J. Math. Soc. Japan
Vol. 64, No. 3 (2012) pp. 927–939
doi: 10.2969/jmsj/06430927

On representations of real Nash groups

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(Received Nov. 30, 2010)
(Revised Jan. 31, 2011)

Abstract. Some basic results on compact affine Nash groups related to their Nash representations are given. So, first a Nash version of the Peter-Weil theorem is proved and then several more results are given: for example, it is proved that an analytic representation of such a group is of class Nash and that the category of the classes of isomorphic embedded compact Nash groups is isomorphic with that of the classes of isomorphic embedded algebraic groups. Moreover, given a compact affine Nash group G, a closed subgroup H and a homogeneous Nash G-manifold X, it is proved that the twisted product $G \times_H X$ is a Nash G-manifold which is Nash G-diffeomorphic to an algebraic G-variety; besides, this algebraic structure is unique.

Introduction.

This paper is dedicated to the study of some basic properties of Nash groups related to their Nash representations (all Nash objects are assumed to be real and analytic). The starting point of our results is a Nash version, that we give in Theorem 1.3, of the Peter-Weil theorem on representative functions on Lie groups. It is well known that this famous theorem states that the representative functions on a compact Lie group H are dense in the space of all continuous functions on H. Well, we prove the following fact: the representative Nash functions on a compact affine Nash group G are dense in the space of all continuous functions on G. As a consequence of this theorem we deduce the existence of a faithful representation of G (Corollary 1.4) and, in turn, this fact allows, on one hand, to obtain that an analytic representation of G is of class Nash (Corollary 1.7); on the other hand, it suggests to compare the category N of compact affine Nash groups with that, A, of compact affine algebraic groups. Now, if one considers the objects of N and of A up to isomorphisms (that is, if one considers the abstract structures), one gets two categories that we prove to be isomorphic (Theorem 1.8).

In Section 2 we deal mainly with the equivariant Nash conjecture. We remember that this conjecture, still open, says that, if G is a compact Lie group, a compact C^∞ G-manifold is C^∞ G- diffeomorphic to a real non singular alge-

2010 *Mathematics Subject Classification.* Primary 20G05; Secondary 57S15, 14P20.
Key Words and Phrases. representation theory, equivariant Nash conjecture.

braic G-variety, where an algebraic G-variety is an invariant algebraic variety in a representation space of G. The main difficulty to obtain algebraic structures on C^∞ G-manifolds lies in a cobordism process. Nevertheless, some special cases are known in which the conjecture is verified. See for example [**4**], [**5**], [**6**]. In this paper we consider the conjecture in Nash setting: given a compact affine Nash group G, a closed subgroup H of G and a homogeneous Nash G-manifold X, first of all we prove that the twisted product $G \times_H X$ is a Nash G-manifold; then we prove that both X and the twisted product have an algebraic G-structure. This means that there exist Nash representations of G and non singular algebraic G-varieties in the spaces of the representations which are Nash G-diffeomorphic to X and to $G \times_H X$, respectively; and more, such structures are unique (Theorem 2.5, Theorem 2.7).

Finally, in Section 3 we recall a question posed by M. Shiota in [**16**]: roughly speaking, he asks whether a Nash map between affine Nash manifolds is, up to Nash diffeomorphisms between the previous Nash manifolds and non singular algebraic varieties, an algebraic map. Now, the existence of an algebraic structure on a homogeneous Nash G-manifold and on a compact affine Nash group, as follows from Theorem 2.5 and Theorem 2.7, permits us to give a positive answer in these cases (Proposition 3.1).

1. Nash representations.

We shall consider only Nash functions and Nash manifolds which are real and analytic. First, let us briefly recall some definitions and facts we shall use later (see e.g. [**1**], [**15**], [**16**]).

