J. Math. Soc. Japan
Vol. 64, No. 3 (2012) pp. 851–883
doi: 10.2969/jmsj/06430851

Energy decay for a nonlinear generalized Klein-Gordon equation in exterior domains with a nonlinear localized dissipative term

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(Received Aug. 10, 2010)
(Revised Jan. 19, 2011)

Abstract. We derive an energy decay estimate for solutions to the initial-boundary value problem of a semilinear wave equation in exterior domains with a nonlinear localized dissipation. Our equation includes an absorbing term like $|u|^\alpha u$, $\alpha \geq 0$, and can be regarded as a generalized Klein-Gordon equation at least if α is closed to 0. This observation plays an essential role in our argument.

1. Introduction.

In this paper we consider the initial-boundary value problem of the nonlinear wave equations of the form:

$$u_{tt} - \Delta u + \rho(x, u_t) + g(u) = 0 \ \text{ in } \ \Omega \times R^+ \tag{1.1}$$

$$u(x, 0) = u_0(x), \ u_t(x, 0) = u_1(x) \ \text{ and } \ u(x, t)|_{\partial\Omega} = 0 \tag{1.2}$$

where Ω is an exterior domain in R^N with a smooth, say C^2, boundary $\partial\Omega$, that is, $\Omega = R^N/V$ with a compact set V in R^N, $\rho(x, v)$ is a function like $\rho(x, v) = a(x)|v|^r v$, $0 \leq r \leq 2/(N-2)^+$, and $g(u)$ is a nonlinear term like $g(u) = k_0|u|^\alpha u$, $0 \leq \alpha \leq 2/(N-2)^+$, $k_0 \geq 0$. When V is empty the boundary condition should be dropped and the problem is reduced to the Cauchy problem in the whole space R^N. We also note that when $N = 1$ and V is not empty, then $\Omega = (-\infty, a)$ or (a, ∞) for some $a \in R$.

The existence and uniqueness of global solutions to the problem (1.1)–(1.2) is standard (see, e.g., [**5**]), and the energy $E(t) \equiv (1/2)(\|u_t(t)\|^2 + \|\nabla u(t)\|^2) + \int_\Omega G(u(t))dx$ is decreasing, where $\|\cdot\|$ denotes L^2 norm in Ω and $G(u) = \int_0^u g(\eta)d\eta$. Here we are interested in the energy decay of the solutions when the effect of $\rho(x, u_t)$ is localized near a portion of the boundary $\partial\Omega$ and near infinity. To

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2010 *Mathematics Subject Classification.* Primary 35B35, 35L70.
Key Words and Phrases. energy decay, nonlinear wave equation, localized dissipation.

explain our problem more precisely let us consider the case $\rho(x, u_t) = a(x)|u_t|^r u_t$, $0 \le r \le 2/(N-2)^+$ and $g(u) = k_0|u|^\alpha u$. We set for $x_0 \in R^N$,

$$\Gamma(x_0) = \{x \in \partial\Omega \mid (x - x_0) \cdot \nu(x) > 0\}, \tag{1.3}$$

where $\nu(x)$ is the outward normal vector at $x \in \partial\Omega$, which is often used in the boundary control theory in bounded domains (cf. Russell [**26**], Chen [**2**], Lions [**4**]). V is star-shaped with respect to x_0 if and only if $\Gamma(x_0)$ is empty. We assume that $a(x)$ is a nonnegative bounded function and there exist x_0 and a (relatively) open set $\omega \subset \bar{\Omega}$ such that

$$\overline{\Gamma(x_0)} \subset \omega \quad \text{and} \quad a(x) \ge \epsilon_0 > 0 \quad \text{for} \quad x \in \omega \cup B(R)^c, \; R \gg 1, \tag{1.4}$$

with some ϵ_0, where $B(R) = \{x \in R^N \mid |x| < R\}$. This is now a rather standard assumption concerning localized dissipative term.

We also employ a stronger assumption

$$\partial\Omega \subset \omega \quad \text{and} \quad a(x) \ge \epsilon_0 > 0 \quad \text{for} \quad x \in \omega \cup B(R)^c, \; R \gg 1, \tag{1.4$'$}$$

where if $\Omega = R^N$ or V is star-shaped with respect to x_0 we drop the condition $\partial\Omega \subset \omega$ in (1.4)$'$ and in these cases (1.4) and (1.4)$'$ are coincide each other.

The problem admits a unique solution $u(\cdot) \in C([0,\infty); H_1^0(\Omega)) \cap C^1([0,\infty); L^2(\Omega))$ for each $(u_0, u_1) \in H_1^0(\Omega) \times L^2(\Omega)$. When $\rho(x, u_t) = a(x)u_t$ with $a(x) \ge \epsilon_0 > 0$ on the whole domain Ω and $g(u) = u$ it is easy to show the exponential decay:

$$E(t) \le CE(0)e^{-\lambda t} \tag{1.5}$$

with some $\lambda > 0$. The estimate (1.5) still holds for the case $g(u) = u + |u|^\alpha u$, $0 \le \alpha \le 2/(N-2)^+$ if $CE(0)$ is replaced by C_0, where C_0 denotes a constant depending on $E(0)$.

In [**28**] Zuazua treated the case: $\Omega = R^N$, $\rho(x, u_t) = a(x)u_t$ with $a(x) \ge \epsilon_0 > 0$, $|x| > R \gg 1$, and $g(u) = u + |u|^\alpha u$, and proved the exponential decay (1.5) with $CE(0)$ replaced by C_0. We note that the linear term u included in $g(u)$ plays an essential role in [**28**] and the argument is not applied to the case $g(u) = |u|^\alpha u$. That is, the equation treated in [**28**] is a semilinear Klein-Gordon equation with a linear localized dissipation near infinity. Subsequently, the present author considered in [**12**] the Cauchy problem for the case: $\rho(x, u_t) = a(x)|u_t|^r u_t$ with $a(x) \ge \epsilon_0 > 0$ for $|x| \ge R \gg 1$ and $g(u) = u$, and proved the estimate

$$E(t) \leq \begin{cases} C_1(1+t)^{-(2-Nr)/r} & \text{if } \ 0 < r < 2/N \\ C_1\{\log(2+t)\}^{-N} & \text{if } \ r = 2/N, \end{cases} \tag{1.6}$$

where we assumed $\operatorname{supp} u(x) \cup \operatorname{supp} u_1(x) \subset B(L)$, $L \gg 1$, and C_1 denotes a constant depending on $\|u_0\|_{H_2} + \|u_1\|_{H_1}$. For the nonlocalized case $\rho(x, u_t) = |u_t|^r u_t$ we know that (1.6) holds with C_1 replaced by C_0 (see [**10**]). Mochizuki and Motai [**7**] extended the result in [**10**] to the case $\operatorname{supp} u_0(x) \cup \operatorname{supp} u_1(x)$ is not compact and further proved for the case $g(u) = 0$ that

$$E(t) \leq C_0\{\log(2+t)\}^{-N}.$$

Further considerations have been done by Todorova and Yordanov [**24**], Todorova, Ugŭryu and Yordanov [**25**] for the case $\rho(x, u_t) = |u_t|^r u_t$ and $g(u) = 0$. See also Motai [**8**], Nakao and Ono [**21**], Matsuyama [**6**] and Sunagawa [**23**] for related topics.

Quite recently we have considered in [**19**] the Cauchy problem for the case like $\rho(x, u_t) = |u_t|^r u_t$, $0 \leq r \leq 2/(N-2)^+$, and $g(u) = |u|^\alpha u$, $0 \leq \alpha \leq 2/(N-2)^+$. The result in [**19**] is stated as follows:

$$E(t) \leq \begin{cases} C_1(1+t)^{-\eta} & \text{if } \ \eta > 0 \\ C_1 \log(2+t)^{-N} & \text{if } \ \eta = 0 \end{cases} \tag{1.7}$$

where we set $\eta = (\alpha+2)/(\alpha + r + \alpha r) - N$. The idea in [**19**] is to consider the equation as a nonlinear generalized Klein-Gordon equation. In earlier papers [**13**], [**20**] we also considered the usual wave equation without mass term u under linear or half-linear localized dissipations and derived some decay estimates of the energy, but, to our knowledge, no result is known for the case of nonlinear localized case: $\rho(x, u_t) = a(x)|u_t|^r u_t$ and $g(u) \equiv 0$. Thus concerning the energy decay problem for the equation (1.1)–(1.2) we can not regard the term $g(u)$ as a perturbation of the wave equation. In other words, any decay estimate of energy is not known for the problem (1.1)–(1.2) even for small amplitude solutions.

The object of this paper is to combine the idea in [**19**] with the arguments in [**28**], [**12**], [**13**], [**20**] to derive some decay estimates of the energy for the problem (1.1)–(1.2) where $\rho(x, u_t)$ is a nonlinear localized dissipation and $g(u)$ is a nonlinear absorbing term. See also [**16**] where the existence of global attractors is discussed for a related problem in exterior domains. We also use some ideas in our recent papers [**11**], [**14**], [**15**], [**18**] where the problems related to (1.1)–(1.2) in bounded domains have been considered. Quite recently, Aloui, Ibrahim and Nakanishi [**1**] have proved an exponential decay for the semilinear Klein-Gordon equation in a domain exterior to a star-shaped obstacle with a linear localized dissipation $\rho(x,u_t)=a(x)u_t$ and an arbitrary order nonlinearity $g(u)=u+f(u)$ by use of Morawetz space-time integral estimate. It seems difficult to apply the method in [**1**] to the case where $\rho(x,u_t)$ is nonlinear.

2. Preliminaries.

We use only familiar function spaces, and their definitions are omitted. We denote by $\|\cdot\|_p$ the L^p norm on Ω. We set $\Omega(R)\equiv\Omega\cap B(R)$. By use of a function $a(x)$ satisfying (1.4) or (1.4)$'$ we make the following assumption on $\rho(x,v)$.

Hyp.A. $\rho(x,v)$ is measurable in $x\in\Omega$ for any $v\in\boldsymbol{R}$ and Lipschitz continuous in v for a.e. $x\in\Omega$ with $\rho_v(x,v)\geq 0$, and satisfies:

(1)
$$k_0a(x)|v|^{r+2}\leq\rho(x,v)v\leq k_1a(x)|v|^{r+2}\quad\text{if }\ |v|\leq 1\ \text{ and }\ x\in\Omega(R),\ R\gg 1,$$

with some $k_0,k_1>0$ and r, $0\leq r<\infty$.

