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Wandering subspaces and the Beurling type theorem, III

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Abstract. Let $H^2(\boldsymbol{D}^2)$ be the Hardy space over the bidisk. Let $\{\varphi_n(z)\}_{n\geq 0}$ and $\{\psi_n(w)\}_{n\geq 0}$ be sequences of one variable inner functions satisfying some additinal conditions. Associated with them, we have a Rudin type invariant subspace $\mathscr{M}$ of $H^2(\boldsymbol{D}^2)$. We study the Beurling type theorem for the fringe operator F_w on $\mathscr{M} \ominus z\mathscr{M}$.

1. Introduction.

Let T be a bounded linear operator on a Hilbert space H. For a subset E of H, we denote by $[E]_H$ the smallest invariant subspace of H for T containing E. Let M be an invariant subspace of H for T. We denote by $M \ominus TM$ the orthogonal complement of TM in M. The space $M \ominus TM$ is called a wandering subspace of M for the operator T. We have $[M \ominus TM]_H \subset M$. We say that the Beurling type theorem for T if $[M \ominus TM]_H = M$ for every invariant subspace M of H for T. Our basic problem is to find operators T on H for which the Beurling type theorem holds.

Let $\boldsymbol{D}$ be the open unit disk in the complex plane $\boldsymbol{C}$. We denote by $H^2(\boldsymbol{D})$ the Hardy space on $\boldsymbol{D}$. A function $\varphi(z)$ in $H^2(\boldsymbol{D})$ is called inner if $|\varphi(z)| = 1$ a.e. on $\partial\boldsymbol{D}$. Let T_z be the multiplication operator on $H^2(\boldsymbol{D})$ by the coordinate function z. For every nonzero invariant subspaces M of $H^2(\boldsymbol{D})$ for T_z, the Beurling theorem [**2**] says that $M \ominus T_z M = \boldsymbol{C}\cdot\varphi(z)$ for an inner function $\varphi(z)$ and $M = [M \ominus T_z M]_{H^2(\boldsymbol{D})}$ (see also [**5**], [**7**]). For a nonzero closed invariant subspace M of the Dirichlet shift T_z on the Dirichlet space $\mathscr{D}$, Richter showed that $\dim(M \ominus T_z M) = 1$ and the Beurling type theorem holds for the Dirichlet shift in [**15**]. Aleman, Richter, and Sundberg proved that the Beurling type theorem also holds for the Bergman shift on the Bergman space $L^2_a(\boldsymbol{D})$ in [**1**]. In [**19**], Shimorin showed the following theorem.

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Shimorin's Theorem. *If $T : H \to H$ satisfies the following conditions*

(a) $\|Tx + y\|^2 \le 2(\|x\|^2 + \|Ty\|^2)$, $x, y \in H$,
(b) $\bigcap\{T^n H : n \ge 0\} = \{0\}$,

then the Beurling type theorem holds for T.

As an application of this theorem, Shimorin gave a simpler proof of the Aleman, Richter, and Sundberg theorem (see also [**6**]). Later, different proofs of the the Beurling type theorem are given in [**12**], [**14**], [**20**]. Recently, the authors [**8**] proved the following theorem.

Theorem A. *If $T : H \to H$ satisfies the following conditions*

(i) $\|Tx\|^2 + \|T^{*2}Tx\|^2 \le 2\|T^*Tx\|^2$, $x \in H$,
(ii) *T is bounded below,*
(iii) $\|T^{*n}x\| \to 0$ *as $n \to \infty$ for every $x \in H$,*

then the Beurling type theorem holds for T.

Also it is pointed out that conditions (a) and (b) in Shimorin's theorem are equivalent to conditions (i)–(iii) in Theorem A.

Let $H^2 := H^2(\boldsymbol{D}^2)$ be the Hardy space over the bidisk $\boldsymbol{D}^2$. We identify a function in H^2 with its boundary function on the distinguished boundary $(\partial \boldsymbol{D})^2$ of $\boldsymbol{D}^2$, so we think of H^2 as a closed subspace of the Lebesgue space $L^2 := L^2((\partial \boldsymbol{D})^2)$. We use z, w as variables in $\boldsymbol{D}^2$. We denote by $H^2(z)$ the z-variable Hardy space, and we think of $H^2(z)$ as a closed subspace of H^2. Then H^2 coincides with the tensor product $H^2(z) \otimes H^2(w)$. Let T_z, T_w be multiplication operators on H^2 by z and w. A closed subspace M of H^2 is called invariant if $T_z M \subset M$ and $T_w M \subset M$. For a subset E of M, we denote by $[E]_{H^2}$ the smallest invariant subspace containing E. For a subspace E of H^2, we denote by P_E the orthogonal projection from L^2 onto E. See books [**3**], [**16**] for the study of the Hardy space H^2 over $\boldsymbol{D}^2$.

Let M be an invariant subspace of H^2. Write $R_z = T_z|_M$ and $R_w = T_w|_M$, the operators on M. Since R_z is an isometry on M, by the Wold decomposition theorem we have

$$M = \sum_{n=0}^{\infty} \oplus (M \ominus zM) z^n.$$

So a lot of information of an invariant subspace M are considered to be encoded in those of $M \ominus zM$. So to study the structure of invariant subspaces M of H^2, $M \ominus zM$ is one of the most important spaces. Note that $P_{M \ominus zM} = I - R_z R_z^*$. To study $M \ominus zM$, Yang [**21**] defined the fringe operator F_w on $M \ominus zM$ by

$$F_w f = P_{M\ominus zM} R_w f, \quad f \in M \ominus zM,$$

and studied the properties of F_w (see [**21**], [**22**], [**23**]).

Let $M_\varphi := [z-\varphi(w)]_{H^2}$ for a nonconstant inner function $\varphi(w)$. In the previous paper [**9**], as applications of Theorem A we studied the Beurling type theorem for the fringe operator F_w on $M_\varphi \ominus zM_\varphi$ and for the compression operator S_z on $H^2 \ominus M_\varphi$, respectively.

In this paper, we shall study invariant subspaces of H^2 based on inner sequences. Let $\{\varphi_n(z)\}_{n\geq 0}$ and $\{\psi_n(w)\}_{n\geq 0}$ be sequences of inner functions such that $\varphi_n(z)/\varphi_{n+1}(z)$ and $\psi_{n+1}(w)/\psi_n(w)$ are inner functions for every $n \geq 0$. Moreover we assume that $\bigcap_{n=0}^\infty \psi_n(w)H^2(w) = \{0\}$. Let

$$\mathscr{M} = \sum_{n=0}^{\infty} \oplus\big(\varphi_n(z)H^2(z)\big) \otimes \big(\psi_n(w)H^2(w) \ominus \psi_{n+1}(w)H^2(w)\big).$$

Then $\mathscr{M}$ is an invariant subspace of H^2. This type of invariant subspaces of H^2 have been studied in [**4**], [**16**], [**17**], [**18**]. We have

$$\mathscr{M} \ominus z\mathscr{M} = \sum_{n=0}^{\infty} \oplus \varphi_n(z)\big(\psi_n(w)H^2(w) \ominus \psi_{n+1}(w)H^2(w)\big).$$

We study the Beurling type theorem for the fringe operator F_w on $\mathscr{M} \ominus z\mathscr{M}$. Without loss of generality, we assume that $\psi_0(w) = 1$. Our strategy of the study is to define an invertible bounded linear operator $\boldsymbol{V} : H^2(w) \to \mathscr{M} \ominus z\mathscr{M}$ satisfying $\boldsymbol{V}T_w = F_w\boldsymbol{V}$ on $H^2(w)$. Using this operator, we study the Beurling type theorem for F_w on $\mathscr{M} \ominus z\mathscr{M}$. In Section 2, we shall study the case $\varphi_0(0) \neq 0$, and in Section 4 we shall study the case $\varphi_0(0) = 0$.

For nonconstant inner functions $\varphi(z)$ and $\psi(w)$, let

$$M = \varphi(z)H^2 + \psi(w)H^2.$$

Then M is an invariant subspace of H^2 and a special case of $\mathscr{M}$. Recently these type of M are studied in [**10**], [**23**]. In Section 3, we study the Beurling type theorem for F_w on $M \ominus zM$. When

$$\psi(w) = \frac{w-\beta}{1-\overline{\beta}w}, \quad |\beta| < 1,$$

we shall show that the Beurling type theorem holds for F_w on $M \ominus zM$ if and only if $|\beta|/(1+|\beta|) \leq |\alpha|^2$, where $\alpha = \varphi(0) \neq 0$.

2. Invariant subspaces based on inner sequences.

Let $\{\varphi_n(z)\}_{n\geq 0}$ and $\{\psi_n(w)\}_{n\geq 0}$ be sequences of inner functions satisfying the following conditions;

(1) $\psi_0(w) = 1$,
(2) $(\varphi_n(z))/(\varphi_{n+1}(z))$ is an inner function and $(\psi_{n+1}(w))/(\psi_n(w))$ is a nonconstant inner function for every $n \geq 0$,
(3) $\bigcap_{n=0}^{\infty} \psi_n(w)H^2(w) = \{0\}$.

Write

$$\frac{\varphi_n(z)}{\varphi_{n+1}(z)} = \zeta_n(z) \quad \text{and} \quad \frac{\psi_{n+1}(w)}{\psi_n(w)} = \xi_n(w).$$

Let

$$\begin{aligned}\mathscr{M} &= \sum_{n=0}^{\infty} \oplus\big(\varphi_n(z)H^2(z)\big) \otimes \big(\psi_n(w)H^2(w) \ominus \psi_{n+1}(w)H^2(w)\big)\\ &= \sum_{n=0}^{\infty} \oplus\big(\varphi_n(z)H^2(z)\big) \otimes \big(\psi_n(w)\big(H^2(w) \ominus \xi_n(w)H^2(w)\big)\big).\end{aligned}$$

By conditions (2) and (3), it is not difficult to see that $\mathscr{M}$ is an invariant subspace of H^2 and

$$\mathscr{M} \ominus z\mathscr{M} = \sum_{n=0}^{\infty} \oplus \varphi_n(z)\psi_n(w)\big(H^2(w) \ominus \xi_n(w)H^2(w)\big).$$

By (1) and (3), we have

$$H^2(w) = \sum_{n=0}^{\infty} \oplus \psi_n(w)\big(H^2(w) \ominus \xi_n(w)H^2(w)\big).$$

For each $n \geq 0$, we write

$$E_n = \psi_n(w)\big(H^2(w) \ominus \xi_n(w)H^2(w)\big).$$

Then

$$H^2(w) = \sum_{n=0}^{\infty} \oplus E_n \quad \text{and} \quad \mathscr{M} \ominus z\mathscr{M} = \sum_{n=0}^{\infty} \oplus \varphi_n(z) E_n.$$

Moreover in this and next sections, we assume that

(4) $0 < |\varphi_0(0)| < 1$.

Let $A_0 = 1$, and for each positive integer n let

$$A_n = \prod_{j=0}^{n-1} \zeta_j(0).$$

By conditions (2) and (4), $\varphi_n(0) \neq 0$, $\zeta_n(0) \neq 0$, and $A_n \neq 0$ for every $n \geq 0$. We have

$$A_n = \prod_{j=0}^{n-1} \frac{\varphi_j(0)}{\varphi_{j+1}(0)} = \frac{\varphi_0(0)}{\varphi_n(0)},$$

so we get $0 < |\varphi_0(0)| \leq |A_{n+1}| \leq |A_n| \leq |A_0| = 1$. Note that $|\zeta_n(0)| \to 1$ as $n \to \infty$. We define an operator $\boldsymbol{V} : H^2(w) \to \mathscr{M} \ominus z\mathscr{M}$ by

$$\boldsymbol{V}(g_n(w)) = A_n \varphi_n(z) g_n(w), \quad g_n(w) \in E_n.$$

Then $\boldsymbol{V}$ is an invertible bounded linear operator.

