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Canal foliations of S^3

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Abstract. The goal of the article is to classify foliations of S^3 by *regular* canal surfaces, that is envelopes of one-parameter families of spheres which are immersed surfaces. We will add some extra information when the leaves are "surfaces of revolution" in a conformal sense.

1. Introduction.

It is well known that codimension-one totally geodesic foliations of closed Riemannian manifolds of constant non-zero curvature do not exist. In fact, totally geodesic foliations of compact negatively curved manifolds do not exist in any codimension, see [**Ze1**], [**Ze2**].

Similarly, codimension-one totally umbilical foliations do not exist on S^n (by purely topological reason) or on negatively curved closed Riemannian manifolds [**LW**]. Since umbilicity is a conformal property of submanifolds, one should look for a weaker conformal property for which enough foliations with all the leaves possessing the property exist on (some) manifolds of non-zero constant sectional curvature.

In [**LW**], we proved also that Dupin foliations do not exist on S^3 or on compact hyperbolic 3-manifolds. This shows that asking that the leaves are all pieces of Dupin cyclides is still a very rigid condition. Since Dupin cyclides can be characterized as surfaces with *zero conformal principal curvatures* (see [**CSW**] for definitions), one should look further for weaker local conformal properties (compare [**CSW**]).

Next, in [**BW**], it has been shown that the condition "all the scalar local conformal invariants are constant" is still too strong: foliations by surfaces with such invariants do not exist on S^3 or on closed hyperbolic 3-manifolds. Surprisingly, all the surfaces with constant conformal principal curvatures must have one of

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these curvatures equal to zero and this means that such surfaces are canals, that is envelopes of one-parameter families of spheres. Note also that canal surfaces are important for computer aided geometric design (see [**Kr**], [**PP**], etc.)

All of that brings us to a question about existence and classification of foliations of $\boldsymbol{S}^3$ and closed hyperbolic 3-manifolds by canal surfaces. Here, we give an answer to this question in the case of $\boldsymbol{S}^3$: we show how to construct such foliations on $\boldsymbol{S}^3$ and we classify all of them. They may look like standard Reeb foliations with the torus leaf being a Dupin cyclide. The others are obtained from these by replacing the toral leaf with a zone $\boldsymbol{T}^2 \times I$ foliated by tori, cylinders and planes (see Theorems 4.2.2, 4.2.1 and 4.2.3).

We consider also a condition imposed to the leaves which is slightly stronger than to be just canal surfaces but still weaker than being Dupin: to be conformal surfaces of revolution.

The foliations we consider are of class at least $\mathscr{C}^2$, in particular their leaves are of class at least $\mathscr{C}^2$.

The authors thank Gil Solanes for pointing to us the fact that our curve in Λ^2, in a first version of Figure 13 was not possible.

2. Canal surfaces and "surfaces of revolution".

2.1. The set of spheres in $\boldsymbol{S}^3$.

The *Lorentz quadratic form* $\mathscr{L}$ on $\boldsymbol{R}^5$ and the associated *Lorentz bilinear form* $\mathscr{L}(\cdot,\cdot)$, are defined by $\mathscr{L}(x_0,\dots,x_4) = x_0^2 - (x_1^2 + \cdots + x_4^2)$ and $\mathscr{L}(u,v) = u_0v_0 - (u_1v_1 + \cdots + u_4v_4)$.

The Euclidean space $\boldsymbol{R}^5$ equipped with this pseudo-inner product $\mathscr{L}$ is called the *Lorentz space* and denoted by $\boldsymbol{L}^5$.

The isotropy cone $\mathscr{L}ight = \{v \in \boldsymbol{R}^5 \mid \mathscr{L}(v) = 0\}$ of $\mathscr{L}$ is called the *light cone*. Its non-zero vectors are also called *light-like*. The light cone divides the set of vectors $v \in \boldsymbol{L}^5, v \notin \{\mathscr{L} = 0\}$ in two classes:

A vector v in $\boldsymbol{R}^5$ is called *space-like* if $\mathscr{L}(v) < 0$ and *time-like* if $\mathscr{L}(v) > 0$.

A straight line is called space-like (or time-like) if it contains a space-like (or, respectively, time-like) vector. A vector subspace is called space-like if all its non-zero vectors are space-like; it is called time-like if it contain space-like and time-like vectors. In particular an hyperplane $P = (\boldsymbol{R}\sigma)^\perp$, $\sigma \in \Lambda^4$ is always time-like.

The points at infinity of the light cone in the upper half space $\{x_0 > 0\}$ form a 3-dimensional sphere. Let it be denoted by $\boldsymbol{S}^3_\infty$.

The quadric Λ^4, usually called *de Sitter space*, is defined by the equation $\mathscr{L} = -1$. To each point $\sigma \in \Lambda^4$ corresponds an oriented sphere $\Sigma = (\boldsymbol{R}\sigma)^\perp \cap \boldsymbol{S}^3_\infty$ (see Figure 1). The intersection of the half-space $\mathscr{L}(\sigma,\cdot) < 0$ with $\boldsymbol{S}^3_\infty$ is a 3-ball

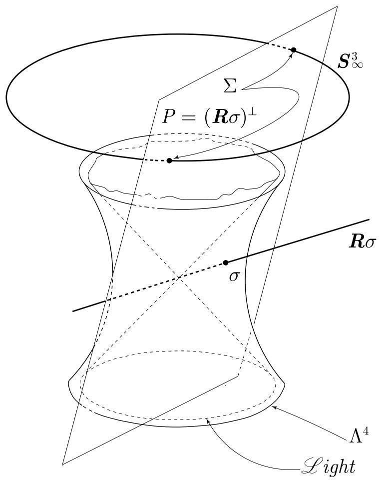

Figure 1. $\boldsymbol{S}^3_\infty$ and the correspondence between points of Λ and spheres.

of boundary Σ, therefore it orients Σ.

Instead of finding the points of $\boldsymbol{S}^3$ "at infinity", we can also consider the section of the lightcone by the space-like affine hyperplane $\{x_0 = 1\}$. The section is a sphere of dimension 3 endowed with a metric of constant curvature 1 which has, in the hyperplane $\{x_0 = 1\}$ the equation $\boldsymbol{S}^3_1 = \{(x_1, \dots, x_4) \mid x_1^2 + \cdots + x_4^2 - 1 = 0\}$. The affine hyperplane $\{x_0 = 1\}$ looks horizontal for our Euclidean eye. We can, instead of $\{x_0 = 1\}$, chose any affine hyperplane H_z tangent at a point z to the upper sheet $\boldsymbol{H}^4$ of the hyperboloid $\mathscr{H} = \{\mathscr{L} = 1\}$. The intersection $\mathscr{L}ight \cap H_z$ is a 3-sphere which also inherits from the Lorentz metric a metric of constant curvature 1 (see [**H-J**], [**LW**], and Figure 2). Changing the point $z \in \boldsymbol{H}^4$ will only change the metric induced from $-\mathscr{L}$ on the sphere $\boldsymbol{S}^3$. The correspondences between the different spheres $\mathscr{L}ight \cap H_z$ obtained using the rays of the lightcone are conformal.

The correspondence between a point $\sigma \in \Lambda^4$ and the sphere $\Sigma = \boldsymbol{S}^3 \cap (\operatorname{Span}\sigma)^\perp$ gives to σ the role of a linear equation of Σ. Therefore the points of a plane $\operatorname{Span}(\sigma_1, \sigma_2)$, $\sigma_1 \in \Lambda^4$, $\sigma_2 \in \Lambda^4$ can be viewed as defining equations $\langle \lambda\sigma_1 + \mu\sigma_2, \cdot\rangle = 0$, linear combinations of the "equations" $\langle \sigma_1, \cdot\rangle = 0$ and $\langle \sigma_2, \cdot\rangle = 0$. It is enough to consider the "normalised" equations corresponding to points of $\operatorname{Span}(\sigma_1, \sigma_2) \cap \Lambda^4$. The corresponding spheres form a (linear) *pencil.*

If the plane $\operatorname{Span}(\sigma_1, \sigma_2)$ is space-like, then the spheres Σ_1 and Σ_2 associated to σ_1 and σ_2 intersect along a circle Γ, and all the spheres of the pencil contain Γ. Such a pencil is called *pencil with base circle.*

If the plane $\operatorname{Span}(\sigma_1, \sigma_2)$ is time-like, then the spheres Σ_1 and Σ_2 associated

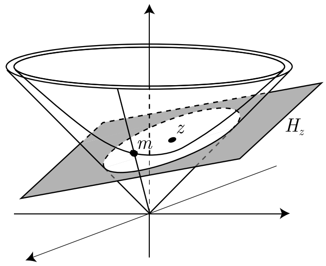

Figure 2. A tangent space to $\boldsymbol{H}^4$ cuts the light cone at a unit sphere.

to σ_1 and σ_2 are disjoint and all the spheres of the pencil are pairwise disjoint; two degenerate spheres of the pencil are points which we call *limit points*. Such a pencil is called *pencil with limit points* or *Poncelet pencil* here.

