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Bilinear estimates in dyadic BMO and the Navier-Stokes equations

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Abstract. We establish bilinear estimates in dyadic BMO as an extension of Kozono and Taniuchi's result on the usual BMO. To establish the bilinear estimates we use sharp maximal functions, while they used the boundedness of pseudo-differential operators by Coifman and Meyer. By this extension we prove that the dyadic BMO norm of the velocity controls the blow-up phenomena of smooth solutions to the Navier-Stokes equations. Moreover, we give an odd function with type II singularity which clarifies the difference between BMO and dyadic BMO.

1. Introduction.

In this paper, we investigate the continuation and blow-up phenomena of smooth solutions to the Navier-Stokes equations in $\boldsymbol{R}^n$, $n \geq 3$:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p = 0, \quad \operatorname{div} u = 0, \quad \text{in } x \in \boldsymbol{R}^n,\ t > 0, \\ u|_{t=0} = a, \end{cases} \tag{N-S}$$

where $u = (u^1(x,t), u^2(x,t), \dots, u^n(x,t))$ and $p = p(x,t)$ denote the unknown velocity vector and the unknown pressure of the fluid at the point $(x,t) \in \boldsymbol{R}^n \times (0,\infty)$, respectively, while $a = (a^1(x), a^2(x), \dots, a^n(x))$ is the given initial velocity vector.

Fujita and Kato [**3**] proved that for every $a \in H^s \equiv W^{s,2}(\boldsymbol{R}^n)$ with $s > n/2 - 1$, there exist $T = T(\|a\|_{H^s})$ and a solution $u(t)$ of (N-S) on $[0,T)$ in the class

$$u \in C([0,T); H^s) \cap C^1((0,T); H^s) \cap C((0,T); H^{s+2}). \tag{CL(s)}$$

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It is an interesting question whether the solution $u(t)$ really loses its regularity at $t = T$. Kozono and Taniuchi [**9**] proved that if

$$\int_0^T \|u(t)\|_{\mathrm{BMO}}^2 \, dt < \infty, \tag{1.1}$$

then the solution $u(t)$ in the class (CL(s)) on $[0, T)$ can be continued to the one with the same regularity on $[0, T')$ for some $T' > T$. Their result is an extension of Giga [**4**] and Beale, Kato and Majda [**1**].

In this paper we extend their result more, that is, we replace (1.1) with

$$\int_0^T \|u(t)\|_{\mathrm{BMO}^{\mathrm{dy}}}^2 \, dt < \infty, \tag{1.2}$$

where $\mathrm{BMO}^{\mathrm{dy}}$ is the dyadic version of BMO. The space $\mathrm{BMO}^{\mathrm{dy}}$ is a little wider than BMO. It is well known that $\log|x| \in \mathrm{BMO} \subset \mathrm{BMO}^{\mathrm{dy}}$. However, for example, an odd function

$$f(x) = \begin{cases} \log|x|, & x_3 > 0, \\ -\log|x|, & x_3 < 0, \end{cases} \qquad x = (x_1, x_2, x_3) \in \boldsymbol{R}^3$$

is in $\mathrm{BMO}^{\mathrm{dy}} \setminus \mathrm{BMO}$. Moreover, we can give an odd function $u \in C([0, T) \times \boldsymbol{R}^3)$ such that $u(t) \in C^\infty(\boldsymbol{R}^3) \cap L^\infty(\boldsymbol{R}^3)$ for $t \in [0, T)$,

$$\int_0^T \|u(t)\|_{\mathrm{BMO}^{\mathrm{dy}}}^2 \, dt < \infty, \quad \int_{T-\epsilon}^T \|u(t)\|_{\mathrm{BMO}}^2 \, dt = \infty \tag{1.3}$$

for small $\epsilon > 0$, and $u \in L^\infty([0, T); L^2(\boldsymbol{R}^3))$ but

$$\lim_{t \to T-0} \|u(t, \cdot)\|_{L^q(B(0,1))} = \infty,$$

for any $q > 2$, where $B(0, 1)$ is the unit ball in $\boldsymbol{R}^3$. The function u has the type II singularity, namely, u doesn't satisfy the condition

$$\sup_x |u(t, x)| \le \frac{C}{\sqrt{T - t}},$$

for any $C > 0$ (see [**8**] for the type of singularity).

In order to consider a possible-blow-up solution for the 3D Navier-Stokes equations, handling $\mathrm{BMO}^{\mathrm{dy}}$ is more reasonable than the usual BMO. To be more precise, if blow-up occurs in the axi-symmetric case, it must be an odd function on the horizontal direction to the axis. Another example is an initial data $u_0 = (u_0^1, u_0^2, u_0^3)$ which satisfies the following symmetric conditions:

$$\begin{cases} \tau_j u_0^j = -u_0^j & (j = 1, 2, 3), \\ \tau_i u_0^j = u_0^j & (i \neq j), \end{cases}$$

where $\tau_1 f(x) = f(-x_1, x_2, x_3)$, $\tau_2 f(x) = f(x_1, -x_2, x_3)$ and $\tau_3 f(x) = f(x_1, x_2, -x_3)$. We easily see that the corresponding solution also holds the same symmetric conditions, namely,

$$\begin{cases} \tau_j u^j(t) = -u^j(t) & (j = 1, 2, 3), \\ \tau_i u^j(t) = u^j(t) & (i \neq j) \text{ for } t > 0. \end{cases}$$

If we assume that the possible blow-up point exists at the origin, the flow must be an odd function. These examples suggest that if there is a blow-up solution to the 3D Navier-Stokes equations with special initial data as described above, it should have an odd structure, and it means that we need to design function spaces suitable for odd functions. In this point of view, $\mathrm{BMO}^{\mathrm{dy}}$ should be better than the usual BMO for such blow-up problem.

To get the result described in [**9**], Kozono and Taniuchi proved the following apriori estimate:

$$\sup_{\epsilon_0 < t < T} \|u(t)\|_{H^{[s]+1}} \leq \|u(\epsilon_0)\|_{H^{[s]+1}} \exp\left(C \int_{\epsilon_0}^{T} \|u\|_{\mathrm{BMO}}^2 \, dt \right), \tag{1.4}$$

where $C = C(n, s)$ is independent of T. If the left hand side of (1.4) is finite, then the strong solution u of (N-S) in the class (CL(s)) on $[0, T)$ can be extended continuously beyond T by the standard argument of continuation of local solutions. They proved (1.4) by showing bilinear estimates in BMO.

To establish the bilinear estimates in BMO they used the boundedness of pseudo-differential operators by Coifman and Meyer. In this paper we prove bilinear estimates in $\mathrm{BMO}^{\mathrm{dy}}$ by using the sharp maximal functions. The method is based on another proof of Kozono-Taniuchi's lemma by Miyachi [**11**]. Then we get the result on the Navier-Stokes equations by the same way as Kozono and Taniuchi's argument.

We state the results on the Navier-Stokes equations and prove the bilinear estimates in $\mathrm{BMO}^{\mathrm{dy}}$ in Sections 2 and 3, respectively. Section 4 is to prove propositions on the sharp maximal functions, which are used to prove the bilinear estimates in $\mathrm{BMO}^{\mathrm{dy}}$. Moreover, we give examples satisfying (1.3) in Section 5.

2. Results on the Navier-Stokes equations.

Let $C_{0,\sigma}^{\infty}$ denote the set of all C^{∞} vector-valued functions $\phi = (\phi^1, \phi^2, \ldots, \phi^n)$ with compact support in $\boldsymbol{R}^n$ such that $\operatorname{div}\phi = 0$. L_{σ}^r is the closure of $C_{0,\sigma}^{\infty}$ with respect to the L^r-norm $\|\cdot\|_r$. L^r stands for the usual (vector-valued) L^r-space over $\boldsymbol{R}^n$, $1 \le r \le \infty$. H_{σ}^s denotes the closure of $C_{0,\sigma}^{\infty}$ with respect to the H^s-norm

$$\|\phi\|_{H^s} = \left\|(1-\Delta)^{s/2}\phi\right\|_2, \quad s \ge 0.$$

Following Kozono and Taniuchi [**9**], our definition of a strong solution of (N-S) is as follows.

