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Kummer's quartics and numerically reflective involutions of Enriques surfaces

To the memory of Professor Masaki Maruyama

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Abstract. A (holomorphic) involution σ of an Enriques surface S is said to be *numerically reflective* if it acts on the cohomology group $H^2(S, \boldsymbol{Q})$ as a reflection. We show that the invariant sublattice $H(S, \sigma; \boldsymbol{Z})$ of the anti-Enriques lattice $H^-(S, \boldsymbol{Z})$ under the action of σ is isomorphic to either $\langle -4 \rangle \perp U(2) \perp U(2)$ or $\langle -4 \rangle \perp U(2) \perp U$. Moreover, when $H(S, \sigma; \boldsymbol{Z})$ is isomorphic to $\langle -4 \rangle \perp U(2) \perp U(2)$, we describe (S, σ) geometrically in terms of a curve of genus two and a Göpel subgroup of its Jacobian.

An automorphism of an Enriques surface S is said to be *numerically trivial* if it acts on the cohomology group $H^2(S, \boldsymbol{Q}) \simeq \boldsymbol{Q}^{10}$ trivially. By [**11**] and [**10**], numerically trivial involutions are classified into three types. An involution of S is called *numerically reflective* if it acts on $H^2(S, \boldsymbol{Q})$ as a reflection, that is, the eigenvalue -1 is of multiplicity one. In this article, we shall study numerically reflective involutions as the next case of the classification of involutions of an Enriques surface.

We first explain a construction, with which we started our investigation. Let C be a (smooth projective) curve of genus two and $J = J(C)$ be its Jacobian variety. As is well known the quotient variety $J(C)/\{\pm 1_J\}$ is realized as a quartic surface with 16 nodes in $\boldsymbol{P}^3$, called *Kummer's quartic*. The minimal resolution of $J(C)/\{\pm 1_J\}$ is called *the Jacobian Kummer surface* of C and denoted by $\mathrm{Km}\, C$.

Let $G \subset J(C)_{(2)}$ be a *Göpel subgroup* which is not *bi-elliptic* (Definitions 1.4 and 1.6). Then the four associated nodes $\bar{G} \subset J(C)/\{\pm 1_J\}$ are linearly independent in $\boldsymbol{P}^3$ (Proposition 5.2). Let $(x : y : z : t)$ be a coordinate of $\boldsymbol{P}^3$ such that the four nodes are the four vertices of the tetrahedron $xyzt = 0$. Then the equation of Kummer's quartic $J(C)/\{\pm 1_J\} \subset \boldsymbol{P}^3$ is of the form

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$$q(xt+yz, yt+zx, zt+xy)+4xyzt=0 \tag{1}$$

for a ternary quadratic form $q(x,y,z) = ax^2+by^2+cz^2+dyz+exz+fxy$ by Hutchinson[**7**]. The standard Cremona transformation $(x:y:z:t) \mapsto (1/x : 1/y : 1/z : 1/t)$ of $\boldsymbol{P}^3$ leaves the quartic invariant and induces a (holomophic) involution of $\mathrm{Km}\,C$, which we denote by ε_G. As is observed in [**8**, Section 3], the involution ε_G has no fixed points and the quotient $(\mathrm{Km}\,C)/\varepsilon_G$ is an Enriques surface (Proposition 5.1 and Remark 5.3).

The projection $(x:y:z:t) \mapsto (x:y:z)$ from the node $(0:0:0:1)$ gives a rational map of degree two from the quartic $\mathrm{Km}\,C$ to $\boldsymbol{P}^2$. The Galois group of this double cover is generated by the involution

$$\beta : (x:y:z:t) \mapsto \left(x:y:z: \frac{q(yz,xz,xy)}{tq(x,y,z)} \right), \tag{2}$$

which commutes with ε_G and descends to an involution σ_G of the Enriques surface $(\mathrm{Km}\,C)/\varepsilon_G$. Our main purpose of this article is to characterize $((\mathrm{Km}\,C)/\varepsilon_G, \sigma_G)$ as an Enriques surface with an involution, making use of the following:

Proposition 1. *σ_G is numerically reflective.*

Let S be an Enriques surface and $\tilde{S}$ the covering K3 surface of S. We denote by ε the covering involution of $\tilde{S} \to S$ and by $H^-(S,\boldsymbol{Z})$, the *anti-Enriques lattice*, that is, the anti-invariant part of $H^2(\tilde{S},\boldsymbol{Z})$ with respect to the action of ε^*. An involution σ of an Enriques surface S uniquely lifts to a symplectic involution σ_K of $\tilde{S}$, of which the associated map σ_K^* of $H^-(S,\boldsymbol{Z})$ acts trivially on $H^{2,0} \subset H^-(S,\boldsymbol{Z}) \otimes \boldsymbol{C}$ (Proposition 2.1). We denote by $H(S,\sigma;\boldsymbol{Z})$ the invariant part of $H^-(S,\boldsymbol{Z})$ under the action of σ_K^*. Both $H^-(S,\boldsymbol{Z})$ and $H(S,\sigma;\boldsymbol{Z})$ carry polarized Hodge structures of weight two.

