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The separable quotient problem and the strongly normal sequences

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Abstract. We study the notion of a *strongly normal sequence* in the dual E^* of a Banach space E. In particular, we prove that the following three conditions are equivalent:
(1) E^* has a strongly normal sequence,
(2) $(E^*, \sigma(E^*, E))$ has a Schauder basic sequence,
(3) E has an infinite-dimensional separable quotient.

Introduction.

We put $S(X) = \{x \in X : \|x\| = 1\}$ and $B(X) = \{x \in X : \|x\| \leq 1\}$ if X is a normed space. Let E be a Banach space. A sequence $(y_n) \subset S(E^*)$ is *normal* in E^* if $\lim_n y_n(x) = 0$ for every $x \in E$; clearly, the normal sequences coincide with the normalized ω^*-null sequences. The excellent Josefson-Nissenzweig theorem states that the dual of any infinite-dimensional Banach space contains a normal sequence ([**5**], [**12**]). It is easy to see that a sequence $(y_n) \subset S(E^*)$ is normal if and only if the subspace $\{x \in E : \lim_n y_n(x) = 0\}$ is dense in E. We will say that a sequence $(y_n) \subset S(E^*)$ is *strongly normal* if the subspace $\{x \in E : \sum_{n=1}^{\infty} |y_n(x)| < \infty\}$ is dense in E ([**18**]). Clearly, every strongly normal sequence in E^* is normal.

One of the most known open problems for Banach spaces is the separable quotient problem: *Does every infinite-dimensional Banach space has an infinite-dimensional separable quotient?* i.e. *Does every infinite-dimensional Banach space E has a closed subspace M such that the quotient space E/M is infinite-dimensional and separable?* ([**1**], [**8**], [**10**], [**11**], [**15**]–[**22**])

Recall that a sequence (x_n) in a locally convex space F is: (1) a *Schauder basis* of F if for each element x of F there is a unique sequence (α_n) of scalars such that $x = \sum_{n=1}^{\infty} \alpha_n x_n$ and the coefficient functionals $x_n^*, n \in \boldsymbol{N}$, defined by $x_n^*(x) = \alpha_n$, are continuous on F; (2) a *Schauder basic sequence* if it is a Schauder basis of its closed linear span X in F.

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We shall prove that a Banach space E has an infinite-dimensional separable quotient iff E^* contains a strongly normal sequence iff $E^*_\sigma = (E^*, \sigma(E^*, E))$ has a Schauder basic sequence (Theorem 3). Before, developing some ideas of [**4**], we shall show that every strongly normal sequence in the dual E^* of a Banach space E contains a Schauder basic subsequence in E^*_σ (Theorem 1).

We state the following.

Problem. Does every normal sequence in the dual E^* of a Banach space E contains a strongly normal subsequence?

If this problem has a positive answer for a given infinite-dimensional Banach space E, then by the Josefson-Nissenzweig theorem and Theorem 3, E has an infinite-dimensional separable quotient.

We show that for every WCG (i.e. weakly compactly generated) Banach space E our problem has a positive answer (Proposition 4). Next we give an example of a normal sequence in the dual E^* of some known non-WCG Banach space E, which is not strongly normal but every subsequence of it contains a strongly normal subsequence (Example).

Finally, we show that a Banach space E has no infinite-dimensional separable quotient iff every continuous linear map from a Banach space to E with dense range is a surjection iff every sequence of continuous linear maps from E to some non-zero (or to every) Fréchet space F, which is point-wise convergent on a dense subspace of E is point-wise convergent on E to some continuous linear map from E to F (Theorem 6).

Results.

Johnson and Rosenthal proved that any normal sequence (y_n) in the dual E^* of a separable Banach space E has a Schauder basic subsequence $(y_{k(n)})$ in E^*_σ ([**4**, Theorem III.1]). Developing some ideas of their proof we shall show the following.

Theorem 1. *Let E be a Banach space. Any strongly normal sequence (y_n) in E^* contains a Schauder basic subsequence $(y_{k(n)})$ in E^*_σ.*

Proof. Let $\varphi : E \to E^{**}$ be the canonical embedding map.

(A1) First we shall show that for every finite-dimensional subspace Y of E^* and every $\varepsilon \in (0, 1/2)$ there exists a finite subset H of $S(E)$ such that for every $f \in S(Y^*)$ there is an $x \in H$ with $\|f - \varphi(x) \mid Y\| < 2\varepsilon$.

Let $\psi : (E/{}^\perp Y) \to (E/{}^\perp Y)^{**}$ be the canonical embedding map; clearly ψ is an isometric isomorphism. Since $({}^\perp Y)^\perp = Y$, the map

$$\alpha : Y \to (E/^{\perp}Y)^*,\ \alpha(y)(x + {}^{\perp}Y) = y(x),\ \text{for } y \in Y, x \in E,$$

is an isometric isomorphism ([**14**, 4.9(b)]). Thus the adjoint map

$$\alpha^* : (E/^{\perp}Y)^{**} \to Y^*,\ \alpha^*(\psi(x + {}^{\perp}Y)) = \varphi(x) \mid Y,\ \text{for } x \in E,$$

is also an isometric isomorphism ([**2**, 8.6.18(a)]).