(2)
$$k_0a(x)|v|^{p+2}\leq\rho(x,v)v\leq k_1a(x)|v|^{p+2}\ \text{ if }\ |v|\geq 1\ \text{ and }\ x\in\Omega(R),\ R\gg 1,$$

with some $k_0,k_1>0$ and p, $0\leq p\leq 2/(N-2)^+$.

(3)
$$k_0|v|^{q+2}\leq\rho(x,v)v\leq k_1|v|^{q+2}\ \text{ if }\ x\in B(R)^c,\ R\gg 1,$$

with $k_0,k_1>0$ and $0\leq q\leq 2/(N-2)^+$.

A typical example is $\rho(x,v)=a(x)|v|^rv$ which satisfies Hyp.A with $p=q=r$. Assume that $a(x)=0$ for $R-1\leq|x|\leq R$, $R\gg 1$. Then a little more complicate example is $\rho(x,v)=\phi(x)a(x)\min\{|v|^r,|v|^p\}v+(1-\phi(x))a(x)|v|^qv$ where we assume $0\leq p\leq r$ and $\phi(x)$ is a function such that $0\leq\phi(x)\leq<1$ with $\phi(x)=0$ for $|x|>R$, $R\gg 1$, and $\phi(x)=1$ for $|x|<R-1$. We could divide the assumption (3) in two cases $|v|\leq 1$ and $|v|\geq 1$ as in (1), (2). Then more general examples would satisfy the conditions. However, to make the essential feature of the argument clear we employ the assumption (3).

Hyp.B. $g(u)$ is a Lipschitz continuous function on R satisfying:

(1)
$$g(0)=0,\quad k_0|u|^{\alpha+2}\leq G(u)\leq\frac{d_0}{2}g(u)u$$

with some $k_0 > 0$ and d_0, $0 < d_0 < 1$, where $G(u) = \int_0^u g(\eta)d\eta$, and

(2)

$$|g'(u)| \le k_2|u|^\alpha$$

with some $k_2 > 0$ and $0 \le \alpha \le 2/(N-2)^+$.

A typical example of $g(u)$ is $g(u) = |u|^\alpha u$ with $0 < \alpha \le 2/(N-2)^+$. Let us define $g(u)$ in the following way: $g(u) = |u|^\alpha u$ if $|u| \le R_1$, $g(u) = R_1^{\alpha-\beta}|u|^\beta u$ if $R_1 \le |u| \le R_2$ and $g(u) = (R_1/R_2)^{\alpha-\beta}|u|^\alpha u$ if $|u| \ge R_2$, where $\alpha, \beta > 0$ and $0 < R_1 < R_2$. This is another simple example. It is clear that we can consider $g(x,u)$ for $g(u)$, and also we could make a more general assumption on $g(u)$ so that the examples $g(u) = |u|^\alpha u + |u|^\beta u$, $g(u) = \max\{|u|^\alpha, |u|^\beta\}u$ may be included. However we employ Hyp.B to avoid inessential difficulties.

Throughout the paper we assume further that

$$\operatorname{supp} u_0(\cdot) \cup \operatorname{supp} u_1(\cdot) \subset B(L) \tag{2.1}$$

with some $L \gg 1$. It is well known that under Hyp.A and Hyp.B the problem (1.1)–(1.2) admits a unique solution $u(\cdot) \in X_{2,loc} \equiv L^\infty_{loc}([0,\infty); H_2) \cap W^{1,\infty}_{loc}([0,\infty); H^0_1) \cap W^{2,\infty}([0,\infty); L^2)$ for each $(u_0, u_1) \in H_2 \cap H^0_1 \times H^0_1$ and further it satisfies

$$\operatorname{supp} u(t,\cdot) \subset B(L+t). \tag{2.2}$$

(See John [**3**].) By density argument we see that the problem admits a unique solution $u(\cdot) \in C([0,\infty); H^0_1) \cap C^1([0,\infty); L^2)$ with $\int_0^\infty \int_\Omega \rho(x,u_t)u_t dxds \le E(0)$ for each $(u_0,u_1) \in H^0_1 \times L^2$ and (2.2) is also valid if (2.1) holds.

Our first result on energy decay reads as follows.

Theorem 2.1. *We assume that $\partial\Omega$ is not empty or $\Omega = R^N$, $N \ge 3$. Assume Hyp.A under* (1.4)′ *with $p = q = 0$ and Hyp.B. We assume further $0 < \alpha \le 2/(N-1)$. Then, for a solution $u(\cdot) \in C([0,\infty); H^0_1) \cap C^1([0,\infty); L^2)$ we have*:

$$E(t) \le C_0(L)(1+t)^{-\gamma} \quad \textit{if} \;\; 0 < \alpha < 2/(N-1) \tag{2.3}$$

with $\gamma = \min\{2/r, 2/\alpha + 1 - N\}$, and

$$E(t) \le C_0(L)(\log(2+t))^{-N} \quad \textit{if} \;\; \alpha = 2/(N-1) \tag{2.4}$$

where $C_0(L)$ denotes constants depending on $E(0)$ and L. When $\Omega = R^N$, $N \geq 3$, or V is star-shaped the above results hold with $\gamma = 2/\alpha + 1 - N$.

Remark 2.1. When $g(u) = k_0 u, k_0 > 0$, linear, the above result holds also for $\alpha = 0$. In this case we see $\gamma = 2/r$, and if further $r = 0$, we have the usual exponential decay $E(t) \leq C_0 e^{-\lambda t}$ for some $\lambda > 0$. This exponential decay estimate is also true even if $r > 0$ when V is star-shaped or $\Omega = R^N$, $N \geq 3$. If $g(u)$ is nonlinear and $\alpha = 0$ the result is delicate (cf. [**27**]).

When $p > 0$ and/or $q > 0$ in Hyp.A, (2), the result becomes more complicated. We set

$$E_1(t) = \frac{1}{2}\left(\|u_{tt}(t)\|^2 + \|\nabla u_t(t)\|^2\right).$$

Theorem 2.2. *Let $N \geq 3$ and assume Hyp.A under $(1.4)'$ with $p > 0$ and/or $q > 0$ and Hyp.B. We make the assumptions on α, r, p and q such that*

$$\frac{\alpha + 2}{q\alpha + q + \alpha} > N \tag{2.5}$$

and

$$\gamma \equiv \min\left\{\frac{2}{r}, \frac{\alpha+2}{q\alpha+q+\alpha} - N, \frac{2(2+2p-Np)}{(N-2)p}, \frac{2(2+2q-Nq)}{(N-2)q}\right\} > \frac{4N}{(N-2)(2+2\alpha-N\alpha)}. \tag{2.6$'$}$$

Then, for a solution $u(\cdot) \in X_{2,loc}$ we have

$$E(t) \leq C_1(L)(1+t)^{-\gamma} \quad \text{and} \quad E_1(t) \leq C_1(L) < \infty \tag{2.7}$$

where $C_1(L)$ denotes constants depending on $\|u_0\|_{H_2} + \|u_1\|_{H_1}$ and L.

When we replace the condition $(1.4)'$ by (1.4) there exists $\epsilon > 0$ such that if $E(0) \leq \epsilon$, then the estimate (2.7) holds under the conditions (2.5) and

$$\gamma > \frac{4}{2 + 4\alpha - N\alpha}. \tag{2.6}$$

When $\Omega = R^N$ or V is star-shaped above results hold with γ replaced by

$$\gamma = \min\left\{\frac{\alpha+2}{q\alpha+q+\alpha} - N, \frac{2(2+2p-Np)}{(N-2)p}, \frac{2(2+2q-Nq)}{(N-2)q}\right\}.$$

We note that the condition (2.6) is weaker than (2.6)$'$.

Theorem 2.3. *We assume $N = 1, 2$ and $\partial\Omega \neq \phi$. Assume Hyp.A under (1.4)$'$ with $p > 0$ and/or $q > 0$ and Hyp.B. We make the assumptions on α, r, p and q such that*

$$\frac{\alpha+2}{q\alpha+q+\alpha} > N$$

and

$$\begin{aligned}\gamma &\equiv \min\left\{\frac{2}{r}, \frac{\alpha+2}{q\alpha+q+\alpha} - N\right\}\\ &> \max\left\{\frac{2(4+2\alpha-N\alpha)}{4+6\alpha-N\alpha},\ pN\min\left\{\frac{r+2}{2r}, \frac{\alpha+2}{2(q\alpha+q+\alpha)} - \frac{N-1}{2}\right\}\right\}.\end{aligned} \tag{2.8$'$}$$

Then, for a solution $u(\cdot) \in X_{2,loc}$ we have:

$$E(t) \leq C_1(L)(1+t)^{-\gamma} \quad \text{and} \quad E_1(t) \leq C_1(L) < \infty. \tag{2.9}$$

When we replace (1.4)$'$ *by* (1.4) *there exists $\epsilon > 0$ such that if $E(0) < \epsilon$, then the estimate* (2.9) *holds under the above conditions with* (2.8)$'$ *replaced by*

$$\gamma > \frac{2(4+2\alpha-N\alpha)}{4+6\alpha-N\alpha}. \tag{2.8}$$

When V is star-shaped we can replace γ by $\gamma = \alpha + 2/(q\alpha+q+\alpha) - N$ and the condition (2.8)$'$ *by*

$$\begin{aligned}\gamma &\equiv \frac{\alpha+2}{q\alpha+q+\alpha} - N\\ &> \max\left\{\frac{2(4+2\alpha-N\alpha)}{4+6\alpha-N\alpha},\ pN\min\left\{\frac{\alpha+2}{2(q\alpha+q+\alpha)} - \frac{N-1}{2}\right\}\right\}.\end{aligned} \tag{2.6}$$

Remark 2.2. We note that the conditions in Theorems 2.2, 2.3 are satisfied if α, p, q are all small.

REMARK 2.3. If $g(u) = k_0 u, k_0 > 0$, linear, the estimates for $E(t)$ in Theorems are valid without any conditions on γ. The result is still valid even for $\Omega = R^N$, $N = 1, 2$.

We use the following lemma concerning a difference inequality which is a generalization of the inequality considered in [**9**].

LEMMA 2.1. *Let $\phi(t)$ be a nonincreasing continuous function defined on $[0, T)$ satisfying*

$$\phi(t) \le \sum_{i=1}^{m} C_i^{1/(1+r_i)} (1+t)^{\theta_i/(1+r_i)} (\phi(t) - \phi(t+1))^{1/(1+r_i)}, \quad 0 \le t < T,$$

with some $C_i > 0$, $0 \le \theta_i < 1$ and $r_i > 0$, $i = 1, \dots, m$. Then we have

$$\phi(t) \le M\left(1 + \sum_{i=1}^{m} C_i^{1/r_i}\right)(1+t)^{-\gamma}, \quad 0 \le t < T, \tag{2.9}$$

where M is a constant depending only on $\phi(0)$ and the exponent $\gamma > 0$ is given by $\gamma = \min_{i=1,\dots,m}\{(1-\theta_i)/r_i\}$.