When the involution σ is numerically reflective, $H(S,\sigma;\boldsymbol{Z})$ is isomorphic to either a) $\langle -4\rangle \perp U(2) \perp U(2)$ or b) $\langle -4\rangle \perp U(2) \perp U$ as a lattice (Proposition 3.2). If σ is σ_G, the involution constructed above, then the case a) occurs, and the converse is also true:

Theorem 2. *Let σ be a numerically reflective involution of an Enriques surface S such that $H(S,\sigma;\boldsymbol{Z})$ is isomorphic to $\langle -4\rangle \perp U(2) \perp U(2)$. Then*

1) *there exists a unique curve C of genus two such that $H(S,\sigma;\boldsymbol{Z})$ and $H^2(J(C),\Theta;\boldsymbol{Z})$ are isomorphic polarized Hodge structures* (*Lemmas* 4.2, 4.3), *and*
2) *(S,σ) is isomorphic to $((\mathrm{Km}\,C)/\varepsilon_G, \sigma_G)$, the pair constructed above.*

See Remark 5.3 for explicit equations of $(\mathrm{Km}\,C)/\varepsilon_G$ and an example appearing as a Hilbert modular surface attached to a certain congruence subgroup of $SL_2(\mathscr{O}_{\boldsymbol{Q}(\sqrt{2})})$. The case b) will be discussed elsewhere.

A Jacobiam Kummer surface $\mathrm{Km}\, C$ is expressed as the intersection of three diagonal quadrics $\sum_{i=1}^{6} x_i^2 = \sum_{i=1}^{6} \lambda_i x_i^2 = \sum_{i=1}^{6} \lambda_i^2 x_i^2 = 0$ in $\boldsymbol{P}^5$ for mutually distinct six constants $\lambda_1, \dots, \lambda_6$. Hence we have 10 fixed-point-free involutions, *e.g.*, $(x_1 : x_2 : x_3 : x_4 : x_5 : x_6) \mapsto (x_1 : x_2 : x_3 : -x_4 : -x_5 : -x_6)$, corresponding to the 10 odd theta characteristics of C. A Jacobian Kummer surface $\mathrm{Km}\, C$ has exactly 15 Göpel subgroups. A general $\mathrm{Km}\, C$ is expressed as the quartic Hessian surfaces in six different ways ([**6**], [**3**]) and accordingly has six involutions of Hutchinson-Weber type which are also free from fixed points.

CONJECTURE 3. *If the Picard group of $J(C)$ is infinitely cyclic, then a fixed-point-free involution ε of $\mathrm{Km}\, C$ is conjugate to one of the above $31(= 10 + 15 + 6)$ involutions*[1].

In the situation of the conjecture, the quotient group of $H^-(\mathrm{Km}\, C, \boldsymbol{Z})$ by the sum of the transcendental lattice and the anti-invariant Picard lattice is of order four. Our proof of Theorem 2 shows that the conjecture holds true when this abelian group is of type $(2, 2)$.

After a preparation on Kummer and Enriques surfaces in Sections 1 and 2, we compute the period of a numerically reflective involution in Sections 3 and 4. In Section 5, we construct a Hutchinson-Göpel involution ε_G of a Jacobian Kummer surface from its planar description. In Section 6, we compute the period of the Enriques surface $(\mathrm{Km}\, C)/\varepsilon_G$ more explicitly, and prove Theorem 2 using an equivariant Torelli theorem for Enriques surfaces (Theorem 2.3).

NOTATIONS. Given an abelian group A, we denote by $A_{(2)}$ the two-torsion subgroup. A free $\boldsymbol{Z}$-module with an integral symmetric bilinear form is simply called *a lattice*. U denotes the lattice of rank two given by the symmetric matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. A_l, D_l and E_l are the *negative* definite root lattices of rank l of type A, D and E, respectively. For a lattice L and a rational number r, we denote by $L(r)$ the lattice obtained by replacing the bilinear form $(\ .\ )$ on L by $r(\ .\ )$.

