J. Math. Soc. Japan
Vol. 64, No. 4 (2012) pp. 1169–1180
doi: 10.2969/jmsj/06441169

Order of Meromorphic Maps and Rationality of the Image Space

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(Received Mar. 23, 2011)

Abstract. Let $\iota : \boldsymbol{C}^2 \hookrightarrow S$ be a compactification of the two dimensional complex space $\boldsymbol{C}^2$. By making use of Nevanlinna theoretic methods and the classification of compact complex surfaces K. Kodaira proved in 1971 ([**2**]) that S is a rational surface. Here we deal with a more general meromorphic map $f : \boldsymbol{C}^n \to X$ into a compact complex manifold X of dimension n, whose differential df has generically rank n. Let ρ_f denote the order of f. We will prove that if $\rho_f < 2$, then every global symmetric holomorphic tensor must vanish; in particular, *if* $\dim X = 2$ *and* X *is kähler, then* X *is a rational surface. Without the kähler condition there is no such conclusion, as we will show by a counter-example using a Hopf surface.* This may be the first instance that the kähler or non-kähler condition makes a difference in the value distribution theory.

1. Introduction and main results.

Let X be a compact hermitian manifold with metric form ω. Let $f : \boldsymbol{C}^n \to X$ be a meromorphic map (cf. [**4**] for this section in general). If the differential df is generically of maximal rank, f is said to be *differentiably non-degenerate*. We set

$$\alpha = dd^c \|z\|^2 \tag{1.1}$$

for $z = (z_j) \in \boldsymbol{C}^n$, where $d^c = (i/4\pi)(\bar{\partial} - \partial)$ and $\|z\|^2 = \sum_{j=1}^n |z_j|^2$. We use the notation:

$$B(r) = \{z \in \boldsymbol{C}^n : \|z\| < r\}, \qquad S(r) = \{z \in \boldsymbol{C}^n : \|z\| = r\} \quad (r > 0).$$

We define the *order function* of f with respect to ω by

$$T_f(r; \omega) = \int_1^r \frac{dt}{t^{2n-1}} \int_{B(t)} f^*\omega \wedge \alpha^{n-1}. \tag{1.2}$$

2010 *Mathematics Subject Classification.* Primary 32H30; Secondary 14M20.

Key Words and Phrases. meromorphic map, order, rationality.

This research was supported in part by Grant-in-Aid for Scientific Research (B) 23340029 and SFB/TR 12 (DFG).

Then the (upper) order is defined by

$$\rho_f = \overline{\lim_{r\to\infty}} \frac{\log T_f(r;\omega)}{\log r}.$$

It is easy to see that ρ_f is independent of the choice of the metric (form ω) on X.

Example 1.3. (i) If $X = \boldsymbol{P}^n(\boldsymbol{C})$ and f is rational, then $\rho_f = 0$.

(ii) Let X be a compact torus. If $f : \boldsymbol{C}^n \to X$ is non-constant, then $\rho_f \geq 2$. If $\lambda : \boldsymbol{C}^n \to X$ $(\dim X = n)$ is the universal covering map, then $\rho_\lambda = 2$.

A compact complex manifold which is bimeromorphic to $\boldsymbol{P}^n(\boldsymbol{C})$ is called a *rational variety.* A two-dimensional compact complex manifold is called a complex *surface.* If it admits a kähler metric, it is called a kähler surface.

The main result of this paper is the following:

Main Theorem 1.4. *Let X be a kähler surface. Assume that there is a differentiably non-degenerate meromorphic map $f : \boldsymbol{C}^2 \to X$. If $\rho_f < 2$, then X is rational.*

The kähler condition is necessary by the following:

Theorem 1.5. *There is a Hopf surface S for which there is a differentiably non-degenerate holomorphic map $f : \boldsymbol{C}^2 \to S$ with $\rho_f = 1$.*

Let Ω_X^k denote the sheaf of germs of holomorphic k-forms over a complex manifold X. We denote by $S^l\Omega_X^k$ its l-th symmetric tensor power. In particular, $K_X = \Omega_X^n$ $(n = \dim X)$ denotes the canonical bundle over X.