If the plane $\mathrm{Span}(\sigma_1, \sigma_2)$ contains only one light-ray, which means that it is tangent to the light-cone along this light-ray, then the two spheres Σ_1 and Σ_2 are tangent at the point m corresponding to the light-ray. All the spheres of the pencil are then tangent at m. Such a pencil is called a *pencil of tangent spheres*.

Chapter one of the book [**H-J**] contains a "projective" presentation of the different types of pencils of spheres.

Notice that the intersection $\gamma \subset \Lambda^4$ of Λ^4 with a space-like plane $\mathrm{Span}(\sigma_1, \sigma_2)$ containing the origin, endowed with the metric induced from $-\mathscr{L}$, is a circle of radius one in the Euclidean plane $\mathrm{Span}(\sigma_1, \sigma_2)$. We have seen that the points of this circle correspond to the spheres of a pencil with base circle. The arc-length of a segment contained in γ is equal to the angle between the spheres corresponding to the extremities of the arc.

It is convenient to have a formula giving the point $\sigma \in \Lambda^4$ in terms of the Riemannian geometry of the corresponding sphere $\Sigma \subset \boldsymbol{S}^3 \subset \mathscr{L}ight$ and a point m on it. In order to recover the point $\sigma \in \Lambda^4$ we need to know a point $m \in \Sigma$, the unit vector $\vec{n}$ tangent to $\boldsymbol{S}^3$ normal to Σ at m and the geodesic curvature of Σ, that is the geodesic curvature k_g of any geodesic circle on Σ.

PROPOSITION 2.1.1. *The point $\sigma \in \Lambda^4$ corresponding to the sphere $\Sigma \subset \boldsymbol{S}^3 \subset \mathscr{L}ight$ is given by*

$$\sigma = k_g m + \vec{n}. \tag{1}$$

REMARK. A similar proposition can be stated for spheres in the Euclidean space $\boldsymbol{E}^3$ seen as a section of the light cone by an affine hyperplane parallel to an hyperplane tangent to the light cone.

The proof of Proposition 2.1.1 can be found in [**H-J**] and [**LO**]. The idea of the proof is shown on Figures 3 and 4: Let H_z be the affine hyperplane such that $\boldsymbol{S}^3 = \mathscr{L}ight \cap H_z$, let P be the hyperplane such that $\Sigma = \boldsymbol{S}^3 \cap P$. Let us consider the hyperplanes tangent at a point $m \in \Sigma$ to the lightcone. They are the hyperplanes orthogonal to the rays $\boldsymbol{R}m$. As the subspace $\mathrm{Span}(\boldsymbol{R}m)$, $m \in \Sigma$, is P, the space $P^\perp$ is the intersection of the hyperplane $(\boldsymbol{R}m)^\perp$, $m \in \Sigma$. The intersection of an hyperplane $(\boldsymbol{R}m)^\perp$ with H_z is the plane tangent at m to $\boldsymbol{S}^3$. Therefore the line $P^\perp$ intersects the affine hyperplane H_z at a point which is the vertex of the cone tangent to $\boldsymbol{S}^3$ along Σ. The intersection of the line $P^\perp$ with Λ^4 is formed of two antipodal points $\sigma \in \Lambda^4$ and $-\sigma \in \Lambda^4$. The point σ corresponds to the orientation of Σ given by the normal vector $\vec{n}$.

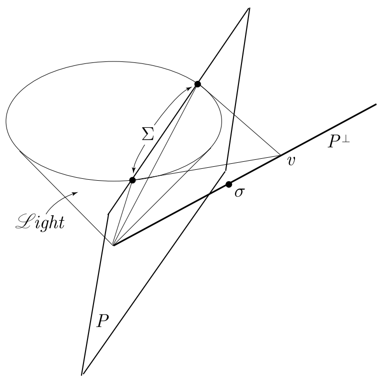

Figure 3. The plane P which contains the sphere Σ, the line $P^\perp = \boldsymbol{R}\sigma$) and the vertex v of the cone tangent to the sphere Σ in $\boldsymbol{R}^4$.

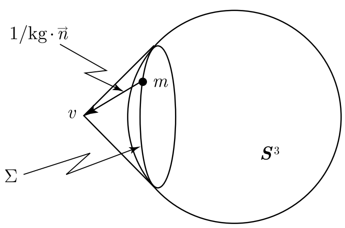

Figure 4. The geodesic curvature k_g, pictured within the affine hyperplane H_z.

Remark. We can in a similar way show that oriented circles of a sphere $\boldsymbol{S}^2$ nicely correspond to the points of the 3-dimensional quadric $\Lambda^3 \subset \boldsymbol{R}^4$ defined by the equation $\mathscr{L}(x) = -1$, where $\mathscr{L}(x) = x_0^2 - x_1^2 - x_2^2 - x_3^2$.

Lowering even more the dimension we see that oriented pairs of points (that is boundary of arcs) in a circle $\boldsymbol{S}^1$ nicely correspond to the points of the 2-dimensional quadric $\Lambda^2 \subset \boldsymbol{R}^3$ defined by the equation $\mathscr{L}(x) = -1$, where $\mathscr{L}(x) = x_0^2 - x_1^2 - x_2^2$.

2.2. Canal surfaces.

A differentiable curve $\gamma = \gamma(t)$ is called space-like if, at each point its tangent vector $\overset{\bullet}{\gamma}(t)$ is space-like, that is $\mathscr{L}(\overset{\bullet}{\gamma}) < 0$. In this case the family of spheres Σ_t associated to the points $\gamma(t)$ defines an envelope which is a surface, a union of circles called the *characteristic circles* of the surface. There is one characteristic circle Γ_{Car} on each sphere Σ_t of the family and it is the intersection of Σ_t and the sphere $\overset{\bullet}{\Sigma}_t = [\mathrm{Span}(\overset{\bullet}{\gamma}(t))]^{\perp} \cap \boldsymbol{S}^3$. From now on we will suppose that the space-like curve γ is parametrized by arc-length, that is $|\mathscr{L}(\overset{\bullet}{\gamma})| = 1$.

The spheres envelopping a canal surface form a curve in Λ^4 that we call the curve *corresponding* to the surface. It is space-like, as the existence of an envelope forces nearby spheres to intersect.

An extra condition is necessary to guarantee that the envelope is immersed. The geodesic acceleration vector $\overrightarrow{k_g} = \overset{\bullet\bullet}{\gamma}(t) + \gamma(t)$ should be time-like. We call the envelope of the spheres Σ_t corresponding to the points of such a curve γ a *regular canal surface.*

Important canal surfaces are the *Dupin cyclides* (see [**Dar**] and [**LW**]). They are envelopes of the spheres of a curve $\beta \subset \Lambda^4$ which is the intersection of Λ^4 with an affine plane. In fact a Dupin cyclide is in two different ways the envelope of a one-parameter family of spheres (see [**LW**] for a complete simultaneous description of the canal surfaces and the corresponding curves in Λ^4).

An embedded Dupin cyclide is the envelope of the spheres of a “large” circle, that is intersection of Λ^4 with a space-like affine plane H which is away from the origin in a time-like direction. This occurs when H is of the form $H = x_H + h$, where h a space-like vectorial plane and x_H is a time-like vector orthogonal to h. The radius of the circle $(\Lambda^4 \cap H) \subset H$, for the Euclidean metric induced on H from $-\mathscr{L}$, is then larger than 1. The point x_H is also the (only) critical point of the function $\mathscr{L}$ restricted to H ($\mathscr{L}(x_H) > 0$).