DEFINITION 2.1. Let $a \in H_{\sigma}^s$ for $s > n/2 - 1$. A measurable function u on $\boldsymbol{R}^n \times (0,T)$ is called a strong solution of (N-S) in the class $CL_s(0,T)$ if

(i) $u \in C([0,T); H_{\sigma}^s) \cap C^1((0,T); H_{\sigma}^s) \cap C((0,T); H_{\sigma}^{s+2})$.
(ii) u satisfies (N-S) with some distribution p such that $\nabla p \in C((0,T); H^s)$.

For the existence of the above strong solution, see [**3**], [**7**], [**4**]. Notice that $u \cdot \nabla u \in H^s$ for $u \in H^{s+2}$ with $s > n/2 - 1$.

Next, we recall the dyadic BMO. We denote by $\mathscr{D}$ the set of all dyadic cubes in $\boldsymbol{R}^n$, that is,

$$\mathscr{D} = \left\{ Q_{j,k} = \prod_{i=1}^{n} [2^{-j}k_i, 2^{-j}(k_i+1)) : j \in \boldsymbol{Z},\ k = (k_1, \ldots, k_n) \in \boldsymbol{Z}^n \right\}.$$

For a cube Q we denote its measure by $|Q|$.

DEFINITION 2.2. Let $\mathrm{BMO}^{\mathrm{dy}}(\boldsymbol{R}^n)$ be the set of all measurable functions f on $\boldsymbol{R}^n$ such that $\|f\|_{\mathrm{BMO}^{\mathrm{dy}}} < \infty$, where

$$\|f\|_{\mathrm{BMO}^{\mathrm{dy}}} = \sup_{Q \in \mathscr{D}} \frac{1}{|Q|} \int_Q |f(x) - f_Q|\,dx, \quad f_Q = \frac{1}{|Q|} \int_Q f(x)\,dx.$$

Then $\mathrm{BMO}(\boldsymbol{R}^n) \subset \mathrm{BMO}^{\mathrm{dy}}(\boldsymbol{R}^n)$ and $\|f\|_{\mathrm{BMO}^{\mathrm{dy}}} \le \|f\|_{\mathrm{BMO}}$. Let

$$f(x) = \begin{cases} -\log x, & x > 0, \\ \log(-x), & x < 0, \end{cases} \quad x \in \boldsymbol{R}.$$

Then $f \in \mathrm{BMO}^{\mathrm{dy}}(\boldsymbol{R}) \setminus \mathrm{BMO}(\boldsymbol{R})$ and also $f(\cdot - z) \in \mathrm{BMO}^{\mathrm{dy}}(\boldsymbol{R}) \setminus \mathrm{BMO}(\boldsymbol{R})$ for all $z \in \cup_{j \in \boldsymbol{Z}} 2^{-j}\boldsymbol{Z}$.

Our result on continuation of strong solutions is the following:

THEOREM 2.1. *Let $s > n/2 - 1$ and let $a \in H^s_\sigma$. Suppose that u is a strong solution of* (N-S) *in the class $CL_s(0,T)$. If*

$$\int_{\epsilon_0}^{T} \|u(t)\|^2_{\mathrm{BMO}^{\mathrm{dy}}}\, dt < \infty, \quad \text{for some } 0 < \epsilon_0 < T,$$

then u can be continued to the strong solution in the class $CL_s(0,T')$ for some $T' > T$.

The theorem above can be proven by the same way as Kozono and Taniuchi [**9**] with the bilinear estimate in dyadic BMO. We prove the estimate in the next section.

An immediate consequence of the theorem above is the following.

COROLLARY 2.2. *Let u be a strong solution of* (N-S) *in the class $CL_s(0,T)$ for $s > n/2 - 1$. Suppose that T is maximal, i.e., u cannot be continued in the class $CL_s(0,T')$ for any $T' > T$. Then*

$$\int_{\epsilon}^{T} \|u(t)\|^2_{\mathrm{BMO}^{\mathrm{dy}}}\, dt = \infty, \quad \text{for all } 0 < \epsilon < T.$$

In particular, we have

$$\limsup_{t \to T-0} \|u(t)\|_{\mathrm{BMO}^{\mathrm{dy}}} = \infty.$$

3. Bilinear estimates in dyadic BMO.

In this section, we prove the following.

THEOREM 3.1.

(i) *Let $p \in (0,\infty)$. For all $f, g \in L^p \cap \mathrm{BMO}^{\mathrm{dy}}$,*

$$\|f \cdot g\|_{L^p} \le C\big(\|f\|_{L^p}\|g\|_{\mathrm{BMO}^{\mathrm{dy}}} + \|f\|_{\mathrm{BMO}^{\mathrm{dy}}}\|g\|_{L^p}\big).$$

(ii) *Let $p \in (1,\infty)$. For all $f, g \in W^{1,p}$ with $\nabla f, \nabla g \in \mathrm{BMO}^{\mathrm{dy}}$,*

$$\|f\cdot\nabla g\|_{L^p}\le C\big(\|f\|_{L^p}\big\|(-\Delta)^{1/2}g\big\|_{\mathrm{BMO^{dy}}}+\big\|(-\Delta)^{1/2}f\big\|_{\mathrm{BMO^{dy}}}\|g\|_{L^p}\big).$$

(iii) *Let* $p\in(1,\infty)$. *For all* $f,g\in W^{|\alpha|+|\beta|,p}\cap\mathrm{BMO^{dy}}$ *with* $|\alpha|\ge1,|\beta|\ge1$,

$$\|\partial^\alpha f\cdot\partial^\beta g\|_{L^p}\le C\big(\|f\|_{\mathrm{BMO^{dy}}}\big\|(-\Delta)^{(|\alpha|+|\beta|)/2}g\big\|_{L^p}+\big\|(-\Delta)^{(|\alpha|+|\beta|)/2}f\big\|_{L^p}\|g\|_{\mathrm{BMO^{dy}}}\big).$$

The above Theorem 3.1 is an improvement of Kozono-Taniuchi's lemma in [**9**] which is for the usual BMO and $p\in(1,\infty)$. We proved the theorem by using the method based on another proof of Kozono-Taniuchi's lemma by Miyachi [**11**]. To do this we need the definitions of the maximal and sharp maximal operators.

For a cube Q we denote its measure and sidelength by $|Q|$ and $\ell(Q)$, respectively. For a nonnegative integer d, let $\mathscr{P}_d$ denote the set of all polynomials having degree at most d. For $r,\lambda\in(0,\infty)$, let

$$M^{(r),\mathrm{dy}}f(x)=\sup_{Q\in\mathscr{D},\,Q\ni x}\left(\frac{1}{|Q|}\int_Q|f(y)|^r\,dy\right)^{1/r},$$

$$M^{\sharp(r),\mathrm{dy}}_{d,\lambda}f(x)=\sup_{Q\in\mathscr{D},\,Q\ni x}\inf_{P\in\mathscr{P}_d}\frac{1}{\ell(Q)^\lambda}\left(\frac{1}{|Q|}\int_Q|f(y)-P(y)|^r\,dy\right)^{1/r}.$$

We simply denote $M^{(1),\mathrm{dy}}$ and $M^{\sharp(1),\mathrm{dy}}_{0,0}$ by M^{dy} and $M^{\sharp,\mathrm{dy}}$, respectively. Then

$$\big\|M^{\sharp(r),\mathrm{dy}}_{0,0}f\big\|_{L^\infty}\sim\big\|M^{\sharp,\mathrm{dy}}f\big\|_{L^\infty}=\|f\|_{\mathrm{BMO^{dy}}}. \tag{3.1}$$

See also [**5**, Theorem 14] for $\|M^{\sharp,\mathrm{dy}}f\|_{L^p}$.

We state four propositions on dyadic sharp maximal operators, which are proven in the next section. These are valid for the usual sharp maximal operators as in Miyachi [**11**].

Proposition 3.2.