The key tool for the proof of the Main Theorem 1.4 is:

Theorem 1.6. *Let X be an n-dimensional compact complex manifold. Assume that there exists a differentiably non-degenerate meromorphic map $f : \boldsymbol{C}^m \to X$ $(m \geq n)$ with $\rho_f < 2$. Then for arbitrary $l_k \geq 0$ with $\sum_{k=1}^n l_k > 0$*

$$H^0\big(X, S^{l_1}\Omega_X^1 \otimes \cdots \otimes S^{l_n}\Omega_X^n\big) = \{0\}.$$

Remark 1.7. So far by our knowledge, the above theorems are the first instance that the kähler or non-kähler condition makes a difference in the value distribution theory.

2. Proof of the Main Theorem.

2.1. Proof of Theorem 1.6.

Assume the existence of an element

$$\tau \in H^0\big(X, S^{l_1}\Omega_X^1 \otimes \cdots \otimes S^{l_n}\Omega_X^n\big) \setminus \{0\}.$$

We take a hermitian metric h on X with the associated form ω. There are induced hermitian metrics on the symmetric powers of the bundles Ω_X^k and their tensor products which by abuse of notation are again denoted by h. Let $\|\tau\|_h$ denote the norm of τ with respect to h. Then there is a constant $c_1 > 0$ such that

$$\|\tau\|_h \leq c_1. \tag{2.1}$$

We denote by ξ_λ the coefficient functions of $f^*\tau$ with respect to the standard coordinate system $(z_1, \ldots, z_m)$ on $\boldsymbol{C}^m$. Since f is meromorphic, $f^*\tau$ is obviously holomorphic outside the indeterminacy set I_f. Because $\operatorname{codim}(I_f) \geq 2$ and because $f^*\tau$ is a section in a globally defined vector bundle, it extends holomorphically to I_f. Thus we may regard $f^*\tau$ as being holomorphic on $\boldsymbol{C}^n$ and the coefficient functions ξ_Λ with respect to the flat frames generated by $dz_1, \ldots, dz_n$ are holomorphic as well. We set

$$\|f^*\tau\|^2_{\boldsymbol{C}^m} = \sum_\Lambda |\xi_\Lambda|^2 \not\equiv 0. \tag{2.2}$$

We define a function ζ on $\boldsymbol{C}^m$ by

$$f^*\omega \wedge \alpha^{m-1} = \zeta\alpha^m.$$

Since f is differentiably non-degenerate, $f^*\tau \not\equiv 0$. By (2.1) there are positive constants c_2 and c_3 such that

$$\zeta \geq c_2 \|f^*\tau\|^{2c_3}_{\boldsymbol{C}^m}. \tag{2.3}$$

By (2.2) $\|f^*\tau\|^{2c_3}_{\boldsymbol{C}^m}$ is plurisubharmonic. Since $f^*\tau \not\equiv 0$ is holomorphic, it follows that

$$\int_{S(1)} \|f^*\tau\|^{2c_3}_{\boldsymbol{C}^m}\gamma = c_4 > 0,$$

where

$$\gamma = \frac{1}{r^{2m-1}}\, d^c\|z\|^2 \wedge \alpha^{m-1}, \tag{2.4}$$

induced on $S(r)$ with $r=1$. Since the plurisubharmonicity of $\|f^*\tau\|_{\boldsymbol{C}^m}^{2c_3}$ implies $dd^c\|f^*\tau\|_{\boldsymbol{C}^m}^{2c_3} \geq 0$ as currents,

$$\int_{S(r)} \|f^*\tau\|_{\boldsymbol{C}^m}^{2c_3}\gamma - \int_{S(s)} \|f^*\tau\|_{\boldsymbol{C}^m}^{2c_3}\gamma = \int_s^r \frac{dt}{t^{2m-1}} \int_{B(t)} dd^c\|f^*\tau\|_{\boldsymbol{C}^m}^{2c_3} \wedge \alpha^{m-1} \geq 0$$