An analytic function $f : U \to \boldsymbol{R}$ on an open semialgebraic subset of $\boldsymbol{R}^n$ is called a Nash function if there exists a non-zero polynomial $P(t_1, \dots, t_{n+1})$ with the property that $P(x_1, \dots, x_n, f(x)) = 0$ for every $(x_1, \dots, x_n) \in U$. A map $f : U \to \boldsymbol{R}^m$ is said to be a Nash map if every component is a Nash function. In this paper a Nash manifold is an analytic manifold with coordinate changes given by Nash maps. We remark that often in the literature such a manifold is said to be a locally Nash manifold [**16**]. If there exists a Nash embedding of a Nash manifold into $\boldsymbol{R}^n$ for some n we say that the manifold is affine. We remember that exists a non affine Nash manifold. A Nash submanifold Y of a Nash manifold X is a topological subspace of X such that for any $y \in Y$ there exist an open neighbourhood U_y of y and local coordinates with the property that $Y \cap U_y = \{x \in U_y; f_1(x) = \cdots f_p(x) = 0\}$, where $f_1, \dots, f_p$ are Nash functions and $\operatorname{rk} \partial(f_1, \dots, f_p)/\partial(x_1, \dots, x_n) = p$ at y.

A (coordinate) Nash bundle $\mathscr{B} = (B, p, X, F, G, \{g_{ij} : U_i \cap U_j \to G\})$ is a bundle where the total and base spaces B and X, and the fibre F are Nash manifolds, the projection $p : B \to X$ is a Nash map, the structure group is a Nash group and the coordinate transformations are Nash maps.

If a Nash manifold is endowed with a group structure and if the group operations are of class Nash, the manifold is said to be a Nash group; it is an affine Nash group if the manifold is affine.

A subgroup H of a Nash group G is said to be a Nash subgroup of G if it is a submanifold of G; of course H is a Nash group and it is closed. Examples of Nash groups are given by the real algebraic groups and by the identity component of a real algebraic group. We remark that the connected one dimensional Nash groups has been classified by J. J. Madden and C. M. Stanton [**11**]. One can define as usual the notion of Nash action of a Nash group. In particular, we have:

Definition 1.1. Let G be a Nash group.

1) A (linear) Nash representation of G is a Nash homomorphism $G \to GL(n)$; this means a homomorphism of groups which is a Nash map;
2) if X is a Nash manifold with G action and if the action $G \times X \to X$ is a Nash map, X is called a Nash G-manifold;
3) an equivariant Nash map between G-manifolds is called a Nash G-map;
4) let $p_1, \dots, p_k : \boldsymbol{R}^n \to \boldsymbol{R}$ be polynomials. The set of common zeros of these polynomials is a (real) algebraic variety. An affine Nash G-submanifold (resp. an algebraic G-variety) is a Nash submanifold (resp. an algebraic variety) in a representation space of G which is G-invariant;
5) a Nash manifold with G action is said to have an affine Nash G-structure (resp. an algebraic G-structure) if it is Nash G-diffeomorphic to an affine Nash G-submanifold (resp. to a non singular algebraic G-variety).

From now on in this section we shall consider only compact affine Nash groups and G stands for such a group. We are interested in representative Nash functions on G. On this subject, we recall that left translations in G induce an action of G on the space of real valued C^∞ functions $f : G \to \boldsymbol{R}$ as follows: $f \mapsto L(g, f)$, $L(g, f)(x) = f(g^{-1}x)$. The function f is said to be representative if the functions $L(g, f)$ generate a finite dimensional subspace of the space of all real valued C^∞ functions on G. It is well known the Peter-Weil theorem about representative functions on a compact Lie group, which asserts the density of these functions in the space of all continuous functions on such a group. Our first aim is to give a Nash version of this theorem, considering representative Nash functions. Before we need the following results, due to T. Kawakami:

Theorem 1.2.

1) *Let X be a compact C^∞ G-manifold. Then there exists an affine Nash G-*

submanifold Y and a C^∞ G-diffeomorphism $X \to Y$.

2) *A C^∞ G-diffeomorphism between homogeneous Nash G-manifolds is a Nash map.*

Proof. See [**9**]. $\square$

The result on the representative Nash functions we are talking about is the following

Theorem 1.3. *The representative Nash functions on G are dense in the strong topology in the space of all continuous functions on G.*