Hence for every $f \in S(Y^*)$ there is an $x \in S(E)$ with $\|f - \varphi(x) \mid Y\| < \varepsilon$. Indeed, for every $f \in S(Y^*)$ there exist $v \in E$ and $z \in {}^{\perp}Y$ such that $\varphi(v) \mid Y = f$, $\|v + {}^{\perp}Y\| = 1$ and $1 \le \|v + z\| < 1 + \varepsilon$. Thus for $u = v + z$ and $x = u/\|u\|$ we have $x \in S(E)$ and $\|f - \varphi(x) \mid Y\| = 1 - \|u\|^{-1} < \varepsilon$.

The set $S(Y^*)$ is compact, so there exists a finite subset $\{f_1, \ldots, f_n\}$ of $S(Y^*)$ with $S(Y^*) \subset \bigcup_{m=1}^n K(f_m, \varepsilon)$. Let $x_1, \ldots, x_n \in S(E)$ with $\|f_m - \varphi(x_m) \mid Y\| < \varepsilon$ for $1 \le m \le n$. Put $H = \{x_1, \ldots, x_n\}$. Then for every $f \in S(Y^*)$ there is an $x \in H$ with $\|f - \varphi(x_m) \mid Y\| < 2\varepsilon$.

(A2) Since $\lim_n y_n(x) = 0$ for every $x \in E$, using (A1) we can choose inductively a strictly increasing sequence $(k(n)) \subset \boldsymbol{N}$ and an increasing sequence (H_n) of finite subsets of $S(E)$ such that for every $n \in \boldsymbol{N}$ we have

- (i) for every $f \in S(Y_n^*)$ there is an $x \in H_n$ with $\|f - \varphi(x) \mid Y_n\| < 2^{-n-1}$, where Y_n is the linear span of the set $\{y_{k(i)} : 1 \le i \le n\}$;
- (ii) $|y_{k(n+1)}(x)| < 2^{-n-2}$ for every $x \in H_n$.

(A3) For every $n \in \boldsymbol{N}$ and for all $\alpha_1, \ldots, \alpha_{n+1} \in \boldsymbol{K}$ we have

$$\Big\| \sum_{i=1}^n \alpha_i y_{k(i)} \Big\| \le (1 + 2^{1-n}) \Big\| \sum_{i=1}^{n+1} \alpha_i y_{k(i)} \Big\|.$$

Indeed, let $n \in \boldsymbol{N}$ and $\alpha_1, \ldots, \alpha_{n+1} \in \boldsymbol{K}$. Put $y = \sum_{i=1}^n \alpha_i y_{k(i)}$ and $z = \alpha_{n+1} y_{k(n+1)}$. Then there is $f \in S(Y^*)$ with $f(y) = \|y\|$ ([**14**, 3.3]). By (A2) there is an $x \in H_n$ with $\|f - \varphi(x) \mid Y_n\| < 2^{-n-1}$ and $|y_{k(n+1)}(x)| < 2^{-n-2}$. If $\|z\| > 2\|y\|$, then $\|y + z\| > \|y\|$. If $\|z\| \le 2\|y\|$, then $\|y + z\| \ge |(y + z)(x)| \ge |y(x)| - |z(x)| \ge |f(y)| - |f(y) - y(x)| - |z(x)| = \|y\| - |(f - \varphi(x) \mid Y_n)(y)| - \|z\||y_{k(n+1)}(x)| \ge (1 - 2^{-n})\|y\| \ge (1 + 2^{1-n})^{-1}\|y\|$.

Since $\prod_{n=1}^{\infty}(1+2^{1-n}) < \infty$, using [**9**, 4.1.24], we infer that $(y_{k(n)})$ is a Schauder basic sequence in E^* such that $\|P_n\| \le \prod_{k=n}^{\infty}(1 + 2^{1-k}) < 1 + 2^{4-n}$, $n \in \boldsymbol{N}$, where $P_n : Y \to Y$, $\sum_{i=1}^{\infty} \alpha_i y_{k(i)} \to \sum_{i=1}^{n} \alpha_i y_{k(i)}$ and Y is the closed linear span of $(y_{k(n)})$.

(A4) The operator $T : E \to Y^*$, $(Tx)(y) = y(x)$, $x \in E$, $y \in Y$, is well defined, linear and continuous. Let $(f_n) \subset Y^*$ be the sequence of coefficient functionals associated with the Schauder basis $(y_{k(n)})$ in Y. Clearly, (f_n) is a Schauder basis of its closed linear span F in Y^* ([**9**, 4.4.1]). Put $G = \{x \in E : \sum_{n=1}^{\infty} |y_n(x)| < \infty\}$.