Affine planes of the form $H = x_H + h$, where h a space-like vectorial plane and x_H is a space-like vector orthogonal to h of norm less than 1 intersect Λ^4 in circles of radius less than 1. The corresponding Dupin cyclide has then two singular points; an example is the surface of revolution obtained rotating a sphere around an axis which intersects it.

The limit case, an affine plane $H = x_H + h$, where h a space-like vectorial plane and x_H is a light-like vector orthogonal to h, cuts Λ^4 in a circle of radius 2π. The corresponding spheres are tangent to a line at the point corresponding to the lightray generated by x_H.

Let us prove a geometric property of canal surfaces which is in some sense

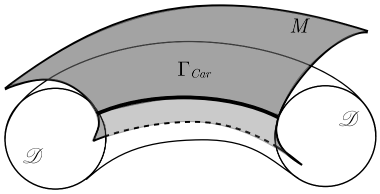

Figure 5. The Dupin necklace $\mathscr{D}$ and the canal surface M crossing it along a characteristic circle Γ_{Car}.

the existence of an osculating Dupin cyclide to the canal along any characteristic circle. Figure 5 shows how the canal surface M may cross its Dupin necklace along a characteristic circle Γ_{Car}.

THEOREM 2.2.1. *Let M be a canal surface. We denote by k_1 the principal curvature of the surface which coincides with the principal curvature of the spheres of the family* (*the lines of principal curvature for k_1 are the characteristic circles of the envelope*). *The osculating spheres $\Sigma_2(\varphi)$ for the principal curvature k_2 along a characteristic circle Γ_{Car}* (*parametrized by φ*) *of a canal surface have an envelope which is a Dupin cyclide $\mathscr{D}$; in other terms the points $\sigma_2(\varphi) \in \Lambda^4$ corresponding to the spheres $\Sigma_2(\varphi)$ form a circle δ.*

DEFINITION 2.2.2. We call Dupin necklace, and denote by $\mathscr{D}$, the Dupin cyclide formed by the osculating spheres $\Sigma_2(\varphi)$ for the principal curvature k_2 along a characteristic circle Γ_{Car}. We denote by δ the corresponding curve in Λ^4, and call it also Dupin necklace when there is no risk of confusion.

PROOF. Here, writing formula (1), we get $\sigma_2(\varphi) = k_2 m(\varphi) + N(m(\varphi))$. Differentiating with respect to φ we get, $\sigma_2' = k_2' m + k_2 m' - k_1 m'$, so, using the fact that $k_1' = 0$, $\sigma_2' = (k_2 - k_1)' m + (k_2 - k_1) m' = [(k_2 - k_1)m]'$. Therefore

$$\sigma_2(\varphi) = (k_2 - k_1) m(\varphi) + \sigma_{2,0}. \tag{2}$$

Recall that the intersection $T_{\sigma_{2,0}} \cap \Lambda^4$ is a shifted light cone (of dimension 3) with vertex at $\sigma_{2,0}$. As the spheres we consider are tangent to the sphere $\Sigma_{2,0}$, corresponding to the point $\sigma_{2,0} \in \Lambda^4$, along the characteristic circle, these spheres belong to a subcone $\mathscr{L}C$ of dimension 2; one can see that from Equation (2) as the points $m(\varphi)$ belong to the characteristic circle.

On the other hand, the following circle of the cone $\mathscr{L}C$ is a candidate to be the set of osculating circles along the characteristic circle Γ_{Car}: the circle $\beta^*(t)$ "conjugate" to the osculating circle $\beta(t)$ to the curve $\gamma \subset \Lambda^4$ at the point $\sigma(t)$, that is the intersection $\Lambda^4 \cap T_{\sigma_1}\Lambda^4 \cap T_{\sigma_2}\Lambda^4 \cap T_{\sigma_3}\Lambda^4$ (the affine plane $(T_{\sigma_1}\Lambda^4 \cap$

$T_{\sigma_2}\Lambda^4 \cap T_{\sigma_3}\Lambda^4$) is independent of the choice of three different points $\sigma_1, \sigma_2, \sigma_3$ on the osculating circle $\beta(t)$ to γ at $\sigma(t)$). The circle $\beta^*(t)$ is the limit as $h \to 0$ of the circles (in Λ) formed by the spheres simultaneously tangent to three spheres $\Sigma(t)$, $\Sigma(t+h)$ and $\Sigma(t+2h)$ corresponding to three points $\sigma(t)$, $\sigma(t+h)$ and $\sigma(t+2h)$ of the curve $\gamma \subset \Lambda^4$. On the same light-ray as the points corresponding to the two osculating spheres at a point $m(t) \in \Gamma_{Car}$ (one is $\sigma(t)$) there is a point of the circle $\beta^*(t)$: the point $\tilde{\sigma}(t)$ corresponding to an osculating sphere to the Dupin cyclide defined by $\beta(t)$ and $\beta^*(t)$. We need to prove it cannot have a contact of centre or saddle type at $m(t)$ with the canal. The sphere $\tilde{\Sigma}(t)$ can also be seen as a limit of spheres $\tilde{\Sigma}(t,h)$ tangent at $m(t)$ to $\Sigma(t)$ and tangent to another nearby sphere $\Sigma(t+h)$. By definition, such a sphere does not intersect the interior of a ball of boundary $\Sigma(t+h)$. The transverse distance of the sphere $\tilde{\Sigma}(t)$ and the sphere $\tilde{\Sigma}(t,h)$ is of the order of h^3 at the point of contact of $\tilde{\Sigma}(t,h)$ and $\Sigma(t+h)$. This contradicts the fact that along a line of principal curvature which is not a characteristic circle the transverse distance between the canal and a generic sphere tangent at $m(t)$ to the canal is of the order of h^2. This proves that the spheres corresponding to points of $\beta^*(t)$ are osculating spheres to the canal tangent to the characteristic circle Γ_{Car}. $\square$

Before constructing canal surfaces which are topologically planes, we will need to understand Dupin cyclides corresponding to small circles in Λ^4. An example is the cyclide obtained rotating a sphere of $\boldsymbol{R}^3$ around an axis close to a diameter; if the axis were the diameter, the envelope would degenerate into the sphere itself, and the circle in Λ^4 into the corresponding point.

PROPOSITION 2.2.3 (Small circles). *Let $\beta \subset \Lambda^4$ be the intersection $\Lambda^4 \cap H$ where H is an affine space-like plane of the form $x_H + h$, where h a space-like vector plane and x_H is a space-like vector orthogonal to h which satisfies $-1 < \mathscr{L}(x_H) < 0$. Then*

- 1. *The radius of the circle β is smaller than* 1,
- 2. *The envelope of the spheres corresponding to points of β is also the envelope corresponding to another curve $\beta^* \subset \Lambda^4$, $\beta^* = \Lambda^4 \cap H^*$, where H^* is an affine space of mixed type, orthogonal to* $\mathrm{Span}(O,H)$, *and intersecting the line generated by $\overrightarrow{Ox_H}$ at a point x_{H^*}, also critical for the Lorentz norm of $y, y \in H^*$, satisfying $\mathscr{L}(x_H, x_{H^*}) = -1$.*
- 3. *Moreover, when $\mathscr{L}(x_H)$ and $\mathscr{L}(x_{H^*})$, tend both to -1, the curve $\beta = \Lambda^4 \cap H$ is a circle of radius going to zero, and the curve $\beta^* = \Lambda^4 \cap H^*$ is a hyperbola the geodesic curvature of which also goes to infinity.*

PROOF. The proof of the first two points is given in [**LW**, Section 5.2].

As the Lorentz norm $\mathscr{L}(x_H)$ is close to -1 and greater than -1, the intersection $\Lambda^4 \cap H$ is contained in $\Lambda^4 \cap \mathrm{Span}(O, H)$, a 2-sphere of radius 1 centred at the origin. The intersection of this sphere with H is a small circle of radius $r = \sqrt{1 + \mathscr{L}(x_H)}$. Its geodesic curvature, in the sphere or in Λ^4, is therefore large (see Figure 6).

The situation of the hyperbola $\Lambda^4 \cap H^*$ is symmetric. The hyperbola is very close to its asymptotes and for our Euclidean eye has a sharp turn. Its curvature in the Lorentz plane H^*, and also its geodesic curvature in Λ^4, are therefore constant and large (see Figure 6). □

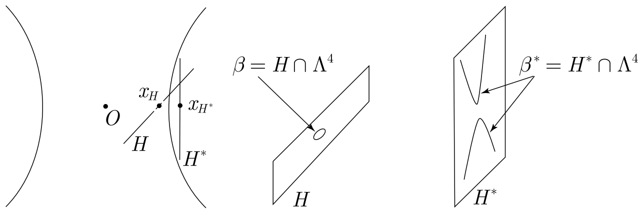

Figure 6. Small circle and acute hyperbola.

