J. Math. Soc. Japan
Vol. 64, No. 3 (2012) pp. 885–902
doi: 10.2969/jmsj/06430885

Commutators of BMO functions with spectral multiplier operators

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(Received June 5, 2010)
(Revised Jan. 27, 2011)

Abstract. Let L be a non-negative self adjoint operator on $L^2(X)$ where X is a space of homogeneous type. Assume that L generates an analytic semigroup e^{-tL} whose kernel satisfies the standard Gaussian upper bounds. By the spectral theory, we can define the spectral multiplier operator $F(L)$. In this article, we show that the commutator of a BMO function with $F(L)$ is bounded on $L^p(X)$ for $1 < p < \infty$ when F is a suitable function.

1. Introduction.

Suppose that L is a non-negative self-adjoint operator on $L^2(X)$. Let $E(\lambda)$ be the spectral resolution of L. By the spectral theorem, for any bounded Borel function $F : [0, \infty) \to \boldsymbol{C}$, one can define the operator

$$F(L) = \int_0^\infty F(\lambda) dE(\lambda), \tag{1}$$

which is bounded on $L^2(X)$. The L^p-boundedness of spectral multipliers is a well-known problem which has been studied extensively, see for example [**A1**], [**A2**], [**B**], [**C**], [**D**], [**DeM**], [**DOS**], [**FS**], [**Ho**], [**He**] and [**MSt**].

Assume that L generates an analytic semigroup e^{-tL} whose kernel $p_t(x, y)$ satisfies the standard Gaussian upper bounds. The Gaussian upper bound will transform to an upper bound on the kernel of $F(L)$. However, we do not assume any regularity on the space variables of the kernels $p_t(x, y)$. Hence, under only the standard Gaussian upper bound condition, $F(L)$ might not be a Calderón-Zygmund operator.

Let us remind the reader that if T is a standard Calderón-Zygmund singular integral operator, i.e., T is a bounded operator on $L^2(\boldsymbol{R}^n)$ and the associated

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2000 *Mathematics Subject Classification.* Primary 42B20; Secondary 42B35, 47B38.
Key Words and Phrases. spectral multiplier theorems, non-negative self-adjoint operator, heat semigroup, space of homogeneous type.
The author was supported by a Macquarie University scholarship.

kernel $k(x,y)$ of T satisfies the Hölder continuity condition in both variables x and y (or at least the Hörmander condition), then T is bounded from the Hardy space $H^1(\boldsymbol{R}^n)$ to $L^1(\boldsymbol{R}^n)$. By interpolation, T is bounded on $L^p(\boldsymbol{R}^n)$ for all p, $1 < p < 2$. For such a standard Calderón-Zygmund operator T, it is well known that the commutator operator of a BMO function b and T given by $[b,T]f = T(bf) - bTf$ is also bounded on $L^p(\boldsymbol{R}^n)$. See, for example [**St**].

It is natural to raise the question for the boundedness of $F(L)$ and the commutator $[b, F(L)]$ of $F(L)$ and a BMO function b. The boundedness of $F(L)$ was studied in [**DOS**]. It was proved that the spectral multiplier operator $F(L)$ is bounded on $L^p(X)$ if the following condition holds for $q \in [2,\infty]$ and appropriate s,

$$\sup_{t>0} \|\eta\, \delta_t F\|_{W_s^q} < \infty, \tag{2}$$

where $\delta_t F(\lambda) = F(t\lambda)$, $\|F\|_{W_s^q} = \|(I - d^2/dx^2)^{s/2} F\|_{L^q}$ and η is an auxiliary non-zero cut-off function such that $\eta \in C_c^\infty(\boldsymbol{R}_+)$. We note that (2) is actually independent of the choice of η.

The boundedness of the commutator of $F(L)$ and a BMO function b will be investigated in this paper. Precisely, in the same setting as in [**DOS**], we will show that the commutator $[b, F(L)]$ is bounded on $L^p(X)$ for all $1 < p < \infty$. Note that due to the absence of smoothness condition on the heat kernels, Fefferman-Stein's maximal function may not be suitable. However, we can overcome this obstacle by using the sharp maximal function in [**M**].

The organization of this paper is as follows. In Section 2, we recall some important properties of BMO functions and some maximal functions. The main results will be addressed in Section 3. Section 4 gives the proof of the main results.

2. Preliminaries and notations.

Let (X,d,μ) be a space endowed with a distance d and a nonnegative Borel measure μ on X. Set $B(x,r) = \{y \in X : d(x,y) < r\}$ and $V(x,r) = \mu(B(x,r))$. We shall often just use B instead of $B(x,r)$. Assume that (X,d,μ) satisfies the volume doubling property, that is, there exists a constant $C > 0$ such that

$$V(x,2r) \le CV(x,r) \quad \forall\, r > 0,\ x \in X. \tag{3}$$

By (3) there exist positive constants n and C_n such that

$$\frac{V(x,r)}{V(x,s)} \le C_n \left(\frac{r}{s}\right)^n, \quad \forall\, r \ge s > 0,\ x \in X. \tag{4}$$

The parameter n is a measure of the dimension of the space. It also follows form the doubling condition that there exist C and N, $0 \le N \le n$ so that

$$V(y,r) \le C\left(1+\frac{d(x,y)}{r}\right)^N V(x,r), \quad \forall\, r>0,\ x,y \in X. \tag{5}$$

uniformly for all $x, y \in X$ and $r > 0$. Indeed, property (5) with $N = n$ is a direct consequence of the triangle inequality for the metric d and (4). In many cases like for example the Euclidean space $\boldsymbol{R}^n$ or Lie groups of polynomial growth, N can be chosen to be 0.