For $x \in E$ we have $Tx = \sum_{n=1}^{\infty} y_{k(n)}(x) f_n$. Indeed, let $x \in G$. For $n \geq 2$ we get $\|f_n\| = \|f_n\| \|y_{k(n)}\| = \|P_n - P_{n-1}\| \leq 2 + 2^{6-n} \leq 18$, so the series $\sum_{n=1}^{\infty} y_{k(n)}(x) f_n$ is convergent in F. For $y \in Y$ we have $(Tx)(y) = y(x) = (\sum_{n=1}^{\infty} f_n(y) y_{k(n)})(x) = \sum_{n=1}^{\infty} f_n(y) y_{k(n)}(x) = (\sum_{n=1}^{\infty} y_{k(n)}(x) f_n)(y)$, so $Tx = \sum_{n=1}^{\infty} y_{k(n)}(x) f_n \in F$. Hence $T(E) = T(\overline{G}) \subset \overline{T(G)} \subset F$. Let $x \in E$. Then $Tx = \sum_{j=1}^{\infty} \alpha_j f_j$ for some scalars $\alpha_1, \alpha_2, \ldots$. Hence $\alpha_n = (\sum_{j=1}^{\infty} \alpha_j f_j)(y_{k(n)}) = (Tx)(y_{k(n)}) = y_{k(n)}(x)$, $n \in \boldsymbol{N}$, so $Tx = \sum_{n=1}^{\infty} y_{k(n)}(x) f_n$.

(A5) For every $g \in F$ and every $\varepsilon > 0$ there is $x \in E$ with $\|x\| = \|g\|$ such that $\|g - Tx\| < \varepsilon$. Indeed, for every $g \in S(F)$ there is a sequence $(g_n) \subset S(F)$ with $\lim g_n = g$ such that $g_n \in F_n$ for $n \in \boldsymbol{N}$, where F_n is the linear span of the set $\{f_1, \ldots, f_n\}$. Thus it is enough to show that for every $n \in \boldsymbol{N}$ and every $g \in S(F_n)$ there is $x \in S(E)$ with $\|g - Tx\| \leq 2^{7-n}$. Let $n \in \boldsymbol{N}$, $g \in S(F_n)$ and $h = \|g|Y_n\|^{-1} g$.

Since $h \mid Y_n \in S(Y_n^*)$, by (A2) there is an $x \in H_n$ with $\|h \mid Y_n - \varphi(x) \mid Y_n\| < 2^{-n-1}$. Put $f = \sum_{i=1}^{n} y_{k(i)}(x) f_i$. For $y \in Y_n$ we have $f(y) = \sum_{i=1}^{n} y_{k(i)}(x) f_i(y) = (\sum_{i=1}^{n} f_i(y) y_{k(i)})(x) = y(x) = \varphi(x)(y)$, so $f \mid Y_n = \varphi(x) \mid Y_n$.

By (A4) and (A2) we get $\|Tx - g\| = \|\sum_{i=1}^{\infty} y_{k(i)}(x) f_i - g\| \leq \|f - g\| + \sum_{i=n+1}^{\infty} |y_{k(i)}(x)| \|f_i\| \leq \|f - g\| + \sum_{i=n+1}^{\infty} 2^{-i-1}(2 + 2^{6-i}) \leq (\|f - h\| + \|h - g\|) + 2^{6-n}$. For $u \in F_n$ we have $\|u\| = \sup\{|u(P_n y)| : y \in S(Y)\} \leq \|u \mid Y_n\| \|P_n\|$, so $\|f - h\| \leq \|f \mid Y_n - h \mid Y_n\| \|P_n\| = \|\varphi(x) \mid Y_n - h \mid Y_n\| \|P_n\| < 2^{-n-1}(1 + 2^{4-n}) \leq 2^{4-n}$. Moreover $\|h - g\| = \|g|Y_n\|^{-1} - 1 \leq \|g\|^{-1} \|P_n\| - 1 \leq 2^{4-n}$. Thus $\|Tx - g\| \leq 2^{7-n}$.

(A6) We show that $T(E) = F$. Let $g \in F$. Using (A5) we choose an element $x_1 \in E$ with $\|x_1\| = \|g\|$ such that $\|g - Tx_1\| < 2^{-1}$. Next we choose an element $x_2 \in E$ with $\|x_2\| = \|g - Tx_1\|$ such that $\|g - Tx_1 - Tx_2\| < 2^{-2}$. This way we can obtain a sequence $(x_n) \subset E$ such that $\|x_{n+1}\| = \|g - \sum_{j=1}^{n} Tx_j\|$ and $\|g - \sum_{j=1}^{n+1} Tx_j\| < 2^{-n-1}$ for $n \in \boldsymbol{N}$. Clearly, the series $\sum_{j=1}^{\infty} x_j$ is convergent in E to some x and $Tx = g$.