2.3. Umbilics and spherical caps.

In order to obtain the planar leaves of a foliation of $\boldsymbol{S}^3$ we need to understand how to "close" a canal cylinder. There are two ways, and we will use both of them:

1) add a spherical cap tangent to the cylindrical part,
2) shrink the characteristic circles ending with an umbilic.

The price to pay will be to accept a singular point at the extremity of the curve $\gamma \subset \Lambda^4$, in the sense that the geodesic curvature of γ goes to infinity when approaching the extremity.

We consider now a parametrization by $t \in]t_1, T]$ of an arc of γ of extremity at the end point of γ; as we want to include the extremity $\gamma(T)$, we have to give up parametrizations by arc-length, as we will prove that the geodesic curvature on an arc $[\gamma(t), \gamma(T)[$ is unbounded. The characteristic circle $\Gamma_{Car}(t)$ on the sphere Σ_t corresponding to the point $\gamma(t) \in \gamma$ is $\Gamma_{Car}(t) = \Sigma_t \cap [\mathrm{Span}(\dot{\gamma}(t))]^{\perp}$.

Let us first suppose that, at the endpoint $\gamma(T)$, the curve $\gamma \subset \Lambda^4$ has a space-like tangent direction. The point $\gamma(T)$ of γ corresponds to a sphere $\Sigma(T)$. It provides the spherical cap contained in $\Sigma(T)$ of boundary the characteristic circle $[\mathrm{Span}(\sigma(T), \dot{\sigma}(T))]^{\perp} \cap \boldsymbol{S}^3$. Let us suppose that the envelope completed with the spherical cap is a $\mathscr{C}^2$ surface. As the principal curvatures are continuous functions on the envelope, the curvature of the two osculating spheres at points of a characteristic circle both have the same limit $k = k_1(T)$, the mean curvature of the sphere Σ_T corresponding to the endpoint $\gamma(T)$. Therefore, the Dupin necklaces δ_t associated to the characteristic circles $\Gamma_{Car}(t), t \to T$ have to be smaller and smaller. This is only possible if the geodesic curvature of the curve γ goes to infinity when t goes to T.

Reciprocally, a curve $\gamma : [-\infty, T] \to \Lambda^4$ such that $\dot{\gamma}$ is space-like, $\overrightarrow{k_g}$ is time-like and satisfying $lim_{t\to T}|\mathscr{L}(k_g)| = +\infty$ corresponds to an envelope such that the points of the last characteristic circle $\Gamma_{Car}(T)$ are umbilics, as the condition $\mathscr{L}(k_g) > 0$, $|\mathscr{L}(k_g)| \to \infty$ implies that the corresponding Dupin necklaces have to shrink to a point.

To ensure that the foliation of the surface by characteristic circles is transversally $\mathscr{C}^1$, and that the leaves of the other foliation by lines of principal curvature are $\mathscr{C}^1$ up to the last characteristic circle $[\mathrm{Span}(\sigma(T), \dot{\sigma}(T))]^\perp \cap \boldsymbol{S}^3$ we also demand that the plane $\mathrm{Span}(\dot{\sigma}(t), \ddot{\sigma}(t))$ has a limit. This guarantees that the family of circles $\Sigma(T) \cap \dot{\Sigma}(t)$ is tangent to a pencil of circles (in the set Λ^3 of circles of the sphere $\Sigma(T)$).

The second possibility demands that the tangent vector at the endpoint of γ is light-like. In fact, as, in this case, the radii of the characteristic circle tend to zero, the spheres $\dot{\Sigma}(t)$, which are orthogonal to the spheres $\Sigma(t)$, therefore almost orthogonal to the limit sphere $\Sigma(T)$, need to have radii going to zero, and moreover need to shrink to a point m_0. This point m_0 has to be an umbilic as all the lines of principal curvature orthogonal to the characteristic circles converge to m_0. For our Euclidean eye the tangent vector $\dot{\gamma}(t)$ needs to lean over the line $\mathrm{Span}(m_0)$ when t goes to T. The Dupin necklaces δ_t again have radii going to zero. Therefore, as in the previous case, the geodesic curvature of γ has to go to infinity when t goes to T. Again, in order to ensure that the foliation of the surface by characteristic circles is transversally $\mathscr{C}^1$, and that the leaves of the other foliation by lines of principal curvature are $\mathscr{C}^1$ up to the umbilic, we also demand that the plane $\mathrm{Span}(\dot{\sigma}(t), \ddot{\sigma}(t))$ has a limit. This guarantees that the family of circles $\Sigma(T) \cap \dot{\Sigma}(t)$ has, when renormalised by the composition of a stereographic projection of pole at the umbilic and suitable homotheties, a limit which is a pencil of concentric circles.

2.4. Canal leaves.

When a canal surface is the leaf of a foliation of $\boldsymbol{S}^3$ it has to be regular. The corresponding curve γ can be

1) a segment
2) a line
3) a half-line
4) a closed curve

In the first case the leaf is a sphere; that is impossible if it is a leaf of a foliation of $\boldsymbol{S}^3$. In the second case the leaf is a cylinder, and in the fourth a torus (A leaf of a foliation of $\boldsymbol{S}^3$ cannot be a Klein bottle, as a Klein bottle cannot be embedded in $\boldsymbol{S}^3$). In the third case it is a plane. In the first and third case we should demand that the tangent direction at an extremity of the curve is light-like; then the point of the envelope corresponding to the extremity is an umbilical point of the leaf.

As we supposed that the foliation is of class $\mathscr{C}^2$, the principal directions of the leaves vary continuously in the complement of umbilical points. As the characteristic circles are lines of curvature, they also vary continuously in the complement of umbilical points of leaves. Let $\gamma \subset \Lambda^4$ be the curve corresponding to a leaf L. Let us consider a value of the parameter t which is not an extremity of the domain of definition of γ. Then the characteristic circle $C_t \subset \Sigma_t = \gamma(t)^\perp$ is the boundary of two spherical caps of the sphere $\dot{\Sigma}_t = \dot{\gamma}(t)^\perp \cap \boldsymbol{S}^3_\infty$. The condition $\mathscr{L}(\overrightarrow{k_g}) > 0$ implies that either the spheres $\dot{\gamma}(t+h)^\perp \cap \boldsymbol{S}^3_\infty$, for h small enough, are nested, or that their intersection occurs "only in one side" of the characteristic circles. In the latter case, the vector $\ddot{\gamma}(t)$ is space-like but the condition $\mathscr{L}(\overrightarrow{k_g}) > 0$ implies that the sphere $\ddot{\gamma}(t)^\perp \cap \boldsymbol{S}^3_\infty$ intersects the sphere $\dot{\gamma}(t)^\perp \cap \boldsymbol{S}^3_\infty$ on one side of the characteristic circle. Therefore we can continuously chose a component D_t of $\dot{\gamma}(t)^\perp \cap \boldsymbol{S}^3_\infty \setminus C_t$. For small enough values of h, all the discs D_{t+h} are disjoint. Their union when h belongs to a small interval $[-h_0, h_0]$ form a solid cylinder $\mathscr{C}yl_{h_0}$.

Let us now consider a leaf L', envelope of the spheres corresponding to the point of a curve $\gamma' \subset \Lambda^4$, through a point $m' \in \boldsymbol{S}^3 \setminus \mathscr{C}yl_{h_0}$ close enough from a point $m \in C_t$. Let us call C_τ the characteristic circle containing the point m'. It cannot cross the "vertical" part of the boundary of $\mathscr{C}yl_{h_0}$ when m' is close enough from m. Therefore it has to "go around" it (see Figure 7). More precisely, the characteristic circle C_τ bounds on the sphere $\dot{\Sigma}'_t = \dot{\gamma}'(t)^\perp \cap \boldsymbol{S}^3_\infty$ a disc which intersects the vertical part of $\partial\mathscr{C}yl_{h_0}$ in a closed curve close (in the $\mathscr{C}^1$-topology) to C_t.