The standard Hardy-Littlewood maximal function $M_r f$, $1 \le r < \infty$ is defined by

$$M_r f(x) = \sup_{x \in B}\left(\frac{1}{V(B)}\int_B |f(y)|^r d\mu(y)\right)^{1/r},$$

where the sup is taken over all balls containing x. If $r = 1, M_1 f$ will be denoted by Mf. The standard Fefferman-Stein sharp maximal function of f is defined by

$$M^\sharp f(x) = \sup_{x\in B}\frac{1}{V(B)}\int_B |f(y) - f_B| d\mu(y),$$

where $f_B = \frac{1}{V(B)}\int_B f(x)d\mu(x)$. We will say $f \in \mathrm{BMO}(X)$ if $f \in L^1_{\mathrm{loc}}(X)$ and $M^\sharp f \in L^\infty$. If $f \in \mathrm{BMO}$, the BMO semi-norm of f is given by

$$\|f\|_{\mathrm{BMO}} = \sup_x M^\sharp f(x) = \sup_x \sup_{B \ni x}\frac{1}{V(B)}\int_B |f(y)-f_B|d\mu(y).$$

We will work with a class of integral operators $\{A_t\}_{t>0}$, which plays the role of generalized approximations to the identity. We assume that for each $t > 0$, the operator A_t is defined by its kernel $a_t(x, y)$ in the sense that

$$A_t f(x) = \int_X a_t(x,y) f(y) d\mu(y)$$

for every function $f \in \cup_{p\ge 1} L^p(X)$.

We also assume that the kernel $a_t(x, y)$ of A_t satisfies the following condition

$$|a_t(x,y)| \le h_t(x,y) = \frac{1}{V(x,t^{1/m})} g\left(\frac{d(x,y)}{t^{1/m}}\right) \tag{6}$$

in which g is a positive, bounded, decreasing function satisfying

$$\lim_{r\to\infty} r^{n+N+\kappa} g(r) = 0 \tag{7}$$

for some $\kappa > 0$.

Associated to $\{A_t\}_{t>0}$, the new sharp maximal function $M_A^\sharp f$ is defined by

$$M_A^\sharp f(x) = \sup_{x\in B} \frac{1}{V(B)} \int_B |(I - A_{t_B})f(y)| d\mu(y), \tag{8}$$

where $t_B = r_B^m$ and $f \in \cup_{p\ge 1} L^p(X)$, see [**M**].

We recall some results which will be used in the sequel.

(a) Assume $b \in \mathrm{BMO}$ and $k > 1$. Then for any ball B we have

$$|b_B - b_{kB}| \le C\|b\|_{\mathrm{BMO}} \log k.$$

(b) (John-Nirenberg inequality) Let $1 \le p < \infty$. Then $b \in \mathrm{BMO}$ if and only if

$$\left(\frac{1}{V(B)} \int_B |b(y) - b_B|^p d\mu(y)\right)^{1/p} \le C\|b\|_{\mathrm{BMO}}$$

for all balls $B \subset X$.

The following estimates concerning on the generalized approximations of identity and the new sharp maximal function $M_A^\sharp$ provide the necessary ingredients for the proof of the main results, see [**M**], [**DY**].

LEMMA 2.1. *Let $\{A_t\}_{t>0}$ be a "generalized approximations of identity". Then,*

(a) *For $f \in L^p(X)$, $1 < p < \infty$, we have*

(i)
$$\|f\|_{L^p} \le \|Mf\|_{L^p} \le C\big\|M_A^\sharp f\big\|_{L^p}, \quad \textit{if} \ \ \mu(X) = \infty; \tag{9}$$

(ii)
$$\|f\|_{L^p} \le \|Mf\|_{L^p} \le C\big\|M_A^\sharp f\big\|_{L^p} + \|f\|_{L^1}, \quad \textit{if} \ \ \mu(X) < \infty; \tag{10}$$

(b) *For $b \in \mathrm{BMO}$, $f \in L^p(X)$, $1 < p < \infty$, $x \in X$ and $1 < r < \infty$, we have*

$$\sup_{x\in B} \frac{1}{V(B)} \int_B \big|A_{t_B}(b - b_B)f(y)\big| d\mu(y) \le C\|b\|_{\mathrm{BMO}} M_r f(x). \tag{11}$$

3. Main results.

Unless otherwise specified in the sequel we always assume that L is a non-negative self-adjoint operator on $L^2(X)$ and that the semigroup e^{-tL}, generated by $-L$ on $L^2(X)$, has the kernel $p_t(x,y)$ which satisfies the following Gaussian upper bound

$$|p_t(x,y)| \le \frac{C}{V(x,t^{1/m})} \exp\left(- \frac{d(x,y)^{m/(m-1)}}{ct^{1/(m-1)}} \right) \tag{GE}$$

for all $t>0$, and $x,y \in X$, where C, c and m are positive constants and $m \ge 2$.

Such estimates are typical for elliptic or sub-elliptic differential operators of order m (see for instance, [**DOS**], [**R**] and [**VSC**]).

Let T be a bounded linear operator on $L^2(X)$. By the kernel $K(x,y)$ associated to T we shall mean that

$$Tf(x) = \int_X K(x,y)f(y)d\mu(y) \quad \text{for all} \quad x \notin \operatorname{supp} f$$

holds for all $f \in L^\infty(X)$ with bounded support. In what follows, by $K_{F(\sqrt[m]{L})}(x,y)$ we denote the associated kernel of $F(\sqrt[m]{L})$.

The main new results, which we describe in this paper, are Theorems 3.1 and 3.3 below.