for $r > s > 0$ (cf. [**4**]). Thus,

$$\int_{S(r)} \|f^*\tau\|_{\boldsymbol{C}^m}^{2c_3}\gamma$$

is monotone increasing in $r > 0$. Then,

$$\frac{1}{r^{2m-1}} \int_{S(r)} \|f^*\tau\|_{\boldsymbol{C}^m}^{2c_3}\, d^c\|z\|^2 \wedge \alpha^{m-1} = \int_{S(r)} \|f^*\tau\|_{\boldsymbol{C}^m}^{2c_3}\gamma \geq \int_{S(1)} \|f^*\tau\|_{\boldsymbol{C}^m}^{2c_3}\gamma = c_4$$

for $r > 1$, so that

$$\int_{S(r)} \|f^*\tau\|_{\boldsymbol{C}^m}^{2c_3} d^c\|z\|^2 \wedge \alpha^{m-1} \geq c_4 r^{2m-1}, \quad r > 1.$$

Therefore

$$\int_{B(r)} \|f^*\tau\|_{\boldsymbol{C}^m}^{2c_3}\alpha^m \geq \int_1^r c_4 t^{2m-1} dt = \frac{c_4}{2m}(r^{2m}-1), \quad r > 1.$$

We deduce from this that

$$\begin{aligned} T_f(r,\omega) &= \int_1^r \frac{dt}{t^{2m-1}} \int_{B(t)} \zeta\alpha^m \geq c_2 \int_1^r \frac{dt}{t^{2m-1}} \int_{B(t)} \|f^*\tau\|_{\boldsymbol{C}^m}^{2c_3}\alpha^m \\ &\geq \frac{c_2 c_4}{2m} \int_1^r \left(t - \frac{1}{t^{2m-1}}\right) dt = \frac{c_2 c_4}{4m} r^2 + C_m(r), \end{aligned}$$

where $C_1(r) = O(\log r)$ and $C_m(r) = O(1)$ for $m \geq 2$. Thus,

$$\rho_f = \overline{\lim_{r\to\infty}} \frac{\log T_f(r,\omega)}{\log r} \geq 2.$$

This is a contradiction. □

Corollary 2.5. *If X in Theorem 1.6 is 1-dimensional, then X is biholomorphic to $\boldsymbol{P}^1(\boldsymbol{C})$.*

2.2. Proof of the Main Theorem 1.4.

There is a fine classification theory of complex surfaces (cf. Kodaira [**2**], Barth-Peters-Van de Ven [**1**]). According to it, we know the following fact, where $b_1(X) = \dim H_1(X, \boldsymbol{R})$ denotes the first Betti number of X.

Theorem 2.6 (Kodaira [**68**, Theorem 54]). *If a complex surface X satisfies $b_1(X) = 0$ and $H^0(X, K_X^l) = \{0\}$ for all $l > 0$, then X is rational.*

This enables us to prove Theorem 1.4 as follows. By Theorem 1.6 $\dim H^0(X, \Omega_X^1) = 0$. Due to the kähler assumption, we have $b_1(X) = 2\dim H^0(X, \Omega_X^1) = 0$. Moreover, $H^0(X, K_X^l) = \{0\}$ for all $l > 0$ again by Theorem 1.6. It follows from Theorem 2.6 that X is rational. □

3. Proof of Theorem 1.5.

Let $\lambda \in \boldsymbol{C}$ with $|\lambda| > 1$. Then a Hopf surface S is defined as the quotient of $\boldsymbol{C}^2 \setminus \{(0,0)\}$ under the $\boldsymbol{Z}$-action given by $n : (x, y) \mapsto (\lambda^n x, \lambda^n y)$. Such a surface S is known to be diffeomorphic to $S^1 \times S^3$. As a consequence $b_1(S) = 1$ and S is *not kähler*.