(A7) The sequence $(g_n) \subset F^*$ of coefficient functionals associated with the Schauder basis (f_n) in F is a Schauder basis in F^*_σ. The adjoint map $T^* : F^* \to E^*$ is an isomorphism of F^*_σ and the closed subspace $T^* F^*$ of E^*_σ ([**14**, 4.14 and 4.15]). Thus the sequence $(T^* g_n)$ is a Schauder basic sequence in E^*_σ. We have $(T^* g_n)(x) = g_n(Tx) = g_n(\sum_{i=1}^{\infty} y_{k(i)}(x) f_i) = y_{k(n)}(x)$ for $x \in E$ and $n \in \boldsymbol{N}$, so $T^* g_n = y_{k(n)}$ for $n \in \boldsymbol{N}$. We have shown that $(y_{k(n)})$ is a Schauder basic sequence in E^*_σ. □

Let E be a Banach space. By the Banach-Steinhaus theorem every sequence $(y_n) \subset E^*$ which is point-wise bounded on E is bounded. We will say that a sequence $(y_n) \subset E^*$ is *pseudobounded* if it is point-wise bounded on a dense subspace of E and $\sup_n \|y_n\| = \infty$.

For Schauder basic sequences in E^*_σ we have the following.

Proposition 2. *Let E be a Banach space and let (y_n) be a Schauder basic sequence in E^*_σ. If $(y_n) \subset S(E^*)$, then (y_n) is strongly normal in E^*. If $\sup_n \|y_n\| = \infty$, then (y_n) is pseudobounded in E^*. Every pseudobounded sequence (z_n) in E^* has a Schauder basic subsequence in E^*_σ.*

Proof. Denote by Y the closure of the linear span of the set $\{y_n : n \in \boldsymbol{N}\}$ in E^*_σ. Then there is a sequence $(x_n) \subset E$ such that $y_n(x_m) = \delta_{n,m}$ for all $n, m \in \boldsymbol{N}$ and $y(x) = \sum_{n=1}^\infty y(x_n) y_n(x)$ for all $y \in Y, x \in E$. For the linear span X of the set $\{x_n : n \in \boldsymbol{N}\}$ we have

$$(X + {}^\perp Y)^\perp = (X \cup {}^\perp Y)^\perp = X^\perp \cap ({}^\perp Y)^\perp = X^\perp \cap Y = \{0\}.$$

Thus $X + {}^\perp Y$ is dense in E, so the subspaces $\{x \in E : \sum_{n=1}^\infty |y_n(x)| < \infty\}$ and $\{x \in E : \sup_n |y_n(x)| < \infty\}$ are dense in E, too.

Let $(k(n)) \subset \boldsymbol{N}$ be a strictly increasing sequence with $\|z_{k(n)}\| \geq n^2$ for $n \in \boldsymbol{N}$. Put $v_n = z_{k(n)}/\|z_{k(n)}\|$ for $n \in \boldsymbol{N}$. The sequence (v_n) is strongly normal in E^*, since $\{x \in E : \sup_n |z_n(x)| < \infty\} \subset \{x \in E : \sum_{n=1}^\infty |v_n(x)| < \infty\}$. Using Theorem 1 we infer that the sequence $(z_{k(n)})$ has a Schauder basic subsequence in E^*_σ. $\square$

Using the last proposition we get the following.

Theorem 3. *Let E be a Banach space. Then the following conditions are equivalent*:

(1) *E has an infinite-dimensional separable quotient*;
(2) *E^* has a strongly normal sequence*;
(3) *E^*_σ has a Schauder basic sequence*;
(4) *E^* has a pseudobounded sequence.*

Proof. (1) $\Rightarrow$ (2). By [**6**, Proposition 1], there exists a biorthogonal sequence $((x_n, y_n)) \subset E \times E^*$ such that $A = (\operatorname{lin}\{x_n : n \in \boldsymbol{N}\} + \bigcap_{n=1}^\infty \ker y_n)$ is a dense subspace in E; clearly we can assume that $(y_n) \subset S(E^*)$. The sequence (y_n) is strongly normal in E^*, since $\{x \in E : \sum_{n=1}^\infty |y_n(x)| < \infty\} \supset A$.

Using Theorem 1 we get (2) $\Rightarrow$ (3). By [**20**, Proposition 1], we obtain (3) $\Rightarrow$ (1). Using Proposition 2 we get the equivalence (3) $\Leftrightarrow$ (4). $\square$

It is known that every infinite-dimensional WCG Banach space has an infinite-dimensional separable quotient. We shall show the following ([**18**]).

Proposition 4. *Let E be a WCG Banach space. Then every normal sequence (y_n) in E^* contains a strongly normal subsequence.*

Proof.
Case 1: E is separable. Let $X = \{x_n : n \in \boldsymbol{N}\}$ be a countable dense subset of E. For every $n \in \boldsymbol{N}$ we choose $k(n) \in \boldsymbol{N}$ with $|y_{k(n)}(x_i)| < n^{-2}$ for $1 \le i \le n$; we can assume that the sequence $(k(n))$ is strictly increasing. Then the sequence $(y_{k(n)})$ is strongly normal in E^*, since $\{x \in E : \sum_{n=1}^{\infty} |y_{k(n)}(x)| < \infty\} \supset X$.