Lemma 2.1. *Let*

$$g = \sum_{n=0}^{\infty} \oplus \varphi_n(z) g_n(w) \in \sum_{n=0}^{\infty} \oplus \varphi_n(z) E_n = \mathscr{M} \ominus z\mathscr{M}$$

and

$$f(w) = \sum_{n=0}^{\infty} \oplus f_n(w) \in \sum_{n=0}^{\infty} \oplus E_n = H^2(w).$$

Then we have the following.

(i) $\boldsymbol{V}^* g = \sum_{n=0}^{\infty} \oplus \overline{A}_n g_n(w) \in H^2(w)$.
(ii) $\boldsymbol{V}^{-1} g = \sum_{n=0}^{\infty} \oplus A_n^{-1} g_n(w) \in H^2(w)$.
(iii) $(\boldsymbol{V}^*)^{-1} f(w) = \sum_{n=0}^{\infty} \oplus \overline{A}_n^{(-1)} \varphi_n(z) f_n(w) \in \mathscr{M} \ominus z\mathscr{M}$.

(iv) $(\boldsymbol{V}^*\boldsymbol{V})^{-1}f(w) = \sum_{n=0}^{\infty} \oplus |A_n|^{-2} f_n(w) \in H^2(w)$.

Proof. (i) We have

$$\begin{aligned}\langle \boldsymbol{V}^* g, f(w)\rangle &= \langle g, \boldsymbol{V} f(w)\rangle \\ &= \left\langle \sum_{n=0}^{\infty} \oplus \varphi_n(z) g_n(w), \sum_{n=0}^{\infty} \oplus A_n \varphi_n(z) f_n(w) \right\rangle \\ &= \sum_{n=0}^{\infty} \langle \overline{A}_n g_n(w), f_n(w)\rangle \\ &= \left\langle \sum_{n=0}^{\infty} \oplus \overline{A}_n g_n(w), \sum_{n=0}^{\infty} \oplus f_n(w) \right\rangle.\end{aligned}$$

Thus we get (i).

It is easy to get (ii)–(iv) from (i) and the definition of $\boldsymbol{V}$. □

The following is a key theorem in this paper.

Theorem 2.2. *$\boldsymbol{V} T_w = F_w \boldsymbol{V}$ on $H^2(w)$.*

Proof. Let k be a nonnegative integer. We have

$$\sum_{n=k}^{\infty} \oplus E_n = \psi_k(w) H^2(w),$$

so $\sum_{n=k}^{\infty} \oplus E_n$ is an invariant subspace of $H^2(w)$ for T_w. Let $f(w) \in E_k$. Then we may write $wf(w)$ as

$$wf(w) = \sum_{n=k}^{\infty} \oplus f_n(w) \in \sum_{n=k}^{\infty} \oplus E_n.$$

Hence

$$\boldsymbol{V} T_w f(w) = \sum_{n=k}^{\infty} \oplus A_n \varphi_n(z) f_n(w).$$

We have also

$$
\begin{aligned}
F_w \boldsymbol{V} f(w) &= F_w A_k \varphi_k(z) f(w) \\
&= A_k P_{\mathscr{M}\ominus z\mathscr{M}} \biggl(\varphi_k(z) \sum_{n=k}^{\infty} \oplus f_n(w) \biggr) \\
&= A_k \sum_{n=k}^{\infty} \oplus P_{\mathscr{M}\ominus z\mathscr{M}} (\varphi_k(z) f_n(w)) \\
&= A_k \sum_{n=k}^{\infty} \oplus \bigl\langle \varphi_k(z) f_n(w), \varphi_n(z) f_n(w) \bigr\rangle \frac{\varphi_n(z) f_n(w)}{\|f_n\|^2} \\
&= A_k \sum_{n=k}^{\infty} \oplus \langle \varphi_k(z), \varphi_n(z) \rangle \varphi_n(z) f_n(w) \\
&= A_k \varphi_k(z) f_k(w) \oplus \sum_{n=k+1}^{\infty} \oplus A_k \biggl\langle \frac{\varphi_k}{\varphi_{k+1}} \frac{\varphi_{k+1}}{\varphi_{k+2}} \cdots \frac{\varphi_{n-1}}{\varphi_n}, 1 \biggr\rangle \varphi_n(z) f_n(w) \\
&= A_k \varphi_k(z) f_k(w) \oplus \sum_{n=k+1}^{\infty} \oplus \biggl(\prod_{j=0}^{k-1} \zeta_j(0) \biggr) \biggl(\prod_{j=k}^{n-1} \zeta_j(0) \biggr) \varphi_n(z) f_n(w) \\
&= \sum_{n=k}^{\infty} \oplus A_n \varphi_n(z) f_n(w).
\end{aligned}
$$

Therefore $\boldsymbol{V} T_w f(w) = F_w \boldsymbol{V} f(w)$ for every $f(w)$ in E_k and $k \geq 0$. This shows the assertion. □

The following corollary follows directly from Theorem 2.2.

COROLLARY 2.3. *For every inner function $\theta(w)$, $\boldsymbol{V}(\theta(w) H^2(w))$ is an invariant subspace of $\mathscr{M} \ominus z\mathscr{M}$ for F_w.*

THEOREM 2.4. *Let L be a nonzero invariant subspace of $\mathscr{M} \ominus z\mathscr{M}$ for F_w. Then we have the following.*

- (i) *$L = \boldsymbol{V}(\theta(w) H^2(w))$ for an inner function $\theta(w)$.*
- (ii) *$\boldsymbol{V}\theta(w)$ is a cyclic vector of L for F_w.*
- (iii) $\dim(L \ominus F_w L) = 1$.
- (iv) $(\mathscr{M} \ominus z\mathscr{M}) \ominus L = (\boldsymbol{V}^*)^{-1} \bigl(H^2(w) \ominus \theta(w) H^2(w) \bigr)$.
- (v) *Let $g \in L$ satisfy $L \ominus F_w L = \boldsymbol{C} \cdot g$. Then $[L \ominus F_w L]_{\mathscr{M}\ominus z\mathscr{M}} = L$ if and only if $(\boldsymbol{V}^{-1} g)/\theta(w)$ is an outer function. If $(\boldsymbol{V}^{-1} g)/\theta(w)$ is not outer, let $\theta_1(w)$ be its inner factor, then*

$$\boldsymbol{V}((\theta\theta_1)(w) H^2(w)) = [L \ominus F_w L]_{\mathscr{M}\ominus z\mathscr{M}}.$$

(vi) $g = P_L(\boldsymbol{V}^*)^{-1}\theta(w)$ *for* g *in* (v).

Proof.

(i) By Theorem 2.2, we have $\boldsymbol{V}T_w\boldsymbol{V}^{-1}L = F_wL \subset L$. Then $T_w\boldsymbol{V}^{-1}L \subset \boldsymbol{V}^{-1}L$ and $\boldsymbol{V}^{-1}L$ is a nonzero closed subspace of $H^2(w)$. By the Beurling theorem, $\boldsymbol{V}^{-1}L = \theta(w)H^2(w)$ for an inner function $\theta(w)$. Thus we get $L = \boldsymbol{V}(\theta(w)H^2(w))$.

(ii) By Theorem 2.2, $\boldsymbol{V}(T_w^k\theta(w)) = F_w^k\boldsymbol{V}(\theta(w))$ for every $k \geq 0$. Since $\theta(w)$ is a cyclic vector in $\theta(w)H^2(w)$ for T_w, by (i) we get (ii).

(iii) By (i) and Theorem 2.2,

$$F_wL = F_w\boldsymbol{V}(\theta(w)H^2(w)) = \boldsymbol{V}T_w(\theta(w)H^2(w)) = \boldsymbol{V}(w\theta(w)H^2(w)).$$

Since $\boldsymbol{V} : H^2(w) \to \mathscr{M} \ominus z\mathscr{M}$ is invertible,

$$F_wL \subsetneqq \boldsymbol{C}\cdot\boldsymbol{V}\theta(w) + \boldsymbol{V}(w\theta(w)H^2(w)) = L$$

and F_wL is closed. Thus we get (iii).

(iv) Let $f \in \mathscr{M} \ominus z\mathscr{M}$. By (i), $f \perp L$ if and only if $\boldsymbol{V}^*f \perp \theta(w)H^2(w)$. Hence

$$\boldsymbol{V}^*\big((\mathscr{M} \ominus z\mathscr{M}) \ominus L\big) = H^2(w) \ominus \theta(w)H^2(w).$$

Thus we get the assertion.

(v) By Theorem 2.2, $\boldsymbol{V}T_w^k\boldsymbol{V}^{-1} = F_w^k$ for every $k \geq 0$. So $[L \ominus F_wL]_{\mathscr{M}\ominus z\mathscr{M}} = L$ if and only if the linear span of $\{w^k(\boldsymbol{V}^{-1}g)(w) : k \geq 0\}$ is dense in $\theta(w)H^2(w)$. Thus we get the first assertion.

Suppose that $(\boldsymbol{V}^{-1}g)(w)/\theta(w)$ is not outer. Let $\theta_1(w)$ be its inner factor. By the above argument, we have

$$\boldsymbol{V}((\theta\theta_1)(w)H^2(w)) = [L \ominus F_wL]_{\mathscr{M}\ominus z\mathscr{M}}.$$

(vi) We have $(\boldsymbol{V}^*)^{-1}\theta(w) \perp F_wL$. For, by (i) we have

$$\big\langle(\boldsymbol{V}^*)^{-1}\theta(w), F_wh\big\rangle = \big\langle\theta(w), \boldsymbol{V}^{-1}F_wh\big\rangle = \big\langle\theta(w), T_w\boldsymbol{V}^{-1}h\big\rangle = 0$$

for every $h \in L$. Also we have $(\boldsymbol{V}^*)^{-1}\theta(w) \not\perp L$. For, by (i) we have $\theta(w) \not\perp \theta(w)H^2(w) = \boldsymbol{V}^{-1}L$. Hence by (iii), we may take $g(w) = P_L(\boldsymbol{V}^*)^{-1}\theta(w)$. □

For arbitrary inner function $\theta(w)$, it seems difficult to compute $g = P_L(\boldsymbol{V}^*)^{-1}\theta(w)$. But for some special cases, we may compute it. For each $k \geq 0$, let

$$\mathscr{M}_k = \sum_{n=k}^{\infty} \oplus \big(\varphi_n(z) H^2(z)\big) \otimes \big(\psi_n(w)\big(H^2(w) \ominus \xi_n(w) H^2(w)\big)\big).$$

Then we have

$$\mathscr{M}_k \ominus z\mathscr{M}_k = \sum_{n=k}^{\infty} \oplus \varphi_n(z) E_n = \boldsymbol{V}(\psi_k(w) H^2(w)).$$

Hence by Corollary 2.3, $\mathscr{M}_k \ominus z\mathscr{M}_k$ is an invariant subspace of $\mathscr{M} \ominus z\mathscr{M}$ for F_w, and by Theorem 2.4 (ii) $\boldsymbol{V}\psi_k(w)$ is a cyclic vector of $\mathscr{M}_k \ominus z\mathscr{M}_k$ for F_w.

Corollary 2.5.

$$(\mathscr{M}_k \ominus z\mathscr{M}_k) \ominus F_w(\mathscr{M}_k \ominus z\mathscr{M}_k) = \boldsymbol{C} \cdot (\boldsymbol{V}^*)^{-1} \psi_k(w)$$

for every $k \geq 0$.