Remark. If a leaf L accumulates on a toral leaf, the family of characteristic circles of L converges to a family of characteristic circles on the toral leaf.

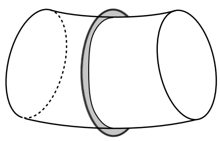

Figure 7. How a nearby characteristic circle encircles the solid cylinder bounded by a piece of leaf.

2.5. Conformal axis of a surface of revolution.

Surfaces of revolution of $\boldsymbol{R}^3$ are envelopes of spheres centred on the axis of revolution of the surface. The characteristic circles are contained in planes orthogonal to this axis.

In this article we will adopt a wider, conformal definition.

Definition 2.5.1. A *surface of revolution* is a regular canal such that the spheres that define it are all orthogonal to a given circle C called the *axis* of the surface of revolution.

Remarks.

- The image of C by a stereographic projection of $\boldsymbol{S}^3$ on $\boldsymbol{R}^3$ sending a point of the axis C to infinity is the usual rotational axis of the image of the envelope which is a surface of revolution of $\boldsymbol{R}^3$ in the usual sense.
- The spheres orthogonal to a given circle C form a totally geodesic surface $\Lambda_C \subset \Lambda^4$. All the surfaces Λ_C are isometric copies of the 2-dimensional de Sitter space $\Lambda^2 \subset \boldsymbol{R}^3$, where $\boldsymbol{R}^3$ is endowed with the Lorentz "metric" $\mathscr{L}(x) = x_0^2 - x_1^2 - x_2^2$, and Λ^2 has the equation $\mathscr{L}(x) = -1$. Therefore, when a regular canal is a surface of revolution the curve $\gamma \subset \Lambda^4$ defining it has to be contained in some Λ_C.

Lemma 2.5.2. *A regular canal is a surface of revolution if and only if the curve γ corresponding to the spheres of the family is contained in a totally geodesic 2-dimensional de Sitter space $\Lambda^2_C \subset \Lambda^4$.*

Proof. Consider the pencil of spheres containing the axis C. It corresponds to a geodesic circle of Λ^4 of the form $h_C \cap \Lambda^4$, where h_C is a space-like plane of $\boldsymbol{L}^5$. The totally geodesic Λ^2_C is then $h_C^{\perp} \cap \Lambda^4$. □

Remarks.

- The curve $\gamma \subset \Lambda^4$ corresponding to a surface of revolution determines the "axis" C. If γ is a geodesic, then the spheres form a pencil and do not define a regular envelope.
- The only surfaces of revolution that have two axes are the Dupin cyclides. Notice that, to find the two axes, we need to use the two different curves of Λ^4 which give the Dupin cyclide as an envelope in two different ways.

Another property of leaves of a foliation by surfaces of revolution is:

Lemma 2.5.3. *Two asymptotic leaves have a common axis.*

Proof. We consider now a pairs of points $(x_t, y_t), x_t \in L_1, y_t \in L_2, t \in \boldsymbol{R}$, such that $d(x_t, y_t) \to 0$ when $t \to \infty$. We suppose also that the two points stay away from umbilical points. As the two leaves L_1 and L_2 are asymptotic, and as the foliation is of class $\mathscr{C}^2$, therefore a fortiori of class $\mathscr{C}^1$, the tangent planes through x_t and y_t respectively, get arbitrarily close. Moreover, as the foliation is of class $\mathscr{C}^2$, the principal directions at the points x_t and y_t get close. The continuity of the principal directions implies that the integral curves of each of these line field also get close. Therefore, in particular the characteristic circles $C_{1,t}$ at the points x_t and $C_{2,t}$ at y_t get close. The characteristic circle at a point of a canal surface is the intersection of the two spheres $\Sigma_t = \gamma(t)^\perp \cap \boldsymbol{S}^3$ and $\dot{\Sigma}_t = \dot{\gamma}(t)^\perp \cap \boldsymbol{S}^3$; the first sphere is tangent to the surface, the second normal to it. At the points x_t we get this way a sphere $\Sigma_{1,t} \supset C_{1,t}$ tangent to L_1 at x_t and a sphere $\dot{\Sigma}_{1,t} \supset C_{1,t}$ normal to L_1. Similarly we get two spheres $\Sigma_{2,t} \supset C_{2,t}$ and $\dot{\Sigma}_{2,t} \supset C_{2,t}$ respectively tangent and normal to L_2 at y_t. The corresponding spheres get close when $x_t \in L_1$ and $y_t \in L_2$ do. Therefore the two curves $\gamma_1 \subset \Lambda^4$ and $\gamma_2 \subset \Lambda^4$ corresponding to the two families of spheres having L_1 and L_2 as envelopes are asymptotic or have the same limit which should be a point where they are tangent. This last possibility is not allowed when L_1 and L_2 are leaves of a foliation: if the tangent vector to the limit point of the two curves is space-like, the two leaves should share a circle where they are tangent to the same sphere, if the limit tangent is light-like, the two leaves share a point where they are tangent (see Proposition 3.2.2).

As the two leaves are surfaces of revolution, the first is the envelope corresponding to a curve $\gamma_1 \subset \Lambda_{C_1}$, and the second the envelope corresponding to a curve $\gamma_2 \subset \Lambda_{C_2}$. If Λ_{C_1} and Λ_{C_2} are different, these two surfaces can intersect in

1. two antipodal points,
2. a geodesic hyperbola of Λ^4 or two parallel light-rays,
3. a geodesic circle γ_0 of Λ^4.

The last case is the only one which is compatible with the existence of such asymptotic curves $\gamma_1 \subset \Lambda_{C_1}$ and $\gamma_2 \subset \Lambda_{C_2}$. Fortunately, in that case the envelopes would not be of bounded geometry (for any of the conformally equivalent metrics of $\boldsymbol{S}^3$). To see that, first recall that a geodesic circle in Λ^4 corresponds to a pencil of spheres with a base circle.

Our two curves γ_1 and γ_2 are now asymptotic to the circle γ_0. The Dupin necklaces on the two leaves therefore get thinner and thinner when t goes to infinity, contradicting the fact that leaves of a smooth foliation (without singularities) have bounded geometry.

Therefore one needs to have $\Lambda_{C_1} = \Lambda_{C_2}$ that is $C_1 = C_2 = C$. The common axis of the two leaves is the circle C. $\square$

3. Canal Reeb components.

3.1. Toy examples in dimension 2 and 3.

As foliations with leaves which are conformal surfaces of revolution and mutually asymptotic are envelopes of spheres which are orthogonal to the same circle $\Gamma \subset \boldsymbol{S}^3$, it is natural to consider the intersection of the foliation with the spheres of the pencil of axis Γ. This way we obtain one-dimensional foliations on two-spheres.

In dimension 2, the envelope of the circles corresponding to a circle $\beta \subset \Lambda^3$ is the union of two circles. When the circle β is of radius larger than 1, that is when $\beta = \Lambda^3 \cap H$, where H is an affine space-like 2-space such that, using the notations of the paragraph about Dupin cyclides in Subsection 2.2, $\mathscr{L}(Ox_H) < 0$, the two envelope circles are disjoint. The intersection $\Lambda^3 \cap \mathrm{Span}(O, H)$ is a two-dimensional de Sitter space Λ^2 which also contains β. The interpretation of β as a set of pairs of points on a circle is not enlightening here. Nevertheless, given a pair of points on a circle $\Gamma_0 \subset \Sigma_0 \subset \boldsymbol{S}^3$, there is a unique circle in Σ_0 orthogonal to Γ_0 and containing the two points. There is also a unique sphere orthogonal to Γ_0 containing the same two points (and the previous circle).

Therefore, we can construct simultaneously a Poincaré component of the annulus and a Reeb component of a solid torus bounded by a regular Dupin cyclide from a family of curves in Λ^2 (see Figure 8).

More generally, we can consider a foliation of a sphere Σ by curves with leaves which are the envelope of circles and extend it to a domain of $\boldsymbol{S}^3$ foliated by the envelope of the spheres containing the circles and orthogonal to Σ. This would provide a family of foliations with a constraint weaker that the requirement that all leaves are conformally surfaces of revolution, and so we will not consider them in this article.