When $0 \le \theta_i \le 1$, $i = 1, \dots, m$, and $\theta_i = 1$ for some i we have, instead of (2.9), that

$$\phi(t) \le \tilde{M}\{\log(2+t)\}^{-\tilde{\gamma}}, \tag{2.10}$$

where $\tilde{M}$ depends on $\phi(0)$ and $C_i, i = 1, \dots, m$ and the exponent $\tilde{\gamma} > 0$ is given by $\tilde{\gamma} = \min_{i=1,\dots,m}\{1/r_i\}$.

PROOF. For a proof of (2.9) see [**17**] or [**19**], where the case $m = 2$ is proved. The general case $m \ge 3$ is essentially the same. □

3. A basic inequality for $E(t)$.

In this section we derive a basic inequality on $E(t)$ for a solution $u(\cdot) \in X_{2,loc}$. We start from the following standard identities.

$$\frac{d}{dt}E(t) + \int_\Omega \rho(x, u_t) u_t dx = 0, \tag{3.1}$$

$$\frac{d}{dt}(u_t, \eta^2 u) + \int_\Omega \eta^2(x)|\nabla u|^2 dx - \int_\Omega \eta^2(x)|u_t|^2 dx$$
$$+ \int_\Omega \eta^2(x) g(u) u dx + 2\int_\Omega \nabla u \cdot \nabla\eta\eta u dx + \int_\Omega \eta^2(x)\rho(x, u_t) u dx = 0 \tag{3.2}$$

and

$$\frac{d}{dt}\int_\Omega (u_t(t), \boldsymbol{h}(x)\cdot\nabla u(t))dx + \frac{1}{2}\int_\Omega \nabla\cdot\boldsymbol{h}(x)(|u_t(t)|^2 - |\nabla u(t)|^2)dx$$
$$+\sum_{i,j}\int_\Omega \frac{\partial h_i}{\partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_i}dx - \frac{1}{2}\int_{\partial\Omega}\boldsymbol{h}\cdot\nu\left|\frac{\partial u}{\partial \nu}\right|^2 dS + \int_\Omega \rho(x,u_t)\boldsymbol{h}\cdot\nabla u dx$$
$$-\int_\Omega G(u)\nabla\cdot\boldsymbol{h}dx = 0, \tag{3.3}$$

where $\boldsymbol{h}(x) = (h_1(x), \dots, h_n(x))$.

These identities are derived by multiplying the equation by u_t, $\eta^2(x)u$ and $\boldsymbol{h}(x)\cdot\nabla u(t)$, respectively. We take a function $\phi(r)$ such that

$$\phi(r) = \begin{cases} \epsilon_0 & \text{if } \ 0 \le r \le R + |x_0| \\ \epsilon_0(R+|x_0|)/r & \text{if } \ r \ge R + |x_0|. \end{cases}$$

Proposition 3.1. *It holds that*

$$\frac{d}{dt}\chi_k(t) + \epsilon_1 E(t) + k\int_\Omega \rho(x,u_t)u_t dx$$
$$\le \frac{1}{2}\int_{\Gamma(x_0)}\left|\frac{\partial u}{\partial\nu}\right|^2 \nu\cdot\phi(|x-x_0|)(x-x_0)dx + \int_\Omega |\rho(x,u_t)|^2 dx + \int_{\Omega(R)^c}|u_t|^2 dx \tag{3.4}$$

for some $\epsilon_1 > 0$, where $k > 0$ is a large number and we set

$$\chi_k(t) = \int_\Omega u_t\phi(|x-x_0|)(x-x_0)\cdot\nabla u dx + kE(t) + m\int_\Omega u_t u dx \tag{3.5}$$

with a constant $m > 0$.

Proof. The proof is rather standard (cf. [**27**], [**13**], [**14**], [**20**] etc.) and we give an outline of it.

Combining (3.1) $\times k$, (3.3) with $\eta^2(x) = m = const. > 0$ and (3.4) with $\boldsymbol{h}(x) = x - x_0$ we have

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \chi(t) + k\int_\Omega \rho(x,u_t)u_t dx + \left(\frac{N\phi(r)+\phi'(r)r}{2} - m\right)\int_\Omega |u_t|^2 dx \\
&\quad + \left(m + \phi(r) - \frac{N\phi(r)+\phi'(r)r}{2}\right)\int_\Omega |\nabla u|^2 dx + \int_\Omega \frac{\phi'(r)((x-x_0)\cdot\nabla u)^2}{r dx} \\
&\quad + \int_\Omega \big(mg(u)u - G(u)(N\phi(r)+\phi'(r)r)\big)dx + m\int_\Omega \rho(x,u_t)u dx \\
&= \frac{1}{2}\int_{\partial\Omega}\left|\frac{\partial u}{\partial \nu}\right|^2 \nu(x)\cdot(x-x_0)dS. \qquad (3.6)
\end{aligned}
$$

Note that by Hyp.B, (1), and $\phi'(r) \le 0$,

$$
mg(u)u - G(u)(N\phi(r)+\phi'(r)r) \ge \frac{2}{d_0}\left(m - \frac{d_0 N}{2}\right)G(u), \quad d_0 < 1.
$$

We take $m > 0$, $k > 1$ such that

$$
\max\left\{\frac{d_0 N}{2}, \frac{N-1}{2}\right\}\epsilon_0 < m < \frac{N\epsilon_0}{2} < k.
$$

Then we see that

$$
\frac{2}{d_0}\left(m - \frac{d_0 N}{2}\right) \ge \epsilon_1,
$$

$$
m - \frac{N\phi(r)+\phi'(r)r}{2} + \phi(r) + \phi'(r)r \ge \epsilon_1
$$

and

$$
l(r) + \frac{N\phi(r)+\phi'(r)r}{2} - m \ge \epsilon_1
$$

for some $\epsilon_1 > 0$, where we set $l(r) = 1$ if $r \ge R + |x_0|$ and $l(r) = 0$ if $r \le R + |x_0|$. Further, since $\nu(x)\cdot\phi(r)(x-x_0) \le 0$ for $x \in \Gamma(x_0)^c \cap \partial\Omega$, we see

$$
\int_{\partial\Omega}\left|\frac{\partial u}{\partial \nu}\right|^2 \nu(x)\cdot\phi(r)(x-x_0)dS \le \int_{\Gamma(x_0)}\left|\frac{\partial u}{\partial \nu}\right|^2 \nu(x)\cdot\phi(r)(x-x_0)dS. \qquad (3.7)
$$

Thus, (3.4) follows from (3.6) and (3.7). $\square$

To control the boundary integral on the right-hand side of (3.4) we consider a vector field $\boldsymbol{h} \in (W^{1,\infty}(\Omega))^N$ such that

$$\boldsymbol{h} = \nu \ \text{ on } \ \Gamma(x_0), \ \boldsymbol{h} \cdot \nu \geq 0 \ \text{ on } \ \partial\Omega \ \text{ and } \ \boldsymbol{h}(x) = 0 \ \text{ on } \ \boldsymbol{R}^N \setminus \tilde{\omega},$$

where $\tilde{\omega}$ is an open set in $\boldsymbol{R}^N$ such that $\overline{\Gamma(x_0)} \subset \tilde{\omega} \cap \overline{\Omega} \subset \omega$. Then, from (3.3) we find

$$\begin{aligned}\int_{\Gamma(x_0)} \left|\frac{\partial u}{\partial \nu}\right|^2 dS \leq 2\frac{d}{dt}\int_\Omega u_t \boldsymbol{h} \cdot \nabla u dx + 2\int_\omega \big(|u_t(t)|^2 + |\rho(x,u_t)|^2\big)dx \\ + C\int_{\tilde{\omega}} \big(G(u) + |\nabla u|^2\big)dx. \qquad (3.8)\end{aligned}$$

Further we introduce a function

$$\eta(x) = \begin{cases} 1 & \text{on } \ \tilde{\omega} \cap \Omega \\ 0 & \text{on } \ \overline{\Omega} \cap \omega^c. \end{cases}$$

Then we see by (3.3),

$$\begin{aligned}&\int_{\Omega \cap \tilde{\omega}} (|\nabla u|^2 + g(u)u)dx \\ &\quad \leq -\frac{d}{dt}\int_\Omega \eta(x)^2 u_t u dx + C\int_\omega \big(|u|^2 + |u_t|^2 + |\rho(x,u_t)|^2\big)dx. \qquad (3.9)\end{aligned}$$

From (3.4), (3.8) and (3.9) we obtain the following.

PROPOSITION 3.2.

$$\begin{aligned}&\frac{d}{dt}\tilde{\chi}_k(t) + k\int_\Omega \rho(x,u_t)u_t dx + \epsilon_1 \int_\Omega (|u_t|^2 + |\nabla u|^2 + G(u))dx \\ &\quad \leq C\int_\omega (|u_t|^2 + |u(t)|^2)dx + C\int_{\Omega(R)^c} |u_t|^2 dx \\ &\quad\quad + C\int_\Omega \big(|\rho(x,u_t)u(t)| + |\rho(x,u_t)|^2\big)dx \qquad (3.10)\end{aligned}$$

where we set

$$\tilde{\chi_k}(t) = \chi_k(t) + C(\eta^2 u, u_t) - C(\boldsymbol{h} \cdot \nabla u, u_t).$$

We note that if $\Omega = R^N$ or V is star-shaped, then $\omega = \phi$, empty, and the last two terms in the definition of $\tilde{\chi_k}(t)$ can be dropped.

To control the L^2 norm of $u(x,t)$ on $\Omega(R)$ we prepare the following proposition.

Proposition 3.3. *Let $u(t)$ be a solution of (1.1)–(1.2) with $E(0) \le R_0$. Then, under Hyp.A with $(1.4)'$ and Hyp.B there exist $T_0 >$ independent of R_0 such that if $T > T_0$, for any $\epsilon > 0$ we have*

$$\int_t^{t+T} \int_{\Omega(R)} |u|^2 dx ds \le C_\epsilon \int_t^{t+T} \left(\int_\Omega |\rho(x,u_t)|^2 dx + \int_\omega |u_t|^2 dx \right) ds + \epsilon E(t), \tag{3.11}$$

with a constant C_ϵ depending on ϵ and R_0, where we except for the case $\Omega = R^N$, $N = 1, 2$, or V is star-shaped.