(i) *There exists a constant* $C>0$, *dependent only on* n, *such that*

$$M^{\sharp(1),\mathrm{dy}}_{1,1}f(x)\le C\sum_{j=1}^n M^{\sharp(1),\mathrm{dy}}_{0,0}(\partial_j f)(x)$$

for all $x\in\boldsymbol{R}^n$.

(ii) *Let* $r \in (1,\infty)$ *and* $1/r + 1/r' = 1$. *Then*

$$M^{\sharp(1),\mathrm{dy}}_{1,0}(f\partial_1 g)(x) \le 4M^{(r),\mathrm{dy}}f(x)M^{\sharp(r'),\mathrm{dy}}_{0,0}(\partial_1 g)(x) + 9M^{\sharp(1),\mathrm{dy}}_{1,1}f(x)M^{\mathrm{dy}}g(x)$$

for all $x \in \boldsymbol{R}^n$.

PROPOSITION 3.3. *Let* $k, m \in \boldsymbol{N}$. *Then there exists a constant* $C > 0$, *dependent only on* n, k *and* m, *such that*

$$\sum_{|\alpha|=m} |\partial^\alpha f(x)| \le C\big(M^{\sharp,\mathrm{dy}}f(x)\big)^{k/(k+m)}\bigg(M\bigg(\sum_{|\gamma|=k+m}|\partial^\gamma f|\bigg)(x)\bigg)^{m/(k+m)}$$

for all $x \in \boldsymbol{R}^n$. *In the above* M *is the usual Hardy-Littlewood maximal operators.*

PROPOSITION 3.4. *Let* $p \in (0,\infty)$ *and* $k, m \in \boldsymbol{N}$.

(i) *There exists a constant* $C > 0$, *dependent only on* n *and* p, *such that*

$$\|f\|_{L^{2p}} \le C\|f\|^{1/2}_{\mathrm{BMO^{dy}}}\|f\|^{1/2}_{L^p}.$$

(ii) *If* $1 < \tilde{p} = mp/(k+m) \le \infty$, *then there exists a constant* $C > 0$, *dependent only on* n, p, k *and* m, *such that*

$$\bigg\|\sum_{|\alpha|=m}|\partial^\alpha f|\bigg\|_{L^p} \le C\|f\|^{k/(k+m)}_{\mathrm{BMO^{dy}}}\bigg\|\sum_{|\gamma|=k+m}|\partial^\gamma f|\bigg\|^{m/(k+m)}_{L^{\tilde{p}}}.$$

Let $\epsilon = (\epsilon_1,\dots,\epsilon_n) \in E \equiv \{-1,1\}^n$ and let $\{D(\epsilon)\}_{\epsilon\in E}$ be 2^n quadrants, that is,

$$D(\epsilon) = \big\{x = (x_1,\dots,x_n) \in \boldsymbol{R}^n : \epsilon_j x_j > 0,\ j = 1,\dots,n\big\}.$$

PROPOSITION 3.5. *Let* $p, r \in (0,\infty)$ *and* $d \in \{0\} \cup \boldsymbol{N}$. *If* $M^{\sharp(r),\mathrm{dy}}_{d,0}f \in L^p(\boldsymbol{R}^n)$, *then there exist polynomials* $\pi_\epsilon \in \mathscr{P}_d$, $\epsilon \in E$, *such that*

$$\bigg\|f - \sum_{\epsilon\in E}\pi_\epsilon\chi_{D(\epsilon)}\bigg\|_{L^p} \le C\big\|M^{\sharp(r),\mathrm{dy}}_{d,0}f\big\|_{L^p},$$

where the constant $C > 0$ *is dependent only on* n, p, r *and* d. *Moreover, if there exist* $s \in (0,\infty)$ *and a sequence of cubes* $\{Q_j\}_j \subset \mathscr{D}$ *such that* $Q_1 \subset Q_2 \subset \cdots \subset$

$D(\epsilon)$, $\sup_j |Q_{j+1}|/|Q_j| < \infty$, $\cup_j Q_j = D(\epsilon)$ and

$$\lim_{j\to\infty} \frac{1}{|Q_j|} \int_{Q_j} |f(x)|^s \, dx = 0,$$

then $\pi_\epsilon = 0$. In particular, if $f \in L^s(\boldsymbol{R}^n)$ for some $s \in (0,\infty)$, then

$$\|f\|_{L^p} \le C \big\| M_{d,0}^{\sharp(r),\mathrm{dy}} f \big\|_{L^p}.$$

Now we prove the main result in this section by using the propositions above.

Proof of Theorem 3.1.

(i) By Hölder's inequality and Proposition 3.4 (i) we have

$$\begin{aligned} \|fg\|_{L^p} &\le \|f\|_{L^{2p}} \|g\|_{L^{2p}} \\ &\lesssim \big(\|f\|_{\mathrm{BMO^{dy}}} \|f\|_{L^p} \|g\|_{\mathrm{BMO^{dy}}} \|g\|_{L^p}\big)^{1/2} \\ &\le \big(\|f\|_{L^p} \|g\|_{\mathrm{BMO^{dy}}} + \|f\|_{\mathrm{BMO^{dy}}} \|g\|_{L^p}\big). \end{aligned}$$

(ii) Take r so that $1 < r < p$. By Proposition 3.2 and (3.1) we have

$$\begin{aligned} & M_{1,0}^{\sharp(1),\mathrm{dy}} (f\partial_1 g)(x) \\ &\quad \le C\bigg(M^{(r),\mathrm{dy}} f(x) M_{0,0}^{\sharp(r'),\mathrm{dy}} (\partial_1 g)(x) + \sum_{j=1}^n M_{0,0}^{\sharp(1),\mathrm{dy}} (\partial_j f)(x) M^{\mathrm{dy}} g(x) \bigg) \\ &\quad \le C\bigg(M^{(r),\mathrm{dy}} f(x) \|\partial_1 g\|_{\mathrm{BMO^{dy}}} + \sum_{j=1}^n \|\partial_j f\|_{\mathrm{BMO^{dy}}} M^{\mathrm{dy}} g(x) \bigg). \end{aligned}$$

Since $f\partial_1 g \in L^{p/2}(\boldsymbol{R}^n)$, by Proposition 3.5 and the L^p-boundedness of the operators $M^{(r),\mathrm{dy}}$ and M^{dy} we have

$$\begin{aligned} \|f\partial_1 g\|_{L^p} &\le C \big\| M_{1,0}^{\sharp(1),\mathrm{dy}} (f\partial_1 g) \big\|_{L^p} \\ &\le C\bigg(\|f\|_{L^p} \|\partial_1 g\|_{\mathrm{BMO^{dy}}} + \sum_{j=1}^n \|\partial_j f\|_{\mathrm{BMO^{dy}}} \|g\|_{L^p} \bigg). \end{aligned}$$

This shows the conclusion.

(iii) Let $|\alpha| = m$ and $|\beta| = k$. Define q and r as

$$\frac{mq}{m+k} = \frac{kr}{m+k} = p.$$

Then $1/p = 1/q + 1/r$. By Proposition 3.4 (ii) we have

$$\begin{aligned}
&\|\partial^\alpha f \cdot \partial^\beta g\|_{L^p} \le \|\partial^\alpha f\|_{L^q} \|\partial^\beta g\|_{L^r} \\
&\lesssim \|f\|_{\mathrm{BMO^{dy}}}^{k/(k+m)} \Big\| \sum_{|\gamma|=k+m} |\partial^\gamma f| \Big\|_{L^p}^{m/(k+m)} \|g\|_{\mathrm{BMO^{dy}}}^{m/(k+m)} \Big\| \sum_{|\gamma|=k+m} |\partial^\gamma g| \Big\|_{L^p}^{k/(k+m)} \\
&= \Bigg(\|f\|_{\mathrm{BMO^{dy}}} \Big\| \sum_{|\gamma|=k+m} |\partial^\gamma g| \Big\|_{L^p} \Bigg)^{k/(k+m)} \\
&\qquad \cdot \Bigg(\Big\| \sum_{|\gamma|=k+m} |\partial^\gamma f| \Big\|_{L^p} \|g\|_{\mathrm{BMO^{dy}}} \Bigg)^{m/(k+m)} \\
&\lesssim \|f\|_{\mathrm{BMO^{dy}}} \Big\| \sum_{|\gamma|=k+m} |\partial^\gamma g| \Big\|_{L^p} + \Big\| \sum_{|\gamma|=k+m} |\partial^\gamma f| \Big\|_{L^p} \|g\|_{\mathrm{BMO^{dy}}}.
\end{aligned}$$

The proof is complete. □

4. Proofs of Propositions.

In this section, we write $f \lesssim g$ if $f \le Cg$ for some positive constant C and we write $f \sim g$ if $f \lesssim g \lesssim f$. We denote $\int_{\boldsymbol{R}^n} f(x)dx$ and $(1/|Q|)\int_Q f(x)dx$ by $\int f$ and $⨍_Q f$, respectively.