Theorem 3.1. *Let L be a non-negative self-adjoint operator such that the corresponding heat kernels satisfy the Gaussian bounds* (GE). *Let $s > n/2$. Assume that for any $R>0$ and all Borel functions F such that* $\operatorname{supp} F \subseteq [0,R]$,

$$\int_X \left|K_{F(\sqrt[m]{L})}(x,y)\right|^2 d\mu(x) \le \frac{C}{V(y,R^{-1})} \|\delta_R F\|_{L^q}^2 \tag{12}$$

for some $q \in [2,\infty]$. Then for any bounded Borel function F such that $\sup_{t>0} \|\eta\, \delta_t F\|_{W_s^q} < \infty$, the commutator $[b, F(L)]$ of a BMO *function b and $F(L)$ is bounded on $L^p(X)$ for all $1<p<\infty$,*

$$\|[b,F(L)]f\|_{L^p(X)} \le C\|b\|_{\mathrm{BMO}} \|f\|_{L^p(X)}.$$

Theorem 3.1 implies the following result which gives the boundedness of the commutator $[b, F(L)]$.

Theorem 3.2. *Let L be a non-negative self-adjoint operator such that the* *corresponding heat kernels satisfy the Gaussian bounds* (GE). *Let* $s > n/2$. *Then for any bounded Borel function* F *such that* $\sup_{t>0} \|\eta\, \delta_t F\|_{W_s^\infty} < \infty$, *the commutator* $[b, F(L)]$ *is bounded on* $L^p(X)$ *for all* $1 < p < \infty$,

$$\|[b, F(L)]f\|_{L^p(X)} \le C\|b\|_{\mathrm{BMO}}\|f\|_{L^p(X)}.$$

Proof. Note that it was proved in Lemma 2 of [**DOS**], that for any Borel function F such that $\operatorname{supp} F \subset [0, R]$,

$$\begin{aligned}\left\|K_{F(\sqrt[m]{L})}(\cdot, y)\right\|^2_{L^2(X)} &= \left\|K_{\bar{F}(\sqrt[m]{L})}(y, \cdot)\right\|^2_{L^2(X)} \\ &\le \frac{C}{V(y, R^{-1})}\|F\|^2_{L^\infty}. \end{aligned} \tag{13}$$

This shows that (12) always holds for $q = \infty$, and Theorem 3.2 follows from Theorem 3.1. □

For further discussion and rationale for (12), see [**DOS**, p. 451]. Note that for elliptic operators on compact manifolds, (12) cannot hold for any $q < \infty$, see [**DOS**, (3.3)]. To be able to study these operators as well, following the ideas in [**CS**], [**DOS**], we introduce some variation of (12). For a Borel function F such that $\operatorname{supp} F \subseteq [-1, -2]$ we define the norm $\|F\|_{N,q}$ by the formula

$$\|F\|_{N,q} = \left(\frac{1}{3N}\sum_{\ell=1-N}^{2N} \sup_{\lambda\in[(\ell-1)/N,\ell/N)} |F(\lambda)|^q\right)^{1/q},$$

where $q \in [1, \infty)$ and $N \in \boldsymbol{Z}_+$. For $q = \infty$, we put $\|F\|_{N,\infty} = \|F\|_{L^\infty}$. It is obvious that $\|F\|_{N,q}$ increases monotonically in q. The next theorem is a variation of Theorem 3.1. This variation can be used in case of operators with nonempty pointwise spectrum, see also [**CS**, Theorem 3.6] and [**DOS**, Theorem 3.2].

Theorem 3.3. *Assume that* $\mu(X) < \infty$. *Let* L *be a non-negative self-adjoint operator such that the corresponding heat kernels satisfy the Gaussian bounds* (GE). *Let* $s > n/2$. *Suppose that for any* $N \in \boldsymbol{Z}_+$ *and for all Borel functions* F *such that* $\operatorname{supp} F \subseteq [-1, N+1]$,

$$\int_X \left|K_{F(\sqrt[m]{L})}(x, y)\right|^2 d\mu(x) \le \frac{C}{V(y, N^{-1})}\|\delta_N F\|^2_{N,\,q} \tag{14}$$

for some $q \ge 2$. *Then for any bounded Borel function* F *such that* $\sup_{t>1}$

$\|\eta\,\delta_t F\|_{W^q_s} < \infty$, *the commutator* $[b, F(L)]$ *is bounded on* $L^p(X)$ *for all* $p > 1$.

Let us consider several operators which satisfy (12) or (14), see [**DOS**], [**DSY**].

(a) *Homogeneous groups.* Let $\boldsymbol{G}$ be a Lie group of polynomial growth and let $X_1, \dots, X_k$ be a system of left-invariant vector fields on $\boldsymbol{G}$ satisfying the Hörmander condition. We define the Laplace operator L acting on $L^2(\boldsymbol{G})$ by the formula

$$L = -\sum_{i=1}^{k} X_i^2. \tag{15}$$

If $B(x, r)$ is the ball defined by the distance associated with system $X_1, \dots, X_k$ (see e.g. [**VSC**, Chapter III.4]), then there exist natural numbers $n_0, n_\infty \geq 0$ such that $V(x, r) \sim r^{n_0}$ for $r \leq 1$ and $V(x, r) \sim r^{n_\infty}$ for $r > 1$ (see e.g. [**VSC**, Chapter III.2]). We call $\boldsymbol{G}$ a homogeneous group if there exists a family of dilations on $\boldsymbol{G}$. A family of dilations on a Lie group $\boldsymbol{G}$ is a one-parameter group $(\tilde{\delta}_t)_{t>0}$ $(\tilde{\delta}_t \circ \tilde{\delta}_t = \tilde{\delta}_{ts})$ of automorphisms of $\boldsymbol{G}$ determined by

$$\tilde{\delta}_t Y_j = t^{n_j} Y_j, \tag{16}$$

where $Y_1, \dots, Y_\ell$ form a linear basis of Lie algebra of $\boldsymbol{G}$ and $n_j \geq 1$ for $1 \leq j \leq \ell$ (see [**FS**]). We say that an operator L defined by (15) is homogeneous if $\tilde{\delta}_t X_i = tY_i$ for $1 \leq i \leq k$. For the homogeneous Laplace operator $n_0 = n_\infty = \sum_{j=1}^{\ell} n_j$ (see [**FS**]).
Spectral multiplier theorems for the homogeneous Laplace operators acting on homogeneous groups were investigated by Hulanicki and Stein [**HS**] ([**FS**, Theorem 6.25]) and De Michele and Mauceri [**DeM**]. See also [**C**] and [**MM**]. It is well known that such an operator L is self-adjoint nonnegative and satisfies (GE). If L is a left-invariant operator acting on the unimodular Lie group $\boldsymbol{G}$ then (12) holds for $q = 2$. In particular case when L is a nonnegative self-adjoint left-invariant operator on homogeneous group $\boldsymbol{G}$, if L is homogeneous of order m, i.e, $\tilde{\delta}_t L = t^m L$, then (12) holds for $q = \infty$, see [**DOS**, p. 467].