Proof. Similar inequalities are proved in [**27**], [**28**] and [**11**], [**13**], [**20**], and we show an outline of the proof.

If the assertion is not true there exist $\{t_n\}$ and solutions $\{u_n(t)\}$ such that

$$\begin{aligned} &\int_{t_n}^{t_n+T} \int_{\Omega(R)} |u_n(s)|^2 dx ds \\ &\qquad \ge n \int_{t_n}^{t_n+T} \left(\int_\omega |u_{n,t}(s)|^2 dx + \int_\Omega |\rho(x, u_{n,t}(t))|^2 \right) ds + \epsilon E_n(t), \end{aligned} \tag{3.12}$$

and $E_n(t) \le E_n(0) \le R_0$, where $E_n(t)$ is defined by $E(t)$ with $u(t)$ replaced by $u_n(t)$. We set

$$\int_{t_n}^{t_n+T} \int_{\Omega(R)} |u_n(s)|^2 dx ds = \lambda_n^2$$

and

$$u_n(\cdot + t_n)/\lambda_n = v_n(t), \quad 0 \le t \le T.$$

If λ_n does not tend to 0 we may assume $\lambda_n^2 \ge \epsilon_0 > 0$ for some $\epsilon_0 > 0$. Then we see that $\{u_n(t + t_n)\}$ is bounded in $L^\infty([0,T]; H_{1,loc}(\Omega)) \cap W^{1,\infty}([0,T]; L^2(\Omega))$ and, along a subsequence,

$u_n(\cdot + t_n) \to \tilde{u}(\cdot)$ strongly in $L^2_{loc}([0,T] \times \Omega)$ and weakly* in $L^\infty([0,T]; H_{1,loc}(\Omega))$

and

$$u_{n,t}(\cdot + t_n) \to \tilde{u}_t(\cdot) \text{ weakly}^* \text{ in } L^\infty([0,T]; L^2(\Omega)).$$

Note that

$$\int_0^T \int_{\Omega(R)} |\tilde{u}(x,s)|^2 dxds \geq \epsilon_0 > 0. \tag{3.13}$$

Further,

$$\int_0^T \int_\omega |u_{n,t}(s+t_n)|^2 dxds + \int_0^T \int_\Omega |\rho(x, u_{n,t}(s+t_n))|^2 dxds \to 0 \text{ as } n \to \infty$$

and

$$g(u_n(t_n+t)) \to g(\tilde{u}(t)) \ \text{ in } \ L^1_{loc}(\Omega \times [0,T]) \text{ as } n \to \infty.$$

Hence, the limit function $\tilde{u}(t) \in L^\infty([0,T]; \dot{H}_1(\Omega) \cap L^{\alpha+2}(\Omega)) \cap W^{1,\infty}([0,T]; L^2_{loc}(\Omega))$ satisfies the equation

$$\tilde{u}_{tt} - \Delta\tilde{u} + g(\tilde{u}) = 0 \ \text{ in } \ \Omega \times (0,T) \tag{3.14}$$

and

$$\tilde{u}_t(x,t) = 0 \ \text{ for } \ (x,t) \in \omega \cup \Omega(R)^c \times [0,T].$$

When $\partial\Omega \subset \omega$ (see $(1.4)'$), we can apply the unique continuation theorem due to Ruiz [**22**] (cf. Zuazua [**27**]) to see that there exists a certain constant $T_0 > 0$ such that if $T > T_0$, $u(x,t) = u(x) \equiv 0$ on $\Omega(R) \times [0,T]$, which contradicts to (3.13).

If λ_n tends to 0 $\{v_n(t)\}$ defined above satisfies $\|v_{n,t}(t)\|^2 + \|\nabla v_n(t)\|^2 \leq 2/\epsilon < \infty$ and very similar properties as $u_n(t+t_n)$. In particular, by the assumption $|g(u_n)/u_n| \leq C|u_n|^\alpha$, $0 < \alpha \leq 2/(N-2)^+$, we see

$$\frac{g(u_n(t_n+t))}{\lambda_n} = \frac{g(u_n)}{u_n} v_n \to 0.$$

Hence, the limit function $v \in L^\infty([0,T]; H_{1,loc} \cap \dot{H}_1(\Omega))$ with $v_t \in L^\infty([0,T]; L^2(\Omega))$ satisfies

$$v_{tt} - \Delta v = 0 \ \text{ in } \ \Omega \times (0,T) \tag{3.15}$$

and

$$v_t(x,t) = 0 \ \text{ for } \ (x,t) \in \omega \cup \Omega(R)^c \times [0,T].$$

Thus by a rather simple unique continuation theorem we see that if $T > T_0$, $v_t(x,t) \equiv 0$ on $\Omega \times [0,T]$, which implies $v(x,t) = const. = 0$ if $\partial\Omega$ is not empty or $\Omega = R^N$, $N \geq 3$. This contradicts to $\|v(t)\|_{L^2([0,T]\times\Omega(R))} = 1$. $\square$

Under the weaker assumption (1.4) we replace Proposition 3.3 by the following:

PROPOSITION 3.4. *Let $u(t)$ be a solution of* (1.1)–(1.2) *with $E(0) \leq R_0$, satisfying additional condition*

$$\|u_{tt}(t)\| + \|\nabla u_t(t)\| \leq K$$

for some $K > 0$. Then, under Hyp.A with (1.4) *and Hyp.B, there exist a large $T_0 > 0$ and a small $\delta > 0$ such that if $T > T_0$ and $E(0) < \delta$, we have the estimate* (3.11) *for any $0 < \epsilon \ll 1$, where T_0 is independent of R_0 and K.*

PROOF. By the same argument as above we obtain (3.14) if λ_n^2 does not tend to 0. Under the weaker assumption $\Gamma(x_0) \subset \omega$ (see (1.4)) it seems difficult to apply the unique continuation theorem by Ruiz. However, under the additional assumption we see that $\tilde{u}(t) \in \tilde{X}_2(T) \equiv L^\infty([0,T];\dot{H}_2 \cap L^{\alpha+2}) \cap W^{1,\infty}([0,T];\dot{H}_1) \cap W^{2,\infty}([0,T];L^2_{loc}(\Omega))$ and if $E(0) < \delta \ll 1$ we can use a simpler unique continuation theorem (see Appendix) and conclude again $\tilde{u}(x,t) \equiv 0$ on $\Omega(R) \times [0,T]$, $T > T_0$. Thus, we have a contradiction. $\square$

REMARK 3.1. If $\alpha = 0$ we have,instead of (3.14),

$$v_{tt} - \Delta v + m(x,t)v = 0 \ \text{ in } \ (0,T) \times \Omega$$

with $m \in L^\infty((0,T) \times \Omega(R))$. It is delicate whether we can conclude $v(x,t) \equiv 0$ on $[0,T] \times \Omega(R)$ or not. When $g(u) = u$, linear, we see $m(x,t) \equiv 1$ and the assertion holds even for the case $\Omega = R^N$, $N = 1,2$.

Now, we take $T, T > \max\{T_0, 1\}$. Then we arrive at the following basic inequality for $E(t)$.

PROPOSITION 3.5. *For $T > T_0$, we have*

$$\tilde{\chi_k}(t+T) - \tilde{\chi_k}(t) + k\int_t^{t+T}\int_\Omega \rho(x,u_t)u_t dxds + \epsilon_1 \int_t^{t+T} E(s)ds$$
$$\leq C\int_t^{t+T}\Big(\int_\omega |u_t|^2dx + \int_{\Omega(R)^c}|u_t|^2dx$$
$$+\int_\Omega |\rho(x,u_t)|^2dx + \int_\Omega |\rho(x,u_t)||u|dx\Big)ds \tag{3.16}$$

where we recall

$$\tilde{\chi_k}(t) = kE(t) + \int_\Omega u_t(t)\phi(r)(x-x_0)\cdot\nabla u(t)dx + m\int_\Omega u_t(t)u(t)dx$$
$$+ C(\eta^2 u(t), u_t(t)) - C(\boldsymbol{h}\cdot\nabla u(t), u_t).$$

We note that if V is star-shaped, the last two terms appearing in the definition of $\tilde{\chi}_k(t)$ should be dropped. Under the weaker condition (1.4) we assume in addition that $E(0) < \delta \ll 1$ and $\|u_{tt}(t)\| + \|\nabla u(t)\| \leq K < \infty$.

Remark 3.2. When $\rho(x,u_t) = a(x)u_t$ with (1.4), linear, we can show instead of (3.10),

$$\hat{\chi}_k(t+T) - \hat{\chi}_k(t) + k\int_t^{t+T}\int_\Omega \rho(x,u_t)u_t dxds + \epsilon_1 \int_t^{t+T} E(s)ds$$
$$\leq C\int_t^{t+T}\Big(\int_\omega |u_t|^2dx + \int_\Omega a(x)|u_t|^2\Big)dx, \tag{3.10$'$}$$

where $\hat{\chi}_k(t) = \chi_k(t) + \int_\Omega a(x)|u(t)|^2dx$. From $(3.10)'$ and the fact $\int_0^\infty\int_\Omega a(x)\cdot|u_t|^2dxdt \leq E(0) < \infty$ we see for a large $k > 0$,

$$\int_0^\infty E(t)dt \leq \tilde{\chi}_k(0) + C_0 \leq C_0 < \infty.$$

Since

$$\frac{d}{dt}\{(1+t)E(t)\} = E(t) + (1+t)\frac{d}{dt}E(t) \leq E(t)$$

we obtain

$$E(t) \leq C_0(1+t)^{-1}.$$

This is true for $0 < \alpha \leq 2/(N-2)^+$, which is a new result for our semilinear wave equation (cf. [**13**]).