Now

$$\omega = \frac{i}{2\pi} \cdot \frac{dx \wedge d\bar{x} + dy \wedge d\bar{y}}{|x|^2 + |y|^2} = \frac{dd^c \|(x,y)\|^2}{\|(x,y)\|^2}$$

is a positive $(1,1)$-form on $\boldsymbol{C}^2 \setminus \{(0,0)\}$ which is invariant under the above given $\boldsymbol{Z}$-action. Therefore it induces a positive $(1,1)$-form on the quotient surface S which by abuse of notation is again denoted by ω.

Let α and γ be as in (1.1) and (2.4), respectively. We claim that the holomorphic map $f : \boldsymbol{C}^2 \to S$ induced by

$$(z, w) \mapsto (z, 1 + zw)$$

is of order 1. By definition this means

$$\rho_f = \overline{\lim_{r \to \infty}} \frac{\log T_f(r, \omega)}{\log r} = 1,$$

i.e.,

$$\overline{\lim_{r\to\infty}} \frac{1}{\log r} \log \int_1^r \frac{dt}{t^3} \int_{B(t)} f^*\omega \wedge \alpha = 1.$$

Note that

$$f^*\omega \wedge \alpha = \frac{1+|z|^2+|w|^2}{2(|z|^2+|1+zw|^2)}\alpha^2.$$

We define

$$I_r = \int_{S(r)} \frac{r^2}{|z|^2+|1+zw|^2} dV, \qquad r = \|(z,w)\|.$$

Here dV is the euclidean volume element on $S(r)$, and therefore a constant multiple of $r^3\gamma$. It is sufficient to show

$$I_r = O(r^{2+\varepsilon}), \quad \forall\varepsilon > 0, \quad \text{and} \quad r^2 = O(I_r). \tag{3.1}$$

Indeed, assume that this holds. Because of $\lim_{r\to\infty}(1+r^2)/r^2 = 1$, (3.1) is equivalent to the assertion

$$I'_r = O(r^{2+\varepsilon}), \quad \text{and} \quad r^2 = O(I'_r).$$

with

$$I'_r = \int_{S(r)} \frac{1+r^2}{|z|^2+|1+zw|^2} dV.$$

From this we first obtain

$$\int_{B(r)} \frac{1+r^2}{|z|^2+|1+zw|^2}\alpha^2 = O\left(\int^r I'_r dr\right) = O(r^{3+\varepsilon}), \quad \forall\varepsilon > 0,$$

implying

$$T_f(r) = \frac{1}{2}\int_1^r \frac{dt}{t^3} \int_{B(r)} \frac{1+r^2}{|z|^2+|1+zw|^2}\alpha^2 = O(r^{1+\varepsilon}), \quad \forall\varepsilon > 0,$$

and

$$\rho_f = \overline{\lim_{r \to infty}} \frac{\log T_f(r)}{\log r} \leq 1.$$

In the same way from the second estimate of (3.1) we get the opposite estimate $\rho_f \geq 1$, and therefore $\rho_f = 1$. Hence it suffices to show (3.1).

We define

$$\eta = \frac{r^2}{|z|^2 + |1 + zw|^2}.$$

Thus we have to show

$$I_r = \int_{S(r)} \eta dV = O(r^{2+\varepsilon}).$$

We set

$$\eta = \frac{r^2}{\phi(z, w)}, \qquad \phi(z, w) = |z|^2 + |1 + zw|^2.$$

3.1. Geometric estimates.

For $(z, w) \in S(r)$ let $\theta \in [0, 2\pi)$ such that $e^{i\theta}|zw| = zw$. Let $K > 0$, $-\infty < \lambda < 1$ and $\mu \geq 0$. We set

$$\Omega_{K,\lambda,\mu} = \left\{(z, w) \in S(r) : |z| \leq Kr^\lambda,\ |\sin\theta| \leq r^{-\mu}\right\}.$$

We need some volume estimates.