1. Preliminary.

We recall some basic facts on the cohomology of Kummer surfaces. Let T be a two-dimensional complex torus. The minimal resolution of the quotient $T/\{\pm 1_T\}$ is called the *Kummer* surface of T and denoted by $Km(T)$. $Km(T)$ contains 16 mutually disjoint (-2) curves $N_a, a \in T_{(2)}$, parametrized by the two-torsion

[1] This conjecture has been solved by H. Ohashi [**13**].

subgroup $T_{(2)} \simeq (\boldsymbol{Z}/2\boldsymbol{Z})^4$ of T. We denote by Γ_{Km} the primitive hull of the lattice generated by the 16 N_a's. Let Λ be the orthogonal complement of Γ_{Km} in $H^2(Km(T), \boldsymbol{Z})$. Then Λ is the image of $H^2(T, \boldsymbol{Z})$ by the quotient morphism from the blow-up of T at $T_{(2)}$ onto $Km(T)$. The following is well known ([**1**, Chapter VIII, Section 5]):

Lemma 1.1. *$\Lambda \subset H^2(Km(T))$ is isomorphic to $H^2(T, \boldsymbol{Z})$ as a Hodge structure and to $H^2(T, \boldsymbol{Z})(2) \simeq U(2) \perp U(2) \perp U(2)$ as a lattice.*

The discriminant group A_Λ of Λ is $((1/2)\Lambda)/\Lambda \simeq H^2(T, \boldsymbol{Z}/2\boldsymbol{Z})$ and the discriminant form q_Λ is essentially the cup product, that is, $q_\Lambda(\bar{y}) = (y \cup y)/2 \bmod 2$ for $y \in H^2(T, \boldsymbol{Z})$.

Let $P \subset T_{(2)}$ be a subgroup of order four, or equivalently, a two-dimensional subspace of $T_{(2)}$ over the finite field $\boldsymbol{F}_2$. We put $N_{P'} = \sum_{a \in P'} N_a \in \Gamma_{Km}$ for a coset P' of $P \subset T_{(2)}$. Noting that a one-dimensional vector space over $\boldsymbol{F}_2$ is identified with its (unique) basis, we denote the Plücker coordinate of $P^\perp \subset T_{(2)}^\vee$ by $\pi_P \in \bigwedge^2 T_{(2)}^\vee \simeq H^2(T, \boldsymbol{Z}/2\boldsymbol{Z})$ and regard it as an element of $\Lambda/2\Lambda$. The following is known ([**1**, Chaper VIII, Section 5]):

Lemma 1.2. *$(N_{P'} \bmod 2) + \pi_P = 0$ holds in $H^2(Km(T), \boldsymbol{Z}/2\boldsymbol{Z})$ for every coset P' of $P \subset T_{(2)}$.*

Let (A, Θ) be a *principally polarized* abelian surface, that is, Θ is an ample divisor with $(\Theta^2) = 2$. The orthogonal complement of $[\Theta]$ in $H^2(A, \boldsymbol{Z})$ is equipped with a polarized Hodge structure. We denote it by $H^2(A, \Theta; \boldsymbol{Z})$. As a lattice it is isomorphic to $\langle -2 \rangle \perp U \perp U$.

Proposition 1.3. *A polarized Hodge structure of weight two on the lattice $\langle -2 \rangle \perp U \perp U$ is isomorphic to $H^2(A, \Theta; \boldsymbol{Z})$ for a principally polarized abelian surface (A, Θ). Moreover, such (A, Θ) is unique up to isomorphisms.*

Proof. A Hodge structure of weight 2 on the lattice $U \perp U \perp U$ is isomorphic to $H^2(T, \boldsymbol{Z})$ for a 2-dimensional complex torus T. Moreover, such T is unique up to an isomorphism and taking the dual (Shioda [**14**]). Our proposition is a direct consequence of these results. □

Let $e^{2\Theta} : K(2\Theta) \times K(2\Theta) \to \boldsymbol{C}^*$ be the Weil pairing with respect to 2Θ ([**2**, Chaper 6]). The group $K(2\Theta)$ coincides with the two-torsion group $A_{(2)}$ and is naturally identified with $H_1(A, \boldsymbol{Z}/2\boldsymbol{Z})$. Via this identification, $e^{2\Theta}(\alpha, \beta) = 1$ is equivalent to $([\Theta], \alpha \wedge \beta) = 0$, where $(\ ,\ )$ is the natural pairing between cohomology and homology.

DEFINITION 1.4. A subgroup G of the two-torsion group $A_{(2)}$ is *Göpel* if it is of order four and totally isotropic with respect to the Weil pairing $e^{2\Theta}$.