4. Difference inequalities for $E(t)$.

We have by (3.1),

$$k_0 \int_t^{t+T} \int_{\Omega(R)^c} |u_t(s)|^{q+2} dxds \leq \int_t^{t+T} \int_\Omega \rho(x, u_t) u_t dxds$$
$$= E(t) - E(t+T) \equiv D(t)^2 \tag{4.1}$$

and

$$\int_t^{t+T} \int_{\Omega(R)^c} |u_t(s)||u(s)| dxds$$
$$\leq \left(\int_t^{t+T} \int_{\Omega(R)^c} |u_t(s)|^{q+2} dxds \right)^{1/(q+2)}$$
$$\times \left(\int_t^{t+T} \int_{\Omega(R)^c} |u(s)|^{(q+2)/(q+1)} dxds \right)^{(q+1)/(q+2)}$$
$$\leq CD(t)^{2/(q+2)} \left(\int_t^{t+T} \int_{\Omega(R)^c} |u(s)|^{(q+2)/(q+1)} dxds \right)^{(q+1)/(q+2)}. \tag{4.2}$$

Here, by the fact $\operatorname{supp} u(t) \subset B(L+t)$,

$$\left(\int_t^{t+T} \int_{\Omega(R)^c} |u(s)|^{(q+2)/(q+1)} dxds \right)^{(q+1)/(q+2)}$$
$$\leq \left(\int_t^{t+T} \int_\Omega |u(s)|^{\alpha+2} dx \right)^{1/(\alpha+2)} \left(\int_{B(L+t)} 1 dx \right)^{(q\alpha+\alpha+q)/(q+2)(\alpha+2)}$$
$$\leq C(L)(1+t)^{N(q\alpha+q+\alpha)/(q+2)(\alpha+2)} E(t)^{1/(\alpha+2)}.$$

Hence we have

$$\int_t^{t+T} \int_{\Omega(R)^c} |u_t(s)||u(s)| dxds$$
$$\leq C(L)(1+t)^{N(q\alpha+q+\alpha)/(q+2)(\alpha+2)} D(t)^{2/(q+2)} E(t)^{1/(\alpha+2)} \equiv A_1(t)^2. \tag{4.3}$$

We know from (4.3) that there exist $t_1 \in [t, t+T/4]$, $t_2 \in [t+3T/4, t+T]$ such that

$$\int_{\Omega(R)^c} |(u_t(t_i), u(t_i))| dx \leq \frac{4}{T} A_1(t)^2, \quad i = 1, 2. \tag{4.4}$$

Thus, by Proposition 3.5 with $t = t_1$, $t+T = t_2$, $\epsilon = \epsilon_1/2$ and (4.4) we have

$$\begin{aligned}
&\epsilon_1 \int_{t_1}^{t_2} E(s) ds \\
&\leq C \int_{t_1}^{t_2} \left(\int_\omega |u_t|^2 dx + \int_{\Omega(R)^c} |u_t|^2 dx + \int_\Omega |\rho(x, u_t)|^2 dx + \int_\Omega |\rho(x, u_t)||u| dx \right) ds \\
&\quad + 2kE(t_1) + 2 \sum_{i=1,2} \bigg\{ \int_\Omega |u_t(t_i) \phi(r)(x - x_0) \cdot \nabla u(t_i)| dx \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + C|(\boldsymbol{h} \cdot \nabla u(t_i), u_t(t_i))| \bigg\} \\
&\quad + m \sum_{i=1,2} \left\{ \int_\Omega |u_t(t_i) u(t_i)| dx + C(\eta^2 u(t_i), u_t(t_i)) \right\} \\
&\leq C \int_{t_1}^{t_2} \left(\int_\omega |u_t|^2 dx + \int_{\Omega(R)^c} |u_t|^2 dx + \int_\Omega |\rho(x, u_t)|^2 dx + \int_\Omega |\rho(x, u_t)||u| dx \right) ds \\
&\quad + \frac{5}{2} kE(t_1) + CA_1(t)^2 + C \sum_{i=1,2} \int_{\Omega(R)} |u_t(t_i)||u(t_i)| dx
\end{aligned} \tag{4.5}$$

for a large $k > 0$.

Further, if $\partial\Omega \neq \phi$ we see, by Poincare's inequality,

$$C \sum_{i=1,2} \int_{\Omega(R)} |u_t(t_i) u(t_i)| dx \leq C \sum_{i=1,2} \|u_t(t_i)\| \|\nabla u(t_i)\| \leq CE(t_1) \tag{4.6}$$

and if $\Omega = R^N$, $N \geq 3$,

$$\begin{aligned}
C \sum_{i=1,2} \int_{\Omega(R)} |u_t(t_i) u(t_i)| dx &\leq C \|u_t(t_i)\| \left\{ \int_\Omega |u(t_i)|^{2N/(N-2)} dx \right\}^{(N-2)/2N} \\
&\leq C \sum_{i=1,2} \|u_t(t_i)\| \|\nabla u(t_i)\| \leq CE(t_1).
\end{aligned} \tag{4.6'}$$

Moreover,

$$
\begin{aligned}
3kE(t_1) &= 3kE(t_2) + 3k \int_{t_1}^{t_2} \int_\Omega \rho(x, u_t) u_t dx ds \\
&\le \frac{3k}{t_2 - t_1} \int_{t_1}^{t_2} E(s) ds + 3kD(t)^2 \\
&\le \frac{\epsilon_1}{2} \int_{t_1}^{t_2} E(s) ds + 3kD(t)^2 \qquad (4.7)
\end{aligned}
$$

where we take further $T > 3k\epsilon_1^{-1}$. Then, we have from (4.5), (4.6) (or (4.6)$'$) and (4.7) that

$$
\int_{t_1}^{t_2} E(s) ds \le C\big(I_1 + I_2 + I_3 + D(t)^2 + A_1(t)^2\big) \qquad (4.8)
$$

where we set

$$
\begin{aligned}
I_1 &= C \int_t^{t+T} \left(\int_\omega |u_t|^2 dx + \int_{\Omega(R)^c} |u_t|^2 dx \right) ds, \\
I_2 &= C \int_t^{t+T} \int_\Omega |\rho(x, u_t)|^2 dx ds
\end{aligned}
$$

and

$$
I_3 = C \int_t^{t+T} \int_\Omega |\rho(x, u_t)||u| dx ds.
$$

(For the definition of $A_1(t)^2$ see (4.3).)

Further we see from (3.1) that

$$
E(t) \le E(t_2) + \int_t^{t+T} \int_\Omega \rho(x, u_t) u_t dx ds \le \frac{1}{T} \int_{t_1}^{t_2} E(s) ds + D(t)^2
$$

and hence, recalling the definition of $A_1(t)^2$,

$$
\begin{aligned}
E(t) &\le C(I_1 + I_2 + I_3 + D(t)^2) \\
&\quad + C(L)(1+t)^{N(q\alpha+q+\alpha)/(q+2)(\alpha+1)} D(t)^{2(\alpha+2)/(q+2)(\alpha+1)}. \qquad (4.9)
\end{aligned}
$$

Our task is to estimate the terms $I_i, i = 1, 2, 3$. For this we introduce the following notations:

$$\Omega_1(t) = \{x \in \Omega \mid |u_t(x,t)| \le 1\}, \quad \Omega_2(t) = \{x \in \Omega \mid |u_t(x,t)| \ge 1\},$$
$$\omega_i(t) = \Omega_i(t) \cap \omega, \quad i = 1, 2,$$

and

$$\Omega_i(t, R) = \Omega_i(t) \cap B(R), \quad i = 1, 2.$$

Then,

$$\begin{aligned}
I_1 &\le C\Big(\int_t^{t+T}\int_{\omega_1(s)} |u_t|^{r+2} dxds\Big)^{2/(r+2)} + C\int_t^{t+T}\int_{\omega_2(s)} |u_t|^{p+2} dxds \\
&\quad + C(L)(1+t)^{Nq/(q+2)}\Big(\int_t^{t+T}\int_{\Omega(R)^c} |u_t|^{q+2} dxds\Big)^{2/(q+2)} \\
&\le C\big(D(t)^{4/(r+2)} + D(t)^2 + C(L)(1+t)^{Nq/(q+2)} D(t)^{4/(q+2)}\big). \qquad (4.10) \\
I_2 &\le C\int_t^{t+T}\int_{\Omega_1(s,R)} a(x)|u_t|^{2(r+1)} dxds + C\int_t^{t+T}\int_{\Omega_2(s,R)} a(x)|u_t|^{2(p+1)} dxds \\
&\quad + C\int_t^{t+T}\int_{\Omega(R)^c} |u_t|^{2(q+1)} dxds \\
&\le C\int_t^{t+T}\int_{\Omega_1(s,R)} a(x)|u_t|^{r+2} dxds \\
&\quad + C\int_t^{t+T}\left(\int_{\Omega_2(s,R)} a(x)|u_t|^{p+2} dx\right)^{2(p+1)(1-\theta_1)/(p+2)} \|u_t(s)\|_\beta^{2(p+1)\theta_1} ds \\
&\quad + C\int_t^{t+T}\left(\int_{\Omega(R)^c} |u_t|^{q+2} dx\right)^{2(q+1)(1-\theta_2)/(q+2)} \|\nabla u_t(s)\|_\beta^{2(q+1)\theta_2} ds
\end{aligned}$$

where $\beta = 2N/(N-2)^+$ and

$$\theta_1 = \frac{Np}{(p+1)(2N-(p+2)(N-2)^+)}$$

and

$$\theta_2 = \frac{Nq}{(q+1)(2N-(q+2)(N-2)^+)}.$$

(A trivial modification is needed if $N = 2$.) Hence, by Gagliardo-Nirenberg inequality and the definition of $D(t)^2$, we see

$$\begin{aligned} I_2 &\le CD(t)^2 + CD(t)^{4(p+1)(1-\theta_1)/(p+2)} E(t)^{(p+1)\theta_1(1-\tilde{\theta})} \|\nabla u_t(s)\|^{2(p+1)\theta_1\tilde{\theta}} \\ &\quad + CD(t)^{4(q+1)(1-\theta_2)/(q+2)} E(t)^{(q+1)\theta_2(1-\tilde{\theta})} \|\nabla u_t(s)\|^{2(q+1)\theta_2\tilde{\theta}} \qquad (4.11) \end{aligned}$$

where

$$\tilde{\theta} = (1/2 - 1/\beta)N = \begin{cases} 1 & \text{if } N \ge 3, \\ 1-\delta,\ 0<\delta \ll 1, & \text{if } N = 2, \\ 1/2 & \text{if } N = 1. \end{cases}$$

Finally,

$$\begin{aligned} I_3 &\le C \int_t^{t+T} \biggl(\int_{\Omega_1(s,R)} a(x)|u_t|^{r+1}|u|dx \\ &\qquad + \int_{\Omega_2(s,R)} a(x)|u_t|^{p+1}|u|dx + \int_{\Omega(R)^c} |u_t|^{q+1}|u|dx \biggr) ds \\ &\equiv I_{3,1} + I_{3,2} + I_{3,3}. \qquad (4.12) \end{aligned}$$

Here, we see

$$\begin{aligned} I_{3,1} &\le C\biggl(\int_t^{t+T} \int_{\Omega_1(s,R)} a(x)|u_t|^{2(r+1)} dxds \biggr)^{1/2} \biggl(\int_t^{t+T} \int_{\Omega_1(s,R)} |u|^2 dxds \biggr)^{1/2} \\ &\le C\biggl(\int_t^{t+T} \int_{\Omega_1(s,R)} a(x)|u_t|^{r+2} dxds \biggr)^{1/2} \biggl(\int_t^{t+T} \int_{\Omega(R)} |u|^2 dxds \biggr)^{1/2} \\ &\le CD(t)\biggl(\int_t^{t+T} \int_{\Omega(R)} |u|^2 dxds \biggr)^{1/2}. \qquad (4.13) \end{aligned}$$

Further,

$$\left(\int_{\Omega(R)}|u|^2dx\right)^{1/2} \le C\left(\int_{\Omega(R)}|u|^{2N/(N-2)}dx\right)^{(N-2)/2N}$$
$$\le C\|\nabla u(t)\| \le C\sqrt{E(t)} \ \text{ if } \ N \ge 3.$$

When $\partial\Omega \ne \phi$ the result is also true for the case $N = 1, 2$ due to Poincare's inequality. (When $\alpha = 0$ the inequality $\|u(t)\| \le C\sqrt{E(t)}$ is trivial even for the case $\Omega = R^N$, $N = 1, 2$.)