Let us first construct a foliation of an annulus contained in $\boldsymbol{S}^2$ or $\boldsymbol{R}^2$, that we will call *Poincaré component*; we will call the leaves of a Poincaré component which are lines *Poincaré leaves*. The starting point is the graph of an even function

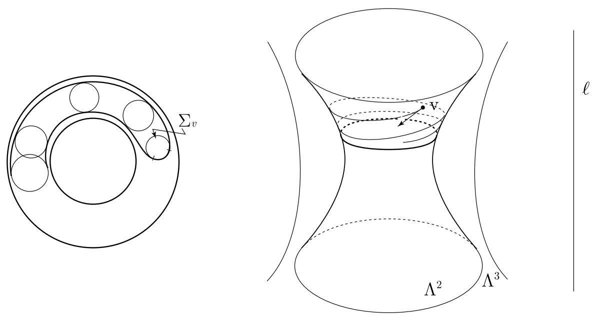

Figure 8. The circles defining a Poincaré leaf ending with a circular cap.

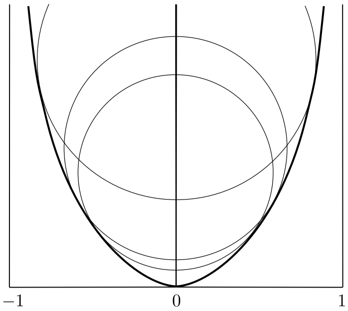

Figure 9. Poincaré leaf, first step.

$f : (-1, 1) \to \boldsymbol{R}$ whose graph has two vertical asymptotes $x = \pm 1$. We can now fill the vertical strip with Poincaré leaves translating vertically the previous curve (see Figure 9). The quotient $([-1, 1] \times \boldsymbol{R})/(0, 1) \cdot \boldsymbol{Z}$ is a foliated annulus. The two boundary circles are leaves. The other leaves are lines which spiral to the two boundary circles. In dimension 3, we should follow the classical construction of Reeb component of the solid torus $(D^2 \times \boldsymbol{R}))/\boldsymbol{Z}$. We recall it here. Then, in dimension 3, from the graph of our function f that we see contained in the vertical (x, z)-plane, we can obtain a surface of revolution of axis Oz asymptotic to the cylinder of equation $x^2 + y^2 = 1$. This surface has an umbilic point at the origin. The images by vertical translations of this surface and the boundary cylinder foliate the solid cylinder $\{x^2 + y^2 \leq 1\}$. The quotient of this solid cylinder by the unit vertical translation is a foliated solid torus $D^2 \times S^1$. In the interior of the torus, the leaves are planes asymptotic to the torus. Any foliation obtained from the above by a diffeomorphism is called a *Reeb component* (see [**Re**], [**CC**] etc.).

Let us now construct a Poincaré component the leaves of which are envelopes of one parameter family of circles which are orthogonal to a given circle (see Figure 8).

The set of circles of a sphere $\boldsymbol{S}^2$ or the set of circles-or-planes of the Euclidean plane $\boldsymbol{R}^2$ forms a quadric $\Lambda^3 \subset \boldsymbol{L}^4$. Therefore we can repeat what has been done for spheres in $\boldsymbol{S}^3$ for circles in $\boldsymbol{S}^2$. We could have considered also zero dimensional circles, that is pairs of points, in $\boldsymbol{S}^1$. These form a two dimensional quadric $\Lambda^2 \subset \boldsymbol{L}^3$. Circles orthogonal to a given circle $C \subset \boldsymbol{S}^2$ intersect it in two points. Notice also that the circles orthogonal to a given circle C correspond to the points of the intersection of Λ^3 with the hyperplane $P \subset \boldsymbol{L}^4$ orthogonal to the line generated by the point $\sigma_C \in \Lambda^3$ corresponding to the circle C. This intersection Λ^2_C is the quadric of the 3-dimensional affine subspace P given by the equation $\mathscr{L}|_P = 1$, where $\mathscr{L}|_P$ is the restriction to P of the Lorentz form $\mathscr{L}$ of $\boldsymbol{L}^4$.

Consider the intersection of Λ^2_C with an affine space-like plane $Q \subset P$ which does not contain the origin. It is a circle $\beta \subset \Lambda^2_C$. The envelope of the circles corresponding to the points of β is the union of two circles bounding an annulus of $\boldsymbol{S}^2$ (see [**LW**]). Let us denote by ℓ the time-like line orthogonal to Q. It is oriented as the x_4-axis of $\boldsymbol{L}^4$.

Let us now consider a curve $\delta : (-\infty, T] \to \Lambda^2_\Gamma$; the curve δ is contained in the region above β starting at a point with light-like tangent, space-like everywhere else and asymptotic to β (see Figure 8 again). We suppose that the projection of δ on ℓ is increasing, and that the absolute value of the geodesic curvature of δ goes to infinity when t approaches T. For the choice of the metric on $\boldsymbol{S}^3$ associated to the time-like line ℓ the two circles boundary of the annulus are equidistant from a geodesic and the circles of the family are centred on this geodesic.

As was explained in Subsection 2.3, we have

Proposition 3.1.1. *The extremity v of the curve δ, where the tangent vector is light-like, corresponds to a circle which is the osculating circle to the envelope of the circles corresponding to δ at the only vertex U of this envelope.*

We can also consider a circle $\Gamma \subset \boldsymbol{S}^3$. The two-dimensional de Sitter quadric Λ^2_Γ is now embedded in Λ^4. The curve $\beta \subset \Lambda^2_\Gamma$ corresponds to spheres which envelope a Dupin cyclide. The curve $\delta \subset \Lambda^4$ corresponds to a family of spheres which envelopes a planar leaf contained in a solid torus. The only umbilic of a planar leaf is its intersection with the circle Γ.

Figure 13 shows how to obtain a more complicated Reeb leaf: the curve in Λ^2 may cross the limit circle and may be immersed.

3.2. General Reeb components and turbulization.

It is now easy to visualize a Poincaré component such that the boundary of the annulus is a pair of disjoint closed curves in $\boldsymbol{S}^2$ (see Figure 10, in our example the circles the envelope of which is the boundary of the annulus are all orthogonal to an equator Γ, so we had drawn the curve γ_{out} corresponding to these circles on $\Lambda_\Gamma \subset \Lambda^3$).

To the boundary of an annulus A contained in a 2-sphere $\Sigma \subset \boldsymbol{S}^3$, which is the envelope of the circles of a curve $\gamma_0 \subset \Lambda^3_\Sigma \subset \Lambda^4$, we can associate, considering γ_0 as a curve of Λ^4, a torus boundary of a solid torus. The torus intersects Σ orthogonally along ∂A.

To recognize, looking just at two curves $\gamma_1 \subset \Lambda^4$ and $\gamma_2 \subset \Lambda^4$, if the corresponding envelopes intersect is not easy. The condition that a one-parameter family $\{\gamma_a(t)\}$ of curves should satisfy in order to guarantee that the envelopes are the leaves of a foliation is easier, as it is enough to check that locally leaves do not intersect. Locally the one-parameter family of curves can be seen as a map

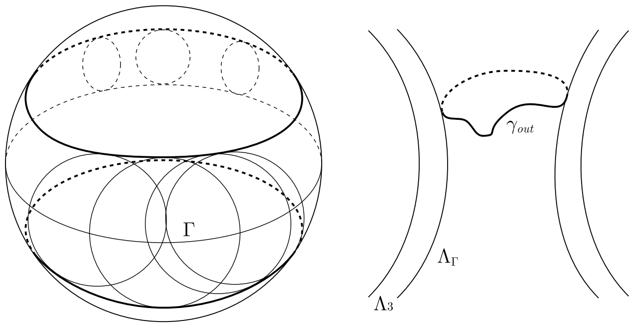

Figure 10. The circles defining the boundary of a Poincaré component.

from an open disc of $\boldsymbol{R}^2$ to Λ.

PROPOSITION 3.2.1. *If the plane* $\mathrm{Span}(\dot{\gamma}_{a_0}(t_0), (\partial\gamma_a/\partial a)(t_0))$ *is of mixed type, then the envelopes of the spheres of the curves* γ_a *form a foliation around a point of the characteristic circle* $C(a_0, t_0)$ *of the envelope of the spheres of the curve* γ_{a_0}.