4.1. Proof of Proposition 3.2.

We state two lemmas. The first lemma is on a kind of test functions. For $f \in L^1_{loc}(\boldsymbol{R}^n)$, we write $f \perp \mathscr{P}_d$ if $\int fP = 0$ for all $P \in \mathscr{P}_d$.

Lemma 4.1 ([**10**, Lemma 2], [**14**, Lemma 2.5]). *For a cube Q and $t \in (0,\infty)$, let*

$$A(Q,t) = \{\varphi \in C^\infty(\boldsymbol{R}^n) : \operatorname{supp}\varphi \subset Q,\ \varphi \perp \mathscr{P}_1,\ \|\varphi\|_\infty \le t\ell(Q)^{-n-1}\},$$

$$B(Q,t) = \Bigg\{\varphi \in C^\infty(\boldsymbol{R}^n) : \varphi = \sum_{i=1}^n \partial_i \psi_i,\ \psi_i \in C^\infty(\boldsymbol{R}^n),\ \operatorname{supp}\psi_i \subset Q,$$
$$\psi_i \perp \mathscr{P}_0,\ \|\psi_i\|_\infty \le t\ell(Q)^{-n}\Bigg\}.$$

Then there exists a constant $C > 0$, dependent only on n, such that, for all Q, $A(Q,1) \subset B(Q,C)$.

Lemma 4.2. *Let $r \in [1,\infty)$ and $\delta \in (0,1]$. For a cube Q and a measurable set $E \subset Q$ with $|E| \ge \delta|Q|$,*

$$\left(⨍_Q |f - f_E|^r \right)^{1/r} \le (1 + \delta^{-1/r}) \inf_{c \in \boldsymbol{C}} \left(⨍_Q |f - c|^r \right)^{1/r}.$$

Proof. For any $c \in \boldsymbol{C}$,

$$|c - f_E| = \left| ⨍_E (f - c) \right| \le \left(⨍_E |f - c|^r \right)^{1/r} \le \delta^{-1/r} \left(⨍_Q |f - c|^r \right)^{1/r}.$$

Then

$$\left(⨍_Q |f - f_E|^r \right)^{1/r} \le \left(⨍_Q |f - c|^r \right)^{1/r} + |c - f_E| \le (1 + \delta^{-1/r}) \left(⨍_Q |f - c|^r \right)^{1/r}.$$

Taking the infimum over all $c \in \boldsymbol{C}$, we have the conclusion. □

Now we prove Proposition 3.2.

Proof of (i). Let Q be any cube. By the duality

$$\left(L^1(Q)/\mathscr{P}_1 \right)^* = \{ g \in L^\infty(Q) : g \perp \mathscr{P}_1 \},$$

we have

$$\begin{aligned}
&\inf_{P \in \mathscr{P}_1} \|f - P\|_{L^1(Q)} \\
&\quad = \|f\|_{L^1(Q)/\mathscr{P}_1} \\
&\quad = \sup \left\{ \left| \int_Q fg \right| : g \in L^\infty(Q),\ g \perp \mathscr{P}_1,\ \|g\|_{L^\infty(Q)} \le 1 \right\} \\
&\quad = \sup \left\{ \left| \int_Q f\varphi \right| : \varphi \in C^\infty(\boldsymbol{R}^n),\ \operatorname{supp} \varphi \subset Q,\ \varphi \perp \mathscr{P}_1,\ \|\varphi\|_{L^\infty(Q)} \le 1 \right\}.
\end{aligned}$$

This implies

$$\inf_{P\in\mathscr{P}_1}\frac{1}{\ell(Q)}\frac{1}{|Q|}\int_Q|f-P| = \sup_{\varphi\in A(Q,1)}\left|\int_Q f\varphi\right|.$$

By Lemma 4.1, for any $\varphi\in A(Q,1)$, we can take $\psi_j\in B(Q,C)$, $j=1,\dots,n$, for some $C>0$ such that

$$\int_Q f\varphi = \int_Q f\sum_{j=1}^n \partial_j\psi_i = -\int_Q\sum_{j=1}^n(\partial_j f)\psi_i = -\int_Q\sum_{j=1}^n(\partial_j f - c_j)\psi_j,$$

with any $c_j\in\boldsymbol{C}$. Then

$$\left|\int_Q f\varphi\right| \le C\sum_{j=1}^n ⨍_Q|\partial_j f - c_j|.$$

Hence

$$\inf_{P\in\mathscr{P}_1}\frac{1}{\ell(Q)}\frac{1}{|Q|}\int_Q|f-P| \le C\sum_{j=1}^n\inf_{c\in\boldsymbol{C}}⨍_Q|\partial_j f - c|.$$

Taking the supremum over all $Q\in\mathscr{D}$ with $Q\ni x$, we have the conclusion.

Proof of (ii). First we show that, for any cube Q, $f\in L^r(Q)$ and $g\in L^{r'}(Q)$,

$$\begin{aligned}&\inf_{P\in\mathscr{P}_1}⨍_Q|f(\partial_1 g)-P|\\ &\quad\le 4\Big(⨍_Q|f|^r\Big)^{1/r}\inf_{c\in\boldsymbol{C}}\Big(⨍_Q|\partial_1 g - c|^{r'}\Big)^{1/r'} + 9\inf_{P\in\mathscr{P}_1}\frac{1}{\ell(Q)}⨍_Q|f-P|⨍_Q|g|.\end{aligned}\tag{4.1}$$

Write $Q=I\times Q'$ with $I=[a,a+h]\subset\boldsymbol{R}$ and $Q'\subset\boldsymbol{R}^{n-1}$. Divide the interval I into three intervals $I_i=[a+(i-1)h/3,a+ih/3]$, $i=1,2,3$. Take $a_i\in I_i$, $i=1,3$, so that

$$\left|\int_{Q'}g(a_i,x')dx'\right| \le \frac{1}{|I_i|}\int_{I_i}\left|\int_{Q'}g(x_1,x')dx'\right|dx_1,$$

where $x=(x_1,x')$, $x'=(x_2,\dots,x_n)$. Let $E=[a_1,a_3]\times Q'$. Then

$$(\partial_1 g)_E = \frac{1}{|E|}\int_{Q'}\int_{a_1}^{a_3} \partial_1 g(x_1,x')dx_1dx' = \frac{1}{|E|}\int_{Q'}(g(a_3,x')-g(a_1,x'))dx',$$

and

$$\begin{aligned}|(\partial_1 g)_E| &\le \sum_{i=1,3}\frac{1}{|E|}\frac{1}{|I_i|}\int_{I_i}\left|\int_{Q'} g(x_1,x')dx'\right|dx_1 \\ &\le \frac{3}{|Q|}\frac{3}{|I|}\int_Q |g(x)|dx = \frac{9}{\ell(Q)}⨍_Q |g(x)|dx. \qquad (4.2)\end{aligned}$$

For any $P \in \mathscr{P}_1$, by Hölder's inequality, Lemma 4.2 and (4.2),

$$\begin{aligned}&⨍_Q |f(\partial_1 g)-(\partial_1 g)_E P| \\ &\le ⨍_Q |f((\partial_1 g)-(\partial_1 g)_E)| + ⨍_Q |(f-P)(\partial_1 g)_E| \\ &\le \left(⨍_Q |f|^r\right)^{1/r}\left(⨍_Q |\partial_1 g-(\partial_1 g)_E|^{r'}\right)^{1/r'} + ⨍_Q |f-P||(\partial_1 g)_E| \\ &\le 4\left(⨍_Q |f|^r\right)^{1/r}\inf_{c\in \boldsymbol{C}}\left(⨍_Q |\partial_1 g-c|^{r'}\right)^{1/r'} + \frac{9}{\ell(Q)}⨍_Q |f-P|⨍_Q |g|.\end{aligned}$$