Case 2: E is not separable. By [**3**, Proposition 1], there is a continuous linear projection $Q : E \to E$ with $\|Q\| = 1$ such that $F = Q(E)$ is a separable closed subspace of E and $(y_n) \subset Q^*(E^*)$. Let $i : F \to E$ be the identity embedding. Put $P : E \to F, x \to Qx$. Then $Q = iP$ and $Q^*(E^*) = P^*(i^*(E^*)) \subset P^*(F^*)$, so $(y_n) \subset P^*(F^*)$. Moreover $P(B(E)) = B(F)$. Therefore for every $z \in F^*$ we have

$$\begin{aligned}\|P^*z\| &= \sup\{|(P^*z)(x)| : x \in B(E)\} = \sup\{|z(Px)| : x \in B(E)\} \\ &= \sup\{|z(x)| : x \in B(F)\} = \|z\|.\end{aligned}$$

Since $(y_n) \subset P^*(F^*) \cap S(E^*)$, there is $(z_n) \subset S(F^*)$ with $P^*z_n = y_n$, $n \in \boldsymbol{N}$. Thus (z_n) is a normal sequence in F^*. By Case 1, (z_n) contains a strongly normal subsequence $(z_{k(n)})$ in F^*. Then the subspace $(\{x \in F : \sum_{n=1}^{\infty} |z_{k(n)}(x)| < \infty\} + \ker P)$ is dense in E, so the subspace $\{x \in E : \sum_{n=1}^{\infty} |y_{k(n)}(x)| < \infty\}$ is dense in E. Thus $(y_{k(n)})$ is strongly normal in E^*. $\square$

Example. The linear space $E = \{(x_n) \in c_0 : \sup_k |\sum_{n=1}^{k} x_n| < \infty\}$ with the norm $\|x\| = \sup_k |\sum_{n=1}^{k} x_n|$, $x = (x_n)$, is a Banach space and it is not WCG ([**17**]). Let $f_n : E \to \boldsymbol{K}$, $x = (x_k) \to x_n$, $n \in \boldsymbol{N}$. Then $(f_n) \subset E^*$, $\lim_n f_n(x) = 0$ for every $x \in E$ and $1 \le \|f_n\| \le 2$ for $n \in \boldsymbol{N}$. Put $y_n = f_n/\|f_n\|$, $n \in \boldsymbol{N}$; clearly (y_n) is a normal sequence in E^*. We shall prove that a subsequence $(y_{k(n)})$ of (y_n) is strongly normal in E^* if and only if the sequence $(k(n)) \subset \boldsymbol{N}$ does not contain arbitrary long series of successive integers. In particular the normal sequence (y_n) is not strongly normal but every subsequence of it contains a strongly normal subsequence.

Proof. Let $(k(n)) \subset \boldsymbol{N}$ be a strictly increasing sequence.

Assume that $(k(n))$ contains arbitrary long series of successive integers. Then for every $s \in \boldsymbol{N}$ there is $n(s) \in \boldsymbol{N}$ such that $k(n(s)+1); \ldots; k(n(s)+2s)$ are successive integers; we can assume that $n(s+1) > n(s) + 2s$ for $s \in \boldsymbol{N}$. Put

$$z_l = \begin{cases} s^{-1} & \text{if } k(n(s)+1) \le l \le k(n(s)+s) \text{ for some } s \in \boldsymbol{N}; \\ -s^{-1} & \text{if } k(n(s)+s+1) \le l \le k(n(s)+2s) \text{ for some } s \in \boldsymbol{N}; \\ 0 & \text{for all other } l \in \boldsymbol{N}. \end{cases}$$

Clearly $z = (z_l) \in E$. Let $x \in E$ with $\sum_{n=1}^{\infty} |y_{k(n)}(x)| < \infty$. Then $\sum_{n=1}^{\infty} |x_{k(n)}| = \sum_{n=1}^{\infty} |f_{k(n)}(x)| = \sum_{n=1}^{\infty} \|f_{k(n)}\| |y_{k(n)}(x)| < \infty$. For $s \in \boldsymbol{N}$ we have

$$1 = \sum_{l=k(n(s)+1)}^{k(n(s)+s)} z_l = \Bigg| \sum_{l=1}^{k(n(s)+s)} (z_l - x_l) - \sum_{l=1}^{k(n(s)+1)-1} (z_l - x_l) + \sum_{l=k(n(s)+1)}^{k(n(s)+s)} x_l \Bigg|$$

$$\le \|z - x\| + \|z - x\| + \sum_{m=n(s)+1}^{n(s)+s} |x_{k(m)}|.$$

Hence for $s \in \boldsymbol{N}$ we get $1 \le 2\|z - x\| + \sum_{m=n(s)+1}^{n(s)+s} |x_{k(m)}|$. Since $\sum_{m=1}^{\infty} |x_{k(m)}| < \infty$ we have $\lim_s \sum_{m=n(s)+1}^{n(s)+s} |x_{k(m)}| = 0$. Thus $\|z - x\| \ge 1/2$. It follows that the set $\{x \in E : \sum_{m=1}^{\infty} |y_{k(m)}(x)| < \infty\}$ is not dense in E, so the subsequence $(y_{k(n)})$ of (y_n) is not strongly normal in E^*.