PROOF. Let $\Sigma_D(a_0, t_0)$ be the sphere $(\dot{\gamma}_{a_0}(t_0))^\perp \cap \boldsymbol{S}^3$. The condition guarantees that the circles $C(a_0, t_0)$ and $\Sigma_D(a_0, t_0) \cap \gamma_a(t)$ are disjoint, and that their Hausdorff distance is of the order of $|a - a_0|$ for a close enough to a_0 and t close enough to t_0 that is $(|t - t_0| \leq M|a - a_0|$ for some constant $M)$. As the distance between the two circles $C(a, t)$ and $\Sigma_D(a_0, t_0) \cap \gamma_a(t)$ is negligible compared to $|a - a_0|$, this guarantees that locally the envelope of γ_a does not meet the envelope of γ_{a_0} near $C(a_0, t_0)$. $\square$

Let us now start with a closed curve contained in Λ such that $\mathscr{L}(\dot{\gamma}_0) < 0$ and $\mathscr{L}(\vec{k}_g(\gamma_0)) > 0$; the associated envelope is a torus. We can construct an annulus contained in Λ^4 as the union of arcs of time-like geodesics "centred" at the points of γ_0, We then view this annulus as an interval bundle over γ_0. Proposition 3.2.1 shows that it is enough to construct disjoint curves in this annulus which spiral to γ_0 on one end, and have an end point satisfying the conditions of Subsection 2.3 and which are local sections of the bundle.

We can directly perform the construction in Λ^2_Γ if we want to obtain a foliation by leaves which are conformally surfaces of revolution (with "axis" Γ).

Let us now state a proposition which gives a negative answer in the particular case of canal foliation to a question of the second author (see [**Bo**])

PROPOSITION 3.2.2. *To an unbounded curve with one extremity with tangent light-like direction corresponds a canal planar leaf. This extremity corresponds to* *the only umbilical point of the planar leaf.*

Curves corresponding to leaves of a *turbulized* (that is, obtained by the procedure of turbulization described in details in, for example, [**CC**, vol. I, Section 3.3]) Reeb component (see Figures 12 and 13) cannot be seen as “monotone in the fibre direction” in some interval bundle over the curve γ_{out} corresponding to the boundary of the component. Nevertheless we can observe that these curves will spiral towards two closed curves γ_{out} and γ_{in}. The situation is not too difficult to analyse as the solid “larger” foliated torus is the union of the balls corresponding to the oriented spheres of γ_{out} and the smaller is the union of the balls corresponding to the oriented spheres of $-\gamma_{in}$ (note the sign) (see [**H-J–P**, p. 26]). The orientations of the limit spheres of a leaf are reversed as the orientations of the two limit circles of a Poincaré component. But, as when we wanted to “close” a planar leaf, there will necessarily be some singular points on the curve γ_L corresponding to the leaf L. Let us first prove that the leaf has to cross one of its defining spheres at a tangency point.

Let us first consider a “baby case”: a piece of foliation of the plane. The envelope of a one-parameter family of circles of the plane is, as in Figure 11, formed of two sheets, each one performing a U-turn. We claim that the curve in Λ^3 corresponding to this family of circles, parametrized by the arc-length of one of the sheets of the envelope, should have at least one singular point. First notice that, between Γ_1 and Γ_2, one circle Γ_{osc} of the family should cross the sheet L_1, therefore be an osculating circle. This implies that Γ_{osc} is also an osculating circle of the second sheet L_2 of the envelope at the point of contact of Γ_{osc} and L_2 obtained from the definition of $L_1 \cup L_2$ as an envelope. This can be seen using Proposition 2.1.1. The point $\sigma \in \Lambda^3$ can be expressed, using the curvature k of the circle Γ corresponding to σ and the point of contact $m^1 \in L_1$ by $\sigma = km^1 + n$. Differentiating, we get $\overset{\bullet}{\sigma} = \overset{\bullet}{k}m^1 + T_1(k - \kappa_1)^1$, where κ_1 is the curvature of L_1 at

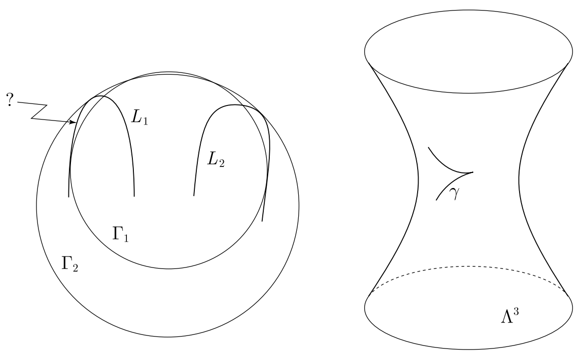

Figure 11. U-turn and a circle of the family which is osculating.

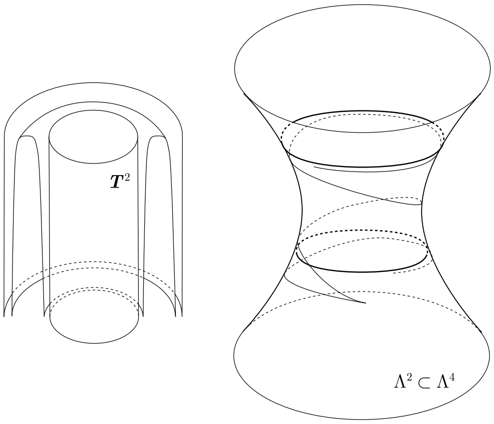

Figure 12. The leaf turns, the curve in Λ has a cusp.

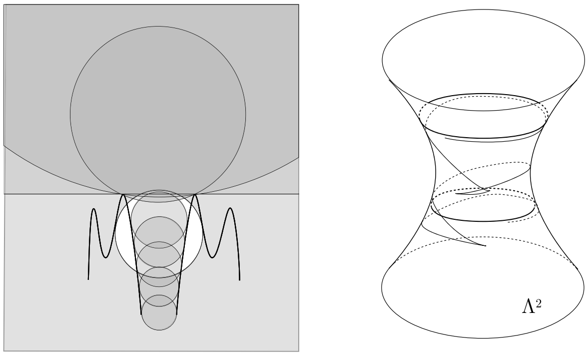

Figure 13. The spheres defining a leaf after turbulization (leaves are surfaces of revolution).

m^1 and T_1 a unit vector tangent to L_1 at m^1. At the point $\sigma_{osc} \in \Lambda^3$, $m^1_{osc} \in L_1$ corresponding to the osculating circle Γ_{osc} we have $\dot{\sigma}_{osc} = \dot{k} m^1_{osc}$. If $\dot{k}$ is not zero at m^1_{osc} the tangent to the curve $\gamma = \{\sigma\} \subset \Lambda^3$ is lightlike. Using now the second sheet L_2 of the envelope to parametrize γ (and a $'$ to note derivatives with respect to some arc-length on L_2) we get $\sigma'_{osc} = k' m^2 + (k - \kappa_2) T_2$, where κ_2 is the curvature of L_2 at m^2 and T_2 a unit vector tangent to L_2 at m^2. Therefore, as the tangent direction to γ does not depend on the choice of the parametrization if the latter is proper, we have $\kappa_2 = k$ at the point m^2 corresponding to Γ_{osc}. As m^1_{osc} and m^2 (or rather m^2_{osc}) are different points we conclude that $\dot{\sigma}_{osc} = \sigma'_{osc} = 0$, proving the parametrized curve γ is singular. Once more we can prove that, generically, this singularity is a cuspidal singularity of the curve γ.

Coming back to leaves of foliations of $\boldsymbol{S}^3$, let us suppose that, travelling on γ_L (that is for $t \in (-\infty - A])$, we start from spheres close to the one defining the inward torus $\boldsymbol{T}_2$, and suppose these are oriented as boundary of the balls $B(t)$, the union of which is the solid torus $S\boldsymbol{T}_2$. Because of the turn, the end of γ_L (that is $t \in [+A, +\infty))$ is formed of spheres $B(t)$ bounding the exterior of the balls B_{int}, the union of which is the solid torus $S\boldsymbol{T}_1$. On the way, the complement $Ext(t)$ of a ball $B(t)$ has to intersect the torus $\boldsymbol{T}_1$ in at least an annulus. The leaf L is now tangent from the inside to the ball $Ext(t)$. Letting t increase, the leaf L will eventually be tangent to the ball $Ext(t)$ from the inside and will approach the balls B_{int}, the union of which is the solid torus $S\boldsymbol{T}_1$. This implies the existence of a value t_0 of t such that the leaf crosses the defining sphere at a tangency point. This point m_0 therefore has to be an umbilic. Then (see Subsection 2.3) the other points of the characteristic circle $\Gamma_{Car}(t_0)$ have to be umbilics too (we could have used Proposition 2.1.1 and a computation similar to that of the baby case to show that the whole characteristic circle through an umbilical point is made of umbilics). In fact, we know that the “other” osculating sphere along $\Gamma_{Car}(t_0)$ form a Dupin cyclide, the Dupin necklace of $\Gamma_{Car}(t_0)$. As this Dupin necklace shares the point $\gamma(t_0)$ with γ, it has to degenerate in a point, and the geodesic curvature of γ has to go to ∞ when t goes to t_0. We have in particular proved that the lines of principal curvature of L of second family cross simultaneously the defining sphere $\Sigma(t_0)$.