First we note that $(\sin\theta)/\theta \geq 2/\pi$ for all $\theta \in [0, \pi/2]$, because sin is concave on $[0, \pi/2]$. It follows that for every $C \in]0, 1]$ we have the following bound for the Lebesgue measure:

$$\mathrm{vol}\left(\{\theta \in [0, 2\pi] : |\sin\theta| \leq C\}\right) \ \leq\ 4(C\pi/2) = 2C\pi. \tag{3.2}$$

Second we define a map $\zeta : \boldsymbol{C}^2 \to \boldsymbol{C} \times \boldsymbol{R}^2$ as follows:

$$\zeta : (z, w) \mapsto (z, r\arg(zw), r),$$

where $r = \|(z, w)\| = \sqrt{|z|^2 + |w|^2}$.

To compute the Jacobian J of this map we set the coordinates so that $z = x + iy$, $w = u + iv$, and write

$$r = \sqrt{x^2 + y^2 + u^2 + v^2},$$
$$\zeta : (x, y, u, v) \mapsto (x, y, r(\arg z + \arg w), r) \in \boldsymbol{R}^4.$$

Then the Jacobian J takes the following form:

$$J = \begin{vmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & \frac{u}{r}(\arg z + \arg w) + r\frac{\partial}{\partial u}\arg w & \frac{u}{r} \\ 0 & 0 & \frac{v}{r}(\arg z + \arg w) + r\frac{\partial}{\partial v}\arg w & \frac{v}{r} \end{vmatrix}$$
$$= \begin{vmatrix} r\frac{\partial}{\partial u}\arg w & \frac{u}{r} \\ r\frac{\partial}{\partial v}\arg w & \frac{v}{r} \end{vmatrix}.$$

An easy practice of computation implies that "$J \equiv -1$".

Furthermore the gradient $\mathrm{grad}(r)$ is of length one and normal on the level set $S(r)$. Correspondingly the map ζ is volume preserving and $S(r)$ has the same volume as its image

$$\zeta(S(r)) = \{z \in \boldsymbol{C} : |z| \leq r\} \times [0, 2\pi r) \times \{r\}, \tag{3.3}$$

namely $2\pi^2 r^3$.

Similarly the euclidean volume of $\Omega_{K,\lambda,\mu}$ agrees with the euclidean volume of

$$\zeta(\Omega_{K,\lambda,\mu}) = \{z \in \boldsymbol{C} : |z| \leq Kr^\lambda\} \times \{\theta r : \theta \in [0, 2\pi), |\sin\theta| \leq r^{-\mu}\} \times \{r\}.$$

Using (3.2) it follows that for $r \geq 1$ the volume of $\Omega_{K,\lambda,\mu}$ is bounded by

$$\pi(Kr^\lambda)^2 \cdot 2r^{-\mu}\pi r = 2K^2\pi^2 r^{2\lambda+1-\mu}.$$

In particular,

$$\mathrm{vol}(\Omega_{K,\lambda,\mu}) = O(r^{2\lambda+1-\mu}). \tag{3.4}$$

3.2. Arithmetic estimates.

Besides the Landau O-symbols, we also use the notation "$\gtrsim$": If f, g are functions of a real parameter r, then $f(r) \gtrsim g(r)$ indicates that

$$\liminf_{r\to+\infty} \frac{f(r)}{g(r)} \geq 1.$$

Similarly $f \sim g$ indicates

$$\lim_{r\to+\infty} \frac{f(r)}{g(r)} = 1.$$

In the sequel, we will work with domains $\Omega \subset S(r)$ (i.e. for each $r > 0$ some subset $\Omega = \Omega_r \subset S(r)$ is chosen). In this context, given functions f, g on $\boldsymbol{C}^2$ we say "$f(z,w) \gtrsim g(z,w)$ holds on Ω" if for every sequence $(z_n, w_n) \in \boldsymbol{C}^2$ with $\lim \|(z_n, w_n)\| = +\infty$ and $(z_n, w_n) \in \Omega_r$ $(r = \|(z_n, w_n)\|)$ we have

$$\liminf_{n\to\infty} \frac{f(z_n, w_n)}{g(z_n, w_n)} \geq 1.$$

We develop some estimates for $\phi(z,w) = |z|^2 + |1+zw|^2$. Fix $\mu > 0$, $-\infty < \lambda < 1$.