Thus, we have from (4.13) that

$$I_{3,1} \le CD(t)\sqrt{E(t)}. \tag{4.14}$$

Similarly, we have

$$\begin{aligned} I_{3,2} &\le C\left(\int_t^{t+T}\int_{\Omega_2(s,R)} a(x)|u_t|^{p+2}dxds\right)^{(p+1)/(p+2)} \\ &\quad\times\left(\int_t^{t+T}\int_{\Omega(R)}|u|^{p+2}dxds\right)^{1/(p+2)} \\ &\le CD(t)^{2(p+1)/(p+2)}\sqrt{E(t)}. \end{aligned} \tag{4.15}$$

The treatment of the term $I_{3,3}$ is a little more delicate and we need the fact $\operatorname{supp} u(t) \subset B(L + t)$. We see

$$I_{3,3} \le C\left(\int_t^{t+T}\int_{\Omega(R)^c}|u_t|^{q+2}dxds\right)^{(q+1)/(q+2)}\left(\int_t^{t+T}\int_{\Omega(R)^c}|u|^{q+2}dxds\right)^{1/(q+2)}.$$

Here, if $\alpha \ge q$,

$$\begin{aligned}\int_{\Omega(R)^c}|u|^{q+2}dx &\le \left(\int_{B(L+t)}|u|^{\alpha+2}dx\right)^{(q+2)/(\alpha+2)}\left(\int_{B(L+t)}1dx\right)^{(\alpha-q)/(\alpha+2)} \\ &\le C(L)(1+t)^{N(\alpha-q)/(\alpha+2)}E(t)^{(q+2)/(\alpha+2)}\end{aligned}$$

and if $\alpha \le q$,

$$\begin{aligned}\int_{\Omega(R)^c}|u|^{q+2}dx &\le C\|u(t)\|_{\alpha+2}^{(q+2)(1-\theta_3)}\|\nabla u(t)\|^{(q+2)\theta_3} \\ &\le CE(t)^{(q+2)(2+\alpha\theta_3)/2(\alpha+2)}\end{aligned}$$

with

$$\theta_3 = \frac{1/(\alpha+2) - 1/(q+2)}{1/N + 1/(\alpha+2) - 1/2} = \frac{2N(q-\alpha)}{(q+2)(4+2\alpha-N\alpha)}.$$

Hence we have

$$I_{3,3} \le C(L) D(t)^{2(q+1)/(q+2)} E(t)^{1/(\alpha+2)} w(t) \tag{4.16}$$

where we set

$$w(t) = \begin{cases} (1+t)^{N(\alpha-q)/(\alpha+2)(q+2)} & \text{if } \alpha \ge q, \\ E(t)^{N\alpha(q-\alpha)/(\alpha+2)(q+2)(4+2\alpha-N\alpha)} & \text{if } \alpha \le q. \end{cases}$$

Summarizing above we obtain from (4.9) that

$$\begin{aligned} E(t) &\le C\big(D(t)^{4/(r+2)} + D(t)^2\big) + C(L)(1+t)^{Nq/(q+2)} D(t)^{4/(q+2)} \\ &\quad + C D(t)^{4(p+1)(1-\theta_1)/(p+2)} E(t)^{(p+1)\theta_1(1-\tilde{\theta})} \|\nabla u_t(s)\|^{2(p+1)\theta_1\tilde{\theta}} \\ &\quad + C D(t)^{4(q+1)(1-\theta_2)/(q+2)} E(t)^{(q+1)\theta_2(1-\tilde{\theta})} \|\nabla u_t(s)\|^{2(q+1)\theta_2\tilde{\theta}} \\ &\quad + C(D(t) + D(t)^{2(p+1)/(p+2)})\sqrt{E(t)} + C(L) D(t)^{2(q+1)/(q+2)} E(t)^{1/(\alpha+2)} w(t) \\ &\quad + C(L)(1+t)^{N(q\alpha+q+\alpha)/(q+2)(\alpha+1)} D(t)^{2(\alpha+2)/(q+2)(\alpha+1)}. \end{aligned} \tag{4.17}$$

Noting that

$$w(t)^{(\alpha+2)/(\alpha+1)} \le C_0(L)(1+t)^{N(q\alpha+q+\alpha)/(q+2)(\alpha+1)}$$

and absorbing $\sqrt{E(t)}$ appearing in the right-hand side of (4.16) into the left-hand side we arrive at the difference inequality for $E(t)$.

Proposition 4.1.

$$\begin{aligned} E(t) &\le C\big(D(t)^{4/(r+2)} + D(t)^2 + D(t)^{4(p+1)/(p+2)}\big) + C(L)(1+t)^{Nq/(q+2)} D(t)^{4/(q+2)} \\ &\quad + C D(t)^{4(p+1)(1-\theta_1)/(p+2)} E(t)^{(p+1)\theta_1(1-\tilde{\theta})} \|\nabla u_t(s)\|^{2(p+1)\theta_1\tilde{\theta}} \\ &\quad + C D(t)^{4(q+1)(1-\theta_2)/(q+2)} E(t)^{(q+1)\theta_2(1-\tilde{\theta})} \|\nabla u_t(s)\|^{2(q+1)\theta_2\tilde{\theta}} \\ &\quad + C_0(L)(1+t)^{N(q\alpha+q+\alpha)/(q+2)(\alpha+1)} D(t)^{2(\alpha+2)/(q+2)(\alpha+1)}, \end{aligned} \tag{4.18}$$

where we recall

$$\theta_1 = \frac{Np}{(p+1)(2N-(p+2)(N-2)^+)}, \quad \theta_2 = \frac{Nq}{(q+1)(2N-(q+2)(N-2)^+)}$$

and

$$\tilde{\theta} = (1/2 - 1/\beta)N = \begin{cases} 1 & \textit{if} \ \ N \geq 3, \\ 1-\delta, \ 0 < \delta \ll 1, & \textit{if} \ \ N = 2, \\ 1/2 & \textit{if} \ \ N = 1. \end{cases}$$

When $\Omega = R^N$ *or* V *is star-shaped the term* $D(t)^{4/(r+2)}$ *in* (4.18) *should be dropped.*

5. Proof of Theorem 2.1.

We assume $p = q = 0$. Then $\theta_1 = \theta_2 = 0$ and (4.17) is reduced to the simple difference inequality

$$E(t) \leq \big(C_0 D(t)^{4/(r+2)} + C_0(L)(1+t)^{N\alpha/2(\alpha+1)} D(t)^{(\alpha+2)/(\alpha+1)}\big). \tag{5.1}$$

Applying Lemma 2.1 to (5.1) we have

$$E(t) \leq C_0(L)(1+t)^{-\gamma} \ \text{ if } \ (N-1)\alpha < 2 \tag{5.2}$$

with $\gamma = \min\{2/r, 2/\alpha + 1 - N\}$ and

$$E(t) \leq C_0(L)(\log(2+t))^{-N} \ \text{ if } \ \alpha = 2/(N-1). \tag{5.3}$$

When $\Omega = R^N$ or V is star-shaped the term $D(t)^{4/(r+2)}$ is ignored and we have the estimate (5.2) with $\gamma = 2/\alpha + 1 - N$ and also (5.4).

6. Proof of Theorems 2.2, 2.3.

We employ a 'loan' method. Since $D(t) \leq C_0 < \infty$ (4.18) is reduced to a little simpler form

$$
\begin{aligned}
E(t) \le\ & C_0(L)\big\{D(t)^{4/(r+2)} + (1+t)^{N(q\alpha+q+\alpha)/(q+2)(\alpha+1)} D(t)^{2(\alpha+2)/(q+2)(\alpha+1)}\big\} \\
& + CD(t)^{4(p+1)(1-\theta_1)/(p+2)} E(t)^{(p+1)\theta_1(1-\tilde{\theta})} \|\nabla u_t(s)\|^{2(p+1)\theta_1\tilde{\theta}} \\
& + CD(t)^{4(q+1)(1-\theta_2)/(q+2)} E(t)^{(q+1)\theta_2(1-\tilde{\theta})} \|\nabla u_t(s)\|^{2(q+1)\theta_2\tilde{\theta}}. \qquad (6.1)
\end{aligned}
$$

We fix T such that $T > 4T_0$ and take any $\tilde{T} > T > 4T_0$. We assume for a moment that

$$
\|u_{tt}(t)\| + \|\nabla u_t(t)\| \le K, \quad 0 \le t \le \tilde{T} + T. \qquad (6.2)
$$

In fact, this is true for $0 \le t \le \tilde{T}+T$ if we choose a large $K = K(\tilde{T}) \gg 1$. We must show that K can be chosen independently of $\tilde{T}$. Anyway, under the assumption of (6.2) we have from (6.1) that

$$
\begin{aligned}
E(t) \le C_0(L)\big\{& D(t)^{4/(r+2)} + (1+t)^{N(q\alpha+q+\alpha)/(q+2)(\alpha+1)} D(t)^{2(\alpha+2)/(q+2)(\alpha+1)} \\
& + CK^{2(p+1)\theta_1\tilde{\theta}} D(t)^{4(p+1)(1-\theta_1)/(p+2)} E(t)^{(p+1)\theta_1(1-\tilde{\theta})} \\
& + CK^{2(q+1)\theta_2\tilde{\theta}} D(t)^{4(q+1)(1-\theta_2)/(q+2)} E(t)^{(q+1)\theta_2(1-\tilde{\theta})}\big\}, \\
& \qquad 0 \le t \le \tilde{T} + T. \qquad (6.3)
\end{aligned}
$$