Taking the infimum over all $P \in \mathscr{P}_1$, we have (4.1). Further, taking the supremum over all $Q \in \mathscr{D}$ with $Q \ni x$, we have the conclusion. □

4.2. Proof of Proposition 3.3.

Let α be a multi-index with $|\alpha| = m$. Fix $x = (x_1, x_2, \dots, x_n) \in \boldsymbol{R}^n$ and take a cube $Q \in \mathscr{D}$ containing x. Take one more cube $\tilde{Q} = \tilde{I} \times \tilde{Q}'$, $\tilde{I} \subset \boldsymbol{R}$, $\tilde{Q}' \subset \boldsymbol{R}^{n-1}$, such that $\tilde{Q} \subset Q$, $\ell(\tilde{Q}) = \ell(Q)/4$ and $\mathrm{dist}(x_1, \tilde{I}) \ge \ell(Q)/4$.

By the Taylor expansion formula we have

$$\begin{aligned}\partial^\alpha f(x) &= \sum_{|\beta|\le k-1}\frac{1}{\beta!}\partial^{\beta+\alpha} f(y)(x-y)^\beta \\ &\quad + k\int_0^1 (1-\theta)^{k-1}\sum_{|\gamma|=k}\frac{1}{\gamma!}\partial^{\gamma+\alpha} f(y+\theta(x-y))(x-y)^\gamma d\theta. \qquad (4.3)\end{aligned}$$

Take $\varphi \in C^\infty(\boldsymbol{R}^n)$ such that

$$\operatorname{supp}\varphi \subset \tilde{Q}, \quad \int \varphi(y)dy = 1, \quad |\partial^\beta \varphi(y)| \le c_\beta \ell(\tilde{Q})^{-n-|\beta|}.$$

Multiplying $\varphi(y)$ to both sides in (4.3) and integrating with respect to y on $\tilde{Q}$, we have

$$\begin{aligned}\partial^\alpha f(x) &= \sum_{|\beta|\le k-1} \frac{1}{\beta!}\int_{\tilde{Q}} \partial^{\beta+\alpha} f(y)(x-y)^\beta \varphi(y)dy \\ &\quad + k\sum_{|\gamma|=k}\frac{1}{\gamma!}\int_{\tilde{Q}}\int_0^1 (1-\theta)^{k-1}\partial^{\gamma+\alpha} f(y+\theta(x-y))(x-y)^\gamma\varphi(y)d\theta\, dy \\ &= L + J.\end{aligned}$$

For L, by the integral by part we have

$$L = \sum_{|\beta|\le k-1} \frac{(-1)^{|\beta+\alpha|}}{\beta!}\int_{\tilde{Q}} (f(y)-f_Q)\partial_y^{\beta+\alpha}\big[(x-y)^\beta\varphi(y)\big]dy.$$

Since

$$\big|\partial_y^{\beta+\alpha}[(x-y)^\beta\varphi(y)]\big| \lesssim \ell(Q)^{|\beta|-n-|\beta+\alpha|} = \ell(Q)^{-n-m},$$

we get

$$|L| \lesssim \int_Q |f(y)-f_Q|\ell(Q)^{-n-m}dy \lesssim \ell(Q)^{-m} M^{\sharp,\mathrm{dy}} f(x).$$

For J, let $dy = dy_1 dy'$, $dy' = dy_2 \dots dy_n$ and

$$J_\gamma = \int_{\tilde{Q}'}\int_0^1 (1-\theta)^{k-1}\partial^{\gamma+\alpha} f(y+\theta(x-y))(x-y)^\gamma\varphi(y)d\theta\, dy'.$$

Then

$$J = k\sum_{|\gamma|=k}\frac{1}{\gamma!}\int_{\tilde{I}} J_\gamma\, dy_1.$$

Let us use the change of valuables $z = y + \theta(x-y)$. Then $y = (z-\theta x)/(1-\theta)$, $x - y = (x-z)/(1-\theta)$ and

$$J_\gamma = \int_{\tilde{R}} (1-\theta)^{k-1} \partial^{\gamma+\alpha} f(z)(x-y)^\gamma \varphi(y) \frac{D(\theta, y_2, \dots, y_n)}{D(z_1, z_2, \dots, z_n)} dz$$
$$= \int_{\tilde{R}} (1-\theta)^{k-1} \partial^{\gamma+\alpha} f(z)(x-y)^\gamma \varphi(y) \frac{1}{(x_1-y_1)(1-\theta)^{n-1}} dz$$
$$= \int_{\tilde{R}} (1-\theta)^{k-n} \partial^{\gamma+\alpha} f(z) \frac{(x-y)^\gamma}{(x_1-y_1)} \varphi(y) dz,$$

where $\tilde{R}$ is the corresponding area to $[0,1] \times \tilde{Q}'$. Note that $\tilde{R} \subset Q$, since $x, y \in Q$ implies $z \in Q$. Using the relations

$$|x-y| \sim |x_1-y_1| \sim \ell(Q), \quad 1-\theta = \frac{|z-x|}{|x-y|} \sim \frac{|z-x|}{\ell(Q)}, \quad |\varphi(y)| \lesssim \ell(Q)^{-n},$$

we have

$$|J_\gamma| \lesssim \int_Q |z-x|^{k-n} |\partial^{\gamma+\alpha} f(z)| dz \; \ell(Q)^{-1},$$

and

$$|J| \le k \sum_{|\gamma|=k} \frac{1}{\gamma!} \int_{\tilde{I}} |J_\gamma| dy_1 \le k \sum_{|\gamma|=k} \frac{1}{\gamma!} |J_\gamma| \ell(\tilde{Q})$$
$$\lesssim \int_Q |z-x|^{k-n} \sum_{|\gamma|=k} |\partial^{\gamma+\alpha} f(z)| dz$$
$$\lesssim \sum_{2^j < (\sqrt{n}/2)\ell(Q)} \int_{2^j < |z-x| \le 2^{j+1}} (2^j)^{k-n} \sum_{|\gamma|=k} |\partial^{\gamma+\alpha} f(z)| dz$$
$$\lesssim \sum_{2^j < (\sqrt{n}/2)\ell(Q)} (2^j)^k M\Bigg(\sum_{|\gamma|=k} |\partial^{\gamma+\alpha} f| \Bigg)(x)$$
$$\le \ell(Q)^k M\Bigg(\sum_{|\gamma|=k} |\partial^{\gamma+\alpha} f| \Bigg)(x).$$

Therefore,

$$|\partial^\alpha f(x)| \le |L| + |J| \lesssim \ell(Q)^{-m} M^{\sharp,\mathrm{dy}} f(x) + \ell(Q)^k M\Bigg(\sum_{|\gamma|=k} |\partial^{\gamma+\alpha} f| \Bigg)(x).$$

Since $Q \in \mathscr{D}$ is arbitrary,

$$
\begin{aligned}
|\partial^\alpha f(x)| &\lesssim \inf_{j \in \boldsymbol{Z}} \left\{ (2^j)^{-m} M^{\sharp,\mathrm{dy}} f(x) + (2^j)^k M\Big(\sum_{|\gamma|=k} |\partial^{\gamma+\alpha} f| \Big)(x) \right\} \\
&\lesssim \big(M^{\sharp,\mathrm{dy}} f(x) \big)^{k/(k+m)} \left(M\Big(\sum_{|\gamma|=k} |\partial^{\gamma+\alpha} f| \Big)(x) \right)^{m/(k+m)}.
\end{aligned}
$$