Proof. Write

$$\psi_k(w) = \sum_{n=k}^{\infty} \oplus f_n(w) \in \sum_{n=k}^{\infty} \oplus E_n.$$

By Lemma 2.1 (iii),

$$(\boldsymbol{V}^*)^{-1} \psi_k(w) = \sum_{n=k}^{\infty} \oplus \overline{A}_n^{(-1)} \varphi_n(z) f_n(w) \in \mathscr{M}_k \ominus z\mathscr{M}_k.$$

By Theorem 2.4 (vi),

$$(\mathscr{M}_k \ominus z\mathscr{M}_k) \ominus F_w(\mathscr{M}_k \ominus z\mathscr{M}_k) = \boldsymbol{C} \cdot (\boldsymbol{V}^*)^{-1} \psi_k(w). \qquad \square$$

Corollary 2.6. *For each* $k \geq 0$, $\big((\boldsymbol{V}^*\boldsymbol{V})^{-1}\psi_k(w)\big)/\psi_k(w)$ *is an outer function if and only if*

$$\big[(\mathscr{M}_k \ominus z\mathscr{M}_k) \ominus F_w(\mathscr{M}_k \ominus z\mathscr{M}_k)\big]_{\mathscr{M} \ominus z\mathscr{M}} = \mathscr{M}_k \ominus z\mathscr{M}_k.$$

Proof. Since $(\boldsymbol{V}^*\boldsymbol{V})^{-1}\psi_k(w) = \boldsymbol{V}^{-1}(\boldsymbol{V}^*)^{-1}\psi_k(w)$, by Theorem 2.4 (v) and Corollary 2.5 we get the assertion. $\square$

Corollary 2.7. *Let* $k \geq 0$. *If* $\xi_k(0) = 0$, *then*

$$\big[(\mathscr{M}_k \ominus z\mathscr{M}_k) \ominus F_w(\mathscr{M}_k \ominus z\mathscr{M}_k)\big]_{\mathscr{M}\ominus z\mathscr{M}} = \mathscr{M}_k \ominus z\mathscr{M}_k.$$

Proof. We have $\psi_{k+1}(w) = \psi_k(w)\xi_k(w)$. Since $\xi_k(0) = 0$, we have $\psi_k(w) \in E_k$. So we have $(\boldsymbol{V}^*)^{-1}\psi_k(w) = \overline{A}_k^{(-1)}\varphi_k(z)\psi_k(w)$. Moreover $\boldsymbol{V}^{-1}(\boldsymbol{V}^*)^{-1}\psi_k(w) = |A_k|^{-2}\psi_k(w)$. Thus $(\boldsymbol{V}^*\boldsymbol{V})^{-1}\psi_k(w)/\psi_k(w) = |A_k|^{-2}$ is outer. Then by Corollary 2.6, we get the assertion. □

We note that

$$(\mathscr{M} \ominus z\mathscr{M}) \ominus F_w(\mathscr{M} \ominus z\mathscr{M}) = \mathscr{M} \ominus (z\mathscr{M} + w\mathscr{M}).$$

Corollary 2.8. *If $(\boldsymbol{V}^*\boldsymbol{V})^{-1}1$ is not an outer function, then*

$$[\mathscr{M} \ominus (z\mathscr{M} + w\mathscr{M})]_{\mathscr{M}} \neq \mathscr{M}.$$

Proof. By Corollary 2.6,

$$\big[(\mathscr{M} \ominus z\mathscr{M}) \ominus F_w(\mathscr{M} \ominus z\mathscr{M})\big]_{\mathscr{M}\ominus z\mathscr{M}} \subsetneqq \mathscr{M} \ominus z\mathscr{M}.$$

Hence

$$[\mathscr{M} \ominus (z\mathscr{M} + w\mathscr{M})]_{\mathscr{M}\ominus z\mathscr{M}} \subsetneqq \mathscr{M} \ominus z\mathscr{M}.$$

Therefore

$$(\mathscr{M} \ominus z\mathscr{M}) \ominus [\mathscr{M} \ominus (z\mathscr{M} + w\mathscr{M})]_{\mathscr{M}\ominus z\mathscr{M}} \perp [\mathscr{M} \ominus (z\mathscr{M} + w\mathscr{M})]_{\mathscr{M}}.$$

Thus we get the assertion. □

The converse of Corollary 2.8 does not hold, see Theorem 3.1 (v).

Theorem 2.9. *Suppose that $\xi_{k+1}(0) = 0$ for some $k \geq 0$. Let $\alpha = \zeta_k(0)$ and $\beta = \xi_k(0)$. Then we have the following.*

(i) *If $1/2 \leq |\alpha|^2 \leq 1$, then*

$$\big[(\mathscr{M}_k \ominus z\mathscr{M}_k) \ominus F_w(\mathscr{M}_k \ominus z\mathscr{M}_k)\big]_{\mathscr{M}\ominus z\mathscr{M}} = \mathscr{M}_k \ominus z\mathscr{M}_k.$$

(ii) *If $0 < |\alpha|^2 < 1/2$ and $|\beta|/(1 + |\beta|) \leq |\alpha|^2$, then*

$$\big[(\mathscr{M}_k \ominus z\mathscr{M}_k) \ominus F_w(\mathscr{M}_k \ominus z\mathscr{M}_k)\big]_{\mathscr{M}\ominus z\mathscr{M}} = \mathscr{M}_k \ominus z\mathscr{M}_k.$$

(iii) *If* $0 < |\alpha|^2 < 1/2$ *and* $|\alpha|^2 < |\beta|/(1+|\beta|)$, *then*

$$\big[(\mathscr{M}_k \ominus z\mathscr{M}_k) \ominus F_w(\mathscr{M}_k \ominus z\mathscr{M}_k)\big]_{\mathscr{M}\ominus z\mathscr{M}} \neq \mathscr{M}_k \ominus z\mathscr{M}_k,$$

so in this case the Beurling type theorem does not hold for F_w *on* $\mathscr{M}\ominus z\mathscr{M}$.

Proof. Recall that $E_k = \psi_k(w)\big(H^2(w) \ominus \xi_k(w)H^2(w)\big)$ for each $k \geq 0$. Since $\xi_{k+1}(0) = 0$, $\psi_k(w) \perp \psi_{k+2}(w)H^2(w)$ and $\psi_{k+1}(w) \in E_{k+1}$, so we have

$$\begin{aligned}\psi_k(w) &= \psi_k(w)\big(1 - \overline{\xi_k(0)}\xi_k(w)\big) \oplus \overline{\xi_k(0)}\psi_k(w)\xi_k(w) \\ &= \psi_k(w)\big(1 - \overline{\xi_k(0)}\xi_k(w)\big) \oplus \overline{\xi_k(0)}\psi_{k+1}(w) \\ &\in E_k \oplus E_{k+1}.\end{aligned}$$

Note that $A_{k+1} = A_k\zeta_k(0) = \alpha A_k$. By Lemma 2.1 (iv),

$$\begin{aligned}(\boldsymbol{V}^*\boldsymbol{V})^{-1}\psi_k(w) &= \frac{1}{|A_k|^2}\psi_k(w)\big(1 - \overline{\xi_k(0)}\xi_k(w)\big) \oplus \frac{\overline{\xi_k(0)}\psi_k(w)\xi_k(w)}{|\alpha|^2|A_k|^2} \\ &= \frac{\psi_k(w)}{|A_k|^2}\bigg(1 + \frac{1-|\alpha|^2}{|\alpha|^2}\overline{\beta}\xi_k(w)\bigg).\end{aligned}$$

Hence $\big((\boldsymbol{V}^*\boldsymbol{V})^{-1}\psi_k(w)\big)/\psi_k(w)$ is an outer function if and only if

$$\frac{1-|\alpha|^2}{|\alpha|^2}|\beta| \leq 1, \quad \text{that is,} \quad \frac{|\beta|}{1+|\beta|} \leq |\alpha|^2.$$

Therefore if $1/2 \leq |\alpha|^2 \leq 1$, then $(1-|\alpha|^2)/|\alpha|^2 \leq 1$, so $\big((\boldsymbol{V}^*\boldsymbol{V})^{-1}\ \psi_k(w)\big)/\psi_k(w)$ is outer. If $0 < |\alpha|^2 < 1/2$ and $|\beta|/(1+|\beta|) \leq |\alpha|^2$, then $\big((\boldsymbol{V}^*\boldsymbol{V})^{-1}\psi_k(w)\big)/\psi_k(w)$ is outer. If $0 < |\alpha|^2 < 1/2$ and $|\alpha|^2 < |\beta|/(1+|\beta|)$, then $\big((\boldsymbol{V}^*\boldsymbol{V})^{-1}\psi_k(w)\big)/\psi_k(w)$ is not outer. By Corollary 2.6, we get the assertion. □

Theorem 2.10. *Suppose that* $\xi_{k+1}(0) = 0$, $0 < |\zeta_k(0)|^2 < 1/2$, *and* $\xi_k(w)$ *is not a finite Blaschke product for some* $k \geq 0$. *Then the Beurling type theorem does not hold for* F_w *on* $\mathscr{M}\ominus z\mathscr{M}$.

Proof. Write $\alpha = \zeta_k(0)$. Then $0 < |\alpha|^2 < 1/2$, so $|\alpha|^2/(1-|\alpha|^2) < 1$. By our assumption, there is an inner subfactor $\eta_0(w)$ of $\xi_k(w)$ such that $|\alpha|^2/(1-|\alpha|^2) < |\eta_0(0)| < 1$. Write $\xi_k(w) = \eta_0(w)\eta_1(w)$. Let $\theta(w) = \psi_k(w)\eta_1(w)$. Since $\psi_{k+1}(w) = \psi_k(w)\xi_k(w)$, $\psi_{k+1}(w)/\eta_0(w) = \theta(w)$. Hence we have

$$\psi_{k+1}(w)H^2(w) \subset \theta(w)H^2(w) \subset \psi_k(w)H^2(w).$$

We shall show that

$$\big[\boldsymbol{V}(\theta(w)H^2(w)) \ominus F_w\boldsymbol{V}(\theta(w)H^2(w))\big]_{\mathscr{M}\ominus z\mathscr{M}} \neq \boldsymbol{V}(\theta(w)H^2(w)),$$

so we get the assertion.

We note that $\psi_n(w)/\theta(w)$ is an inner function for every $n \geq k+1$. Let

$$p_n(z) = \varphi_{k+n}(z), \quad n \geq 0$$

and

$$q_0(w) = 1 \quad \text{and} \quad q_n(w) = \frac{\psi_{k+n}(w)}{\theta(w)}, \quad n \geq 1.$$

Write

$$\mu_n(w) = \frac{q_{n+1}(w)}{q_n(w)}.$$

Then

$$\mu_0(w) = \frac{\psi_{k+1}(w)}{\theta(w)} \quad \text{and} \quad \mu_n(w) = \frac{\psi_{k+n+1}(w)}{\psi_{k+n}(w)} = \xi_{k+n}(w), \quad n \geq 1.$$

We note that $\{p_n(z)\}_{n\geq 0}$ and $\{q_n(w)\}_{n\geq 0}$ satisfy conditions (1)–(4). Let

$$\mathscr{L} = \sum_{n=0}^{\infty} \oplus\big(p_n(z)H^2(z)\big) \otimes \big(q_n(w)\big(H^2(w) \ominus \mu_n(w)H^2(w)\big)\big).$$

Then $\mathscr{L}$ is an invariant subspace of H^2 and

$$\begin{aligned}\mathscr{L} \ominus z\mathscr{L} &= \sum_{n=0}^{\infty} \oplus p_n(z)q_n(w)\big(H^2(w) \ominus \mu_n(w)H^2(w)\big)\\ &= \varphi_k(z)\bigg(H^2(w) \ominus \frac{\psi_{k+1}(w)}{\theta(w)}H^2(w)\bigg)\\ &\quad \oplus \sum_{n=1}^{\infty} \oplus \varphi_{k+n}(z)\frac{\psi_{k+n}(w)}{\theta(w)}\big(H^2(w) \ominus \xi_{k+n}(w)H^2(w)\big).\end{aligned}$$

We have

$$\begin{aligned}\theta(w)H^2(w) &= \big(\theta(w)H^2(w) \ominus \psi_{k+1}(w)H^2(w)\big)\\ &\quad \oplus \sum_{n=1}^{\infty} \oplus \psi_{k+n}(w)\big(H^2(w) \ominus \xi_{k+n}(w)H^2(w)\big)\\ &\subset E_k \oplus \sum_{n=k+1}^{\infty} \oplus E_n.\end{aligned}$$

