where $\boldsymbol{Y} \in \mathscr{V}$. Then, $\int_{M^3} \omega^s(x, \boldsymbol{Y}) d\mu_{P,\boldsymbol{Y}}(x)$ and $\int_{M^3} \omega^u(x, \boldsymbol{Y}) d\mu_{P,\boldsymbol{Y}}(x)$ both are Lyapunov exponents of $\boldsymbol{Y}$ at P.*

Next, we are going to verify that $\boldsymbol{X}$ possesses the C^1-weak-star property on Λ_1. We suppose, on the contrary, that there is a sequence of vector fields $\boldsymbol{X}_\ell \to \boldsymbol{X}$ in $\mathscr{V}$ with $P_\ell \in \mathscr{O}_{\mathrm{per}}(\boldsymbol{X} \restriction \Lambda_1) \cap \mathscr{O}_{\mathrm{per}}(\boldsymbol{X}_\ell \restriction \Lambda_1)$ such that $\boldsymbol{X}_{\restriction P_\ell} = \boldsymbol{X}_{\ell \restriction P_\ell}$ and $\boldsymbol{X}_\ell$ has at least one $\boldsymbol{X}_\ell$-transversal Lyapunov exponent zero at P_ℓ. By choosing some subsequence of $\{\boldsymbol{X}_\ell\}$ if necessary, there is no loss of generality in assuming that $\mu_{P_\ell,\boldsymbol{X}_\ell}$ converges weakly-$*$ to some probability measure μ on M^3 with $P_\ell \to \mathrm{supp}(\mu) \subseteq \Lambda_1$. From Claim 4, it may be assumed, without loss of generality, that

$$\int_{K_{\boldsymbol{X}}} \omega^s(x, \boldsymbol{X}|\boldsymbol{X}_\ell) d\mu_{P_\ell,\boldsymbol{X}_\ell}(x) = \int_{M^3} \omega^s(x, \boldsymbol{X}_\ell) d\mu_{P_\ell,\boldsymbol{X}_\ell}(x) = 0$$

for all $\ell \geq 1$; otherwise, we consider $\omega^u(x, \boldsymbol{X}|\boldsymbol{X}_\ell)$ instead of $\omega^s(x, \boldsymbol{X}|\boldsymbol{X}_\ell)$. As $\omega^s(\cdot, \boldsymbol{X}|\boldsymbol{X}_\ell)$ converges uniformly to $\omega^s(\cdot, \boldsymbol{X})$ as $\ell \to \infty$ by Claim 3, it follows that

$$\int_{K_{\boldsymbol{X}}} \omega^s(x, \boldsymbol{X}) d\mu(x) = \lim_{\ell \to \infty} \int_{K_{\boldsymbol{X}}} \omega^s(x, \boldsymbol{X}|\boldsymbol{X}_\ell) d\mu_{P_\ell,\boldsymbol{X}_\ell}(x).$$

Thus, noting that $\mu_{P_\ell,\boldsymbol{X}_\ell} = \mu_{P_\ell,\boldsymbol{X}}$ for all $\ell \geq 1$ because $\boldsymbol{X}_{\restriction P_\ell} = \boldsymbol{X}_{\ell \restriction P_\ell}$, we could obtain that

$$\lim_{\ell\to\infty}\int_{M^3}\omega^s(x,\boldsymbol{X})d\mu_{P_\ell,\boldsymbol{X}}(x)=0.$$

From Claim 4, this contradicts condition (3) of Theorem 5.1. Thus, the proof of Theorem 5.1 is completed. □

We note here that conditions (1)–(3) of Theorem 5.1 are more stronger than the following one:

(1)′ There exists a continuous invariant splitting over $\Delta := \mathrm{Cl}(\mathscr{O}_{\mathrm{per}}(\boldsymbol{X}\restriction\Lambda))$

$$T_\Delta M = E\oplus F \text{ such that } E(p)=\mathrm{D}_-(p),\ \ F(p)=\mathrm{D}_+(p)\ \forall p\in \mathrm{Per}(\boldsymbol{X}\restriction\Lambda).$$

Proof of Theorem B. The sufficiency part of Theorem B follows from Theorem 5.1 and the necessity part from Theorem A. Therefore, the proof of Theorem B is completed. □

Acknowledgements. It is a pleasure to thank Professors Shaobo Gan and Lan Wen for some helpful discussion on this topic. The author is also grateful to the anonymous referees who suggested many improvements.

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