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(κ, θ)-weak normality

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Abstract. We deal with the property of weak normality (for non-principal ultrafilters). We characterize the situation of $|\prod_{i<\kappa} \lambda_i / D| = \lambda$. We have an application for a question of Depth in Boolean algebras.

0. Introduction.

The motivation of this article, emerged out of a question about the Depth of Boolean algebras. We found that a necessary condition to a positive answer on a question of Monk (appears in [**8**]) depends on the following condition. We need a sequence of cardinals $\bar{\lambda} = \langle \lambda_i : i < \kappa \rangle$ with limit λ (or just $\lambda = \lim_D(\bar{\lambda})$, see Definition 0.4 below, and for simplicity $i < \kappa \Rightarrow \lambda_i \leq \lambda$), and an ultrafilter D on $\mathrm{cf}(\lambda) = \kappa$, such that $|\prod_{i<\kappa} \lambda_i / D| = \lambda$ (see [**12**] and [**3**], about the connection to Boolean algebras; We give new results about the Depth, in Section 2).

These requirements are purely set-theoretical, and they depend on the nature of κ and λ, and also on the properties of D. On one hand, if D is a regular ultrafilter then $|\prod_{i<\kappa} \lambda_i / D| = \lambda^\kappa$. Notice that $\lambda^\kappa > \lambda$ in our case, since $\mathrm{cf}(\lambda) \leq \kappa$. On the other hand, having a measurable cardinal $\kappa = \mathrm{cf}(\lambda)$ (or just $\mathrm{cf}(\lambda) \leq \kappa$, κ is measurable) and a normal ultrafilter D, we can choose a sequence as above, with $|\prod_{i<\kappa} \lambda_i / D| = \lambda$.

Regular ultrafilters and normal ultrafilters are two poles. The question is, what happens to other creatures in the zoo of ultrafilters. We will introduce here the notion of weak normality (the basic notion appears in [**4**], and the general notion is taken from [**11**]), and prove two theorems. First, $|\prod_{i<\kappa} \lambda_i / D| = \lambda$ implies that D is weakly normal (in the sense of Definition 0.3 below). Second, that under the assumption of weak normality one can find $\bar{\lambda} = \langle \lambda_i : i < \kappa \rangle$ with the properties above.

Recall that a normal ultrafilter on κ is closed under diagonal intersections of κ sets from the ultrafilter. It follows, that any regressive function f on κ, has a

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suitable set S_f in the ultrafilter, such that f is constant on S_f. In other words, one can find a (unique) ordinal α_*, such that $\{i < \kappa : f(i) = \alpha_*\} \in D$ (when D is the normal ultrafilter).

This property of regressive functions, leads us to another notion of normality. It might happen that for no α_* one can get $\{i < \kappa : f(i) = \alpha_*\} \in D$, but for some $\alpha_* < \kappa$ we have $\{i < \kappa : f(i) \leq \alpha_*\} \in D$.

Definition 0.1 (Weak normality). Let κ be an infinite cardinal, D a uniform ultrafilter on κ. We say that D is weakly normal, when:

(*) For every regressive function f on κ, one can find $\alpha_* < \kappa$, such that $\{i < \kappa : f(i) \leq \alpha_*\} \in D$.

Every normal ultrafilter is also weakly normal. The opposite need not to be true. If D satisfies the weak normality condition of $\leq \alpha_*$, but not the requirement of $= \alpha_*$, then D is not κ-complete, so it is not a normal ultrafilter.

For our needs, we would like to generalize the notion of weak normality. So far, we focused on regressive functions from κ into κ. Let us define the property of regressiveness, in a more general context.

Definition 0.2 (Regressive pairs). Let (κ, θ) be a pair of cardinals, D an ultrafilter on κ. Let $g : \kappa \to \theta$ be any function. We say that $f : \kappa \to \theta$ is (κ, g)-regressive, if $i < \kappa \Rightarrow f(i) < g(i)$.