(b) *Compact manifolds.* For a general non-negative self-adjoint elliptic operator on a compact manifold, the Gaussian bound (GE) holds by general elliptic regularity theory. Let L be a non-negative elliptic pseudo-differential operator of order m on a compact manifold X of dimension n. Then L satisfies (GE); moreover, (14) hold for $q = 2$, see [**DOS**, p. 469].

(c) *Schrödinger operators.* Let X be a connected complete Riemannian manifold. We consider the Schrödinger operator $L = -\Delta + V$ where $V : X \to \boldsymbol{R}$,

$V \in L^1_{\text{loc}}(X)$ and $V \geq 0$. The operator L is defined by the quadratic form technique. If $p_t(x,y)$ denotes the heat kernel corresponding to L, then as a consequence of the Trotter product formula,

$$|p_t(x,y)| \leq \tilde{p}_t(x,y), \tag{17}$$

where $\tilde{p}_t(x,y)$ denotes the heat kernel corresponding to Δ.
The inequality (17) holds also for heat kernel $p_t(x,y)$ of Schrödinger operator with electromagnetic potentials, see [**Si**, Theorem 2.3] and [**DOS**, (7.9)]. For the Schrödinger operator in this setting, (12) holds for $q = \infty$, see [**DSY**].

4. Proof of main results.

Recall that $B = B(x_B, r_B)$ is the ball of radius r_B and center at x_B. Given $\lambda > 0$, we will write λB for the λ-dilated ball, which is the ball with the same center as B and with radius $r_{\lambda B} = \lambda r_B$. We set

$$S_0(B) = B, \quad \text{and} \quad S_j(B) = 2^j B \backslash 2^{j-1} B \quad \text{for} \quad j = 1, 2, \ldots. \tag{18}$$

The following sharp maximal function $M^\sharp_{L,M}$ will play an important role in the proof of Theorem 3.1:

$$M^\sharp_{L,M} f(x) = \sup_{x \in B} \frac{1}{V(B)} \int_B \left|(I - e^{-r_B^m L})^M f(y)\right| d\mu(y), \tag{19}$$

where $f \in \cup_{p \geq 1} L^p(X)$ and $M \in \boldsymbol{N}$.

In fact we can think of the maximal function $M^\sharp_{L,M}$ as the sharp maximal function $M^\sharp_A$ in Section 2 with $A_t = I - (I - e^{-tL})^M = \sum_{k=1}^M (-1)^k C_n^k e^{-ktL}$. This expression together with the fact that the kernels of $\{e^{-tL}\}_{t>0}$ satisfy (GE) implies that the kernels $a_t(x,y)$ of $A_t = \sum_{k=1}^M (-1)^k C_n^k e^{-ktL}$ satisfy (GE) too. Therefore, as consequence of Lemma 2.1, we obtain the following lemma.

LEMMA 4.1. *Let L be a non-negative self-adjoint operator such that the corresponding heat kernels satisfy the Gaussian bounds* (GE).

(a) *For $f \in L^p(X)$, $1 < p < \infty$ and $M \in \boldsymbol{N}$, we have*

(i)
$$\|f\|_{L^p} \leq \|Mf\|_{L^p} \leq C\left\|M^\sharp_{L,M} f\right\|_{L^p}, \quad \text{if} \;\; \mu(X) = \infty; \tag{20}$$

(ii)
$$\|f\|_{L^p} \leq \|Mf\|_{L^p} \leq C\left\|M^\sharp_{L,M} f\right\|_{L^p} + \|f\|_{L^1}, \quad \text{if} \;\; \mu(X) < \infty; \tag{21}$$

(b) *For $b \in \mathrm{BMO}$, $f \in L^p(X)$, $1 < p < \infty$, $x \in X$ and $1 < r < \infty$, we have*

$$\sup_{x\in B} \frac{1}{V(B)} \int_B \left|(I - (I - e^{-r_B^m L})^M)(b - b_B)f(y)\right| d\mu(y)$$
$$\leq C\|b\|_{\mathrm{BMO}} M_r f(x). \tag{22}$$

As a preamble to the proof of Theorems 3.1 and 3.3, we record a useful auxiliary result. For a proof, see pp. 453–454, Lemma 4.3 of [**DOS**].

Lemma 4.2.

(a) *Suppose that L satisfies* (12) *for some $q \in [2, \infty]$ and that $R > 0$, $s > 0$. Then for any $\epsilon > 0$, there exists a constant $C = C(s, \epsilon)$ such that*

$$\int_X \left|K_{F(\sqrt[m]{L})}(x, y)\right|^2 \big(1 + Rd(x, y)\big)^s d\mu(x) \leq \frac{C}{V(y, R^{-1})} \|\delta_R F\|^2_{W^q_{s/2+\epsilon}} \tag{23}$$

for all Borel functions F such that supp $F \subseteq [R/4, R]$.