Proof. Consider the Nash G-manifold underlying G and call it again G. By Theorem 1.2 there exist a linear representation of G, with $\boldsymbol{R}^m$ as representation space, a Nash G-submanifold $Y \subset \boldsymbol{R}^m$ and a C^∞ G-diffeomorphism $G \to Y$ between homogeneous Nash G-manifolds. Always by Theorem 1.2 this diffeomorphism is a Nash map. So we have the Nash G-map $f : G \to \boldsymbol{R}^m$ between Nash G-manifolds which is a diffeomorphism onto the compact Nash manifold $f(G)$. Now let $p : \boldsymbol{R}^m \to \boldsymbol{R}$ be a polynomial function. Because f is equivariant and G acts linearly on $\boldsymbol{R}^m$, the Nash function $p \circ f : G \to \boldsymbol{R}$ is representative on G [**12**, p. 107]. Then, let $r : G \to \boldsymbol{R}$ be a continuous function and consider the function $r \circ f^{-1} : f(G) \to \boldsymbol{R}$; we can approximate this function by a polynomial function $q : \boldsymbol{R}^m \to \boldsymbol{R}$. The function $q \circ f : G \to \boldsymbol{R}$ is representative on G, of class Nash and approximates r. $\square$

Corollary 1.4.

1) *There exists a faithful Nash representation $G \to GL(n)$ of G.*
2) *G is Nash isomorphic to an affine algebraic group and this algebraic structure is unique.*

Proof.

1) Repeat the proof of the analogous result in Lie case, using Theorem 1.3 instead of Peter-Weil theorem. See e.g. [**3**, Theorem 4.1, p. 136].
2) By 1) G is Nash isomorphic to a compact subgroup H of $GL(n)$, for some n, and H is algebraic [**13**, Theorem 5, p. 133]; moreover, two compact algebraic real groups which are C^∞-diffeomorphic are polynomially isomorphic [**13**, Corollary, p. 246]. $\square$

We know that in the category of Lie groups a closed subgroup of a given Lie group is a Lie group itself. The same fact occurs for G, as the next Corollary shows.

Corollary 1.5. *A closed subgroup of G is a Nash group.*

Proof. Let K be such a subgroup. By Corollary 1.4 there exists a Nash isomorphism $f : G \to H$, where H is an algebraic linear group, $H \subset GL(n)$ for a suitable n. The image $f(K)$ of K is a compact subgroup of $GL(n)$ and hence it is algebraic in $GL(n)$; so, $f(K)$ is an algebraic subgroup of H and then it is a Nash submanifold of H. It follows that K is a Nash submanifold of G. □

Another consequence of Corollary 1.4 deals with the homomorphisms between compact affine Nash groups. We have

Theorem 1.6. *An analytic homomorphism between compact affine Nash groups is of class Nash.*

Proof. Let $f : G \to H$ be such a homomorphism. By Corollary 1.4 there exist Nash isomorphisms $p : G \to G_1$, $q : H \to H_1$, where G_1 and H_1 are real algebraic compact groups. By [**13**, Theorem 11, p. 246] the homomorphism $q \circ f \circ p^{-1} : G_1 \to H_1$ is polynomial. It follows that f is of class Nash. □

Remark 1. A more general version of Theorem 1.6 is proved in [**10**], by a different technique and in the o-minimal structures setting.

Corollary 1.7.

1) *A compact affine Nash group is Nash isomorphic to a closed subgroup of an orthogonal group.*
2) *An analytic representation of a compact affine Nash group is a Nash map.*

Since by Corollary 1.4 we have found on G an algebraic affine structure which is unique, we want to compare the category N of compact affine Nash groups with that, A, of compact affine algebraic groups. In order to do this, let G, H be two objects of N and let $[G]$, $[H]$ be the classes of all compact affine Nash groups Nash isomorphic to G and H, respectively. Therefore, if $G_1 \in [G]$, $H_1 \in [H]$, there exist Nash isomorphisms $\varphi : G \to G_1$, $\psi : H \to H_1$. Now let $f : G \to H$ be a Nash homomorphism; it induces the Nash homomorphism $\psi \circ f \circ \varphi^{-1} : G_1 \to H_1$. Consider the class $[f]$ of all Nash homomorphisms induced by f in this way. We obtain a category, say $N/\sim$, whose objects are the classes of kind $[G]$ and the morphisms are the classes of kind $[f]$.

Similarly one constructs the category, $A/\sim$, whose objects and morphisms are, respectively, classes of compact affine algebraic groups and classes of polynomial homomorphisms, up to polynomial isomorphisms.

Recall now that two categories C_1 and C_2 are said to be isomorphic if there exists a covariant functor $F : C_1 \to C_2$ such that: 1. For any object Z of C_2, there exists a unique object X of C_1 such that $F(X) = Z$; 2. For any pair (X, Y) of objects of C_1, the map $F(X, Y) : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(F(X), F(Y))$ which associates to each morphism $u : X \to Y$ the morphism $F(u) : F(X) \to (Y)$ is a bijection.