In the light of Definition 0.1, taking $\theta = \kappa$ and $g \equiv \mathrm{id}_\kappa$ gives the familiar notion of a regressive function on κ. We would like to form the new concept of weak normality, based on the regressive functions of 0.2. But look, if we choose $g(i) = 0$ for any $i < \kappa$, or even $g : \kappa \to \theta$ bounded, then we will have an uninteresting definition. That's the reason for demand (i) in part (a) below:

Definition 0.3 ((κ, θ)-weak normality). Let (κ, θ) be a pair of cardinals, $g : \kappa \to \theta$, and D an ultrafilter on κ.

(a) D is (κ, g)-weakly normal, if
  (i) $\epsilon < \theta \Rightarrow \{i < \kappa : g(i) \geq \epsilon\} \in D$.
  (ii) For any (κ, g)-regressive f, there is $j_f < \theta$, such that $\{i < \kappa : f(i) < j_f\} \in D$.

(b) D is (κ, θ)-weakly normal if there is a function $g : \kappa \to \theta$ such that D is (κ, g)-weakly normal.

Two remarks about the definition. First, we speak about an ultrafilter (that's what we need for our claims), but the definition (with some modifications) applies also to a filter. Second, we use $f(i) < j_f$ (instead of $\leq$ in 0.2), but there is no essential difference.

The last definition that we need, adapts the notion of limit for sequence of cardinals to the notion of an ultrafilter.

Definition 0.4 ($\lim_D(\bar{\lambda})$). Let $\bar{\lambda} = \langle \lambda_i : i < \kappa \rangle$ be a sequence of cardinals, D an ultrafilter on κ. $\mu := \lim_D(\bar{\lambda})$ is the (unique) cardinal such that $\{i < \kappa : \beta < \lambda_i \leq \mu\} \in D$, for every $\beta < \mu$.

We conclude this section with some elementary facts.

Claim 0.5. *Assume $\bar{\mu} = \langle \mu_j : j < \theta \rangle$ is an increasing sequence of cardinals, with limit λ. Let D be a (κ, θ)-weakly normal ultrafilter on κ, and $g : \kappa \to \theta$ a witness. Let $\lambda_i = \mu_{g(i)}$, for every $i < \kappa$. Then $\lambda = \lim_D(\langle \lambda_i : i < \kappa \rangle)$.*

Proof. Easy, by the definition of $\lim_D$. □

In the following claim we learn something about the relationship between $\lim_D(\langle \lambda_i : i < \kappa \rangle)$ and $|\prod_{i<\kappa} \lambda_i / D|$:

Claim 0.6. $|\prod_{i<\kappa} \lambda_i / D| \geq \lim_D(\langle \lambda_i : i < \kappa \rangle)$.

Proof. Assume to contradiction, that $|\prod_{i<\kappa} \lambda_i / D| = \mu < \lim_D(\bar{\lambda})$. Choose $\beta < \lim_D(\bar{\lambda})$ such that $\mu < \beta$. By 0.4 we have:

$$A := \{i < \kappa : \mu < \beta < \lambda_i \leq \lim_D(\bar{\lambda})\} \in D.$$

Define $\chi = \mathrm{Min}\{\lambda_i : i \in A\}$. Easily, one can define a sequence $\langle a_\alpha : \alpha < \chi \rangle$ of members in $\prod_{i<\kappa} \lambda_i / D$, such that $a_\alpha <_D a_\beta$ (notice that one needs to define the a_α-s only on the set A, and 0 on $\kappa \setminus A$ is alright). But $\chi > \beta > \mu$, contradicting the fact that $|\prod_{i<\kappa} \lambda_i / D| = \mu$. □

We say that $(\prod_{i<\kappa} \lambda_i, \leq_D)$ is θ-directed, if any $A \subseteq \prod_{i<\kappa} \lambda_i / D$ satisfies $|A| < \theta \Rightarrow A$ has an upper bound in $\prod_{i<\kappa} \lambda_i / D$. We say that θ is κ-strong when $\alpha < \theta \Rightarrow |\alpha|^\kappa < \theta$. The following useful claim draws a line between θ-directness and the cardinality of $\prod_{i<\kappa} \lambda_i / D$.