Assume now that $(k(n))$ does not contain arbitrary long series of successive integers. Then there are two strictly increasing sequences $(t(n)), (w(n)) \subset \boldsymbol{N}$ and $m \in \boldsymbol{N}$ such that

(1) $t(n) \le w(n) \le t(n) + m - 2$ for $n \in \boldsymbol{N}$;
(2) $w(n) + 1 < t(n+1)$ for $n \in \boldsymbol{N}$;
(3) $\bigcup_n \{l \in \boldsymbol{N} : t(n) \le l \le w(n)\} = \{k(n) : n \in \boldsymbol{N}\}$.

Let $z \in E$. For $s \in \boldsymbol{N}$ we put $x_s = (x_{s,l})$, where

$$x_{s,l} = \begin{cases} 0 & \text{if } t(n) \le l \le w(n) \text{ for some } n \ge s; \\ \displaystyle\sum_{i=t(n)}^{w(n)+1} z_i & \text{if } l = w(n) + 1 \text{ for some } n \ge s; \\ z_l & \text{for all other } l \in \boldsymbol{N}. \end{cases}$$

Since $|\sum_{i=t(n)}^{w(n)+1} z_i| \le m \max\{|z_i| : i \ge t(n)\}$, $n \in \boldsymbol{N}$ and $\lim_n \max\{|z_i| : i \ge t(n)\} = 0$, we have $x_s \in c_0$. Moreover for $l \in \boldsymbol{N}$ we have $\sum_{i=1}^{l} x_{s,i} = \sum_{i=1}^{t(n)-1} z_i$ if $t(n) \le l \le w(n)$ for some $n \ge s$, and $\sum_{i=1}^{l} x_{s,i} = \sum_{i=1}^{l} z_i$ for all other $l \in \boldsymbol{N}$. Thus $x_s \in E$. Since $x_{s,k(n)} = 0$ if $k(n) \ge t(s)$, we have

$$\sum_{n=1}^{\infty}|y_{k(n)}(x_s)|=\sum_{n=1}^{\infty}\frac{|f_{k(n)}(x_s)|}{\|f_{k(n)}\|}=\sum_{n=1}^{\infty}\frac{|x_{s,k(n)}|}{\|f_{k(n)}\|}<\infty;$$

so $(x_s)\subset\{x\in E:\sum_{n=1}^{\infty}|y_{k(n)}(x)|<\infty\}$. For $s\in\boldsymbol{N}$ we have $\sum_{i=1}^{l}(z_i-x_{s,i})=\sum_{i=t(n)}^{l}z_i$, if $t(n)\le l\le w(n)$ for some $n\ge s$; and $\sum_{i=1}^{l}(z_i-x_{s,i})=0$ for all other $l\in\boldsymbol{N}$. Thus $\|z-x_s\|\le m\max\{|z_i|:i\ge t(s)\}$ for $s\in\boldsymbol{N}$; so $\lim_s\|z-x_s\|=0$. Hence the set $\{x\in E:\sum_{n=1}^{\infty}|y_{k(n)}(x)|<\infty\}$ is dense in E. Therefore $(y_{k(n)})$ is strongly normal in E^*. □

By the equivalence (1) ⇔ (4) in Theorem 3 we obtain the following well known result ([**1**], [**17**]); our proof is quite different from the the original one.

Corollary 5. *A Banach space has an infinite-dimensional separable quotient if and only if it contains a dense non-barrelled subspace.*

Proof. Assume that a Banach space E has an infinite-dimensional separable quotient. By Theorem 3, the space E^* has a pseudobounded sequence (y_n). Put $G=\{x\in E:\sup_n|y_n(x)|<\infty\}$ and $V=\{x\in E:\sup_n|y_n(x)|\le 1\}$. Using the Banach-Steinhaus theorem we infer that G is a proper and dense subspace of E. The set V is a barrell in G and it is not a neighbourhood of zero in G, since V is closed in E. Thus G is not barrelled.