We have

Theorem 1.8. *The categories $N/\sim$ and $A/\sim$ are isomorphic.*

Proof. We construct a covariant functor $F : N/\sim \to A/\sim$ in the following way. Let $[G]$ be an object of $N/\sim$. By Corollary 1.4 there exists an algebraic compact affine group L which is Nash isomorphic to G and the algebraic structure is unique; then we set $F([G]) = [L]$. Because an algebraic group is a Nash group, in this way we obtain a bijection between the objects of $N/\sim$ and the objects of $A/\sim$. Moreover, let $f : G \to H$ be a morphism in N and $\alpha : G \to L$, $\beta : H \to M$ Nash isomorphisms between G, H and the algebraic compact affine groups L, M, respectively. The morphism f induces in A, by [**13**, Theorem 11, p. 246], the morphism $h = \beta \circ f \circ \alpha^{-1} : L \to M$. Then we set $F([f]) = [h]$. So we get a bijection $\mathrm{Hom}([G], [H]) \to \mathrm{Hom}(F[G], F[H])$ for any pair of objects of $N/\sim$. □

2. On equivariant Nash conjecture.

In this section we deal with the equivariant Nash conjecture in Nash setting; at the same time we consider the problem of the linearization of a Nash action. About this problem we give the following definition:

Definition 2.1. Let G be a Nash group and X a Nash G-manifold. We say that the action of G on X is Nash (resp. analytically) linearizable if there exist a Nash representation of G, with space $\boldsymbol{R}^n$, a Nash G-submanifold Y of $\boldsymbol{R}^n$, and a Nash (resp. an analytic) G-diffeomorphism from X onto Y.

If the group G is compact and affine and the manifold X is compact, its action is analytically linearizable, as the next Proposition shows.

Proposition 2.2. *Let G be a compact affine Nash group and X a compact Nash G-manifold. The G action is analytically linearizable.*

Proof. We have to show that there exist a Nash representation of G, with representation space $\boldsymbol{R}^n$, and an analytic G-embedding $f : X \to \boldsymbol{R}^n$ such that $f(X)$ is an affine Nash G-submanifold of $\boldsymbol{R}^n$. To do this, we use Theorem 1.2: there exist a representation space $\boldsymbol{R}^n$ of a Nash representation of G, an affine Nash G-submanifold Y of $\boldsymbol{R}^n$ and a C^∞ G-diffeomorphism $f : X \to Y$. Now we use the following result about the Whitney topology in the space of C^∞ G-

maps $X \to \boldsymbol{R}^n$: "Every Whitney neighborhood of a G-equivariant C^∞ map to a representation space contains a G-equivariant real analytic map" [**8**, Zusatz, p. 233]. Therefore we can approximate f by an analytic G-map $h : X \to \boldsymbol{R}^n$; then, by a G-tubular neighbourhood (see e.g. [**9**]) we get the desired embedding. $\square$

Now we recall the equivariant Nash conjecture.

Equivariant Nash Conjecture. "Let G be a compact Lie group. Every compact C^∞ G-manifold is C^∞ G-diffeomorphic to a non-singular algebraic G-variety".

Some partial results are known, see for example [**4**], [**5**] and [**6**]. However, the general case remains open.

When working in Nash category may be natural and of interest to modify slightly the conjecture supposing that all objects involved are of class Nash. That is:

Equivariant Nash Conjecture in Nash setting. "Let G be a compact affine Nash group. Every compact non-equivariantly affine Nash G-manifold is Nash G-diffeomorphic to a non-singular algebraic G-variety".

We will give a positive answer to this conjecture in some cases and, at the same time, we will find some G actions which are Nash linearizable. Before doing this, we need some preliminary results.

Theorem 2.3. *Let G be a Nash group and H a Nash subgroup of G. Then*

1) *$p : G \to G/H$, $g \mapsto gH$, is a principal Nash H-bundle.*
2) *A map f defined on G/H is a Nash map if and only if the map $f \circ p$ is.*
3) *The natural action of G on G/H, $(g_1, gH) \mapsto g_1 gH$, gives rise to a Nash map $G \times G/H \to G/H$.*

Proof. See [**7**]. $\square$

Proposition 2.4. *Let G be a Nash group and X a homogeneous Nash G-manifold. Let x be a point of X and G_x the isotropy subgroup of x. Then G_x is a Nash subgroup of G and for any $x \in X$ the map $G/G_x \to X$, $gG_x \mapsto gx$, is a Nash G-diffeomorphism.*

Proof. As in the Lie case (see e.g. [**13**, Theorem 4, p. 12]). $\square$

And now we can give a first positive answer to the equivariant Nash conjecture in Nash setting.