Claim 0.7 (Simple properties of cardinal products). *Let D be an ultrafilter on κ.*

(a) *If $(\prod_{i<\kappa} \lambda_i, \leq_D)$ is θ-directed, then $|\prod_{i<\kappa} \lambda_i / D| \geq \theta$.*
(b) *If $\kappa_i = \mathrm{cf}(\kappa_i)$ for every $i < \kappa$, and $\beta < \theta \Rightarrow \{i < \kappa : \beta < \kappa_i\} \in D$, then $(\prod_{i<\kappa} \kappa_i, \leq_D)$ is θ-directed.*

Proof.

(a) Easy, since if $|\prod_{i<\kappa}\lambda_i/D| = \theta_* < \theta$, then there exists an unbounded sequence of members in $\prod_{i<\kappa}\lambda_i/D$, of length θ_*, contradicting the θ-directness.
(b) Having $A \subseteq \prod_{i<\kappa}\kappa_i/D$, $|A| = \theta_* < \theta$, just take the supremum of $g(i)$ for every $g \in A$, on the set $\{i < \kappa : \theta_* < \kappa_i\}$ (and 0 on the rest of the i-s). By our assumptions, we get an upper bound for the set A which belongs to $\prod_{i<\kappa}\kappa_i/D$. □

The last proposition that we need, is about the connection between $\theta = \mathrm{cf}(\lambda)$ and λ. We defined the property of (κ,θ)-weak normality, when $\theta = \mathrm{cf}(\lambda)$. We concentrated in (κ, g)-regressive functions, when $g : \kappa \to \theta$. But sometimes we want to pass from θ to λ in our treatment.

Claim 0.8. *Let $\theta = \mathrm{cf}(\lambda) \leq \kappa < \lambda, D$ an ultrafilter on $\kappa, \langle\mu_j : j < \theta\rangle$ increasing continuous with limit $\lambda, g : \kappa \to \theta$ and $\lambda_i = \mu_{g(i)}$ for every $i < \kappa$ such that* $\lim_D(\langle\lambda_i : i < \kappa\rangle) = \lambda$. *Assume that*

$$f \in \prod_{i<\kappa}\lambda_i \Rightarrow (\exists\gamma_f < \lambda)(\{i < \kappa : f(i) < \gamma_f\} \in D).$$

Then D is (κ, g)-weakly normal (hence (κ,θ)-weakly normal).

Proof. We will show that D is (κ, g)-weakly normal. Let $h : \kappa \to \theta$ be any (κ, g)-regressive function. For every $i < \kappa$ define $f(i) = \mu_{h(i)} + 1$. Clearly $f \in \prod_{i<\kappa}\lambda_i$ since $\lambda_i = \mu_{g(i)}$ and $\mu_{h(i)} < \mu_{g(i)}$ for every $i < \kappa$. Let $\gamma_f < \lambda$ be such that $\{i < \kappa : f(i) < \gamma_f\} \in D$. Define j_h to be the first ordinal such that $\mu_{j_h} > \gamma_f$. By that, we have $\{i < \kappa : h(i) < j_h\} \in D$, so we are done. □

We have defined some notions of normality, for ultrafilters. The other side of the coin is regular ultrafilters. A good source to this subject is [**1**]. Let us start with the definition:

Definition 0.9 (Regular ultrafilters). Let D be an ultrafilter on $\kappa, \alpha \leq \kappa$.

(a) D is α-regular if there exists $E \subseteq D, |E| = \alpha$, and for every $i < \kappa$ we have $|\{e \in E : i \in e\}| < \aleph_0$.
(b) D is regular, when $\alpha = \kappa$.

Notice that every ultrafilter is α-regular for any $\alpha < \aleph_0$, so the definition is interesting only when α is an infinite cardinal. But even in the first infinite cardinal, i.e. $\alpha = \aleph_0$, we have a useful result for our needs.