4. Classification of foliations by canal surfaces.

4.1. Reeb decomposition of $\boldsymbol{S}^3$.

A decomposition of $\boldsymbol{S}^3$ as the union of two solid tori $\boldsymbol{T}_a$ and $\boldsymbol{T}_b$ is obtained considering $\boldsymbol{S}^3$ as the unit sphere of equation $|z_1|^2 + |z_2|^2 = 1$ in the complex two-dimension space $\boldsymbol{C}^2$. For two real numbers $0 < a < 1$; $0 < b < 1$; $a^2 + b^2 = 1$ we can define $\boldsymbol{T}_a = \{(z_1, z_2) \in \boldsymbol{S}^3 \mid |z_1| \leq a\}$ and $\boldsymbol{T}_b = \{(z_1, z_2) \in \boldsymbol{S}^3 \mid |z_2| \leq b\}$. The intersection $\boldsymbol{T}_a \cap \boldsymbol{T}_b$ is the Dupin cyclide $D_{a,b}$ of equation $|z_1| = a$, $|z_2| = b$. The cores of the Dupin cyclide are not only defined using the standard metric of $\boldsymbol{S}^3 \subset \boldsymbol{C}^2$ as the circles C_1 of equations $|z_1| = 0$, $|z_2| = 1$ and C_2 of equations $|z_2| = 0$, $|z_1| = 1$, but can also be defined in a purely conformal way (see [**LW**]). The Dupin cyclide $D_{a,b}$ is in two different ways the envelope of a one-parameter family of spheres; let γ_1 and γ_2 be the corresponding curves in Λ^4. Say that $\boldsymbol{T}_a$ is the union of the spheres corresponding to the points of γ_1. The curve γ_1 is continued in an affine plane $P_1 \subset \boldsymbol{L}^5$. The intersection of the vectorial plane p_1 parallel to P_1 with Λ^4 corresponds to a pencil of spheres the axis of which is the circle C_2. The circle C_1 can be found symmetrically starting with γ_2.

4.2. Foliations of $\boldsymbol{S}^3$.

The leaves are planes, tori and cylinders; the corresponding curves in Λ^4 are closed curves, curves unbounded on both sides, unbounded curves with light-like tangent direction at the extremity. Recall that a foliation of $\boldsymbol{S}^3$ cannot have leaves which are diffeomorphic to spheres. Other topological types are impossible because the leaf is a canal surface, and therefore is a circle bundle with either a point or a spherical cap attached (see Subsection 2.4).

THEOREM 4.2.1. *Any foliation $\mathscr{F}$ of $\boldsymbol{S}^3$ by canal surfaces is either a Reeb foliation with toral leaf a Dupin cyclide, or is obtained from such a Reeb foliation inserting a zone $Z \simeq \boldsymbol{T}^2 \times [0,1]$, a union of toral and cylindrical leaves.*

PROOF. The Novikov theorem (see [**No**] and [**CC**]) implies that at least one leaf of $\mathscr{F}$ is a toral leaf. Let us first suppose that on both sides we get a vanishing circle for the foliation. Then on both sides we get a Reeb component, the characteristic circles on the planar leaves bound discs containing the umbilical points. Characteristic circles through sequences of points approaching the boundary in a Reeb component then converge to a family of meridian (round) circles on the boundary. That way we see that the torus leaf is in two different ways a canal surface and is therefore a Dupin cyclide. Let us now suppose that, at least on one side, the torus leaf is a limit of leaves which are tori or cylinders. It is therefore contained in a connected zone, diffeomorphic to $\boldsymbol{T}^2 \times I$, which is a union of leaves which are tori or cylinders. Consider now a zone $Z_{\max}$ containing the initial torus leaf, and maximal for the above properties. Its boundary is the union of two tori, which out of $Z_{\max}$ are limits of planar leaves; therefore the two tori are boundaries of two Reeb components R_1 and R_2. As the union $R_1 \cup Z_{\max} \cup R_2$ is $\boldsymbol{S}^3$, the two Reeb component are two unknotted solid tori which are linked in $\boldsymbol{S}^3$. $\square$

In order to understand the structure of the foliation of $Z_{\max}$ we need the following

THEOREM 4.2.2. *At least one leaf of a canal foliation is a Dupin cyclide.*

PROOF. The homotopy class of the characteristic circles of a leaf contained in the zone $Z_{\max}$ have to belong to the homotopy class of the meridian of one of the Reeb components. As these classes are the two "canonical" generators of $\pi_1(Z_{\max})$, Remark (2.4) implies that at least one toral leaf contained in Z_{max} has two families of characteristic circles and therefore is a Dupin cyclide. $\square$

To finish the classification of foliations of $\boldsymbol{S}^3$ by canal surfaces just notice that the structure of the maximal zone $Z_{\max}$ is very similar to the structure of foliation of the 2-torus by curves (see [**Kn**] and [**Go**]). It contains a finite number

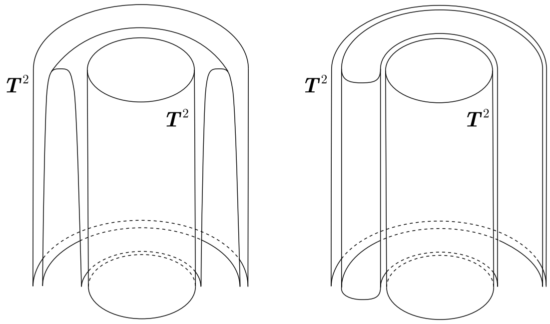

Figure 14. The two types of essential zones.

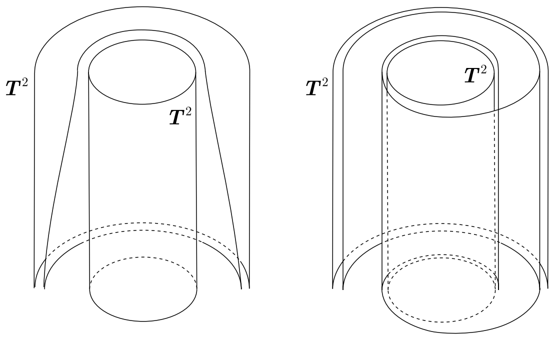

Figure 15. The two of ways of spiralling.

of *essential* zones $Z_i \simeq \boldsymbol{T}^2 \times [0,1]$ containing cylindrical leaves which give to the two limit tori opposite orientations (see Figure 14). The class of the characteristic circles of the cylindrical leaves of Z_i can be the meridian class of any of the Reeb components. We can now summarize all the information on the foliation in the zone $Z_{\max}$ in the following

Theorem 4.2.3. *The zone $Z_{\max}$ contains a maximal product lamination $\boldsymbol{T}^2 \times K$, K a compact subset of the interval*; *the toral leaves are nested, that is all the toral leaves are unknotted, and each toral leaf is contained in one solid torus bounded by the other toral leaves. The complement of this lamination is a union of*:

- *a finite number of* essential zones (*see Figure* 14) *where cylindrical leaves accumulate on the two boundary tori and induce there different orientations.*
- *a finite or countable number of spiralling components* (*see Figure* 15) *where cylindrical leaves accumulate on the two boundary tori and induce there the same*

orientation.

REMARK. In [**Wa**], the *Hausdorffized leaf space* of a foliation of a Riemannian manifolds has been defined as a metric space obtained from a suitable pseudodistance function on the space of leaves by the indentification of the leaves which lie in the pseudodistance zero. It is shown there that if the foliation is of codimension one, then this space is isometric to a metric graph. In this setting, canal foliations of S^3 are represented by a very simple graph: a segment with a finite or countable number of points distinguished on it.

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