Theorem 2.5. *Let G be a compact affine Nash group and X a homogeneous Nash G-manifold. Then there exists a non-singular algebraic G-variety which is Nash G-diffeomorphic to X and two such algebraic G-varieties are algebraically G-isomorphic. Thus X is affine and the G action on X is Nash linearizable. In particular, G is Nash G-diffeomorphic to a non-singular algebraic G-variety.*

Proof. By Proposition 2.4 X is Nash G-diffeomorphic to G/H, for some Nash subgroup $H \subset G$. By [**2**, Theorem 0.5.2, p. 24] there exist an analytic representation $\rho : G \to O(n)$, for some n, and a point $b \in \boldsymbol{R}^n$ such that the isotropy subgroup of b is H:

$$H = G_b = \{g \in G; \rho(g)b = b\}.$$

Because of Corollary 1.7 the map ρ is of class Nash. Therefore let us consider the Nash map $F : G \to \boldsymbol{R}^n$, $g \mapsto \rho(g)b$, and the induced G-map $f : G/H \to \boldsymbol{R}^n$, $gH \mapsto \rho(g)b$. If $p : G \to G/H$ is the canonical projection, it is $f \circ p = F$ and then, by Theorem 2.3, f is of class Nash; moreover, remark that $f(G/H)$ is the orbit of b: it follows, using Proposition 2.4, that f is a Nash G-diffeomorphism; but the orbit $f(G/H)$ is a non singular algebraic G-variety of $\boldsymbol{R}^n$ by [**5**]. Thus X has an algebraic G-structure and so the G action on X is Nash linearizable. The uniqueness of this algebraic G-structure follows from [**5**]. □

The previous Theorem can be generalized. Before we need some preliminary facts. Let H be a closed subgroup of the compact affine Nash group G (by Corollary 1.5 H is a Nash subgroup of G) and X a Nash H-manifold. Recall that the twisted product $G \times_H X$ is the orbit space of the Nash action of H on $G \times X$ given by $(h, (g, x)) \mapsto (gh^{-1}, hx)$. The H-orbit of (g, x) will be denoted by $[g, x]$. We have

Lemma 2.6.

1) *The twisted product $G \times_H X$ is a Nash G-manifold.*
2) *The canonical map $q : G \times X \to G \times_H X$, $(g, x) \mapsto [g, x]$ is of class Nash.*
3) *Let Z be a Nash manifold. A map $f : G \times_H X \to Z$ is of class Nash if and only if $f \circ q$ is.*

Proof. It is known [**2**, p. 47] that the twisted product $G \times_H X$ is the total space of the fibre bundle over G/H with X as fibre and associated with the principal H-bundle $p : G \to G/H$, which is a Nash fibre bundle by Theorem 2.3. It follows that the bundle $\pi : G \times_H X \to G/H$, $\pi([g, x]) = p(g)$, is of class Nash and the total space is a Nash manifold. The action of G on $G \times_H X$ is given by $(g', [g, x]) \mapsto [g'g, x]$. Before to prove that this is a Nash action, consider the following commutative diagram of canonical projections:

$$\begin{array}{ccc} G \times X & \longrightarrow & G \\ {\scriptstyle q}\big\downarrow & & \big\downarrow \\ G \times_H X & \longrightarrow & G/H \end{array} .$$

Let $U \times H \to p^{-1}(U)$ be a chart of the Nash fibre bundle $p : G \to G/H$, $U \subset G/H$; therefore, locally, the previous diagram becomes [**2**, Lemma 2.5, p. 75]

$$\begin{array}{ccc} U \times H \times X & \longrightarrow & U \times H \\ {\scriptstyle q}\big\downarrow & & \big\downarrow \\ U \times X & \longrightarrow & U \end{array} , \qquad \begin{array}{ccc} (u,h,x) & \longmapsto & (u,h) \\ \big\downarrow & & \big\downarrow \\ (u,hx) & \longmapsto & u \end{array} .$$