CLAIM 0.10. *An ultrafilter D on κ is $\aleph_0$-regular if and only if it is not $\aleph_1$-complete.*

PROOF. If D is $\aleph_0$-regular, let $E \subseteq D$ be an evidence. Every $i < \kappa$ belongs to a finite subset of E, and $|E| = \aleph_0$, so $i \notin \bigcap E$ for any $i < \kappa$. In other words, $\bigcap E = \emptyset \notin D$, so D is not $\aleph_1$-complete.

If D is not $\aleph_1$-complete, we can find a countable $E \subseteq D$, such that $\bigcap E \notin D$. Leaning on the fact that D is an ultrafilter, we can define a countable E' which stands in the demands of the $\aleph_0$ regularity. $\square$

We state the following well-known results, without a proof:

THEOREM 0.11. *Let κ be the first cardinal such that we have a non-principal $\aleph_1$-complete ultrafilter on it. Then κ is a measurable cardinal.*

THEOREM 0.12. *Suppose μ is a compact cardinal, $\chi = \mathrm{cf}(\chi) \geq \mu$, and $\theta < \mu$. Then $\chi^\theta = \chi$.*

The proof of these theorems can be found in [**5**].

We conclude this section with an important cardinal arithmetic result, for $\aleph_0$-regular ultrafilters (the proof can be found in [**1**]):

CLAIM 0.13. *Let A be an infinite set, D an $\aleph_0$-regular ultrafilter on τ. Then $|\prod_\tau A/D| \geq |A|^{\aleph_0}$.*

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1. Weak normality and low cardinality.

The title of this section is not just a rhyme. It captures mathematical information. For showing this, let us start with the simple direction.

PROPOSITION 1.1. *Assume D is a (κ, θ)-weakly normal ultrafilter on κ, $\mathrm{cf}(\lambda) = \theta$ and λ is κ-strong. Then we can find a sequence of cardinals $\bar{\lambda} = \langle \lambda_i : i < \kappa \rangle$ such that $\lambda = \lim_D(\bar{\lambda})$ and $|\prod_{i<\kappa} \lambda_i/D| = \lambda$.*

PROOF. First, we choose our sequence. Let $\bar{\mu} = \langle \mu_j : j < \theta \rangle$ be a continuous increasing sequence of cardinals, with limit λ. Let $g : \kappa \to \theta$ be a witness to the (κ, θ)-weak normality of D. Define $\lambda_i = \mu_{g(i)}$, for any $i < \kappa$. By 0.5 we know that $\lambda = \lim_D(\langle \lambda_i : i < \kappa \rangle)$.

Now, we must prove two inequalities:

(a) $\lambda \leq |\prod_{i<\kappa} \lambda_i/D|$. By 0.6 and the fact that $\lambda = \lim_D(\langle \lambda_i : i < \kappa \rangle)$, we conclude that $\lambda \leq |\prod_{i<\kappa} \lambda_i/D|$.

(b) $|\prod_{i<\kappa} \lambda_i/D| \leq \lambda$. Observe that for every $f \in \prod_{i<\kappa} \lambda_i$ we can find $\gamma_f < \lambda$, such that $\{i < \kappa : f(i) \leq \gamma_f\} \in D$.
Why? Well, $f \in \prod_{i<\kappa} \lambda_i = \prod_{i<\kappa} \mu_{g(i)}$. Define $f^* : \kappa \to \theta$ in the following way: for every $i < \kappa$ let $f^*(i)$ be the first ordinal j such that $f(i) < \mu_j$. f^* is (κ, g)-regressive (truly, we have $f^*(i) \leq g(i)$, but the difference between $\leq$ and $<$ is unimportant here). By the (κ, g)-weak normality assumption, one can find $j < \theta$ such that the set $\{i < \kappa : f^*(i) < j\} \in D$. That means also that the set $\{i < \kappa : f(i) < \mu_j\}$ belongs to D, so choose $\gamma_f = \mu_j$ and the assertion follows.
For $\gamma < \lambda$ let $\mathscr{F}_\gamma$ be the set $\{f \in \prod_{i<\kappa} \lambda_i$, and $\{i < \kappa : f(i) < \gamma\} \in D\}$. Now, we have:

$$\left|\prod_{i<\kappa} \lambda_i/D\right| = \left|\left\{f/D : f \in \prod_{i<\kappa} \lambda_i\right\}\right| \leq \left|\bigcup_{\gamma<\lambda} f/D : f \in \mathscr{F}_\gamma\right|$$
$$\leq \sum_{\gamma<\lambda} \left|\left\{f/D : f \in \mathscr{F}_\gamma\right\}\right| \leq \sum_{\gamma<\lambda} |\gamma|^\kappa \leq \lambda \times \lambda = \lambda. \qquad \square$$

One remark about Proposition 1.1. We took an infinite λ such that λ is κ-strong. Clearly, that assumption is vital, since $\lambda^\kappa > \lambda$ in our case. So under that necessary restriction on λ, all we need for the low cardinality of the product is the (κ, θ)-weak normality of D. We turn now to the opposite direction:

THEOREM 1.2. *Assume*

(a) $\theta = \mathrm{cf}(\lambda) \leq \kappa < \lambda$,
(b) $\langle \lambda_i : i < \kappa \rangle$ *is a sequence of cardinals,*
(c) $\lim_D(\langle \lambda_i : i < \kappa \rangle) = \lambda$,
(d) $\lambda_i^\kappa < \lambda$ *for every* $i < \kappa$,
(e) D *is an ultrafilter on* κ,
(f) D *is not closed to descending sequences of length* θ *(e.g.,* D *is not* $\aleph_1$-*complete),*
(g) $|\prod_{i<\kappa} \lambda_i/D| = \lambda$.

Then D *is* (κ, θ)-*weakly normal.*

PROOF. Let $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$ be a set of representatives to $\prod_{i<\kappa} \lambda_i/D$. Denote $\kappa_i = \mathrm{cf}(\lambda_i)$, for every $i < \kappa$.

$(*)_0$ $\lim_D(\langle \kappa_i : i < \kappa \rangle) < \lambda$. [Why? If $\lim_D(\langle \kappa_i : i < \kappa \rangle) \geq \lambda$ then $\beta < \lambda \Rightarrow \{i < \kappa : \beta < \kappa_i\} \in D$, so $(\prod_{i<\kappa} \kappa_i, \leq_D)$ is λ-directed, by 0.6 and 0.7(b). Since λ is singular, $(\prod_{i<\kappa} \kappa_i, \leq_D)$ is even λ^+-directed, so by 0.7(a) $|\prod_{i<\kappa} \kappa_i/D| \geq \lambda^+$ and consequently $|\prod_{i<\kappa} \lambda_i/D| \geq \lambda^+$ since $\kappa_i \leq \lambda_i$ for every $i < \kappa$, contradicting assumption (f) here.]

It follows from $(*)_0$ that $\kappa_i = \text{cf}(\lambda_i) < \lambda_i$ for a set of i's which belongs to D.

Without loss of generality, we can assume that:

$(*)_1$ $\text{cf}(\lambda_i) < \lambda_i$, for every $i < \kappa$.

For every $i < \kappa$, choose $\langle \lambda_{i,\epsilon} : \epsilon < \kappa_i \rangle$ such that:

(i) $\kappa < \lambda_{i,\epsilon} = \text{cf}(\lambda_{i,\epsilon}) < \lambda_i$,

(ii) $\sum_{\epsilon < \kappa_i} \lambda_{i,\epsilon} = \lambda_i$,

(iii) $\kappa_{i_1} = \kappa_{i_2} \Rightarrow \lambda_{i_1,\epsilon} = \lambda_{i_2,\epsilon}$ for every $\epsilon < \kappa_{i_1} = \kappa_{i_2}$.