Assume that a Banach space E contains a dense non-barrelled subspace G. Let W be a barrell in G which is not a neighbourhood of zero in G. The closure V of W in E is absolutely convex and closed in E. The linear span H of V is a dense proper subspace of E. For every $n\in\boldsymbol{N}$ there is $x_n\in(E\setminus V)$ with $\|x_n\|<n^{-2}$. By the Hahn-Banach theorem for every $n\in\boldsymbol{N}$ there is $z_n\in E^*$ with $|z_n(x_n)|>1$ such that $|z_n(x)|\le 1$ for all $x\in V$. Then $\|z_n\|\ge n^2$ for $n\in\boldsymbol{N}$ and $\sup_n|z_n(x)|<\infty$ for $x\in H$; so (z_n) is pseudobounded in E^*. By Theorem 3, E has an infinite-dimensional separable quotient. □

Applying Corollary 5 we get our last result.

Theorem 6. *Let E be an infinite-dimensional Banach space. Let F be a non-zero locally convex space. Then the following conditions are equivalent*:

(1) *Every separable quotient of E is finite-dimensional*;
(2) *Every continuous linear map from a Banach space to E with dense range is a surjection*;
(3) *Every family $\{T_\gamma:\gamma\in\Gamma\}\subset L(E,F)$ which is point-wise bounded on a dense subspace H of E is equicontinuous*;
(4) *Every sequence $(T_n)\subset L(E,F)$ which is point-wise convergent to zero on a dense subspace G of E is point-wise convergent to zero on E*;

If additionally F is sequentially complete then above conditions are equivalent to the following

(5) *Every sequence $(T_n) \subset L(E,F)$ which is point-wise convergent on a dense subspace G of E is point-wise convergent on E to some $T \in L(E,F)$.*

PROOF.

(1) $\Rightarrow$ (2). Let T be a continuous linear map from a Banach space X to E such that the range $T(X)$ is dense in E. By Corollary 5, $T(X)$ is barrelled. Using the open mapping theorem we infer that the map T is open (i.e. for every open subset U in X the set $T(U)$ is open in $T(X)$). By the Banach-Schauder theorem ([**7**, 15.12(2)]), $T(X)$ is closed in E; so $T(X) = E$.

(2) $\Rightarrow$ (1). By Corollary 5 it is enough to show that every dense subspace M of E is barrelled. Let D be a barrell in M and let B be the closed unit ball in M. Denote by S the closure of the set $C = D \cap B$ in E and by H the linear span of S. Let $p : H \to [0;\infty)$ be the Minkowski functional of S. Since S is a bounded and complete barrell in H, p is a complete norm in H and the embedding map $i : (H,p) \to E$ is a continuous linear map with dense range; so $H = E$. Thus S is a neighbourhood of zero in E. Hence D is a neighbourhood of zero in M, because $D \supset C = S \cap M$. Thus M is a barrelled space.

(1) $\Rightarrow$ (3). By Corollary 5, H is a dense barrelled subspace of E. Using the Banach-Steinhaus theorem we infer that the family $\{T_\gamma|H : \gamma \in \Gamma\}$ is equicontinuous. Let V be a closed neighbourhood of zero in F. For some open neighbourhood U of zero in E we have $T_\gamma(U \cap H) \subset V$ for all $\gamma \in \Gamma$. Hence $T_\gamma(U) \subset T_\gamma(\overline{U \cap H}) \subset \overline{T_\gamma(U \cap H)} \subset V$ for all $\gamma \in \Gamma$. Thus the family $\{T_\gamma : \gamma \in \Gamma\}$ is equicontinuous.

(3) $\Rightarrow$ (4). By (3) the sequence (T_n) is equicontinuous. Let $x \in E$. Let W, V be neighbourhoods of zero in F with $V - V \subset W$. For some neighbourhood U of zero in E we have $T_n(U) \subset V$ for $n \in \boldsymbol{N}$. Moreover there exists $y \in E$ with $y - x \in U$ such that $\lim_n T_n(y) = 0$. For some $n_0 \in \boldsymbol{N}$ we have $T_n(y) \in V$ for $n \geq n_0$. Since $T_n(x) = T_n(y) - T_n(y-x)$ and $V - T_n(U) \subset V - V \subset W$, so $T_n(x) \in W$ for $n \geq n_0$. Thus $\lim_n T_n(x) = 0$ for every $x \in E$.

(4) $\Rightarrow$ (1). Suppose, to the contrary, that E has an infinite-dimensional separable quotient. By Theorem 3, E^*_σ has a Schauder basic sequence (y_n); we can assume that $\lim_n \|y_n\| = \infty$, so (y_n) is pseudobounded in E^* (Proposition 2). Put $z_n = y_n/\sqrt{\|y_n\|}$ for $n \in \boldsymbol{N}$. Then $\lim_n \|z_n\| = \infty$. Let $z \in F$ with $z \neq 0$. For every $n \in \boldsymbol{N}$ the map $T_n : E \to F$, $x \to z_n(x)z$, is linear and continuous. Since $\{x \in E : \sup_n |y_n(x)| < \infty\} \subset \{x \in E : \lim_n z_n(x) = 0\}$, the sequence $(T_n) \subset L(E,F)$ is point-wise convergent to zero on a dense subspace of E. By (4), (T_n) is point-wise convergent to zero on E. By the Banach-Steinhaus theorem, (T_n) is equicontinuous, so $\sup_n \|z_n\| < \infty$; a contradiction.