This is the conclusion. □

4.3. Proof of Proposition 3.4.

PROOF OF (i). Take r so that $0 < r < p < \infty$. Then, by Proposition 3.5, (3.1) and the boundedness of $M^{(r),\mathrm{dy}}$ on $L^p(\boldsymbol{R}^n)$,

$$
\begin{aligned}
\|f\|_{L^{2p}}^{2p} &\lesssim \big\| M_{0,0}^{\sharp(r),\mathrm{dy}} f \big\|_{L^{2p}}^{2p} \\
&= \int \big(M_{0,0}^{\sharp(r),\mathrm{dy}} f \big)^{2p} \\
&\le \big\| M_{0,0}^{\sharp(r),\mathrm{dy}} f \big\|_{L^\infty}^{p} \int \big(M_{0,0}^{\sharp(r),\mathrm{dy}} f \big)^{p} \\
&\sim \|f\|_{\mathrm{BMO}^{\mathrm{dy}}}^{p} \big\| M_{0,0}^{\sharp(r),\mathrm{dy}} f \big\|_{L^p}^{p} \\
&\lesssim \|f\|_{\mathrm{BMO}^{\mathrm{dy}}}^{p} \big\| M^{(r),\mathrm{dy}} f \big\|_{L^p}^{p} \\
&\lesssim \|f\|_{\mathrm{BMO}^{\mathrm{dy}}}^{p} \|f\|_{L^p}^{p}.
\end{aligned}
$$

PROOF OF (ii). By Proposition 3.3 and the boundedness of M on $L^{\tilde{p}}$ with $\tilde{p} = mp/(k+m) > 1$,

$$
\begin{aligned}
\Big\| \sum_{|\alpha|=m} |\partial^\alpha f| \Big\|_{L^p} &\lesssim \big\| M^{\sharp,\mathrm{dy}} f \big\|_{L^\infty}^{k/(k+m)} \left\| \left(M\Big(\sum_{|\gamma|=k+m} |\partial^\gamma f| \Big) \right)^{m/(k+m)} \right\|_{L^p} \\
&= \|f\|_{\mathrm{BMO}^{\mathrm{dy}}}^{k/(k+m)} \left\| M\Big(\sum_{|\gamma|=k+m} |\partial^\gamma f| \Big) \right\|_{L^{\tilde{p}}}^{m/(k+m)} \\
&\lesssim \|f\|_{\mathrm{BMO}^{\mathrm{dy}}}^{k/(k+m)} \Big\| \sum_{|\gamma|=k+m} |\partial^\gamma f| \Big\|_{L^{\tilde{p}}}^{m/(k+m)}.
\end{aligned}
$$

□

4.4. Proof of Proposition 3.5.

First we note some properties of polynomials. Let $0 < p \le q \le \infty$ and $d \in \{0\} \cup \boldsymbol{N}$. Then there exists a constant $C > 0$, dependent only on n, p and d, such that, for any polynomial $P \in \mathscr{P}_d$ and any cube Q,

$$\left(⨍_Q |P|^p \right)^{1/p} \le \left(⨍_Q |P|^q \right)^{1/q} \le C \left(⨍_Q |P|^p \right)^{1/p}, \tag{4.4}$$

see for example [**2**, Lemma 3.1]. In the above, when either $p = \infty$ or $q = \infty$ the corresponding expression is replaced by $\|P\|_{L^\infty}$.

For $r \in (0, \infty)$, $A \in [1, \infty)$ and $d \in \{0\} \cup \boldsymbol{N}$, let $\Pi_d^{(r),A}(f, Q)$ be the set of all $\pi \in \mathscr{P}_d$ such that

$$\left(⨍_Q |f - \pi|^r \right)^{1/r} \le A \inf_{P \in \mathscr{P}_d} \left(⨍_Q |f - P|^r \right)^{1/r}.$$

Then $\Pi_d^{(r),1}(f, Q) \subset \Pi_d^{(r),A}(f, Q)$ for $A \in [1, \infty)$. If $f \in L^r(Q)$, then $\Pi_d^{(r),1}(f, Q) \neq \emptyset$, since $\mathscr{P}_d$ is a finite dimensional space.

To prove Proposition 3.5 we define local versions of $M^{(r),\mathrm{dy}}$ and $M_{d,\lambda}^{\sharp(r),\mathrm{dy}}$ and state two lemmas. For $r, \lambda \in (0, \infty)$ and $\Omega \subset \boldsymbol{R}^n$, let

$$M_\Omega^{(r),\mathrm{dy}} f(x) = \sup_{Q \in \mathscr{D},\, x \in Q \subset \Omega} \left(⨍_Q |f|^r \right)^{1/r},$$

$$M_{d,\lambda,\Omega}^{\sharp(r),\mathrm{dy}} f(x) = \sup_{Q \in \mathscr{D},\, x \in Q \subset \Omega} \inf_{P \in \mathscr{P}_d} \frac{1}{\ell(Q)^\lambda} \left(⨍_Q |f - P|^r \right)^{1/r}.$$

Lemma 4.3 ([**14**, Lemma 3.1]). *Let $r \in (0, \infty)$ and $d \in \{0\} \cup \boldsymbol{N}$. Then there exist constants $C > 0$ and $b > 1$ such that, for any $Q \in \mathscr{D}$, $f \in L^r(Q)$, $\delta \in (0, 1]$ and $\lambda > (⨍_Q |f|^r)^{1/r}$,*

$$\begin{aligned} &\left| \left\{ x \in Q : M_Q^{(r),\mathrm{dy}} f(x) > b\lambda,\ M_{d,0,Q}^{\sharp(r),\mathrm{dy}} f(x) \le \delta\lambda \right\} \right| \\ &\quad \le C \left(\frac{\delta}{b} \right)^r \left| \left\{ x \in Q : M_Q^{(r),\mathrm{dy}} f(x) > \lambda \right\} \right|. \end{aligned} \tag{4.5}$$

Lemma 4.4. *Let $p, r \in (0, \infty)$ and $d \in \{0\} \cup \boldsymbol{N}$. Then there exists a constant $C > 0$, dependent only on n, p, r and d, such that, for any $A \in [1, \infty)$, $Q \in \mathscr{D}$, $f \in L^r(Q)$ and $\pi \in \Pi_d^{(r),A}(f, Q)$,*

$$\|f-\pi\|_{L^p(Q)} \le AC \big\| M^{\sharp(r),\mathrm{dy}}_{d,0,Q} f \big\|_{L^p(Q)}.$$

Proof of Lemma 4.4. By the good λ inequality (4.5) and the standard argument (see for example [**13**, Theorem 1.3]) we have the following boundedness: For $p, r \in (0,\infty)$ and $d \in \{0\} \cup \boldsymbol{N}$, there exists a constant $C > 0$, dependent only on n, p, r and d, such that, for any $Q \in \mathscr{D}$ and $f \in L^r(Q)$,

$$\big\| M^{(r),\mathrm{dy}}_{Q} f \big\|_{L^p(Q)} \le C \biggl(\big\| M^{\sharp(r),\mathrm{dy}}_{d,0,Q} f \big\|_{L^p(Q)} + |Q|^{1/p} \biggl(\fint_Q |f|^r \biggr)^{1/r} \biggr). \tag{4.6}$$

Take $\pi \in \Pi^{(r),A}_d(f,Q)$ and substitute $f-\pi$ for f in (4.6). Then

$$\begin{aligned}\|f-\pi\|_{L^p(Q)} &\le \big\| M^{(r),\mathrm{dy}}_{Q} (f-\pi) \big\|_{L^p(Q)} \\ &\lesssim \big\| M^{\sharp(r),\mathrm{dy}}_{d,0,Q} f \big\|_{L^p(Q)} + |Q|^{1/p} \biggl(\fint_Q |f-\pi|^r \biggr)^{1/r} \\ &\le \big\| M^{\sharp(r),\mathrm{dy}}_{d,0,Q} f \big\|_{L^p(Q)} + A|Q|^{1/p} \inf_{x\in Q} M^{\sharp(r),\mathrm{dy}}_{d,0,Q} f(x).\end{aligned}$$

Since

$$|Q|^{1/p} \inf_{x\in Q} M^{\sharp(r),\mathrm{dy}}_{d,0,Q} f(x) = \biggl(\int_Q \Bigl[\inf_{x\in Q} M^{\sharp(r),\mathrm{dy}}_{d,0,Q} f(x) \Bigr]^p \biggr)^{1/p} \le \big\| M^{\sharp(r),\mathrm{dy}}_{d,0,Q} f \big\|_{L^p(Q)},$$

we get the conclusion. □

Now we prove Proposition 3.5.