$(*)_2$ There is no $h \in \prod_{i<\kappa} \kappa_i$, such that $\lim_D(\langle \lambda_{i,h(i)} : i < \kappa \rangle) = \lambda$. [Why? exactly like $(*)_0$, upon replacing κ_i by $\lambda^+_{i,h(i)}$.]

Let $\bar{\mu} = \langle \mu_j : j < \theta \rangle$ be an increasing continuous sequence of singular cardinals, with limit λ. Notice that $\theta > \aleph_0$ here, by (f) and (g), hence such $\bar{\mu}$ exists. We claim that for D-many i's we have $\lambda_i \in \{\mu_j : j < \theta\}$. Otherwise, define $\zeta(i) = \sup\{j : \mu_j < \lambda_i\}$ for $i < \kappa$. Since $\bar{\mu}$ is continuous, we will get $\mu_{\zeta(i)} < \lambda_i$ for D-many i's, so easily one can create $h \in \prod_{i<\kappa} \kappa_i$ such that $\mu_{\zeta(i)} < \lambda_{i,h(i)} < \lambda_i$. Clearly, we have $\lim_D(\langle \lambda_{i,h(i)} : i < \kappa \rangle) = \lambda$, contradicting $(*)_2$. So, without loss of generality:

$(*)_3$ $\lambda_i \in \{\mu_j : j < \theta\}$, for every $i < \kappa$.

For each $i < \kappa$, let $g(i)$ be the first ordinal $j < \theta$ such that $\lambda_i = \mu_j$. We will show (in $(*)_4$ below) that g is a witness to the (κ, θ)-weak normality of D.

$(*)_4$ For every $f \in \prod_{i<\kappa} \lambda_i$, there is $\gamma_f < \lambda$, such that:

$$\{i < \kappa : f(i) < \gamma_f\} \in D.$$

For every $i < \kappa$, define $P_i = \{\lambda_{i,\epsilon} : \epsilon < \kappa_i\}$. By the choice of the $\lambda_{i,\epsilon}$-s, $\prod_{i<\kappa} P_i/D$ is unbounded in $\prod_{i<\kappa} \lambda_i/D$. Observe that $\text{tcf}(\prod_{i<\kappa} P_i/D) = \text{cf}(\lambda) = \theta$, since $|\prod_{i<\kappa} \lambda_i/D| = \lambda$. Consequently, $\text{tcf}(\prod_{i<\kappa} \kappa_i/D) = \theta$, since $\text{otp}(P_i) = \kappa_i$ for every $i < \kappa$.

Now, let $f \in \prod_{i<\kappa} \lambda_i$ be any function. $\prod_{i<\kappa} P_i/D$ is unbounded in $\prod_{i<\kappa} \lambda_i/D$, so we can find $m \in \prod_{i<\kappa} P_i/D$ such that $f <_D m$. By $(*)_2$ and the observation above, $\gamma := \lim_D(\langle m(i) : i < \kappa \rangle) < \lambda$. Choose $\gamma_f = \gamma$, and the proof of $(*)_4$ is complete.

Now we can finish the proof of the theorem. Just notice that Claim 0.8 asserts, under $(*)_4$, that D is (κ, θ)-weakly normal (with respect to the function g, which is defined above). $\square$

We conclude this section with the case of singular cardinals with countable cofinality. One of the early results about the continuum hypothesis, much before the Cohen era and even before Gödel, asserts that $2^{\aleph_0} \neq \aleph_\omega$. More generally, if $\text{cf}(\lambda) = \aleph_0$, then λ can not realize the continuum (the result belongs to König, and appears in [**6**]).

One of the metamathematical ideas of the pcf theory, suggests to replace the questions of 2^θ by questions of products of cardinals, modulo an ultrafilter. We would like to phrase a similar result about singular λ-s with countable cofinality, this time in the light of the pcf. This result is the content of Corollary 1.4 below.