(b) *Suppose that L satisfies* (14) *for some $q \in [2, \infty]$ and that $N > 8$ is a natural number. Then for any $s > 0$, $\epsilon > 0$ and function $\xi \in C_c^\infty([-1, 1])$ there exists a constant $C = C(s, \epsilon, \xi)$ such that*

$$\int_X \left|K_{F*\xi(\sqrt[m]{L})}(x, y)\right|^2 \big(1 + Nd(x, y)\big)^s d\mu(x) \leq \frac{C}{V(y, R^{-1})} \|\delta_N F\|^2_{W^q_{s/2+\epsilon}} \tag{24}$$

for all Borel functions F such that supp $F \subseteq [N/4, N]$.

Proposition 4.3. *Let L be a non-negative self-adjoint operator such that the corresponding heat kernels satisfy the Gaussian bounds* (GE) *and $M > s/m$. Assume that for any $R > 0$ and all Borel functions F such that* $\operatorname{supp} F \subseteq [0, R]$,

$$\int_X \left|K_{F(\sqrt[m]{L})}(x, y)\right|^2 d\mu(x) \leq \frac{C}{V(y, R^{-1})} \|\delta_R F\|^2_{L^q} \tag{25}$$

for some $q \in [2, \infty]$. Then for any bounded Borel function F such that $\sup_{t>0}$ $\|\eta\, \delta_t F\|_{W^q_s} < \infty$*, the operator $F(\sqrt[m]{L})(I - e^{r_B^m L})^M$, has a kernel $K_{r_B,M}(x, y)$ which satisfies*

$$\left(\int_{S_j(B)} \left|K_{r_B,M}(y, z)\right|^2 d\mu(y)\right)^{1/2} \leq C 2^{-js'} V(B)^{-1/2} \tag{26}$$

for any ball $B \subset X$, all $z \in B$ and $s > s' > n/2$.

Proof. We adapt the arguments used in [**DOS**] (see also [**DSY**]) to our present situation. Let $\varphi \in C_c^\infty(0,\infty)$ be a non-negative function satisfying $\operatorname{supp}\varphi \subseteq [1/4,1]$ and $\sum_{k=-\infty}^{\infty}\varphi(2^{-k}\lambda) = 1$ for any $\lambda > 0$, and let φ_k denote the function $\varphi(2^{-k}\cdot)$. Then

$$F(\lambda) = \sum_{k=-\infty}^{\infty}\varphi(2^{-k}\lambda)F(\lambda) = \sum_{k=-\infty}^{\infty}F^k(\lambda), \quad \forall\,\lambda > 0. \tag{27}$$

For every $k \in \boldsymbol{Z}$ and $r > 0$, we set for $\lambda > 0$,

$$F_{r,M}(\lambda) = F(\lambda)\big(1 - e^{-(r\lambda)^m}\big)^M, \tag{28}$$

$$F^k_{r,M}(\lambda) = F^k(\lambda)\big(1 - e^{-(r\lambda)^m}\big)^M. \tag{29}$$

Therefore, for any ball $B \subset X$ any $z \in B$, we have

$$\begin{aligned}\bigg(\int_{S_j(B)}\big|K_{r_B,M}(y,z)\big|^2 d\mu(y)\bigg)^{1/2} &\le \sum_{k=-\infty}^{\infty}\bigg(\int_{S_j(B)}\big|K_{F^k_{r_B,M}(\sqrt[m]{L})}(y,z)\big|^2 d\mu(y)\bigg)^{1/2}\\ &= \sum_{k=-\infty}^{\infty} I_k,\end{aligned}$$

where $K_{F^k_{r_B,M}(\sqrt[m]{L})}(y,z)$ denotes the associated kernel of operator $F^k_{r_B,M}(\sqrt[m]{L})$.

Let us estimate the term I_k. Choose $k_0 = -\log_2 r_B$ and $s > s' > n/2$. Then we have

$$\begin{aligned}&\int_{S_j(B)}\big|K_{F^k_{r_B,M}(\sqrt[m]{L})}(y,z)\big|^2 d\mu(y)\\ &\quad\le C2^{-2s'j}\big(2^k r_B\big)^{-2s'}\sup_{y\in B}\int_X\big|K_{F^k_{r_B,M}(\sqrt[m]{L})}(y,z)\big|^2\big(1 + 2^k d(y,z)\big)^{2s'} d\mu(y).\end{aligned}$$

We then apply Lemma 4.2 with $F = F^k_{r_B,M}$ and $R = 2^k$ to obtain

$$\begin{aligned}&\int_X\big|K_{F^k_{r_B,M}(\sqrt[m]{L})}(y,z)\big|^2\big(1 + 2^k d(y,z)\big)^{2s'} d\mu(y)\\ &\quad\le \frac{C_s}{V(z,2^{-k})}\big\|\delta_{2^k}\big(F^k_{r_B,M}\big)\big\|^2_{W^q_s}.\end{aligned} \tag{30}$$

For any integer l which is greater than s, we have

$$\begin{aligned}\big\|\delta_{2^k}\big(F^k_{r_B,M}\big)\big\|_{W^q_s} &= \big\|\varphi(t)F(2^k t)(1-e^{-(2^k r_B t)^m})^M\big\|_{W^q_s} \\ &\le C\big\|(1-e^{-(2^k r_B t)^m})^M\big\|_{C^l([1/4,1])}\big\|\delta_{2^k}[\varphi_k F]\big\|_{W^q_s}. \qquad (31)\end{aligned}$$

Therefore,

$$\big\|\delta_{2^k}\big(F^k_{r_B,M}\big)\big\|_{W^q_s} \le \begin{cases} C\big\|\delta_{2^k}[\varphi_k F]\big\|_{W^q_s} & \text{if } \ k \ge k_0 \\ C(2^k r_B)^{mM}\big\|\delta_{2^k}[\varphi_k F]\big\|_{W^q_s} & \text{if } \ k < k_0. \end{cases}$$