Proof of Proposition 3.5. We show that, for each $\epsilon \in E$, there exists $\pi_\epsilon \in \mathscr{P}_d$ such that

$$\|f-\pi_\epsilon\|_{L^p(D(\epsilon))} \lesssim \big\| M^{\sharp(r),\mathrm{dy}}_{d,0} f \big\|_{L^p(D(\epsilon))}. \tag{4.7}$$

Let $\{Q_j\} \subset \mathscr{D}$, $Q_1 \subset Q_2 \subset \cdots \subset D(\epsilon)$, $\sup_j |Q_{j+1}|/|Q_j| < \infty$, $\cup_j Q_j = D(\epsilon)$. Take $\pi^j \in \Pi^{(r),1}_d(f,Q_j)$. Then, by Lemma 4.4, we have

$$\|f-\pi^j\|_{L^p(Q_j)} \lesssim \big\| M^{\sharp(r),\mathrm{dy}}_{d,0} f \big\|_{L^p(Q_j)}. \tag{4.8}$$

On the other hand, we have by (4.4)

$$\begin{aligned}\|\pi^j-\pi^{j+1}\|_{L^\infty(Q_1)} &\le \|\pi^j-\pi^{j+1}\|_{L^\infty(Q_j)} \lesssim \left(\fint_{Q_j}|\pi^j-\pi^{j+1}|^r\right)^{1/r}\\ &\le \left(\fint_{Q_j}|f-\pi^j|^r\right)^{1/r}+\left(\frac{|Q_{j+1}|}{|Q_j|}\fint_{Q_{j+1}}|f-\pi^{j+1}|^r\right)^{1/r}\\ &\lesssim \inf_{x\in Q_j} M^{\sharp(r),\mathrm{dy}}_{d,0}f(x) \le |Q_j|^{-1/p}\big\|M^{\sharp(r),\mathrm{dy}}_{d,0}f\big\|_{L^p(Q_j)}\\ &\le |Q_j|^{-1/p}\big\|M^{\sharp(r),\mathrm{dy}}_{d,0}f\big\|_{L^p(D(\epsilon))}.\end{aligned}$$

Note that we may assume that $2^n \le |Q_{j+1}|/|Q_j|$. Hence

$$\sum_{j=1}^{\infty}\|\pi^j-\pi^{j+1}\|_{L^\infty(Q_1)} \lesssim \sum_{j=1}^{\infty}|Q_j|^{-1/p}\big\|M^{\sharp(r),\mathrm{dy}}_{d,0}f\big\|_{L^p(D(\epsilon))}<\infty.$$

This shows that $\{\pi^j\}_j$ converges and the limit π_ϵ is in $\mathscr{P}_d$, since $\mathscr{P}_d$ is a finite dimensional space. Therefore, letting $j\to\infty$ in (4.8), we get (4.7).

Finally, we assume that $\{Q_j\}_j \subset \mathscr{D}$, $Q_1 \subset Q_2 \subset \cdots \subset D(\epsilon)$, $\sup_j |Q_{j+1}|/|Q_j|<\infty$, $\cup_j Q_j = D(\epsilon)$ and

$$\lim_{j\to\infty}\fint_{Q_j}|f|^s=0.$$

In this case, for $s_1\le s$, by Hölder's inequality,

$$\lim_{j\to\infty}\left(\fint_{Q_j}|f|^{s_1}\right)^{1/s_1} \le \lim_{j\to\infty}\left(\fint_{Q_j}|f|^{s}\right)^{1/s}=0.$$

Hence we may assume that $s\le r$. Let $\tilde\pi^j\in\Pi^{(s),1}_d(f,Q_j)$. Then, for any $P\in\mathscr{P}_d$,

$$\left(\fint_{Q_j}|f-\tilde\pi^j|^r\right)^{1/r}\le\left(\fint_{Q_j}|f-P|^r\right)^{1/r}+\left(\fint_{Q_j}|P-\tilde\pi^j|^r\right)^{1/r},$$

and, by (4.4),

$$\begin{aligned}\Big(\fint_{Q_j}|P-\tilde{\pi}^j|^r\Big)^{1/r} &\lesssim \Big(\fint_{Q_j}|P-\tilde{\pi}^j|^s\Big)^{1/s}\\ &\le \Big(\fint_{Q_j}|f-P|^s\Big)^{1/s}+\Big(\fint_{Q_j}|f-\tilde{\pi}^j|^s\Big)^{1/s}\\ &\le 2\Big(\fint_{Q_j}|f-P|^s\Big)^{1/s}\le 2\Big(\fint_{Q_j}|f-P|^r\Big)^{1/r}.\end{aligned}$$

Hence, for some $A' \ge 1$,

$$\Big(\fint_{Q_j}|f-\tilde{\pi}^j|^r\Big)^{1/r}\le A'\inf_{P\in\mathscr{P}_d}\Big(\fint_{Q_j}|f-P|^r\Big)^{1/r}.$$

That is, $\tilde{\pi}^j \in \Pi_d^{(r),A'}(f,Q_j)$, $j=1,2,\dots$. By the same way as before we get $\lim_{j\to\infty}\tilde{\pi}^j=\tilde{\pi}_\epsilon\in\mathscr{P}_d$ and

$$\|f-\tilde{\pi}_\epsilon\|_{L^p(D(\epsilon))}\lesssim A'\big\|M_{d,0}^{\sharp(r),\mathrm{dy}}f\big\|_{L^p(D(\epsilon))}.$$

Here we note that the polynomial π_ϵ satisfying $\|f-\pi_\epsilon\|_{L^p(D(\epsilon))}<\infty$ is unique. Then it turns out that $\lim_{j\to\infty}\tilde{\pi}^j=\pi_\epsilon$. However, by (4.4)

$$\begin{aligned}\|\tilde{\pi}^j\|_{L^\infty(Q_j)} &\lesssim \Big(\fint_{Q_j}|\tilde{\pi}^j|^s\Big)^{1/s}\\ &\le \Big(\fint_{Q_j}|f-\tilde{\pi}^j|^s\Big)^{1/s}+\Big(\fint_{Q_j}|f|^s\Big)^{1/s}\\ &\le \Big(\fint_{Q_j}|f-0|^s\Big)^{1/s}+\Big(\fint_{Q_j}|f|^s\Big)^{1/s}\to 0,\end{aligned}$$

as $j\to\infty$. This shows that $\pi_\epsilon=0$. □

5. Example.

In this section we prove the following proposition. The function u in the proposition satisfies (1.3).

Proposition 5.1. *Let $p\in(0,\infty)$. Then there exists an odd function $u\in C([0,T)\times\boldsymbol{R}^3)$ such that*

$$u(t) \in C^\infty(\boldsymbol{R}^3) \cap L^\infty(\boldsymbol{R}^3) \quad \textit{for } t \in [0,T),$$
$$u \in L^\infty([0,T); L^p(\boldsymbol{R}^3)) \cap L^2([0,T); \mathrm{BMO}^{\mathrm{dy}}(\boldsymbol{R}^3)),$$
$$\lim_{t \to T-0} \sqrt{T-t}\, \|u(t,\cdot)\|_{L^\infty} = \infty,$$
$$\lim_{t \to T-0} \sqrt{T-t}\, \|u(t,\cdot)\|_{\mathrm{BMO}} = \infty,$$

and, for any $q > p$,

$$\lim_{t \to T-0} \|u(t,\cdot)\|_{L^q(B(0,1))} = \infty.$$

To prove the proposition, we state two lemmas without proofs (see for example [**6**], [**15**]).