Proposition 1.3 ($(\kappa,\aleph_0)$-weak normality). *For any cardinal κ there is no $(\kappa,\aleph_0)$-weakly normal ultrafilter on κ.*

Proof. Suppose that D is an ultrafilter on κ, and $g:\kappa\to\aleph_0$ satisfies condition (i) of Definition 0.3 (a). It means that $\{i\in\kappa : g(i)>j\}\in D$ for every $j\in\omega$. Let $f:\kappa\to\aleph_0$ be defined by $f(i)=g(i)-1$ (and if $g(i)=0$ then $f(i)=g(i)$) for all $i<\kappa$.

Then for every $j\in\omega\setminus\{0\}$ we have $\{i\in\kappa : f(i)<j\}=\{i\in\kappa : g(i)\le j\}=\aleph_0\setminus\{i\in\kappa : g(i)>j\}\notin D$, by (i). □

Corollary 1.4. *Assume*

(a) $\aleph_0=\mathrm{cf}(\lambda)\le\kappa<\lambda$,
(b) $\bar\lambda=\langle\lambda_i : i<\kappa\rangle$ *is a sequence of cardinals,*
(c) $\lim_D(\bar\lambda)=\lambda$,
(d) λ *is κ-strong,*
(e) *D is an ultrafilter on κ,*
(f) *D is not closed to descending sequences of length θ (e.g., D is not $\aleph_1$-complete).*

Then $|\prod_{i<\kappa}\lambda_i/D|\neq\lambda$. □

Remark 1.5 (Measurability and weak normality).

($\aleph$) If D is closed under descending sequences of length $\mathrm{cf}(\lambda)$, then Theorem 1.2 and the former corollary need not to be true. In particular, if κ is a measurable cardinal and $\lambda>\kappa$, $\mathrm{cf}(\lambda)=\aleph_0$, then λ can be realized as $|\prod_{i<\kappa}\lambda_i/D|$. Nevertheless, D is not $(\kappa,\aleph_0)$-weakly normal (see 1.3).
($\beth$) The situation is different for (κ,θ)-weakly normal ultrafilters when $\theta>\aleph_0$. For example, it is consistent to have a weakly normal (uniform) ultrafilter on $\aleph_1$, see [**2**] and the history there, and see also [**13**].

2. Applications to Boolean algebras.

We turn now to the field of Boolean algebras:

Definition 2.1 (Depth and Depth$^+$). Let $\boldsymbol{B}$ be a Boolean algebra.

(a) $\mathrm{Depth}(\boldsymbol{B}) := \sup\{\theta\colon$ there exists $A \subseteq \boldsymbol{B}, |A| = \theta$, A is well ordered by $<_{\boldsymbol{B}}\}$.
(b) $\mathrm{Depth}^+(\boldsymbol{B}) := \sup\{\theta^+\colon$ there exists $A \subseteq \boldsymbol{B}, |A| = \theta$, A is well-ordered by $<_{\boldsymbol{B}}\}$.

Monk raised the following question:

QUESTION 2.2. Let $\langle \boldsymbol{B}_i : i < \theta\rangle$ be a sequence of Boolean algebras, D an ultrafilter on $\theta, \boldsymbol{B} = \prod_{i<\theta} \boldsymbol{B}_i/D$. Can we have, in ZFC, an example of $\mathrm{Depth}(\boldsymbol{B}) > \prod_{i<\theta} \mathrm{Depth}(\boldsymbol{B}_i)/D$?

We try to find a necessary condition for such an example above a compact cardinal. We start with the following claim, from [**12**]:

CLAIM 2.3. *Assume*

(a) $\theta < \mu \leq \lambda$,
(b) μ *is a compact cardinal,*
(c) $\lambda = \mathrm{cf}(\lambda), D$ *is an ultrafilter on* θ,
(d) $(\forall \alpha < \lambda)(|\alpha|^\theta < \lambda)$,
(e) $\mathrm{Depth}^+(\boldsymbol{B}_i) \leq \lambda$ *for every* $i < \theta$.

Then $\mathrm{Depth}^+(\boldsymbol{B}) \leq \lambda$.