Note that for all $z \in B$, we have

$$\frac{1}{V(z,2^{-k})} \le \begin{cases} C\dfrac{(2^k r_B)^n}{V(B)} & \text{if } \ k \ge k_0 \\ C\dfrac{1}{V(B)} & \text{if } \ k < k_0. \end{cases}$$

This together with (31) gives

$$\begin{aligned}\sum_{k=-\infty}^{\infty} I_k &= \sum_{k\ge k_0} I_k + \sum_{k<k_0} I_k \\ &\le C\sum_{k\ge k_0} 2^{-s'j}(2^k r_B)^{-s'}(2^k r_B)^{n/2}V(B)^{1/2}\big\|\delta_{2^k}[\varphi_k F]\big\|_{W^q_s} \\ &\quad + C\sum_{k<k_0} 2^{-s'j}(2^k r_B)^{-s'}(2^k r_B)^{mM}V(B)^{1/2}\big\|\delta_{2^k}[\varphi_k F]\big\|_{W^q_s} \\ &\le C2^{-s'j}V(B)^{-1/2}\sup_k\big\|\delta_{2^k}[\varphi_k F]\big\|_{W^q_s} \\ &\le C2^{-s'j}V(B)^{-1/2}.\end{aligned}$$

The proof is complete.

THE PROOF OF THEOREM 3.1. Set $T = (F(\sqrt[m]{L}))^*$. Then T is bounded on L^p, $p > 1$, see [**DOS**].

Firstly, we will show that the commutator $[b,T]$ is bounded on $L^p(X)$, $p > 2$. By duality, $[b, F(\sqrt[m]{L})]$ is bounded on $L^p(X)$, $p < 2$. On the other hand, since $T = \overline{F}(\sqrt[m]{L})$, $[b,T]$ is bounded on $L^p(X)$, $p > 2$. At this stage, by interpolation, $[b, F(\sqrt[m]{L})]$ is bounded on $L^p(X)$, $p > 1$.

To prove $[b,T]$ is bounded on $L^p(X)$, $p>2$, we will claim that for all $x\in X$ and for all balls B containing x we have

$$\frac{1}{V(B)}\int_B \big|(I-e^{r_B^m L})^M[b,T]f(y)\big|d\mu(y)\leq C\|b\|_{\mathrm{BMO}}(M_r(Tf)(x)+M_rf(x)) \quad (32)$$

for any $r>2$.

Once (32) is obtained, the required conclusion follows readily. Indeed, we consider two cases.

Case 1: $\mu(X)=\infty$.
By Lemma 4.1 and the boundedness of T on $L^q(X)$, $q>1$ we have for any $p>r>2$

$$\begin{aligned}\|[b,T]f\|_{L^p} &\leq C\big\|M^\sharp_{L,M}f\big\|_{L^p}\\ &\leq C\|b\|_{\mathrm{BMO}}\big(\|M_r(Tf)\|_{L^p}+\|M_rf\|_{L^p}\big)\\ &\leq C\|b\|_{\mathrm{BMO}}\|f\|_{L^p}.\end{aligned}$$

Case 2: $\mu(X)<\infty$.
The condition $\mu(X)<\infty$ implies that X is bounded. Then there exists $r>0$ such that $X\subset B(x,r)$. By Lemma 4.1 and the boundedness of T on $L^q(X)$, $q>1$ we have for any $p>r>2$, we have

$$\begin{aligned}\|[b,T]f\|_{L^p} &\leq C\big\|M^\sharp_{L,M}f\big\|_{L^p}+\|[b,T]f\|_{L^1}\\ &\leq C\|b\|_{\mathrm{BMO}}\big(\|M_r(Tf)\|_{L^p}+\|M_rf\|_{L^p}\big)+\|[b,T]f\|_{L^1}\\ &\leq C\|b\|_{\mathrm{BMO}}\|f\|_{L^p}+\|[b,T]f\|_{L^1}.\end{aligned}$$

Set $B_0=B(x,r)$. Applying the Hölder inequality and using the fact that T is bounded on $L^q(X)$, $q>1$ again, we have

$$\begin{aligned}\|[b,T]f\|_{L^1} &\leq \int_{B_0}\big|(b-b_{B_0})Tf\big|d\mu+\int_{B_0}\big|T((b-b_{B_0})f)\big|d\mu\\ &\leq \big\|b-b_{B_0}\big\|_{L^{p'}}\|Tf\|_{L^p}+\mu(X)^{1/2}\big\|T((b-b_{B_0})f)\big\|_{L^2}\\ &\leq \mu(X)^{1/p'}\|b\|_{\mathrm{BMO}}\|f\|_{L^p}+\mu(X)^{1/2}\big\|(b-b_{B_0})f\big\|_{L^2}\\ &\leq C\|b\|_{\mathrm{BMO}}\|f\|_{L^p}+C\big\|b-b_{B_0}\big\|_{L^{2p/(p-2)}}\|f\|_{L^p}\\ &\leq C\|b\|_{\mathrm{BMO}}\|f\|_{L^p}.\end{aligned}$$

We return to prove (32). By standard argument we set $f_1 = f\chi_{2B}$ and $f_2 = f - f_1$. We have

$$[b,T]f = (b - b_B)Tf - T((b - b_B)f_1) - T((b - b_B)f_2)$$

and

$$\begin{aligned}&\big(I - (I - e^{r_B^m L})^M\big)[b,T]f\\&\quad = A_{r_B,M}(b - b_B)Tf - A_{r_B,M}T((b - b_B)f_1) - A_{r_B,M}T((b - b_B)f_2),\end{aligned}$$

where $A_{r_B,M} = (I - (I - e^{r_B^m L})^M)$. Therefore,

$$\begin{aligned}&\frac{1}{V(B)}\int_B \big|(I - e^{r_B^m L})^M [b,T]f(y)\big| d\mu(y)\\&\quad = \frac{1}{V(B)}\int_B (b - b_B)Tf + \frac{1}{V(B)}\int_B T((b - b_B)f_1)\\&\quad + \frac{1}{V(B)}\int_B A_{r_B,M}(b - b_B)Tf + \frac{1}{V(B)}\int_B A_{r_B,M}T((b - b_B)f_1)\\&\quad + \frac{1}{V(B)}\int_B (I - e^{r_B^m L})^M T((b - b_B)f_2)\\&\quad = I_1 + I_2 + I_3 + I_4 + I_5.\end{aligned}$$