It follows that the surjective submersion q is of class Nash and then 2) and 3) are proved. It remains to prove that the action of G on $G \times_H X$ is of class Nash. To this purpose, consider the following commutative diagram:

$$\begin{array}{ccc} G \times G \times X & \longrightarrow & G \times X \\ {\scriptstyle \mathrm{id}\times q}\big\downarrow & & \big\downarrow \\ G \times (G \times_H X) & \longrightarrow & G \times_H X \end{array} , \qquad \begin{array}{ccc} (g',g,x) & \longmapsto & (g'g,x) \\ \big\downarrow & & \big\downarrow \\ (g',[g,x]) & \longmapsto & [g'g,x] \end{array} .$$

The arrow below is the G action and it is of class Nash by 3) and because the other arrows are Nash maps. $\square$

Next Theorem tells that the Nash conjecture in Nash setting is true for twisted products of homogeneous Nash manifolds and compact affine Nash groups.

THEOREM 2.7. *Let H be a closed subgroup of the compact affine Nash group G, X a Nash H-manifold, Y a Nash G-manifold and $f : X \to Y$ a Nash H-embedding. Then*

1) *The map $\alpha : G \times_H X \to (G/H) \times Y$, $[g,x] \mapsto (gH, gf(x))$ is a Nash G-embedding.*
2) *Consider only G, H, Y and suppose Y a homogeneous Nash G-manifold. Then $G \times_H Y$ has an algebraic G-structure, and this structure is unique. So, the G action on $G \times_H Y$ is Nash linearizable.*

Proof.

1) Let us consider the map $\beta : G \times_H X \to G \times_H Y$, $[g,x] \mapsto [g, f(x)]$. The twisted products are Nash G-manifolds by Lemma 2.6 and β is a G-equivariant bijection onto its image. Moreover it is a Nash map, as follows from the following commutative diagram and by Lemma 2.6:

$$\begin{tikzcd} G \times X \arrow[r] \arrow[d] & G \times Y \arrow[d] \\ G \times_H X \arrow[r] & G \times_H Y \end{tikzcd} \quad , \quad \begin{tikzcd} (g,x) \arrow[r, mapsto] \arrow[d, mapsto] & (g, f(x)) \arrow[d, mapsto] \\ {[g,x]} \arrow[r, mapsto] & {[g, f(x)]} \end{tikzcd} .$$

In a similar way one proves that the inverse map β^{-1} is a Nash map. So, β is a Nash G-embedding.

Consider now the G-equivariant map $\gamma : G \times_H Y \to (G/H) \times Y$, $[g,y] \mapsto (gH, gy)$. It is a Nash diffeomorphism. In fact, if $q : G \times Y \to G \times_H Y$ is the canonical projection, the map $\gamma \circ q : G \times Y \to (G/H) \times Y$, $(g,y) \mapsto (gH, gy)$ is of class Nash and then, by Lemma 2.6, γ is of class Nash.

The inverse map γ^{-1} is given by

$$\gamma^{-1} : (G/H) \times Y \to G \times_H Y, \quad (gH, y) \mapsto [g, g^{-1}y].$$

It is of class Nash. Consider, in fact, the following commutative diagram:

$$\begin{tikzcd} G \times Y \arrow[r] \arrow[d, "p \times \mathrm{id}"'] & G \times Y \arrow[d] \\ (G/H) \times Y \arrow[r] & G \times_H Y \end{tikzcd} \quad , \quad \begin{tikzcd} (g,y) \arrow[r, mapsto] \arrow[d, mapsto] & (g, g^{-1}y) \arrow[d, mapsto] \\ (gH, y) \arrow[r, mapsto] & {[g, g^{-1}y]} \end{tikzcd} .$$

The arrow below is γ^{-1} and the others arrows are Nash maps; then the assertion follows from Theorem 2.3. Therefore γ is a Nash G-diffeomorphism. This completes the proof of 1) because $\alpha = \gamma \circ \beta$.