First we consider the case $N \ge 3$. Then,

$$
\theta_1 = Np/(p+1)(4+2p-Np), \quad \theta_2 = Nq/(q+1)(4+2q-Nq) \ \text{ and } \ \tilde{\theta} = 1.
$$

Hence we have from (6.3) that

$$
\begin{aligned}
E(t) \le C_0(L)\big\{& D(t)^{4/(r+2)} + (1+t)^{N(q\alpha+q+\alpha)/(q+2)(\alpha+1)} D(t)^{2(\alpha+2)/(q+2)(\alpha+1)} \\
& + K^{2Np/(4+2p-Np)} D(t)^{4(2+2p-Np)/(4+2p-Np)} \\
& + K^{2Nq/(4+2q-Nq)} D(t)^{4(2+2q-Nq)/(4+2q-Nq)}\big\}. \qquad (6.4)
\end{aligned}
$$

Applying Lemma 2.2 to (6.4) we can derive the decay estimate for $E(t)$ which is stated as follows:

Proposition 6.1. *Let $K \gg 1$ and assume*

$$
\|u_{tt}(t)\| + \|\nabla u_t(t)\| \le K, \quad 0 \le t < \tilde{T} + T.
$$

Then, if $3 \leq N < (\alpha+2)/(q\alpha+q+\alpha)$*, we have*

$$E(t) \leq C_0(L)K^{2N/(N-2)}(1+t)^{-\gamma}, \quad 0 \leq t \leq \tilde{T}+T \tag{6.5}$$

with

$$\gamma = \min\left\{\frac{2}{r}, \frac{\alpha+2}{q\alpha+q+\alpha} - N, \frac{2(2+2p-Np)}{(N-2)p}, \frac{2(2+2q-Nq)}{(N-2)q}\right\}.$$

When $\Omega = R^N$ *or* V *is star-shaped we can drop* $2/r$ *in the definition of* γ.

Next we consider the case $N = 1, 2$. Then we see $\theta_1 = p/2(p+1)$, $\theta_2 = q/2(q+1)$, and using the fact $E(t) \leq E(0)$ we have

$$\begin{aligned} E(t) &\leq C_0(L)\bigl\{D(t)^{4/(r+2)} + (1+t)^{(q\alpha+q+\alpha)/(q+2)(\alpha+1)}D(t)^{2(\alpha+2)/(q+2)(\alpha+1)}\bigr\} \\ &\quad + C\bigl\{K^{p\tilde{\theta}}D(t)^2E(t)^{p(1-\tilde{\theta})/2} + K^{q\tilde{\theta}}D(t)^2E(t)^{q(1-\tilde{\theta})/2}\bigr\} \\ &\leq C_0(L)\bigl\{D(t)^{4/(r+2)} + (1+t)^{(q\alpha+q+\alpha)/(q+2)(\alpha+1)}D(t)^{2(\alpha+2)/(q+2)(\alpha+1)} \\ &\qquad + K^{p\tilde{\theta}}D(t)^{2(1-\nu)} + K^{q\tilde{\theta}}D(t)^{2(1-\nu)}\bigr\} \end{aligned} \tag{6.6}$$

with any $0 \leq \nu < 1$. Applying Lemma 2.1 we have the following estimates.

Proposition 6.2. *Let* $K \gg 1$ *and assume*

$$\|u_{tt}(t)\| + \|\nabla u_t(t)\| \leq K, \quad 0 \leq t \leq \tilde{T}+T.$$

Further assume that $N = 1, 2$ *and* $N < (\alpha+2)/(q\alpha+q+\alpha)$*. Then we have*

$$E(t) \leq C_0(L)K^m(1+t)^{-\gamma}, \quad 0 \leq t \leq \tilde{T}+T \tag{6.7}$$

with

$$m = p\tilde{\theta}\nu^{-1} \quad \textit{and} \quad \gamma = \min\left\{\frac{2}{r}, \frac{\alpha+2}{q\alpha+q+\alpha} - N, \frac{1-\nu}{\nu}\right\},$$

where ν, $0 < \nu < 1$, *is arbitrary.* ($2/r$ *can be dropped when* V *is star-shaped or* $\Omega = R^N$.)

We shall choose ν as

$$\frac{1-\nu}{\nu} = \min\left\{\frac{2}{r}, \frac{\alpha+2}{q\alpha+q+\alpha} - N\right\},$$

that is,

$$\nu = \max\left\{\frac{r}{r+2}, \frac{q\alpha+q+\alpha}{\alpha+2-(N-1)(q\alpha+q+\alpha)}\right\}.$$

Then we see

$$m = p\tilde{\theta}\min\left\{\frac{r+2}{r}, \frac{\alpha+2}{q\alpha+q+\alpha} - (N-1)\right\}.$$

By use of the estimates (6.5) and (6.7) with above m, γ we shall derive the estimate for $\|u_{tt}(t)\| + \|\nabla u_t(t)\|$. We employ a similar argument as in [**17**], [**18**]. We set

$$E_1(t) = \frac{1}{2}\big(\|u_{tt}(t)\|^2 + \|\nabla u_t(t)\|^2\big).$$

PROPOSITION 6.3. *Assume that*

$$E(t) \le CK^m(1+t)^{-\gamma}, \quad 0 \le t \le \tilde{T} + T, \tag{6.8}$$

with some $m \ge 0$ and $\gamma > 0$. Assume further that there exists ϵ, $0 \le \epsilon \le 1$, such that

$$\frac{\epsilon\gamma(2+4\alpha-N\alpha)}{4} > 1 \quad \text{if} \quad N \ge 3 \quad \text{and} \quad \frac{\epsilon\gamma(4+6\alpha-N\alpha)}{2(4+2\alpha-N\alpha)} > 1 \quad \text{if} \quad N = 1,2. \tag{6.9}$$

Then we have

$$E_1(t) \le \big\{C_1 + C(\epsilon)E(0)^\mu K^\eta\big\}^2, \quad 0 \le t \le \tilde{T} + T \tag{6.10}$$

where C_1 is a constant depending on $\|u_0\|_{H_2} + \|u_1\|_{H_1}$ and the exponents μ, η are given by

$$\mu = (1-\epsilon)/(2+4\alpha+N\alpha),$$
$$\eta = \begin{cases} (N-2)\alpha/2 + \epsilon m(2+4\alpha-N\alpha)/4 & \text{if} \quad N \ge 3, \\ \epsilon m(4+6\alpha-N\alpha)/2(4+2\alpha-N\alpha) & \text{if} \quad N = 1,2. \end{cases} \tag{6.11}$$

(*When* $N=2$, *the exponent* η *in* (6.10) *should be replaced by* $\eta+\delta$, $0<\delta\ll 1$.)

PROOF. Differentiating the equation we have formally,

$$u_{ttt}-\Delta u_t+\rho_v(x,u_t)u_{tt}=-g'(u)u_t. \tag{6.12}$$

Multiplying the equation by u_{tt} and integrating we have

$$\frac{d}{dt}E_1(t)\le C\int_\Omega |u|^\alpha|u_t||u_{tt}|dx\le C\Big(\int_\Omega |u|^{2\alpha}|u_t|^2dx\Big)^{1/2}\|u_{tt}(t)\|_2$$

and hence

$$\frac{d}{dt}\sqrt{E_1(t)}\le C\Big(\int_\Omega |u|^{2\alpha}|u_t|^2dx\Big)^{1/2}. \tag{6.13}$$

(6.13) is valid in the distribution sense for the solutions $u(\cdot)\in X_{2,loc}$. Here,

$$\begin{aligned}
&\int_\Omega |u|^{2\alpha}|u_t|^2dx\\
&\le\Big(\int_\Omega |u|^{2N/(N-2)^+}dx\Big)^{(N-2)^+\alpha/N}\Big(\int_\Omega |u_t|^{2N/(N-(N-2)^+\alpha)}dx\Big)^{(N-(N-2)^+\alpha)/N}\\
&\le C\|u(t)\|_{\alpha+2}^{2\alpha(1-\hat\theta)}\|\nabla u(t)\|^{2\alpha\hat\theta}\|u_t(t)\|_2^{2(1-\theta)}\|\nabla u_t(t)\|_2^{2\theta}
\end{aligned} \tag{6.14}$$

with $\theta=(N-2)^+\alpha/2$ and $\hat\theta=(2N-(\alpha+2)(N-2)^+)/(4+2\alpha-N\alpha)$. (A trivial modification is needed if $N=2$.) Hence,

$$\begin{aligned}
\int_\Omega |u|^{2\alpha}|u_t|^2dx &\le CK^{(N-2)^+\alpha}E(t)^{\alpha(2+\hat\theta\alpha)/(\alpha+2)+1-\theta}\\
&\le C\begin{cases}K^{(N-2)\alpha}E(t)^{(2+4\alpha-N\alpha)/2} & \text{if }\ N\ge 3\\ E(t)^{(4+6\alpha-N\alpha)/(4+2\alpha-N\alpha)} & \text{if }\ N=1,2.\end{cases}
\end{aligned} \tag{6.15}$$

We make a simple device

$$E(t)\le E(0)^{1-\epsilon}E(t)^\epsilon,\ 0\le\epsilon\le 1.$$

Then, if $N\ge 3$, it follows from (6.13) and (6.15) that

$$\begin{aligned}\sqrt{E_1(t)} &\le \sqrt{E_1(0)} + CK^{(N-2)\alpha/2}E(0)^{(1-\epsilon)(2+4\alpha-N\alpha)/4}\int_0^t E(s)^{\epsilon(2+4\alpha-N\alpha)/4}ds \\ &\le C_1 + CE(0)^{\mu}K^{(N-2)\alpha/2}\int_0^t K^{\epsilon m(2+4\alpha-N\alpha)/4}(1+s)^{-\gamma\epsilon(2+4\alpha-N\alpha)/4}ds\end{aligned} \tag{6.16}$$

for any ϵ, $0 \le \epsilon \le 1$. Under the assumption (6.9) we have the estimate (6.10) with $C(\epsilon) = C/\{\epsilon\gamma(2+4\alpha-N\alpha)-4\}$. When $N = 1, 2$ we have, instead of (6.16),

$$\begin{aligned}\sqrt{E_1(t)} \le C_1 + CK^{\epsilon m(4+6\alpha-N\alpha)/2(4+2\alpha-N\alpha)} \\ \times \int_0^t (1+s)^{-\gamma\epsilon(4+6\alpha-N\alpha)/2(4+2\alpha-N\alpha)}ds\end{aligned} \tag{6.16$'$}$$

and (6.10) follows, where $C(\epsilon) = C/\{\epsilon\gamma(4+6\alpha-N\alpha)-2(4+2\alpha-N\alpha)\}$. □

Now we are in a position to complete the proof of our Theorems 2.2 and 2.3.