We have also

$$\begin{aligned}\theta(w)(\mathscr{L} \ominus z\mathscr{L}) &= \varphi_k(z)\big(\theta(w)H^2(w) \ominus \psi_{k+1}(w)H^2(w)\big)\\ &\quad \oplus \sum_{n=1}^{\infty} \oplus \varphi_{k+n}(z)\psi_{k+n}(w)\big(H^2(w) \ominus \xi_{k+n}(w)H^2(w)\big)\\ &= \boldsymbol{V}(\theta(w)H^2(w)) \subset \mathscr{M} \ominus z\mathscr{M}.\end{aligned}$$

Let

$$\begin{aligned}\mathscr{K} &= \big(\varphi_k(z)H^2(z)\big) \otimes \big(\theta(w)H^2(w) \ominus \psi_{k+1}(w)H^2(w)\big)\\ &\quad \oplus \sum_{n=1}^{\infty} \oplus \big(\varphi_{k+n}(z)H^2(z)\big) \otimes \big(\psi_{k+n}(w)\big(H^2(w) \ominus \xi_{k+n}(w)H^2(w)\big)\big).\end{aligned}$$

Then $\mathscr{K}$ is an invariant subspace of H^2 and $\mathscr{K} \ominus z\mathscr{K} = \boldsymbol{V}(\theta(w)H^2(w))$. Write $V_z = T_z|_{\mathscr{L}}$, $V_w = T_w|_{\mathscr{L}}$, $W_z = T_z|_{\mathscr{K}}$, and $W_w = T_w|_{\mathscr{K}}$. We define a unitary operator $U : \mathscr{L} \to \mathscr{K}$ by $Uf = \theta(w)f, f \in \mathscr{L}$. Then $UV_z = W_zU$ and $UV_w = W_wU$. Let G_w and H_w be the fringe operators on $\mathscr{L} \ominus z\mathscr{L}$ and $\mathscr{K} \ominus z\mathscr{K}$, respectively. Then $F_w|_{\mathscr{K} \ominus z\mathscr{K}} = H_w$, and for $f \in \mathscr{L} \ominus z\mathscr{L}$ we have

$$\begin{aligned}U^{-1}H_wUf &= U^{-1}P_{\mathscr{K} \ominus z\mathscr{K}}W_wUf\\ &= U^{-1}(I_{\mathscr{K}} - W_zW_z^*)UV_wf\\ &= (I_{\mathscr{L}} - V_zV_z^*)V_wf\\ &= P_{\mathscr{L} \ominus z\mathscr{L}}V_wf\\ &= G_wf,\end{aligned}$$

where $I_{\mathscr{K}}$ is the identity operator on $\mathscr{K}$. Hence $UG_w = F_wU$ on $\mathscr{L} \ominus z\mathscr{L}$. By this fact, we have

$$\big[\boldsymbol{V}(\theta(w)H^2(w)) \ominus F_w \boldsymbol{V}(\theta(w)H^2(w))\big]_{\mathscr{M}\ominus z\mathscr{M}} = \boldsymbol{V}(\theta(w)H^2(w))$$

if and only if

$$\big[(\mathscr{L} \ominus z\mathscr{L}) \ominus G_w(\mathscr{L} \ominus z\mathscr{L})\big]_{\mathscr{L}\ominus z\mathscr{L}} = \mathscr{L} \ominus z\mathscr{L}.$$

Now we work on $\mathscr{L} \ominus z\mathscr{L}$. We have

$$\frac{p_0(0)}{p_1(0)} = \frac{\varphi_k(0)}{\varphi_{k+1}(0)} = \zeta_k(0) = \alpha \quad \text{and} \quad 0 < |\alpha|^2 < \frac{1}{2},$$

and

$$q_2(w) = \frac{\psi_{k+2}(w)}{\theta(w)} = \frac{\psi_{k+1}(w)\xi_{k+1}(w)}{\theta(w)} = \eta_0(w)\xi_{k+1}(w).$$

Since $\xi_{k+1}(0) = 0$, we get $q_2(0) = 0$. We have also

$$q_1(w) = \frac{\psi_{k+1}(w)}{\theta(w)} = \eta_0(w).$$

Hence $q_1(0) = \eta_0(0)$. Moreover we have $|\alpha|^2/(1 - |\alpha|^2) < |\eta_0(0)| < 1$. Therefore by Theorem 2.9 (iii), we get

$$\big[(\mathscr{L} \ominus z\mathscr{L}) \ominus G_w(\mathscr{L} \ominus z\mathscr{L})\big]_{\mathscr{L}\ominus z\mathscr{L}} \neq \mathscr{L} \ominus z\mathscr{L}.$$

This shows that

$$\big[\boldsymbol{V}(\theta(w)H^2(w)) \ominus F_w \boldsymbol{V}(\theta(w)H^2(w))\big]_{\mathscr{M}\ominus z\mathscr{M}} \neq \boldsymbol{V}(\theta(w)H^2(w)).$$

Thus we get the assertion. □

Example 2.11. Let $\alpha = \varphi_0(0)/\varphi_1(0)$. We shall give an example of $\mathscr{M}$ satisfying $1/2 \le |\alpha|^2 < 1$, but the Beurling type theorem does not hold for F_w on $\mathscr{M} \ominus z\mathscr{M}$, compared with the assertion of Theorem 2.9.

Let $\psi_0(w) = 1$, $\psi_1(w)$ be a singular inner function and $\psi_n(w) = w^n\psi_1(w)$ for $n \ge 2$. Let $\varphi_0(z)$ be a singular inner function satisfying $0 < |\varphi_0(0)|^2 < 1/2$. There exists a positive number r_1 with $0 < r_1 < 1$ satisfying $1/2 < |\varphi_0(0)|^2/|\varphi_0(0)^{r_1}|^2 <$

1 and $|\varphi_0(0)|^2 < |\varphi_0(0)^{r_1}|^2 < 1/2$. Let $\varphi_1(z) = \varphi_0(z)^{r_1}$. Then there exists a positive number r_2 with $0 < r_2 < r_1 < 1$ satisfying $0 < |\varphi_1(0)|^2/|\varphi_0(0)^{r_2}|^2 < 1/2$. Let $\varphi_2(z) = \varphi_0(z)^{r_2}$ and $\{r_n\}_{n\geq 3}$ be a sequence of positive numbers satisfying $0 < r_{n+1} < r_n < r_2 < r_1 < 1$. Then $\{\varphi_n(z)\}_{n\geq 0}$ and $\{\psi_n(z)\}_{n\geq 0}$ satisfy conditions (1)–(4). Note that $\xi_0(w) = \psi_1(w)/\psi_0(w) = \psi_1(w)$ is not a finite Blaschke product. Since

$$\xi_2(w) = \frac{\psi_3(w)}{\psi_2(w)} = \frac{w^3\psi_1(w)}{w^2\psi_1(w)} = w,$$

we have $\xi_2(0) = 0$. Also we have $0 < |\zeta_1(0)| = |\varphi_1(0)|^2/|\varphi_2(0)|^2 < 1/2$. Therefore by Theorem 2.10, the Beurling type theorem does not hold for F_w on $\mathscr{M} \ominus z\mathscr{M}$. □

Here we study the case $\psi_n(w) = w^n, n \geq 0$. We write $e_n = \varphi_n(z)w^n$ for $n \geq 0$. Then $\{e_n\}_{n\geq 0}$ is an orthonormal basis for $\mathscr{M} \ominus z\mathscr{M}$. We have

$$F_w e_n = \langle we_n, e_{n+1}\rangle e_{n+1} = \langle \varphi_n(z), \varphi_{n+1}(z)\rangle e_{n+1} = \zeta_n(0)e_{n+1},$$

so F_w on $\mathscr{M} \ominus z\mathscr{M}$ is a unilateral weighted shift operator. The following was pointed out essentially in [**9**, Theorem 2.1] as an application of Theorem A.

Lemma 2.12. *Let H be a separable Hilbert space with an orthonormal basis $\{\tau_n\}_{n\geq 0}$. Let $\{c_n\}_{n\geq 0}$ be a sequence of complex numbers satisfying* $\sup_n |c_n| < \infty$. *Let T be a unilatral weighted shift on H defined by $T\tau_n = c_n\tau_{n+1}$ for $n \geq 0$. If $1/\sqrt{2} \leq |c_0| \leq 1$ and $1 \leq |c_n|^2(2 - |c_{n-1}|^2)$ for every $n \geq 1$, then the Beurling type theorem holds for T.*

By the above lemma, we have the following.

Theorem 2.13. *Suppose that $\psi_n(w) = w^n$ for every $n \geq 0$. If*

$$|\zeta_0(0)|^2 \geq \frac{1}{2} \quad \textit{and} \quad |\zeta_n(0)|^2 \geq \frac{1}{2 - |\zeta_{n-1}(0)|^2} \quad \textit{for every } n \geq 1,$$

then the Beurling type theorem holds for the fringe operator F_w on $\mathscr{M} \ominus z\mathscr{M}$.

3. The case $M = \varphi(z)H^2 + \psi(w)H^2$.

Let $\varphi(z)$ and $\psi(w)$ be nonconstant inner functions with $\varphi(0) \neq 0$ and

$$M = \varphi(z)H^2 + \psi(w)H^2.$$

Then M coincides with $\mathscr{M}$ associated with the sequences of inner functions

$$\varphi_0(z) = \varphi(z), \quad \varphi_n(z) = 1, \quad n \geq 1$$

and

$$\psi_0(w) = 1, \quad \psi_n(w) = w^{n-1}\psi(w), \quad n \geq 1.$$

So to study M we use the same notations as the ones in Section 2. We have $A_0 = 1, A_n = \varphi(0) = \zeta_0(0)$ for $n \geq 1$, $\xi_0(w) = \psi(w)$, and $\xi_n(w) = w$ for $n \geq 1$. We have also

$$E_0 = H^2(w) \ominus \psi(w)H^2(w) \quad \text{and} \quad E_n = \boldsymbol{C} \cdot w^{n-1}\psi(w), \quad n \geq 1.$$

Theorem 3.1. *Let $\varphi(z)$ and $\psi(w)$ be nonconstant inner functions with $\varphi(0) \neq 0$ and $M = \varphi(z)H^2 + \psi(w)H^2$. Let $\alpha = \varphi(0)$ and $\beta = \psi(0)$. Then we have the following.*

(i) *If $1/2 \leq |\alpha|^2$, then*

$$\big[(M \ominus zM) \ominus F_w(M \ominus zM)\big]_{M \ominus zM} = M \ominus zM.$$

(ii) *If $0 < |\alpha|^2 < 1/2$ and $|\beta|/(1+|\beta|) \leq |\alpha|^2$, then*

$$\big[(M \ominus zM) \ominus F_w(M \ominus zM)\big]_{M \ominus zM} = M \ominus zM.$$

(iii) *If $0 < |\alpha|^2 < 1/2$ and $|\alpha|^2 < |\beta|/(1+|\beta|)$, then*

$$\big[(M \ominus zM) \ominus F_w(M \ominus zM)\big]_{M \ominus zM} \neq M \ominus zM,$$

so in this case the Beurling type theorem does not hold for F_w on $M \ominus zM$.