Assume now that F is additionally sequentially complete.

(3) $\Rightarrow$ (5). By (3), the sequence (T_n) is equicontinuous. Let $x \in E$. Let W, V be neighbourhoods of zero in F with $(V - V) - (V - V) \subset W$. For some neighbourhood U of zero in E we have $T_n(U) \subset V$ for $n \in \boldsymbol{N}$. Moreover there exists $y \in E$ with $y - x \in U$ such that the sequence $(T_n(y))$ is convergent in F to some element z. Let $n_0 \in \boldsymbol{N}$ with $T_n(y) - z \in V$ for $n \geq n_0$. For $n, m \geq n_0$ we have $T_n x - T_m x = [((T_n y - z) - T_n(y - x)) - ((T_m y - z) - T_m(y - x))] \in (V - V) - (V - V) \subset W$. It follows that $(T_n x)$ is a Cauchy sequence in F, so it is convergent in F to some T_x for every $x \in E$. Clearly, the map $T : E \to F, x \to T_x$ is linear. If $x \in U$, then $(T_n x) \subset V$; hence $Tx \in W$. Thus $T(U) \subset W$; so T is continuous.

The implication (5) $\Rightarrow$ (4) is obvious. Thus (5) is equivalent to conditions (1)–(4). □

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References

[1] G. Bennett and N. Kalton, Inclusion theorems for K-spaces, Canad. J. Math., **25** (1973), 511–524.

[2] R. E. Edwards, Functional analysis, Theory and applications, Dover Publications, Inc., New York, 1995.

[3] K. John and V. Zizler, Projections in dual weakly compactly generated Banach spaces, Studia Math., **49** (1973), 41–50.

[4] W. B. Johnson and H. P. Rosenthal, On ω^*-basic sequences and their applications to the study of Banach spaces, Studia Math., **43** (1972), 77–95.

[5] B. Josefson, Weak sequential convergence in the dual of a Banach space does not imply norm convergence, Ark. Mat., **13** (1975), 78–89.

[6] J. Kąkol and W. Śliwa, Remarks concerning the separable quotient problem, Note Mat., **13** (1993), 277–282.

[7] G. Köthe, Topological vector spaces I, Springer-Verlag, Berlin, 1969.

[8] H. E. Lacey, Separable quotients of Banach spaces, An. Acad. Brasil. Ciênc., **44** (1972), 185–189.

[9] R. E. Megginson, An Introduction to Banach Space Theory, Springer-Verlag, New York, 1998.

[10] J. Mujica, Separable quotients of Banach spaces, Rev. Mat. Univ. Complut. Madrid, **10** (1997), 299–330.

[11] P. P. Narayanaswami, The separable quotient problem for barrelled spaces, In: Functional analysis and related topics, Springer-Verlag, Berlin, 1993, pp. 289–308.

[12] A. Nissenzweig, On ω^* sequential convergence, Israel J. Math., **22** (1975), 266–272.

[13] H. P. Rosenthal, On quasicomplemented subspaces of Banach spaces with an appendix on compactness of operators from $L^p(\mu)$ to $L^r(\nu)$, J. Funct. Anal., **4** (1969), 176–214.

[14] W. Rudin, Functional analysis, International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., New York, 1991.

[15] S. A. Saxon and P. P. Narayanaswami, Metrizable (LF)-spaces, (db)-spaces and the separable quotient problem, Bull. Austral. Math. Soc., **23** (1981), 65–80.

[16] S. A. Saxon and P. P. Narayanaswami, Metrizable [normable] (LF)-spaces and two classical problems in Fréchet [Banach] spaces, Studia Math., **93** (1989), 1–16.

[17] S. A. Saxon and A. Wilansky, The equivalence of some Banach space problems, Colloq. Math., **37** (1977), 217–226.

[18] W. Śliwa, (LF)-spaces and the separable quotient problem, Thesis, in Polish, unpublished, Poznań, 1996.

[19] W. Śliwa, The separable quotient problem for symmetric function spaces, Bull. Polish Acad. Sci. Math., **48** (2000), 13–27.

[20] W. Śliwa and M. Wójtowicz, Separable quotients of locally convex spaces, Bull. Polish Acad. Sci. Math., **43** (1995), 175–185.

[21] M. Wójtowicz, Generalizations of the $c_0 - l_1 - l_\infty$ Theorem of Bessaga and Pełczyński, Bull. Polish Acad. Sci. Math., **50** (2002), 373–382.

[22] M. Wójtowicz, Reflexivity and the Separable Quotient Problem for a Class of Banach Spaces, Bull. Polish Acad. Sci. Math., **50** (2002), 383–394.

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