LEMMA 5.2. *If* $f, g \in \mathrm{BMO}(\boldsymbol{R}^n)$, *then* $\max(f,g), \min(f,g) \in \mathrm{BMO}(\boldsymbol{R}^n)$ *and*

$$\|\max(f,g)\|_{\mathrm{BMO}}, \quad \|\min(f,g)\|_{\mathrm{BMO}} \le 2(\|f\|_{\mathrm{BMO}} + \|g\|_{\mathrm{BMO}}).$$

LEMMA 5.3. *For* $\eta \in C^\infty_{comp}(\boldsymbol{R}^n)$ *with* $\eta \ge 0$ *and* $\int \eta = 1$,

$$\|f * \eta\|_{\mathrm{BMO}} \le 2\|f\|_{\mathrm{BMO}}.$$

PROOF OF PROPOSITION 5.1. We divide into three parts. In Part 1 we give a parametrized function whose $\mathrm{BMO}^{\mathrm{dy}}$-norm is bounded but BMO-norm inflates. In Part 2 we give a parametrized function whose L^p-norm is bounded but $L^q(B(0,1))$-norm inflates ($p < q$). In Part 3 we interpolate the functions constructed in Part 1 and Part 2.

Part 1: For $N > 1$, let

$$\begin{aligned}\bar{g}_N(x) &= \max\left(0, \min\left(N, \log\left(\frac{1}{|x|}\right), 4N + \log|x|\right)\right)\\ &= \begin{cases} 0, & |x| \le e^{-4N}, \\ 4N + \log|x|, & e^{-4N} \le |x| \le e^{-3N}, \\ N, & e^{-3N} \le |x| \le e^{-N}, \\ \log(1/|x|), & e^{-N} \le |x| \le 1, \\ 0, & 1 \le |x|, \end{cases} \qquad x \in \boldsymbol{R}.\end{aligned}$$

Then $\bar{g}_N \in \mathrm{BMO}(\boldsymbol{R})$ and $\|\bar{g}_N\|_{\mathrm{BMO}} \leq c_0$ for some constant c_0 independent of N by Lemma 5.2. Next, let

$$g_N(x) = g_N(x_1, x_2, x_3) = \min\big(\bar{g}_N(x_1), \bar{g}_N(x_2), \bar{g}_N(x_3)\big).$$

Then $g_N \in \mathrm{BMO}(\boldsymbol{R}^3)$ and $\|g_N\|_{\mathrm{BMO}} \leq 6c_0$. Let $\eta \in C^\infty_{comp}(\boldsymbol{R}^3)$, $\eta \geq 0$, $\int \eta = 1$ and $\operatorname{supp} \eta \subset B(0,1)$, and let

$$\eta_N(x) = e^{5N} \eta(e^{5N} x). \tag{5.1}$$

Then $g_N * \eta_N \in C^\infty_{comp}(\boldsymbol{R}^3) \cap \mathrm{BMO}(\boldsymbol{R}^3)$ and $\|g_N * \eta_N\|_{\mathrm{BMO}} \leq 12c_0$ by Lemma 5.3. Let

$$f_N(x) = \begin{cases} g_N * \eta_N(x), & x_3 > 0, \\ -g_N * \eta_N(x), & x_3 < 0. \end{cases}$$

Then $f_N \in \mathrm{BMO}^{\mathrm{dy}}(\boldsymbol{R}^3)$ and

$$\|f_N\|_{\mathrm{BMO}^{\mathrm{dy}}} \leq 12c_0. \tag{5.2}$$

Observing $g_N * \eta_N(x) = 0$ on the set $\{x = (x_1, x_2, x_3) : \min_j |x_j| < e^{-4N} - e^{-5N}\}$, we see that $f_N \in C^\infty_{comp}(\boldsymbol{R}^3)$. On the other hand, for

$$Q = \prod_{j=1}^{3} [-e^{-N}, e^{-N}], \quad Q_1 = \prod_{j=1}^{3} [0, e^{-N}], \quad Q_2 = \prod_{j=1}^{3} [e^{-3N} + e^{-5N}, e^{-N} - e^{-5N}],$$

$|Q_2|/|Q_1| > 1/2$ and

$$\frac{1}{|Q|} \int_Q |f_N(x) - (f_N)_Q| dx = \frac{1}{|Q_1|} \int_{Q_1} |f_N(x)| dx \geq \frac{1}{|Q_1|} \int_{Q_2} N\, dx \geq \frac{N}{2},$$

since $f_N(x) = N$ on Q_2. That is,

$$\|f_N\|_{L^\infty} = N, \quad \frac{N}{2} \leq \|f_N\|_{\mathrm{BMO}} \leq 2N. \tag{5.3}$$

Note that $|f_N(x)| \leq g_N(x) \leq \min_j \max(0, \log(1/|x_j|))$. Then, for any $p \in (0, \infty)$,

$$\|f_N\|_{L^p} \leq c_1, \tag{5.4}$$

for some constant c_1 independent of N.

Part 2: Let η_N be as in (5.1) and $h_N = \tilde{h}_N * \eta_N$, $N > 1$, where

$$H(r) = \frac{1}{r^3\left(\log\left(r + \frac{1}{r}\right)\right)^2}$$

and

$$\tilde{h}_N(x) = \begin{cases} \min(N,\ H^{1/p}(|x|)), & x_3 > 0, \\ -\min(N,\ H^{1/p}(|x|)), & x_3 < 0, \end{cases} \qquad x = (x_1, x_2, x_3) \in \boldsymbol{R}^3.$$

Then $h_N \in C^\infty(\boldsymbol{R}^3) \cap L^\infty(\boldsymbol{R}^3)$ and $h_N(x) = N$ on the set

$$L = \left\{x = (x_1, x_2, x_3) : e^{-5N} < |x| < N^{-p/3} - e^{-5N}, x_3 > e^{-5N}\right\}.$$

It is easy to see that

$$\|h_N\|_{L^p} \le c_2, \quad \|h_N\|_{\mathrm{BMO^{dy}}} \le \|h_N\|_{\mathrm{BMO}} \le 2\|h_N\|_{L^\infty} = 2N \tag{5.5}$$

where the constant c_2 is independent of N. Moreover, for any $q > p$, if N is large enough, then $|\tilde{h}_N(x)| = H^{1/p}(|x|)$ on $|x| \ge N^{-p/2}$ and

$$\begin{aligned} &\int_{B(0,1)\cap\{x_3 > e^{-5N}\}} h_N^q(x)dx \\ &\quad \ge \int_{B(0,1/2)\cap\{x_3 > 2e^{-5N}\}} \tilde{h}_N^q(x)dx \\ &\quad \gtrsim \int_{N^{-p/2}}^{1/2} H(r)^{q/p} r^2\, dr \gtrsim \int_{N^{-p/2}}^{1/2} r^{-1}\left(\log\left(\frac{1}{r}\right)\right)^q dr \gtrsim (\log N)^{q+1}, \end{aligned}$$

that is,

$$\|h_N\|_{L^q} \gtrsim (\log N)^{1+1/q}. \tag{5.6}$$

Part 3: For $N > 1$, let $\theta(N) = (\log N)^{-1}$ and

$$U_N(x) = (1 - \theta(N)) f_N(x) + \theta(N) h_N(x).$$

Then $\max_x U_N(x) = N$, since $L \cap Q_2 \neq \emptyset$. That is,

$$\|U_N\|_{L^\infty} = N.$$

From (5.2)–(5.6) it follows that

$$\|U_N\|_{L^p} \leq c_1 + c_2, \quad \lim_{N\to\infty} \|U_N\|_{L^q} = \infty,$$
$$\|U_N\|_{\mathrm{BMO^{dy}}} \leq \|f_N\|_{\mathrm{BMO^{dy}}} + \theta(N)\|h_N\|_{\mathrm{BMO^{dy}}} \leq 12c_0 + 2N(\log N)^{-1}.$$

If N is large enough, then $\theta(N)\|h_N\|_{\mathrm{BMO}}/\|f_N\|_{\mathrm{BMO}}$ is small and then

$$\|U_N\|_{\mathrm{BMO}} \sim \|f_N\|_{\mathrm{BMO}} \sim N.$$

Therefore, taking a large $C > 0$, a small $\delta > 0$ and letting $u(t,x) = U_N(x)$ with

$$N = C + \frac{C}{\sqrt{T-t}}\left(\log \frac{C}{\sqrt{T-t}}\right)^{\delta},$$

we have the conclusion. □

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