(iv) *If $0 < |\alpha|^2 < 1/2$ and $\psi(w)$ is not a finite Blaschke product, then the Beurling type theorem does not hold for F_w on $M \ominus zM$.*

(v) *If $\beta \neq 0$, then $[M \ominus (zM + wM)]_M \neq M$. Moreover if $0 < |\beta|/(1+|\beta|) \leq |\alpha|^2$, then $(\boldsymbol{V}^*\boldsymbol{V})^{-1}1$ is outer.*

Proof. (i)–(iii) follow from Theorem 2.9. (iv) follows from Theorem 2.10. (v) Since $\psi_2(0) = 0$, we have

$$1 = \big(1 - \overline{\beta}\psi(w)\big) \oplus \overline{\beta}\psi(w) \in E_0 \oplus E_1.$$

Since $A_0 = 1$ and $A_1 = \alpha$, by Lemma 2.1 (iii) we have

$$\begin{aligned}(\boldsymbol{V}^*)^{-1}1 &= \varphi(z)\big(1 - \overline{\beta}\psi(w)\big) \oplus \frac{\overline{\beta}}{\overline{\alpha}}\psi(w) \\ &= \big(1 - \overline{\beta}\psi(w)\big)\left(\varphi(z) + \frac{\overline{\beta}}{\overline{\alpha}}\frac{\psi(w)}{1 - \overline{\beta}\psi(w)}\right).\end{aligned}$$

Since $\varphi(z)$ and $\psi(w)$ are nonconstant inner functions, we have

$$(-\varphi(\boldsymbol{D})) \cap \left(\frac{\overline{\beta}}{\overline{\alpha}}\frac{\psi(w)}{1 - \overline{\beta}\psi(w)}\right)(\boldsymbol{D}) \neq \emptyset.$$

Hence there is $(z_1, w_1) \in \boldsymbol{D}^2$ such that $\big((\boldsymbol{V}^*)^{-1}1\big)(z_1, w_1) = 0$, so $(\boldsymbol{V}^*)^{-1}1$ vanishes on some complex curve C in $\boldsymbol{D}^2$ containing (z_1, w_1). By Corollary 2.5, $M \ominus (zM + wM) = \boldsymbol{C} \cdot (\boldsymbol{V}^*)^{-1}1$. Therefore any function in $[M \ominus (zM + wM)]_M$ vanishes on C. On the other hand, the common zero set in $\boldsymbol{D}^2$ of $\varphi(z)H^2 + \psi(w)H^2$ equals to

$$\{(z, w) \in \boldsymbol{D}^2 : \varphi(z) = 0, \psi(w) = 0\}$$

and this is a discrete set in $\boldsymbol{D}^2$. Therefore we get $[M \ominus (zM + wM)]_M \neq M$.

We have

$$(\boldsymbol{V}^*\boldsymbol{V})^{-1}1 = \big(1 - \overline{\beta}\psi(w)\big) \oplus \frac{\overline{\beta}}{|\alpha|^2}\psi(w) = 1 + \frac{\overline{\beta}(1 - |\alpha|^2)}{|\alpha|^2}\psi(w).$$

Hence if $|\beta|/(1 + |\beta|) \leq |\alpha|^2$, that is, $|\beta|(1 - |\alpha|^2)/|\alpha|^2 \leq 1$, then $(\boldsymbol{V}^*\boldsymbol{V})^{-1}1$ is outer. □

COROLLARY 3.2. *Let $\varphi(z)$ and $\psi(w)$ be nonconstant inner functions with $\varphi(0) \neq 0$ and $M = \varphi(z)H^2 + \psi(w)H^2$. Then $[M \ominus (zM + wM)]_M \neq M$.*

PROOF. Suppose that $\big[M \ominus (zM + wM)\big]_M = M$ and $\varphi(0) \neq 0$. By Theorem 3.1 (v), we have $\psi(0) = 0$. Hence we have $M \ominus (zM + wM) = \boldsymbol{C} \cdot \varphi(z)$, so $[M \ominus (zM + wM)]_M = \varphi(z)H^2 \neq M$. This is a contradiction. □

REMARK 3.3. Let $M = \varphi(z)H^2 + \psi(w)H^2$ for nonconstant inner functions $\varphi(z)$ and $\psi(w)$ (here we do not assume that $\varphi(0) \neq 0$). We note that $[M \ominus (zM + wM)]_M = M$ if and only if $\varphi(0) = \psi(0) = 0$. For, if either $\varphi(0) \neq 0$ or $\psi(0) \neq 0$, by Corollary 3.2 we have $[M \ominus (zM + wM)]_M \neq M$.

Suppose that $\varphi(0) = \psi(0) = 0$. Then it is easy to see that

$$M \ominus (zM + wM) = \boldsymbol{C} \cdot \varphi(z) \oplus \boldsymbol{C} \cdot \psi(w),$$

so we get $\left[M \ominus (zM + wM)\right]_M = M$ (see also [**11**, Theorem 2.3]). □

Now we study the invariant subspace M under the assumption that

$$\psi(w) = \frac{w - \beta}{1 - \overline{\beta} w}, \quad |\beta| < 1.$$

THEOREM 3.4. *Let $\varphi(z)$ be a nonconstant inner function with $\varphi(0) \neq 0$, $\psi(w) = (w - \beta)/(1 - \overline{\beta} w)$ with $|\beta| < 1$, and $M = \varphi(z)H^2 + \psi(w)H^2$. Let $\alpha = \varphi(0)$. Then the Beurling type theorem holds for F_w on $M \ominus zM$ if and only if $|\beta|/(1 + |\beta|) \leq |\alpha|^2$.*

PROOF. Let L be a nonzero invariant subspace of $M \ominus zM$ for F_w. By Theorem 2.4, $L = \boldsymbol{V}(\theta(w)H^2(w))$ for an inner function $\theta(w)$. Since

$$H^2(w) = \boldsymbol{C} \cdot \frac{1}{1 - \overline{\beta} w} \oplus \frac{w - \beta}{1 - \overline{\beta} w} H^2(w),$$

we have

$$\theta(w)H^2(w) = \boldsymbol{C} \cdot \frac{\theta(w)}{1 - \overline{\beta} w} \oplus \theta(w)\frac{w - \beta}{1 - \overline{\beta} w} H^2(w).$$

Note that

$$A_0 = 1, \quad A_n = \alpha \ (n \geq 1), \quad E_0 = \boldsymbol{C} \cdot \frac{1}{1 - \overline{\beta} w},$$

and

$$E_n = \boldsymbol{C} \cdot w^{n-1} \frac{w - \beta}{1 - \overline{\beta} w}, \quad n \geq 1.$$

Then

$$\boldsymbol{V}(\theta(w)H^2(w)) = \boldsymbol{C} \cdot \boldsymbol{V} \frac{\theta(w)}{1 - \overline{\beta} w} + \theta(w)\frac{w - \beta}{1 - \overline{\beta} w} H^2(w).$$

By Theorem 2.4,

$$\boldsymbol{V}(\theta(w)H^2(w)) \ominus F_w \boldsymbol{V}(\theta(w)H^2(w)) = \boldsymbol{C} \cdot g$$

for some $g \in \boldsymbol{V}(\theta(w)H^2(w))$ with $g \neq 0$. By Theorem 2.4 (vi), we may take $g = P_L(\boldsymbol{V}^*)^{-1}\theta(w)$, where $L = \boldsymbol{V}(\theta(w)H^2(w))$. Since

$$\begin{aligned}\theta(w) &= \left\langle \theta(w), \frac{\sqrt{1-|\beta|^2}}{1-\overline{\beta}w} \right\rangle \frac{\sqrt{1-|\beta|^2}}{1-\overline{\beta}w} \oplus \left(\theta(w) - \left\langle \theta(w), \frac{\sqrt{1-|\beta|^2}}{1-\overline{\beta}w} \right\rangle \frac{\sqrt{1-|\beta|^2}}{1-\overline{\beta}w} \right) \\ &= \theta(\beta)\frac{1-|\beta|^2}{1-\overline{\beta}w} \oplus \left(\theta(w) - \theta(\beta)\frac{1-|\beta|^2}{1-\overline{\beta}w} \right) \\ &\in E_0 \oplus \left(\sum_{n=1}^{\infty} \oplus E_n \right),\end{aligned}$$

by Lemma 2.1 (iii) we have

$$(\boldsymbol{V}^*)^{-1}\theta(w) = \varphi(z)\theta(\beta)\frac{1-|\beta|^2}{1-\overline{\beta}w} \oplus \frac{1}{\overline{\alpha}}\left(\theta(w) - \theta(\beta)\frac{1-|\beta|^2}{1-\overline{\beta}w} \right).$$

Since

$$(\boldsymbol{V}^*)^{-1}\theta(w) \perp \theta(w)\frac{w-\beta}{1-\overline{\beta}w}wH^2(w) = \boldsymbol{V}\left(\theta(w)\frac{w-\beta}{1-\overline{\beta}w}wH^2(w) \right),$$

we may write g as

$$g = P_L(\boldsymbol{V}^*)^{-1}\theta(w) = a\boldsymbol{V}\frac{\theta(w)}{1-\overline{\beta}w} + b\theta(w)\frac{w-\beta}{1-\overline{\beta}w}, \quad a, b \in \boldsymbol{C}. \tag{3.1}$$

Hence

$$g = a\boldsymbol{V}\frac{\theta(w)-\theta(\beta)}{1-\overline{\beta}w} + a\boldsymbol{V}\frac{\theta(\beta)}{1-\overline{\beta}w} + b\theta(w)\frac{w-\beta}{1-\overline{\beta}w},$$

so that

$$g = a\alpha\frac{\theta(w)-\theta(\beta)}{1-\overline{\beta}w} + a\theta(\beta)\varphi(z)\frac{1}{1-\overline{\beta}w} + b\theta(w)\frac{w-\beta}{1-\overline{\beta}w}. \tag{3.2}$$

We have

$$\begin{aligned}
w\theta(w) &= \left\langle w\theta(w), \frac{\sqrt{1-|\beta|^2}}{1-\overline{\beta}w}\right\rangle \frac{\sqrt{1-|\beta|^2}}{1-\overline{\beta}w} \\
&\quad \oplus \left(w\theta(w) - \left\langle w\theta(w), \frac{\sqrt{1-|\beta|^2}}{1-\overline{\beta}w}\right\rangle \frac{\sqrt{1-|\beta|^2}}{1-\overline{\beta}w} \right) \\
&= \beta\theta(\beta)\frac{1-|\beta|^2}{1-\overline{\beta}w} \oplus \left(w\theta(w) - \beta\theta(\beta)\frac{1-|\beta|^2}{1-\overline{\beta}w}\right) \\
&\in E_0 \oplus \left(\sum_{n=1}^{\infty} \oplus E_n \right).
\end{aligned}$$

Hence

$$\boldsymbol{V}(w\theta(w)) = \beta\theta(\beta)\varphi(z)\frac{1-|\beta|^2}{1-\overline{\beta}w} \oplus \alpha\left(w\theta(w) - \beta\theta(\beta)\frac{1-|\beta|^2}{1-\overline{\beta}w}\right),$$

so that

$$\boldsymbol{V}(w\theta(w)) = \alpha w\theta(w) + \frac{\beta\theta(\beta)(1-|\beta|^2)}{1-\overline{\beta}w}(\varphi(z)-\alpha). \tag{3.3}$$

By (3.2), we have

$$\begin{aligned}
\langle g, \alpha w\theta(w)\rangle &= a|\alpha|^2\left\langle \frac{\theta(w)-\theta(\beta)}{1-\overline{\beta}w}, w\theta(w)\right\rangle + a\theta(\beta)\overline{\alpha}\left\langle \frac{\varphi(z)}{1-\overline{\beta}w}, w\theta(w)\right\rangle \\
&\quad + b\overline{\alpha}\left\langle \theta(w)\frac{w-\beta}{1-\overline{\beta}w}, w\theta(w)\right\rangle \\
&= a|\alpha|^2(\overline{\beta} - \overline{\beta}|\theta(\beta)|^2) + a|\alpha|^2\overline{\beta}|\theta(\beta)|^2 + b\overline{\alpha}\left\langle \frac{1-|\beta|^2}{1-\overline{\beta}w}, 1\right\rangle \\
&= a|\alpha|^2\overline{\beta} + b\overline{\alpha}(1-|\beta|^2).
\end{aligned}$$

and

$$\begin{aligned}
\left\langle g, \frac{\beta\theta(\beta)(1-|\beta|^2)}{1-\overline{\beta}w}(\varphi(z)-\alpha)\right\rangle &= a\overline{\beta}|\theta(\beta)|^2(1-|\beta|^2)\left\langle \frac{\varphi(z)}{1-\overline{\beta}w}, \frac{\varphi(z)-\alpha}{1-\overline{\beta}w}\right\rangle \\
&= a\overline{\beta}|\theta(\beta)|^2(1-|\beta|^2)\frac{1-|\alpha|^2}{1-|\beta|^2} \\
&= a(1-|\alpha|^2)\overline{\beta}|\theta(\beta)|^2
\end{aligned}$$

Since $\langle g, \boldsymbol{V}(w\theta(w))\rangle = 0$, by (3.3) we have

$$a|\alpha|^2\overline{\beta} + b\overline{\alpha}(1-|\beta|^2) + a(1-|\alpha|^2)\overline{\beta}|\theta(\beta)|^2 = 0. \tag{3.4}$$

First, we study the case $\beta = 0$. Then trivially $|\beta|/(1+|\beta|) \leq |\alpha|^2$ holds. By (3.4), we have $b\overline{\alpha} = 0$, so $b = 0$. Hence $g = a\boldsymbol{V}\theta(w) = \boldsymbol{V}(a\theta(w)), a \neq 0$. Therefore $(\boldsymbol{V}^{-1}g)(w)/\theta(w)$ equals constant a and thus it is outer. Then by Theorem 2.4 (v), the Beurling type theorem holds for F_w on $M \ominus zM$.