(i) For all z, w: $\phi \geq |z|^2$.

(ii) If $(z,w) \in S(r)$ and $|z| \leq 1/2r$, then

$$|w| \leq r \implies |zw| \leq \frac{1}{2} \implies |1+zw| \geq \frac{1}{2}$$

and therefore $\phi \geq 1/4$.

(iii) For $|z| \leq r^\lambda$ we have $|w| \sim r$, i.e., for fixed λ, μ and any choice of $(z_r, w_r) \in S(r)$ with $|z_r| \leq r^\lambda$ we have $\lim_{r\to\infty} |w_r|/r = 1$.

(iv) For $|z| \geq 3/2r$ and $|z| \leq r^\lambda$ we have that $\phi \gtrsim (1/9)|zw|^2$, because $|w| \sim r$ and $|zw| \gtrsim 3/2$ (equivalently, $1 \lesssim (2/3)|zw|$), implying $|1+zw| \geq |zw| - 1 \gtrsim (1/3)|zw|$.

(v) For all z, w, $\phi \geq |\Im(1+zw)|^2 = (|zw|\sin\theta)^2$.

3.3. Putting things together.

We are going to prove first the claim

$$\text{“}I(r) = O(r^{2+\varepsilon}), \quad \forall \varepsilon > 0\text{”}$$

by dividing $S(r)$ into regions A, B, C, D_{-2}, D_{-1}, D_0, D_1, E, F, each of which is investigated separately.

- Region A consists of those points with $|z| \le 1/2r$, i.e., $A = \Omega_{1/2,-1,0}$. The volume $\mathrm{vol}(A)$ is thus of order $O(r^{-1})$. Due to (ii) the integrand η is bounded by $\eta|_A = O(r^2)$. It follows that

$$\int_A \eta\, dV \le \mathrm{vol}(A) \cdot \sup_{(z,w)\in A} \eta(z,w) = O(r).$$

  Hence the contribution of A to the integral $I_r = \int_{S(r)} \eta\, dV$ is bounded by $O(r)$.
- Region B consists of those points with $1/2r \le |z| \le 3/2r$ and $|\sin\theta| < 1/r$. Thus $B \subset \Omega_{3/2,-1,1}$. Due to (3.4) this implies $\mathrm{vol}(B) = O(r^{-2})$. For the integrand $\eta|_B$ we have the bound $\eta|_B = O(r^4)$ (using (i) and $|z| \ge 1/2r$). Hence

$$\int_B \eta\, dV \le \mathrm{vol}(B) \cdot \sup_{(z,w)\in B} \eta(z,w) = O(r^2);$$

  i.e., the contribution of B to the integral I_r is bounded by $O(r^2)$.
- Region C consists of those points with $1/2r \le |z| \le 3/2r$ and $|\sin\theta| > 1/r$. Since $|w| \sim r$, $1/2 \lesssim |zw| \lesssim 3/2$ We take the volume-compatible parameter $\psi = r\theta$ due to (3.3). Then $1/r < |\sin\psi/r| < \psi/r$, and so $\psi > 1$. Therefore

$$\begin{aligned} J_r &:= \int_{1<\psi<2\pi r,\, |\sin\psi/r|>1/r} \eta\, d\psi \\ &= \int_{1<\psi<2\pi r,\, |\sin\psi/r|>1/r} \frac{2r^2}{(\sin\psi/r)^2} d\psi = O(r^4). \end{aligned}$$

  Here in fact we have that there is a constant $c > 1$ such that

$$\frac{r^4}{c} \le J_r \le cr^4.$$

  Therefore it follows that

$$\frac{r^2}{c'} \le \int_C \eta\, dV = \int_{1/2r\le|z|\le3/2r} J_r \frac{i}{2} dz \wedge d\bar{z} \le c' r^2, \tag{3.5}$$

where c' is a positive constant. Thus the contribution of C to the integral I_r is bounded by $O(r^2)$.