By the Hölder inequality and (b) in Lemma 4.1, it is not difficult to show that

$$I_1 + I_2 + I_3 + I_4 \le C\|b\|_{\mathrm{BMO}}(M_r(Tf)(x) + M_r f(x)).$$

It remains to estimate I_5. Denote by $K^*_{r_B,M}$ the kernel of $(I - e^{r_B^m L})^M T$. Then we have

$$\begin{aligned}&\frac{1}{V(B)}\int_B (I - e^{r_B^m L})^M T((b - b_B)f_2)\\&\quad \le \frac{1}{V(B)}\int_B\int_{X\setminus 2B} \big|K^*_{r_B,M}(y,z)\big||(b(z) - b_B)f(z)| d\mu(z)d\mu(y)\\&\quad \le \sum_{j\ge 1}\frac{1}{V(B)}\int_B\int_{S_j(B)} \big|K^*_{r_B,M}(y,z)\big||(b(z) - b_B)f(z)| d\mu(z)d\mu(y)\\&\quad \le \sum_{j\ge 1} K_j.\end{aligned}$$

By Proposition 4.3, John-Nirenberg inequality and Hölder inequality we have

$$\begin{aligned}
K_j &\le \frac{1}{V(B)}\int_B \bigg[\Big(\int_{S_j(B)} \big|K^*_{r_B,M}(y,z)\big|^2 d\mu(z)\Big)^{1/2} \\
&\qquad\qquad \times \Big(\int_{S_j(B)} |(b(z)-b_B)f(z)|^2 d\mu(z)\Big)^{1/2}\bigg] d\mu(y) \\
&\le \frac{C}{V(B)}\int_B \bigg[2^{-js'}V(B)^{-1/2}\bigg(\Big(\int_{S_j(B)} \big|(b(z)-b_{2^jB})f(z)\big|^2 d\mu(z)\Big)^{1/2} \\
&\qquad\qquad + \Big(\int_{S_j(B)} \big|(b_B-b_{2^jB})f(z)\big|^2 d\mu(z)\Big)^{1/2}\bigg)\bigg] d\mu(y) \\
&\le \frac{C}{V(B)}\int_B \bigg[2^{-j(s'-n/2)}\bigg(\Big(\frac{1}{V(2^jB)}\int_{S_j(B)} \big|(b(z)-b_{2^jB})f(z)\big|^2 d\mu(z)\Big)^{1/2} \\
&\qquad\qquad + j\|b\|_{\mathrm{BMO}}\Big(\frac{1}{V(2^jB)}\int_{S_j(B)} |f(z)|^2 d\mu(z)\Big)^{1/2}\bigg)\bigg] d\mu(y) \\
&\le \frac{C}{V(B)}\int_B \big[\|b\|_{\mathrm{BMO}}2^{-j(s'-n/2)}\big(M_rf(x)+jM_2f(x)\big)\big] d\mu(y) \\
&\le C\|b\|_{\mathrm{BMO}}2^{-j(s'-n/2)}\big(M_rf(x)+jM_2f(x)\big).
\end{aligned}$$

Since $s' > n/2$, we have

$$I_5 \le C\sum_{j\ge 1}\|b\|_{\mathrm{BMO}}2^{-j(s'-n/2)}\big(M_rf(x)+jM_2f(x)\big) \le C\|b\|_{\mathrm{BMO}}M_rf(x).$$

This completes our proof.

PROOF OF THEOREM 3.3. We will exploit some ideas in [**DOS**] to our present situation. Note that the condition $\mu(X) < \infty$ forces X to be bounded. It was proved that the kernel $K_{F(\sqrt[m]{L})}(x,y)$ of the operator $F(\sqrt[m]{L})$ satisfies

$$\sup_{y\in X}\big|K_{F(\sqrt[m]{L})}(x,y)\big| \le C < \infty,$$

see [**DSY**].

Since X is bounded, there exists a constant r such that $X \subset B(x,r)$ for all $x \in X$. For a fixed point $x \in X$, setting $B = B(x,r)$, then we have

$$
\begin{aligned}
\big|\big[b, F(\sqrt[m]{L})\big] f(x)\big| &= \int_X \big|(b(x) - b_B) K_{F(\sqrt[m]{L})}(x,y) f(y)\big| d\mu(y) \\
&\quad + \int_X \big|K_{F(\sqrt[m]{L})}(x,y)(b(y) - b_B) f(y)\big| d\mu(y) \\
&\le C(b(x) - b_B)\|f\|_{L^1} + C\|b\|_{\mathrm{BMO}} M_2 f(x).
\end{aligned}
$$

So, for $p > 2$ we have

$$
\begin{aligned}
\big\|\big[b, F(\sqrt[m]{L})\big] f(x)\big\|_{L^p} &\le C\|b - b_B\|_{L^p}\|f\|_{L^1} + C\|b\|_{\mathrm{BMO}}\|M_2 f\|_{L^p} \\
&\le C\|b\|_{\mathrm{BMO}}\|f\|_{L^p}.
\end{aligned}
$$

Therefore, in order to prove Theorem 3.3, we can assume that supp $F \subset [1, \infty]$. By the same notations as in the proof of Theorem 3.1, we set $F^k(\lambda) = \varphi(2^{-k}\lambda)F(\lambda)$, and

$$
\tilde{F} = \sum_{k=1}^{\infty} F^k * \xi,
$$

where ξ is a function defined in (b) of Lemma 4.2.