Next, suppose that $\beta \neq 0$. By (3.4), we have

$$b\overline{\alpha}(1-|\beta|^2) + a\overline{\beta}\big(|\alpha|^2 + (1-|\alpha|^2)|\theta(\beta)|^2\big) = 0.$$

Hence

$$a = \frac{-b\overline{\alpha}(1-|\beta|^2)}{\overline{\beta}\big(|\alpha|^2 + (1-|\alpha|^2)|\theta(\beta)|^2\big)}.$$

Therefore by (3.1), we have

$$g = b\bigg(\frac{-\overline{\alpha}(1-|\beta|^2)}{\overline{\beta}\big(|\alpha|^2 + (1-|\alpha|^2)|\theta(\beta)|^2\big)}\boldsymbol{V}\frac{\theta(w)}{1-\overline{\beta}w} + \theta(w)\frac{w-\beta}{1-\overline{\beta}w}\bigg).$$

We may assume that $b = 1$. Then

$$g = \frac{-\overline{\alpha}(1-|\beta|^2)}{\overline{\beta}\big(|\alpha|^2 + (1-|\alpha|^2)|\theta(\beta)|^2\big)}\boldsymbol{V}\frac{\theta(w)}{1-\overline{\beta}w} + \theta(w)\frac{w-\beta}{1-\overline{\beta}w}.$$

Hence

$$\begin{aligned}(\boldsymbol{V}^{-1}g)(w) &= \frac{-\overline{\alpha}(1-|\beta|^2)}{\overline{\beta}\big(|\alpha|^2 + (1-|\alpha|^2)|\theta(\beta)|^2\big)}\frac{\theta(w)}{1-\overline{\beta}w} + \frac{1}{\alpha}\theta(w)\frac{w-\beta}{1-\overline{\beta}w} \\ &= \frac{\theta(w)}{\alpha(1-\overline{\beta}w)}\bigg(w - \bigg(\beta + \frac{|\alpha|^2(1-|\beta|^2)}{\overline{\beta}\big(|\alpha|^2 + (1-|\alpha|^2)|\theta(\beta)|^2\big)}\bigg)\bigg).\end{aligned}$$

Therefore $(\boldsymbol{V}^{-1}g)(w)/\theta(w)$ is an outer function if and only if

$$\bigg|\beta + \frac{|\alpha|^2(1-|\beta|^2)}{\overline{\beta}\big(|\alpha|^2 + (1-|\alpha|^2)|\theta(\beta)|^2\big)}\bigg| \geq 1,$$

and this is equivalent to

$$|\beta|^2\big(|\alpha|^2 + (1-|\alpha|^2)|\theta(\beta)|^2\big) + |\alpha|^2(1-|\beta|^2) \geq |\beta|\big(|\alpha|^2 + (1-|\alpha|^2)|\theta(\beta)|^2\big).$$

We may rewrite this inequality as

$$-|\beta|(1-|\alpha|^2)|\theta(\beta)|^2 + |\alpha|^2 \geq 0.$$

By Theorem 2.4, the Beurling type theorem holds for F_w on $M \ominus zM$ if and only if the above inequality holds for every inner function $\theta(w)$. Since $0 \leq |\theta(\beta)|^2 \leq 1$, the Beurling type theorem holds for F_w on $M \ominus zM$ if and only if $-|\beta|(1-|\alpha|^2)+|\alpha|^2 \geq 0$. This is equivalent to $|\beta|/(1+|\beta|) \leq |\alpha|^2$. This completes the proof. □

By the proof of Theorem 3.4, we have the following.

THEOREM 3.5. *Let $\varphi(z)$ be a nonconstant inner function with $\varphi(0) \neq 0$, $\psi(w) = (w-\beta)/(1-\overline{\beta}w)$ with $0 < |\beta| < 1$, and $M = \varphi(z)H^2 + \psi(w)H^2$. Let $\alpha = \varphi(0)$. Suppose that $|\alpha|^2 < |\beta|/(1+|\beta|)$. Let $\theta(w)$ be an inner function. Then*

$$\big[\boldsymbol{V}(\theta(w)H^2(w)) \ominus F_w\boldsymbol{V}(\theta(w)H^2(w))\big]_{M\ominus zM} = \boldsymbol{V}(\theta(w)H^2(w))$$

if and only if $|\theta(\beta)|^2 \leq |\alpha|^2/|\beta|(1-|\alpha|^2)$.

By Theorem 3.1,

$$\big[(M \ominus zM) \ominus F_w(M \ominus zM)\big]_{M\ominus zM} = M \ominus zM$$

if and only if either "$1/2 \leq |\alpha|^2$" or "$0 < |\alpha|^2 < 1/2$ and $|\beta|/(1+|\beta|) \leq |\alpha|^2$", where $\alpha = \varphi(0)$ and $\beta = \psi(0)$. Note that $M \ominus zM = \boldsymbol{V}(H^2(w))$.

Next, we shall study the case

$$\big[\boldsymbol{V}(wH^2(w)) \ominus F_w\boldsymbol{V}(wH^2(w))\big]_{M\ominus zM} = \boldsymbol{V}(wH^2(w)).$$

THEOREM 3.6. *Let $\varphi(z)$ be a nonconstant inner function with $\varphi(0) \neq 0$, $\psi(w) = (w-\beta)/(1-\overline{\beta}w)$ with $|\beta| < 1$, and $M = \varphi(z)H^2 + \psi(w)H^2$. Let $\alpha = \varphi(0)$. Then*

$$\big[\boldsymbol{V}(wH^2(w)) \ominus F_w\boldsymbol{V}(wH^2(w))\big]_{M\ominus zM} = \boldsymbol{V}(wH^2(w))$$

if and only if $|\beta|^3/(1+|\beta|^3) \leq |\alpha|^2$.

Proof. We have

$$M \ominus zM = \boldsymbol{C} \cdot \varphi(z) \frac{\sqrt{1-|\beta|^2}}{1-\overline{\beta}w} + \frac{w-\beta}{1-\overline{\beta}w} H^2(w).$$

Let

$$e_0 = \varphi(z) \frac{\sqrt{1-|\beta|^2}}{1-\overline{\beta}w}, \quad e_n = w^{n-1} \frac{w-\beta}{1-\overline{\beta}w} \quad \text{for } n \geq 1.$$

Then $\{e_n\}_{n \geq 0}$ is an orthonormal basis for $M \ominus zM$.

Let

$$\widetilde{e}_0 = \frac{\sqrt{1-|\beta|^2}}{1-\overline{\beta}w}, \quad \widetilde{e}_n = w^{n-1} \frac{w-\beta}{1-\overline{\beta}w} \quad \text{for } n \geq 1.$$

Then we have $E_n = \boldsymbol{C} \cdot \widetilde{e}_n$ for every $n \geq 0$. By Theorem 2.4, $\boldsymbol{V}(wH^2(w))$ is an invariant subspace and

$$(M \ominus zM) \ominus \boldsymbol{V}(wH^2(w)) = \boldsymbol{C} \cdot (\boldsymbol{V}^*)^{-1} 1.$$

We have

$$1 = \langle 1, \widetilde{e}_0 \rangle \widetilde{e}_0 \oplus \langle 1, \widetilde{e}_1 \rangle \widetilde{e}_1 = \sqrt{1-|\beta|^2}\, \widetilde{e}_0 \oplus (-\overline{\beta} \widetilde{e}_1).$$

Note that $A_0 = 1$ and $A_n = \alpha$ for $n \geq 1$. By Lemma 2.1 (iii), we have

$$(\boldsymbol{V}^*)^{-1} 1 = \sqrt{1-|\beta|^2}\, e_0 \oplus \left(-\frac{\overline{\beta}}{\overline{\alpha}} e_1 \right). \tag{3.5}$$

Take $g \in \boldsymbol{V}(wH^2(w))$ satisfying

$$\boldsymbol{V}(wH^2(w)) \ominus F_w \boldsymbol{V}(wH^2(w)) = \boldsymbol{C} \cdot g.$$

We have

$$F_w^* g \in (M \ominus zM) \ominus \boldsymbol{V}(wH^2(w)) = \boldsymbol{C} \cdot (\boldsymbol{V}^*)^{-1} 1.$$

Here we have $F_w^* g \neq 0$. For, suppose that $F_w^* g = 0$. Then

$$g \in (M \ominus zM) \ominus F_w(M \ominus zM) = \boldsymbol{V}(H^2(w)) \ominus \boldsymbol{V}(wH^2(w)),$$

so $g = 0$. This is a contradiction. Hence we may assume that

$$F_w^* g = (\boldsymbol{V}^*)^{-1} 1. \tag{3.6}$$

Then we may write

$$g = a_0 e_0 \oplus a_1 e_1 \oplus a_2 e_2.$$

Since $g \perp (\boldsymbol{V}^*)^{-1}1$, by (3.5) we have

$$a_0 \sqrt{1 - |\beta|^2} - \frac{\beta}{\alpha} a_1 = 0.$$

We have

$$\begin{aligned} F_w e_0 &= \langle F_w e_0, e_0 \rangle e_0 \oplus \langle F_w e_0, e_1 \rangle e_1 \\ &= (1 - |\beta|^2) \left\langle \frac{w}{1 - \overline{\beta} w}, \frac{1}{1 - \overline{\beta} w} \right\rangle e_0 \oplus \sqrt{1 - |\beta|^2} \langle \varphi(z), 1 \rangle \left\langle \frac{w}{1 - \overline{\beta} w}, \frac{w - \beta}{1 - \overline{\beta} w} \right\rangle e_1 \\ &= \beta e_0 \oplus \alpha \sqrt{1 - |\beta|^2} e_1, \\ F_w^* e_0 &= \langle F_w^* e_0, e_0 \rangle e_0 \\ &= \big\langle e_0, \beta e_0 \oplus \alpha \sqrt{1 - |\beta|^2} e_1 \big\rangle e_0 \\ &= \overline{\beta} e_0, \end{aligned}$$

and

$$\begin{aligned} F_w^* e_1 &= \big\langle F_w^* e_1, e_0 \big\rangle e_0 \\ &= \big\langle e_1, \beta e_0 \oplus \alpha \sqrt{1 - |\beta|^2} e_1 \big\rangle e_0 \\ &= \overline{\alpha} \sqrt{1 - |\beta|^2} e_0. \end{aligned}$$

We have $F_w e_n = e_{n+1}$ and $F_w^* e_{n+1} = e_n$ for $n \geq 1$. Then we have

$$F_w^* g = \big(\overline{\beta} a_0 + \overline{\alpha} \sqrt{1 - |\beta|^2} a_1\big) e_0 \oplus a_2 e_1.$$

By (3.5) and (3.6),

$$\overline{\beta}a_0+\overline{\alpha}\sqrt{1-|\beta|^2}a_1=\sqrt{1-|\beta|^2}\quad\text{and}\quad a_2=-\frac{\overline{\beta}}{\overline{\alpha}}.$$

Therefore

$$a_0=\frac{\beta\sqrt{1-|\beta|^2}}{|\beta|^2+|\alpha|^2(1-|\beta|^2)},\quad a_1=\frac{\alpha(1-|\beta|^2)}{|\beta|^2+|\alpha|^2(1-|\beta|^2)},\quad a_2=-\frac{\overline{\beta}}{\overline{\alpha}}.$$