2) Since Y is homogeneous, it is Nash G-diffeomorphic to G/K, for a suitable Nash subgroup K of G, by Proposition 2.4. By Theorem 2.5, both G/H and G/K have an algebraic G-structure and these structures are unique. Then the assertion follows from the Nash G-diffeomorphism $\gamma : G \times_H Y \to (G/H) \times Y$ (see 1). $\square$

Proposition 2.8. *Let G be a compact affine Nash group, H a closed subgroup of G and X a Nash H-manifold.*

1) *Let us suppose that X has an affine Nash H-structure. Then*

a) *$G \times_H X$ has an affine Nash G-structure. Thus, the action of G on the twisted product is Nash linearizable.*

b) *There is an orthogonal group $O(q)$ such that $O(q) \times_H X$ has an affine Nash $O(q)$-structure.*

2) *Conversely, let us suppose that $G \times_H X$ has an affine Nash G-structure. Then X has an affine Nash H-structure.*

PROOF.

1) a) There exists, by hypothesis, a Nash H-equivariant embedding $f : X \to \boldsymbol{R}^p$ in the space of a Nash representation of H. By Corollary 1.4 we can suppose that G is a matrix group; then, by [**14**], there is a continuous representation $\lambda : G \to GL(m)$, with space $\boldsymbol{R}^m$, such that $\boldsymbol{R}^m$, considered as an H-space by restriction, contains the H-space $\boldsymbol{R}^p$ as an invariant linear subspace. By Theorem 1.6 λ is a Nash homomorphism and hence $\boldsymbol{R}^m$ is a Nash G-space. In this way we obtain the H-equivariant Nash embedding $f : X \to \boldsymbol{R}^m$. Then we can use Theorem 2.7: there is a Nash G-embedding $G \times_H X \to (G/H) \times \boldsymbol{R}^m$. Now, by Theorem 2.5, there are a Nash representation ρ of G with space $\boldsymbol{R}^n$ and a Nash G-embedding $G/H \to \boldsymbol{R}^n$. So we get the Nash G-embedding $G \times_H X \to \boldsymbol{R}^n \times \boldsymbol{R}^m$ and $\boldsymbol{R}^n \times \boldsymbol{R}^m$ is the space of the Nash representation $\rho \oplus \lambda$.

b) By Corollary 1.7 H is Nash isomorphic to a closed subgroup of an orthogonal group $O(q)$. Then the statement follows from a) where $G = O(q)$.

2) First we remark that the canonical H-embedding $X \to G \times_H X$ is a Nash map because it is the composition of the Nash maps $X \to G \times X \to G \times_H X$, $x \mapsto (e, x) \mapsto [e, x]$. Now, by hypothesis there exist a Nash representation $\vartheta : G \to GL(p)$, with space $\boldsymbol{R}^p$, and a Nash G-embedding $\varphi : G \times_H X \to \boldsymbol{R}^p$. Then the restriction $\varphi|X : X \to \boldsymbol{R}^p$ is a Nash H-embedding into the space of the representation $\vartheta|H$. □

3. On a question posed by M. Shiota.

In [**15**, Problem VI.2.12, p. 209] M. Shiota posed the following question: "Let $f : X_1 \to X_2$ be a Nash map between affine Nash manifolds. Do there exist Nash diffeomorphisms p_1, p_2 from X_1 and X_2 respectively to affine non singular algebraic varieties such that $p_2 \circ f \circ p_1^{-1}$ is a smooth rational map?". On this subject, some of the previous results lead to the following Proposition:

PROPOSITION 3.1.

1) *Let G_1, G_2 be compact affine Nash groups and $f : G_1 \to G_2$ a C^∞ homomorphism. There exist compact affine algebraic groups H_1, H_2 and Nash diffeomorphisms $q_i : G_i \to H_i$ $(i = 1, 2)$ such that $q_2 \circ f \circ q_1^{-1} : H_1 \to H_2$ is a polynomial map. So, f is a Nash map.*

2) *Let G be a compact affine Nash group, X_1 and X_2 homogeneous Nash G-manifolds and $f : X_1 \to X_2$ a C^∞ G-map. There exist non-singular algebraic G-varieties Y_i and Nash G-diffeomorphisms $p_i : X_i \to Y_i$ $(i = 1, 2)$ such that $p_2 \circ f \circ p_1^{-1}$ is polynomial. So, f is a Nash map.*

Proof.

1) See Theorem 1.6 and its proof.
2) One can find the algebraic G-varieties X_i and the Nash G-diffeomorphisms $p_i : X_i \to Y_i$ $(i = 1, 2)$ using Theorem 2.5. By [**5**] $p_2 \circ f \circ p_1^{-1}$ is polynomial. □

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