By repeating the proof of Theorem 3.1 and using (24) in place of (23) we can prove that the commutator $[b, \tilde{F}(\sqrt[m]{L})]$ is bounded on $L^p(X)$ for all $p > 1$. On the other hand, we have

$$
\big[b, F(\sqrt[m]{L})\big] = \big[b, (F - \tilde{F})(\sqrt[m]{L})\big] + \big[b, \tilde{F}(\sqrt[m]{L})\big].
$$

Therefore, to complete the proof of Theorem 3.3, we need only to show that $[b, (F - \tilde{F})(\sqrt[m]{L})]$ is bounded on $L^p(X)$ for $1 < p < \infty$. By a careful examination the proof of Theorem 3.1, the L^p boundedness of $[b, (F - \tilde{F})(\sqrt[m]{L})]$ follows immediately if one can prove that: for $M > s/m$, any ball $B \subset X$ the operator $H_{r,M}(\sqrt[m]{L})$, where $H_{r,M}(\lambda) = (F(\lambda) - F * \xi(\lambda))(1 - e^{-(r_B\lambda)^m})^M$, $\lambda > 0$, has a kernel $K_{H_{r,M}(\sqrt[m]{L})}(x,y)$ which satisfies

$$
\bigg(\int_{S_j(B)} \big|K_{H_{r_B,M}(\sqrt[m]{L})}(y,z)\big|^2 d\mu(y)\bigg)^{1/2} \le C 2^{-js'} V(B)^{-1/2} \tag{33}
$$

for all $z \in B$, $j \ge 1$ and $s > s' > n/2$.

Let us prove (33). We have

$$\big(F(\sqrt[m]{L}) - \tilde{F}(\sqrt[m]{L})\big)\big(I - e^{-r_B^m L}\big)^M = \sum_{k=1}^{\infty} H^k_{r_B,M}\big(\sqrt[m]{L}\big), \tag{34}$$

where $H^k_{r,M}(\lambda) = (F^k(\lambda) - F^k * \xi(\lambda))(1 - e^{-(r_B\lambda)^m})^M$, $\lambda > 0$. For $j \geq 1$ and $k \geq 1$, denote by $K_{H^k_{r_B,M}(\sqrt[m]{L})}(y,z)$ the Schwartz kernel of operator $H^k_{r_B,M}(\sqrt[m]{L})$. The Hölder inequality together with condition that X is bounded, leads to for $s > s' > n/2$,

$$\begin{aligned}
&\sup_{z\in B}\Big(\int_{S_j(B)} \big|K_{H_{r_B,M}(\sqrt[m]{L})}(y,z)\big|^2 d\mu(y)\Big)^{1/2} \\
&\leq \sum_{k=1}^{N} \sup_{z\in B}\Big(\int_{S_j(B)} \Big|K_{H^k_{r_B,M}(\sqrt[m]{L})}(y,z)\Big|^2 d\mu(y)\Big)^{1/2} \\
&\leq C\sum_{k=1}^{N} \big(2^{j/2} r_B\big)^{-s'} \sup_{z\in B}\Big(\int_X \Big|K_{H^k_{r_B,M}(\sqrt[m]{L})}(y,z)\Big|^2 d(y,z)^{2s} d\mu(y)\Big)^{1/2} \\
&\leq C_X \big(2^{j/2} r_B\big)^{-s} \sup_{z\in B}\Big(\int_X \Big|K_{H^k_{r_B,M}(\sqrt[m]{L})}(y,z)\Big|^2 d\mu(y)\Big)^{1/2} \\
&\leq \sup_{z\in B}\frac{C_X}{V(y,2^{-k})^{1/2}}\big(2^{j/2} r_B\big)^{-s}\big\|\delta_{2^k}\big(H^k_{r_B,M}\big)\big\|_{2^k,q},
\end{aligned} \tag{35}$$

where in the last inequality we apply (14) with $\operatorname{supp} H^k_{r_B,M} \subseteq [-1, 2^k+1]$ and $N = 2^k$.

The expression $H^k_{r_B,M}(\lambda) = (F^k(\lambda) - F^k * \xi(\lambda))(1 - e^{-(r_B\lambda)^m})^M$ gives

$$\begin{aligned}
\big\|\delta_{2^k}\big(H^k_{r_B,M}\big)\big\|_{2^k,q} &= \big\|\delta_{2^k}[F^k(\lambda) - F^k * \xi(\lambda)](1 - e^{-(2^k r_B t)^m})^M\big\|_{2^k,q} \\
&\leq C\min\big\{1, (2^k r_B)^{mM}\big\}\big\|\delta_{2^k}[F^k(\lambda) - F^\ell * \xi(\lambda)]\big\|_{2^k,q}.
\end{aligned} \tag{36}$$

On the other hand, by Proposition 4.6 of [**DOS**], we obtain

$$\begin{aligned}
\big\|\delta_{2^k}[F^k(\lambda) - F^k * \xi(\lambda)]\big\|_{2^k,q} &= \big\|\delta_{2^k}[\varphi_k F] - \xi_{2^k} * \delta_{2^k}[\varphi_k F]\big\|_{2^k,q} \\
&\leq C 2^{-ks}\big\|\delta_{2^k}[\varphi_k F]\big\|_{W^q_s},
\end{aligned}$$

and thus

$$\left\|\delta_{2^k}\left(H^k_{r_B,M}\right)\right\|_{2^k,q} \le C2^{-ks}\min\left\{1,(2^k r_B)^{mM}\right\}\left\|\delta_{2^k}[\varphi_k F]\right\|_{W^q_s}. \tag{37}$$

Substituting (37) back into (35), and then using the doubling property (4), we obtain (33). This complete the proof.

Acknowledgements. This paper is part of the author's PhD thesis and he thanks the supervisor, Prof. X. T. Duong for helpful comments and suggestions.

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