As a consequence, we have

$$\begin{aligned}(\boldsymbol{V}^{-1}g)(w)&=a_0\widetilde{e}_0\oplus\frac{a_1}{\alpha}\widetilde{e}_1\oplus\frac{a_2}{\alpha}\widetilde{e}_2\\&=\left(\left(a_0\sqrt{1-|\beta|^2}-\frac{\beta}{\alpha}a_1\right)\frac{1}{1-\overline{\beta}w}+\frac{a_1}{\alpha}\frac{w}{1-\overline{\beta}w}\right)\oplus\frac{a_2}{\alpha}w\frac{w-\beta}{1-\overline{\beta}w}\\&=\frac{a_1}{\alpha}\frac{w}{1-\overline{\beta}w}\oplus\frac{a_2}{\alpha}w\frac{w-\beta}{1-\overline{\beta}w}\\&=\frac{w}{1-\overline{\beta}w}\left(\frac{a_1}{\alpha}-\frac{a_2\beta}{\alpha}+\frac{a_2}{\alpha}w\right)\\&=\frac{w}{1-\overline{\beta}w}\left(\frac{1-|\beta|^2}{|\beta|^2+|\alpha|^2(1-|\beta|^2)}+\frac{|\beta|^2}{|\alpha|^2}-\frac{\overline{\beta}}{|\alpha|^2}w\right).\end{aligned}$$

If $\beta=0$, then $(\boldsymbol{V}^{-1}g)(w)=w/|\alpha|^2$. By Theorem 2.4 (v), we get

$$\left[\boldsymbol{V}(wH^2(w))\ominus F_w\boldsymbol{V}(wH^2(w))\right]_{M\ominus zM}=\boldsymbol{V}(wH^2(w)).$$

Suppose that $\beta\neq 0$, then we have

$$(\boldsymbol{V}^{-1}g)(w)=\frac{\overline{\beta}}{|\alpha|^2}\frac{w}{1-\overline{\beta}w}\left(\frac{|\alpha|^2}{\overline{\beta}}\left(\frac{1-|\beta|^2}{|\beta|^2+|\alpha|^2(1-|\beta|^2)}+\frac{|\beta|^2}{|\alpha|^2}\right)-w\right).$$

Then $(\boldsymbol{V}^{-1}g)(w)/w$ is an outer function if and only if

$$\frac{|\alpha|^2}{|\beta|}\left(\frac{1-|\beta|^2}{|\beta|^2+|\alpha|^2(1-|\beta|^2)}+\frac{|\beta|^2}{|\alpha|^2}\right)\geq 1,$$

that is,

$$\frac{1-|\beta|^2}{|\beta|^2+|\alpha|^2(1-|\beta|^2)}+\frac{|\beta|^2}{|\alpha|^2}\geq\frac{|\beta|}{|\alpha|^2}.$$

Rewriting this, we have

$$|\alpha|^2(1-|\beta|^2)+|\beta|^2\big(|\beta|^2+|\alpha|^2(1-|\beta|^2)\big)\geq|\beta|\big(|\beta|^2+|\alpha|^2(1-|\beta|^2)\big),$$

and this is equivalent to $|\beta|^3/(1+|\beta|^3)\leq|\alpha|^2$. By Theorem 2.4 (v), we get our assertion. □

Let $\varphi(z)$ be a nonconstant inner function with $\varphi(0)\neq 0$ and $M=\varphi(z)H^2+wH^2$. Then by Theorem 3.4, the Beurling type theorem holds for F_w on $M\ominus zM$.

Theorem 3.7. *Let $\varphi(z)$ be a nonconstant inner function with $\varphi(0)\neq 0$ and $M=\varphi(z)H^2+w^2H^2$. Let*

$$\gamma_0=\sup\{|\theta(0)|(|\theta'(0)|-|\theta(0)|):\theta(w)\text{ is inner}\}.$$

Then $\gamma_0(1+\gamma_0)\leq|\varphi(0)|^2$ if and only if the Beurling type theorem holds for F_w on $M\ominus zM$.

Proof. We have

$$E_0=\boldsymbol{C}\cdot 1\oplus\boldsymbol{C}\cdot w\quad\text{and}\quad E_n=\boldsymbol{C}\cdot w^{n+1}\quad\text{for } n\geq 1.$$

Let L be a nonzero invariant subspace of $M\ominus zM$ for F_w. By Theorem 2.4 (i), there is an inner function $\theta(w)$ such that $L=\boldsymbol{V}(\theta(w)H^2(w))$. Let

$$\theta(w)=(a_0+a_1w)\oplus\sum_{n=2}^{\infty}a_nw^n\in\sum_{n=0}^{\infty}\oplus E_n,$$

where $a_0=\theta(0)$ and $a_1=\theta'(0)$. We have

$$\boldsymbol{V}\theta(w)=\varphi(z)(a_0+a_1w)\oplus\varphi(0)\sum_{n=2}^{\infty}a_nw^n,$$

$$\boldsymbol{V}(w\theta(w))=a_0\varphi(z)w\oplus\varphi(0)\sum_{n=1}^{\infty}a_nw^{n+1},$$

and

$$\boldsymbol{V}(w^k\theta(w))=\varphi(0)\sum_{n=0}^{\infty}a_nw^{n+k},\quad k\geq 2.$$

Since $\theta(w) \perp w^k\theta(w)$ and $w\theta(w) \perp w^k\theta(w)$ for $k \geq 2$, we have $\boldsymbol{V}\theta(w) \perp \boldsymbol{V}(w^k\theta(w))$ and $\boldsymbol{V}(w\theta(w)) \perp \boldsymbol{V}(w^k\theta(w))$ for $k \geq 2$. These lead us that there is a constant $c \in \boldsymbol{C}$ satisfying

$$\boldsymbol{V}\theta(w) + c\boldsymbol{V}(w\theta(w)) \perp F_w L.$$

This is equivalent to $\boldsymbol{V}\theta(w) + c\boldsymbol{V}(w\theta(w)) \perp \boldsymbol{V}(w\theta(w))$, that is,

$$\overline{a}_0 a_1 + |\varphi(0)|^2 \sum_{n=1}^{\infty} \overline{a}_n a_{n+1} + c\left(|a_0|^2 + |\varphi(0)|^2 \sum_{n=1}^{\infty} |a_n|^2\right) = 0.$$

Since $\|\theta\|^2 = 1$, $\sum_{n=1}^{\infty} |a_n|^2 = 1 - |a_0|^2$. Since $w\theta(w) \perp \theta(w)$, we have $\sum_{n=1}^{\infty} \overline{a}_n a_{n+1} = -\overline{a}_0 a_1$. Hence

$$\begin{aligned} c &= -\frac{\overline{a}_0 a_1 (1 - |\varphi(0)|^2)}{|a_0|^2 + |\varphi(0)|^2(1 - |a_0|^2)} \\ &= -\frac{\overline{\theta(0)}\theta'(0)(1 - |\varphi(0)|^2)}{|\theta(0)|^2 + |\varphi(0)|^2(1 - |\theta(0)|^2)}. \end{aligned}$$

Write $g = \boldsymbol{V}\theta(w) + c\boldsymbol{V}(w\theta(w))$. Then $g \in L \ominus F_w L$. We have $(\boldsymbol{V}^{-1}g)(w) = \theta(w)(1 + cw)$. If $|c| > 1$, then $(\boldsymbol{V}^{-1}g)(w)/\theta(w)$ is not outer, and in this case by Theorem 2.4 (v) we have $[L \ominus F_w L]_{M \ominus zM} \neq L$. If $|c| \leq 1$, then $(\boldsymbol{V}^{-1}g)(w)/\theta(w)$ is outer, so $[L \ominus F_w L]_{M \ominus zM} = L$. Therefore there is an inner function $\theta(w)$ satisfying

$$1 < \frac{|\theta(0)||\theta'(0)|(1 - |\varphi(0)|^2)}{|\theta(0)|^2 + |\varphi(0)|^2(1 - |\theta(0)|^2)} \tag{3.7}$$

if and only if the Beurling type theorem does not hold for F_w on $M \ominus zM$.

We may rewrite condition (3.7) as

$$\big(|\theta(0)||\theta'(0)| + 1 - |\theta(0)|^2\big)|\varphi(0)|^2 < |\theta(0)||\theta'(0)| - |\theta(0)|^2. \tag{3.8}$$

We note that $0 \leq |\theta(0)||\theta'(0)| + 1 - |\theta(0)|^2$, and $|\theta(0)||\theta'(0)| + 1 - |\theta(0)|^2 = 0$ if and only if $|\theta(0)| = 1$. But when $|\theta(0)| = 1$, (3.8) does not hold.

So we have

$$0 < |\theta(0)||\theta'(0)| + 1 - |\theta(0)|^2.$$

Then we may rewrite (3.8) as

$$|\varphi(0)|^2 < \frac{|\theta(0)|(|\theta'(0)| - |\theta(0)|)}{|\theta(0)|(|\theta'(0)| - |\theta(0)|) + 1} \le \frac{\gamma_0}{\gamma_0 + 1}. \tag{3.9}$$

If $|\varphi(0)|^2 < \gamma_0/(\gamma_0+1)$, then there exists an inner function $\theta(w)$ satisfying (3.9). In this case, the Beurling type theorem does not hold for F_w on $M \ominus zM$. If $|\varphi(0)|^2 \ge \gamma_0/(\gamma_0+1)$, then there are no inner functions $\theta(w)$ satisfying (3.9). In this case, the Beurling type theorem holds for F_w on $M \ominus zM$. □

Remark 3.8. Let $\theta(w) = (w-\delta)/(1-\delta w)$ for $0<\delta<1$. Then $\theta(0) = -\delta$ and $\theta'(0) = 1-\delta^2$. Hence

$$\gamma_0 \ge |\theta(0)|(|\theta'(0)| - |\theta(0)|) = \delta(1-\delta-\delta^2),$$

so we have $5/27 \le \gamma_0$.

For an inner function $\theta(w)$, $|\theta(0)|^2 + |\theta'(0)|^2 \le 1$. We have

$$\gamma_0 \le \max\{x(y-x) : x^2+y^2 \le 1, x \ge 0, y \ge 0\} = \frac{\sqrt{2}-1}{2},$$

where the maximum attains at $x = \sqrt{2-\sqrt{2}}/2$ and $y = \sqrt{2+\sqrt{2}}/2$. Thus we get $5/27 \le \gamma_0 \le (\sqrt{2}-1)/2$. We note that there are no inner functions $\theta(w)$ satisfying $|\theta(0)| = \sqrt{2-\sqrt{2}}/2$ and $|\theta'(0)| = \sqrt{2+\sqrt{2}}/2$. But we do not know the exact value of γ_0. □

4. Remarks.

In Sections 2 and 3, we assumed that condition (4) holds, that is, $\varphi_0(0) \ne 0$. In this section, we study the case $\varphi_0(0) = 0$. Write

$$\varphi_0(z) = z^{\ell_0} p_0(z), \quad \ell_0 \ge 1,\ p_0(0) \ne 0.$$

We assume that conditions (1)–(3) hold. We use the same notations as the ones in Section 2, so

$$\mathscr{M} = \sum_{n=0}^{\infty} \oplus \big(\varphi_n(z) H^2(z)\big) \otimes \big(\psi_n(w)\big(H^2(w) \ominus \xi_n(w) H^2(w)\big)\big).$$

First, we assume that $\zeta_n(0) \ne 0$ for every $n \ge 0$. Since $\prod_{j=0}^{n-1} \zeta_j(z) = \varphi_0(z)/\varphi_n(z)$, we may write $\varphi_n(z) = z^{\ell_0} p_n(z), p_n(0) \ne 0$. We have $p_0(z) = p_n(z) \prod_{j=0}^{n-1} \zeta_j(z)$. Let

$$\widetilde{\mathscr{M}} = \sum_{n=0}^{\infty} \oplus \big(p_n(z)H^2(z)\big) \otimes \big(\psi_n(w)\big(H^2(w) \ominus \xi_n(w)H^2(w)\big)\big).$$

Then we have

$$z^{\ell_0}\widetilde{\mathscr{M}} = \mathscr{M} \quad \text{and} \quad z^{\ell_0}(\widetilde{\mathscr{M}} \ominus z\widetilde{\mathscr{M}}) = \mathscr{M} \ominus z\mathscr{M}.$$

If $p_0(z) = \lambda_0$ for $\lambda_0 \in \boldsymbol{C}$ with $|\lambda_0| = 1$, we have $p_n(z) = \lambda_n$ for $\lambda_n \in \boldsymbol{C}$ with $|\lambda_n| = 1$. In this case, we have $\widetilde{\mathscr{M}} = H^2$ and $\mathscr{M} = z^{\ell_0}H^2$. Since $\mathscr{M} \ominus z\mathscr{M} = z^{\ell_0}H^2(w)$, the Beurling type theorem holds for F_w. So, we assume that $p_0(z)$ is nonconstant. Then $\{p_n(z)\}_{n\geq 0}$ satisfies conditions (2) and (4), and the Beurling type theorem holds for F_w on $\mathscr{M} \ominus z\mathscr{M}$ if and only if the Beurling type theorem holds for F_w on $\widetilde{\mathscr{M}} \ominus z\widetilde{\mathscr{M}}$.