- For $\gamma \in \{-2,-1,0,1\}$ let D_γ denote the set of those points where $|z| \geq 3/2r$, $|z| \leq r^{1-\varepsilon}$ and $r^{\gamma/2} \leq |z| \leq r^{(\gamma+1)/2}$. For each γ the integrand η is bounded on D_γ by $O(r^{-\gamma})$ (due to (iv)), and the volume $\mathrm{vol}(D_\gamma)$ is bounded by $O(r^{2+\gamma})$, because $D_\gamma \subset \Omega_{1,(\gamma+1)/2,0}$. Thus the contribution of D_γ to the integral I_r is bounded by $O(r^2)$.
- Let E denote the region where $|z| \geq r^{1-\varepsilon}$, $|w| \geq r^{1/2}$. For the integrand we have that $\eta|_E = O(r^{2\varepsilon-1})$ (using (iv)). The volume of E is bounded by the total volume of $S(r)$, i.e., $\mathrm{vol}(E) = O(r^3)$. Together this shows that the contribution of E to I_r is bounded by $O(r^{2+2\varepsilon})$.
- Let F denote the region where $|w| \leq r^{1/2}$. In analogy to (iii) we have $|z| \sim r$. With (i) it follows that $\sup_{(z,w)\in F} \eta(z,w) = O(1)$. On the other hand the volume of F agrees with the volume of $\{(z,w) \in S(r) : |z| \leq r^{1/2}\}$ which according to (3.4) is bounded by $O(r^2)$. Together this yields that the contribution of F to I_r is bounded by $O(r^2)$.

Thus we have a collection of nine regions (A, B, C, D_{-2}, D_{-1}, D_0, D_1, E, F) covering the sphere $S(r)$. For each such region Ω we have verified

$$\int_\Omega \eta \, dV = O(r^{2+\varepsilon}), \quad \varepsilon > 0.$$

This establishes our claim

$$I_r = O(r^{2+\varepsilon}), \quad \varepsilon > 0.$$

Furthermore, it follows from (3.5) that

$$r^2 = O(I_r).$$

As a consequence, the holomorphic map $f : \boldsymbol{C}^2 \to S$ induced by $f : (z,w) \mapsto (z, 1+zw)$ is of order $\rho_f = 1$. $\square$

4. Problems.

Because of the results presented above it may be interesting to recall some problems (conjectures) from [**3**, Section 1.4]. An n-dimensional compact complex manifold X is said to be *unirational* if there is a surjective meromorphic map $\phi : \boldsymbol{P}^n(\boldsymbol{C}) \to X$; in this case, if $g : \boldsymbol{C}^n \to \boldsymbol{P}^n(\boldsymbol{C})$ is a differentiably non-degenerate meromorphic map with order $\rho_g < 2$, then $\phi \circ g : \boldsymbol{C}^n \to X$ is differentiably non-degenerate and has order less than two. Therefore, the rationality and the unirationality of X cannot be distinguished by the existence of a differentiably non-degenerate meromorphic map $f : \boldsymbol{C}^n \to X$ with $\rho_f < 2$.

PROBLEM 4.1. *Let X be a compact kähler manifold of dimension n. If there is a differentiably non-degenerate meromorphic map $f : \boldsymbol{C}^n \to X$ with order $\rho_f < 2$, is X unirational?*

At least this is true for $\dim X \leq 2$ by Corollary 2.5 and the Main Theorem 1.4.

PROBLEM 4.2. *Let $f : \boldsymbol{C} \to X$ be a non-constant entire curve into a projective (or kähler) manifold X. If $\rho_f < 2$, then does X contain a rational curve?*

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