As a simple conclusion, we can derive our necessary condition in terms of cardinal arithmetic:

CONCLUSION 2.4. Assume

(a) $\theta < \mu < \lambda$,
(b) μ is a compact cardinal,
(c) $\mathrm{Depth}(\boldsymbol{B}_i) \leq \lambda$, for every $i < \theta$,
(d) D is a uniform ultrafilter on θ,
(e) $\mathrm{Depth}(\boldsymbol{B}) > \lambda + \prod_{i<\theta} \mathrm{Depth}(\boldsymbol{B}_i)/D$.

Then $\mathrm{Depth}(\boldsymbol{B}) = \lambda^+$ and $|\prod_{i<\theta} \mathrm{Depth}(\boldsymbol{B}_i)/D| = \lambda$.

PROOF. By (c) we know that $\mathrm{Depth}^+(\boldsymbol{B}_i) \leq \lambda^+$, so clearly $\mathrm{Depth}^+(\boldsymbol{B}_i) \leq \lambda^{++}$ for every $i < \theta$. Now, λ^{++} stands in the demands of Claim 2.3 (remember that μ is compact, so $(\lambda^+)^\theta = \lambda^+$ by Solovay's theorem). Hence $\mathrm{Depth}^+(\boldsymbol{B}) \leq \lambda^{++}$, and consequently $\mathrm{Depth}(\boldsymbol{B}) \leq \lambda^+$.

By (e), $|\prod_{i<\theta} \mathrm{Depth}(\boldsymbol{B}_i)/D|$ is strictly less than $\mathrm{Depth}(\boldsymbol{B})$. But we deal here with a case of $\prod_{i<\theta} \mathrm{Depth}(\boldsymbol{B}_i)/D \geq \lambda$, so the only possibility is $\mathrm{Depth}(\boldsymbol{B}) = \lambda^+$ and $|\prod_{i<\theta} \mathrm{Depth}(\boldsymbol{B}_i)/D| = \lambda$. $\square$

We focus, from now on, in the case of a singular λ with cofinality $\aleph_0$. In general, it seems that those cardinals behave in a unique way around questions of Depth. The following Theorem shows that there is a limitation on examples like 2.4, for a singular λ with countable cofinality:

Theorem 2.5. *Assume*

(a) $\kappa < \mu < \lambda, \mu$ *is a compact cardinal,*
(b) κ *is the first measurable cardinal,* $\theta < \kappa$*,*
(c) λ *is a singular cardinal,* $\mathrm{cf}(\lambda) = \aleph_0$*,*
(d) $\langle \boldsymbol{B}_i : i < \theta \rangle$ *is a sequence of Boolean algebras,*
(e) D *is a uniform ultrafilter on* θ*,*
(f) $\mathrm{Depth}(\boldsymbol{B}_i) \leq \lambda$*, for every* $i < \theta$*.*

Then $\mathrm{Depth}(\boldsymbol{B}) \leq \lambda$*.*

Proof. Assume toward contradiction, that $\mathrm{Depth}(\boldsymbol{B}) > \lambda$. Due to 2.4, we have an example of $|\prod_{i<\theta} \mathrm{Depth}(\boldsymbol{B}_i)/D| < \mathrm{Depth}(\boldsymbol{B})$ above a compact μ, so by virtue of Conclusion 2.4 we must have $|\prod_{i<\theta} \mathrm{Depth}(\boldsymbol{B}_i)/D| = \lambda$. Theorem 1.2 implies, under this consideration, that D is $(\theta, \aleph_0)$-weakly normal. But this is impossible, as shown in 1.3. $\square$

Remark 2.6 (Consistency results).

(a) By [**7**] it is consistent that the first compact is the first measurable. Consequently, there is no example of $|\prod_{i<\theta} \mathrm{Depth}(\boldsymbol{B}_i)/D| < \mathrm{Depth}(\boldsymbol{B})$ for singular λ-s with countable cofinality above the first measurable cardinal, in ZFC.
(b) By [**3**], if $\boldsymbol{V} = \boldsymbol{L}$ there is no example as above. This paper gives (part of) the picture under large cardinals assumptions.

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