Next we assume that there exists a nonnegative integer n_0 such that $\zeta_{n_0}(0) = 0$ and $\zeta_n(0) \neq 0$ for every n with $0 \leq n \leq n_0-1$. Hence $A_0 = 1$, $A_n = \prod_{j=0}^{n-1}\zeta_j(0) \neq 0$ for $1 \leq n \leq n_0$, and $A_n = 0$ for $n \geq n_0+1$. Let

$$\mathscr{K} = \sum_{n=0}^{n_0} \oplus \varphi_n(z)\psi_n(w)\big(H^2(w) \ominus \xi_n(w)H^2(w)\big) = \sum_{n=0}^{n_0} \oplus \varphi_n(z)E_n$$

and

$$K = H^2(w) \ominus \psi_{n_0+1}(w)H^2(w) = \sum_{n=0}^{n_0} \oplus E_n.$$

Then $\mathscr{K} \subset \mathscr{M} \ominus z\mathscr{M}$.

Let $0 \leq n \leq n_0$ and $j \geq n_0+1$. Then we may write

$$\varphi_n(z) = z^{\ell_0}q_n(z), \quad q_n(0) \neq 0$$

and

$$\varphi_j(z) = z^{\ell_j}q_j(z), \quad 0 \leq \ell_{j+1} \leq \ell_j < \ell_0, \ \ q_j(0) \neq 0.$$

Since $\varphi_n(z)/\varphi_j(z)$ is inner, $q_n(z)/q_j(z)$ is also inner and we have

$$\langle \varphi_n(z), \varphi_j(z)\rangle = \left\langle z^{\ell_0-\ell_j}\frac{q_n(z)}{q_j(z)}, 1\right\rangle = 0.$$

This shows that $\varphi_n(z)H^2(w) \perp \varphi_j(z)H^2(w)$. Hence

$$w\varphi_n(z)E_n \perp \sum_{j=n_0+1}^{\infty} \oplus\varphi_j(z)E_j.$$

Since $\mathscr{M} \ominus z\mathscr{M} = \sum_{n=0}^{\infty} \oplus\varphi_n(z)E_n$, we have $F_w\mathscr{K} \subset \mathscr{K}$. Let S_w be the compression operator of T_w on K, that is $S_w = P_K T_w|_K$. For a subset E of $\mathscr{K}$, let $[E]_{\mathscr{K}}$ be the closed linear span of $\{F_w^k E : k \geq 0\}$ in $\mathscr{K}$. Similarly, for $E \subset K$ let $[E]_K$ be the closed linear span of $\{S_w^k E : k \geq 0\}$ in K. We define the operator $\widetilde{\boldsymbol{V}} : K \to \mathscr{K}$ by $\widetilde{\boldsymbol{V}} = \boldsymbol{V}|_K$. As in Section 2, we have the following assertions.

Proposition 4.1.

(i) $\widetilde{\boldsymbol{V}} S_w = F_w \widetilde{\boldsymbol{V}}$ *on* K.
(ii) $F_w\mathscr{K}$ *is dense in* $\mathscr{K}$ *if and only if* $1 \notin K$.

It is known that $f(w)$ is a cyclic vector for S_w in K if and only if the greatest common divisor of the inner factor of $f(w)$ and $\psi_{n_0+1}(w)$ equals to 1 (see [**13**, p. 82]).

Proposition 4.2. *Let* L *be a nonzero invariant subspace of* $\mathscr{K}$ *for* F_w. *Then there is an inner function* $\theta(w)$ *such that* $\psi_{n_0+1}(w)/\theta(w)$ *is inner and*

$$L = \widetilde{\boldsymbol{V}}\big(\theta(w)H^2(w) \ominus \psi_{n_0+1}(w)H^2(w)\big).$$

Moreover F_wL *is dense in* L *if and only if* $(\psi_{n_0+1}/\theta)(0) \neq 0$.

Note that $\theta(w)H^2(w) \ominus \psi_{n_0+1}(w)H^2(w)$ is an invariant subspace of K for S_w. The following is the main result in this section.

Theorem 4.3. *The Beurling type theorem holds for* F_w *on* $\mathscr{K}$ *if and only if* $\psi_{n_0+1}(w) = cw^k$ *for some* $k \geq n_0 + 1$ *and* $c \in \boldsymbol{C}$ *with* $|c| = 1$.

Proof. Suppose that $\psi_{n_0+1}(w) \neq cw^\ell$ for every $\ell \geq 1$ and $c \in \boldsymbol{C}$ with $|c| = 1$. Write $\psi_{n_0+1}(w) = w^k\theta(w)$, where $k \geq 0$ and $\theta(w)$ is a nonconstant inner function with $\theta(0) \neq 0$. Let

$$L = \widetilde{\boldsymbol{V}}\big(w^kH^2(w) \ominus \psi_{n_0+1}(w)H^2(w)\big).$$

By Proposition 4.2, L is an invariant subspace of $\mathscr{K}$ for F_w and F_wL is dense in L. Hence $[L \ominus F_wL]_{\mathscr{K}} = \{0\} \neq L$. Thus the Beurling type theorem does not hold for F_w on $\mathscr{K}$. Note that if $\psi_{n_0+1}(w) = cw^k$, by condition (2) we have $k \geq n_0 + 1$.

Suppose that $\psi_{n_0+1}(w) = cw^k$ for some $k \geq n_0 + 1$ and $c \in \boldsymbol{C}$ with $|c| = 1$. Then $\psi_j(w) = c_j w^{k_j}$ for some k_j and $c_j \in \boldsymbol{C}$ with $|c_j| = 1$, $0 \leq j \leq n_0 + 1$, satisfying

$$k_0 = 0 < k_1 < k_2 < \cdots < k_{n_0+1} = k.$$

Let L be a nonzero invariant subspace of $\mathscr{K}$ for F_w. By Proposition 4.2, there is an invariant subspace L_1 of K for S_w satisfying $L = \widetilde{\boldsymbol{V}} L_1$. Since $K = H^2(w) \ominus w^k H^2(w)$, we have

$$L_1 = \boldsymbol{C} \cdot w^m \oplus \boldsymbol{C} \cdot w^{m+1} \oplus \cdots \oplus \boldsymbol{C} \cdot w^{k-1}, \quad 0 \leq m \leq k-1.$$

Since $L_1 \ominus S_w L_1 = \boldsymbol{C} \cdot w^m$, we have $\widetilde{\boldsymbol{V}} L_1 \ominus F_w \widetilde{\boldsymbol{V}} L_1 = \boldsymbol{C} \cdot \widetilde{\boldsymbol{V}} w^m$. Since $[L_1 \ominus S_w L_1]_K = L_1$, we have $[\widetilde{\boldsymbol{V}} L_1 \ominus F_w \widetilde{\boldsymbol{V}} L_1]_{\mathscr{K}} = \widetilde{\boldsymbol{V}} L_1$. Thus the Beurling theorem holds. $\square$

REMARK 4.4. Let $q(w)$ be a nonconstant inner function and $K = H^2(w) \ominus q(w) H^2(w)$. Let S_w be the compression operator on K. By the proof of Theorem 4.3, we see that the Beurling type theorem holds for S_w on K if and only if $q(w) = cw^k$ for some $k \geq 1$ and $c \in \boldsymbol{C}$ with $|c| = 1$.

References

[1] A. Aleman, S. Richter and C. Sundberg, Beurling's theorem for the Bergman space, Acta Math., **177** (1996), 275–310.

[2] A. Beurling, On two problems concerning linear transformations in Hilbert space, Acta Math., **81** (1949), 239–255.

[3] X. Chen and K. Guo, Analytic Hilbert Modules, Chapman Hall/CRC Res. Notes Math., **433**, Chapman & Hall/CRC, Boca Raton, FL, 2003.

[4] R. G. Douglas and R. Yang, Operator theory in the Hardy space over the bidisk I, Integral Equations Operator Theory, **38** (2000), 207–221.

[5] J. B. Garnett, Bounded Analytic Functions, Pure Appl. Math., **96**, Academic Press, New York, 1981.

[6] H. Hedenmalm, B. Korenblum and K. Zhu, Theory of Bergman Spaces, Graduate Texts in Math., **199**, Springer-Verlag, New York, 2000.

[7] K. Hoffman, Banach Spaces of Analytic Functions, Prentice Hall, Englewood Cliffs, NJ, 1962.

[8] K. J. Izuchi, K. H. Izuchi and Y. Izuchi, Wandering subspaces and the Beurling type theorem I, Arch. Math. (Basel), **95** (2010), 439–446.

[9] K. J. Izuchi, K. H. Izuchi and Y. Izuchi, Wandering subspaces and the Beurling type theorem II, New York J. Math., **16** (2010), 489–505.

[10] K. J. Izuchi, T. Nakazi and M. Seto, Backward shift invariant subspaces in the bidisc II, J. Operator Theory, **51** (2004), 361–376.

[11] K. H. Izuchi, Rank-one commutators on invariant subspaces of the Hardy space on the bidisk III, Hokkaido Math. J., **38** (2009), 663–681.

[12] S. McCullough and S. Richter, Bergman-type reproducing kernels, contractive divisors, and dilations, J. Funct. Anal., **190** (2002), 447–480.

[13] N. K. Nikol'skiĭ, Treatise on the Shift Operator, Grundlehren Math. Wiss, **273**, Springer-Verlag, Berlin, 1986.

[14] A. Olofsson, Wandering subspace theorems, Integral Equations Operator Theory, **51** (2005), 395–409.

[15] S. Richter, Invariant subspaces of the Dirichlet shift, J. Reine Angew. Math., **386** (1988), 205–220.

[16] W. Rudin, Function Theory in Polydiscs, Benjamin, New York, 1969.

[17] M. Seto, Infinite sequences of inner functions and submodules in $H^2(D^2)$, J. Operator Theory, **61** (2009), 75–86.

[18] M. Seto and R. Yang, Inner sequence based invariant subspaces in $H^2(D^2)$, Proc. Amer. Math. Soc., **135** (2007), 2519–2526.

[19] S. Shimorin, Wold-type decompositions and wandering subspaces for operators close to isometries, J. Reine Angew. Math., **531** (2001), 147–189.

[20] S. Sun and D. Zheng, Beurling type theorem on the Bergman space via the Hardy space of the bidisk, Sci. China Ser. A, **52** (2009), 2517–2529.

[21] R. Yang, Operator theory in the Hardy space over the bidisk III, J. Funct. Anal., **186** (2001), 521–545.

[22] R. Yang, The core operator and congruent submodules, J. Funct. Anal., **228** (2005), 459–489.

[23] R. Yang, Hilbert-Schmidt submodules and issues of unitary equivalence, J. Operator Theory, **53** (2005), 169–184.

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