We first assume (6.2), $\|u_{tt}(t)\| + \|\nabla u_t(t)\| \le K$, $0 \le t \le \tilde{T} + T$. Then, by Propositions 6.1, 6.2, we have the estimate (6.8), where m, γ are given as in Propositions 6.1, 6.2, according to the case $N \ge 3$ and $N = 1, 2$.

Assume (6.9) and take $K \gg 1$ such that $K^2 > 2C_1^2$. Then, if $E(0)$ is sufficiently small, we have from (6.10) with $\epsilon = 1$,

$$E_1(t) \le \frac{1}{2}(K - \tilde{\epsilon})^2$$

with some $\tilde{\epsilon} > 0$. This implies

$$\|u_{tt}(t)\| + \|\nabla u_t(t)\| \le K - \tilde{\epsilon} < K, \quad 0 \le t \le \tilde{T} + T,$$

and we conclude that the estimates (6.2) and (6.5) (or (6.7)) hold in fact on $[0, \infty)$. The condition (6.9) with $\epsilon = 1$ becomes

$$\gamma > \frac{4}{2+4\alpha-N\alpha} \ \text{ if } \ N \ge 3 \ \text{ and } \ \gamma > \frac{2(4+2\alpha-N\alpha)}{4+6\alpha-N\alpha} \ \text{ if } \ N = 1, 2. \tag{6.9$'$}$$

Thus, Theorems 2.2 and 2.3 are proved for the case $E(0)$ is sufficiently small. Next we show the estimates (6.2) and (6.5) (or (6.7)) on $[0, \infty)$ without smallness condition on $E(0)$. Note that (6.10) implies

$$\|u_{tt}(t)\| + \|\nabla u_t(t)\| \le C_1 K^{\eta}, \quad 0 \le t \le \tilde{T} + T. \tag{6.10$'$}$$

Assume that there exists ϵ, $0 \le \epsilon \le 1$ such that (6.9) holds and $\eta < 1$. Then we can conclude that for a large $K \gg 1$, the estimates (6.2) and (6.5) (or (6.7)) hold in fact on $[0, \infty)$. We first consider the case $N \ge 3$. Then the required condition is reduced to

$$\frac{4}{\gamma(2+4\alpha-N\alpha)} < \epsilon < \frac{2(2+2\alpha-N\alpha)}{m(2+4\alpha-N\alpha)}$$

for some $0 \le \epsilon \le 1$. It is easy to see that the condition is equivalent to:

$$\frac{4}{\gamma(2+4\alpha-N\alpha)} < 1$$

and

$$\frac{4}{\gamma(2+4\alpha-N\alpha)} < \frac{2(2+2\alpha-N\alpha)}{m(2+4\alpha-N\alpha)}.$$

Thus, the required condition is further reduced to

$$\gamma > \max\left\{\frac{4}{2+4\alpha-N\alpha}, \frac{2m}{2+2\alpha-N\alpha}\right\} = \frac{4N}{(N-2)(2+2\alpha-N\alpha)}. \tag{6.17}$$

Theorem 2.2 for the case without smallness condition on $E(0)$ is now proved.

When $N = 1, 2$ we see by a similar argument that the required condition is

$$\gamma > \max\left\{\frac{2(4+2\alpha-N\alpha)}{4+6\alpha-N\alpha}, m\right\}. \tag{6.18}$$

We know

$$m = p\tilde{\theta}\min\left\{\frac{r+2}{r}, \frac{\alpha+2}{q\alpha+q+\alpha} - (N-1)\right\}, \quad \tilde{\theta} = \frac{N}{2},$$

and

$$\gamma = \min\left\{\frac{2}{r}, \frac{\alpha+2}{q\alpha+q+\alpha} - N\right\}.$$

Hence (6.18) becomes

$$\min\left\{\frac{2}{r},\frac{\alpha+2}{q\alpha+q+\alpha}-N\right\}$$
$$>\max\left\{\frac{2(4+2\alpha-N\alpha)}{4+6\alpha-N\alpha},pN\min\left\{\frac{r+2}{2r},\frac{\alpha+2}{2(q\alpha+q+\alpha)}-\frac{N-1}{2}\right\}\right\}. \quad (6.20)$$

When V is star-shaped we replace (6.20) by

$$\frac{\alpha+2}{q\alpha+q+\alpha}-N>\max\left\{\frac{2(4+2\alpha-N\alpha)}{4+6\alpha-N\alpha},\frac{pN(\alpha+2)}{2(q\alpha+q+\alpha)}-\frac{N-1}{2}\right\}. \quad (6.20)'$$

Thus we have proved Theorem 2.3 for the case without smallness condition on $E(0)$.

Appendix.

Here we prove the following simple unique continuation theorem used in the proof of Proposition 3.4.

Proposition A.1. *We assume Hyp.B. Let $u(\cdot)\in\tilde{X}_2(T)\equiv L^\infty([0,T];\dot{H}_2\cap L^{\alpha+2})\cap W^{1,\infty}([0,T];\dot{H}_1)\cap W^{2,\infty}([0,T];L^2_{loc}(\Omega))$ be a solution of the problem*

$$u_{tt}-\Delta u+g(u)=0 \quad in \quad \Omega\times[0,T]$$

with $u_t(x,t)=0$ on $\omega\cup\Omega(R)^c$. Then there exists $T_0>0$ and $\epsilon>0$ such that if $T>T_0$ and $E(0)<\epsilon$, we have $u(x,t)\equiv 0$ on $\Omega\times[0,T]$.

Proof. Set $w(x,t)=u_t(x,t)$ and $w_\delta(x,t)=w(x,\cdot)*\rho_\delta(\cdot)$ where $\rho_\delta(t)$ is a mollifier with $\operatorname{supp}\rho_\delta(\cdot)\subset(-\delta,\delta)$, $0<\delta\ll 1$. Then $w_\delta\in C([0,T];H_2(\Omega(2R)))\cap C^1([0,T];H_1(2R))$ is a solution of the problem

$$w_{\delta,tt}-\Delta w_\delta+g'(u)w*\rho_\delta(t)=0 \;\text{ in }\; \Omega(2R)\times[\delta,T-\delta]. \quad (A.1)$$

Now applying the same, in fact, a simpler argument deriving (3.10) to (A.1) and noting that $w_{\delta,t}=w_\delta=0$ on $\omega\cup\Omega(R)^c$ we have

$$\frac{d}{dt}\tilde{\chi}_k(t)+\epsilon_1\int_{\Omega(R)}\left(|w_{\delta,t}|^2+|\nabla w_\delta|^2\right)dx$$
$$\le C\int_{\Omega(R)}|g'(u)w*\rho_\delta(t)|(|w_{\delta,t}|+|w_\delta|)dx$$

$$\leq C \sup_{t-\delta\leq s\leq t+\delta}\left(\int_{\Omega(R)}|u|^{2(\alpha+1)}dx\right)^{\alpha/2(\alpha+1)}\sup_{t-\delta\leq s\leq t+\delta}\|\nabla w(s)\|(\|w_{\delta,t}(t)\|+\|w_\delta(t)\|)$$

$$\leq CE(0)^{\alpha/2}\sup_{t-\delta\leq s\leq t+\delta}\|\nabla w(s)\|(\|w_{\delta,t}(t)\|+\|w_\delta(t)\|), \tag{A.2}$$

where $\tilde{\chi}_k(t)$ is defined with u replaced by w_δ (note that here in the definition of $\tilde{\chi}_k(t)$, we set $E(t)=(1/2)(\|w_{\delta,t}(0)\|^2+\|\nabla w_\delta(t)\|^2)$). Integrating (A.2) in t and letting δ tend to 0 we have

$$\tilde{\chi}_k(t)+\epsilon_1\int_0^T\int_{\Omega(R)}\big(|w_t|^2+|\nabla w|^2\big)dxds$$
$$\leq \tilde{\chi}_k(0)+CE(0)^{\alpha/2}\int_0^T\big(\|w_t(t)\|^2+\|\nabla w(t)\|^2\big)dt, \tag{A.2$'$}$$

where $\tilde{\chi}_k(t)$ is defined with u replaced by w. Further, by the standard energy identity we see

$$\sup_{0\leq t\leq T}\big(\|w_t(t)\|^2+\|\nabla w(t)\|^2\big)$$
$$\leq \inf_{0\leq t\leq T}\big(\|w_t(t)\|^2+\|\nabla w(t)\|^2\big)+2\int_0^T\int_{\Omega(R)}|g'(u)w||w_t|dxdt$$
$$\leq \inf_{0\leq t\leq T}\big(\|w_t(t)\|^2+\|\nabla w(t)\|^2\big)+CE(0)^{\alpha/2}\int_0^T\big(\|w_t(t)\|^2+\|\nabla w(t)\|^2\big)dt. \tag{A.3}$$

It follows from (A.2)$'$, (A.3) and the fact $\tilde{\chi}_k(0)\leq C(\|w_t(0)\|^2+\|\nabla w(0)\|^2)$ that

$$(\epsilon_1-CE(0)^{\alpha/2})\int_0^T\big(\|w_t(t)\|^2+\|\nabla w(t)\|^2\big)dt$$
$$\leq C\inf_{0\leq t\leq T}\big(\|w_t(t)\|^2+\|\nabla w(t)\|^2\big)\leq C\frac{1}{T}\int_0^T\big(\|w_t(t)\|^2+\|\nabla w(t)\|^2\big)dt. \tag{A.4}$$

Thus we conclude that if $E(0)$ is small and T is sufficiently large, then $w(t)\equiv$ *const*. for $0\leq t\leq T$. Since $w(x,t)=0$ for $|x|\geq R$ we have $w(x,t)\equiv 0$ and hence, $u(x,t)=u(x)$, independent of t. Returning to the original equation we see

$$-\Delta u+g(u)=0 \ \text{ in } \ \Omega. \tag{A.5}$$

By the assumption $E(0) < \infty$ and Hyp.B we know $u \in \dot{H}_1 \cap L^{\alpha+2}$, and hence (A.5) implies

$$\|\nabla u\|^2 + \int_\Omega g(u)u dx \leq 0$$

and we conclude $u(x) \equiv 0$. □

Acknowledgements. The author would like to thank the referee for his/her careful reading of the manuscript and useful comments.

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