Let $P \subset A_{(2)} \simeq H_1(A, \boldsymbol{Z}/2\boldsymbol{Z})$ be a subgroup of order four and $\pi_P \in H^2(A, \boldsymbol{Z}/2\boldsymbol{Z})$ be the Plücker coordinate of $P^\perp \subset H^1(A, \boldsymbol{Z}/2\boldsymbol{Z})$. π_P belongs to $H^2(A, \Theta; \boldsymbol{Z}/2\boldsymbol{Z})$ if and only if it is perpendicular to $\Theta \bmod 2$. Hence we have

LEMMA 1.5. *The Plücker coordinate π_P belongs to $H^2(A, \Theta; \boldsymbol{Z}/2\boldsymbol{Z})$ if and only if P is Göpel.*

The Jacobian $J(C)$ of a curve C of genus two is a principally polarized abelian surface. An involution γ of C is called *bi-elliptic* if the quotient C/γ is an elliptic curve E. In this case, E is embedded into $J(C)$ as the fixed locus of the action of γ on $J(C)$. The two-torsion subgroup $E_{(2)}$ is a Göpel subgroup of $J(C)$, and denoted by G_γ.

DEFINITION 1.6. A Göpel subgroup G, or more precisely, a pair (C, G) is *bi-elliptic* if C has a bi-elliptic involution γ with $G = G_\gamma$.

The composite γ' of a bi-elliptic involution γ and the hyper-elliptic involution is again a bi-elliptic involution of C. The Jacobian $J(C)$ contains $E' := C/\gamma'$ as the fixed locus of the action of γ'. The intersection $E \cap E'$ in $J(C)$ coincides with the common two-torsion subgroups $E_{(2)} = E'_{(2)}$. Hence $J(C)$ is the quotient of $E \times E'$ by a subgroup of order four contained in $E_{(2)} \times E'_{(2)}$. The involution γ, or equivalently γ', induces an involution of the Kummer surface $\operatorname{Km} C$ without fixed points outside two $\boldsymbol{P}^1$'s:

LEMMA 1.7. *Let γ be a bi-elliptic involution of C and $\operatorname{Km}\gamma$ (resp. $J(\gamma)$) be the involution of $\operatorname{Km} C$ (resp. $J(C)$) induced by γ. Then the fixed locus of $\operatorname{Km}\gamma$ is the union of two $\boldsymbol{P}^1$'s which are the images of two elliptic curves $E = \operatorname{Fix} J(\gamma)$ and $E' = \operatorname{Fix} J(\gamma')$.*

2. Involutions of Enriques surfaces.

Let S be a (minimal) *Enriques surface*, that is, a compact complex surface with $H^1(\mathscr{O}_S) = H^2(\mathscr{O}_S) = 0$ and $2K_S \sim 0$. Let $\tilde{S}$ be the universal cover, which is a K3 surface, and let ε be the covering involution of $\tilde{S}$. Consider the action ε^* on $H^2(\tilde{S}, \boldsymbol{Z}) \simeq \boldsymbol{Z}^{22}$. The invariant part coincides with the pull-back of $H^2(S, \boldsymbol{Z})$ by $\tilde{S} \to S$, and the anti-invariant part $H^-(S, \boldsymbol{Z})$ is isomorphic to $E_8(2) \perp U(2) \perp U$ as a lattice ([**1**, Chapter VIII]).

Let σ be a (holomorphic) involution of S. σ is lifted to an automorphism $\tilde{\sigma}$ of the covering K3 surface $\tilde{S}$. Its square $\tilde{\sigma}^2$ is either the identiy or the covering involution ε. The latter is impossible since, in this case, $\tilde{\sigma}$ is free from fixed points and its order necessarily divides $\chi(\mathscr{O}_{\tilde{S}}) = 2$. Hence σ is lifted to two involutions $\tilde{\sigma}$ and $\tilde{\sigma}\varepsilon$ of $\tilde{S}$.

An involution of a K3 surface is called *symplectic* (resp. *anti-symplectic*) if it acts trivially (resp. as -1) on the space $H^0(\tilde{S}, \Omega^2)$ of holomorphic 2-forms. Distinguishing the two lifts by their actions on 2-forms, we have

PROPOSITION 2.1. *There exist exactly two lifts $\sigma_K, \sigma_R \in \operatorname{Aut} \tilde{S}$ of $\sigma \in \operatorname{Aut} S$, where σ_K is a symplectic involution and σ_R an anti-symplectic one.*

Let $H(S, \sigma; \boldsymbol{Z})$ (resp. $H_-(S, \sigma; \boldsymbol{Z})$) be the invariant (resp. the anti-invariant) part of the action of σ_K^* on $H^-(S, \boldsymbol{Z})$. $H(S, \sigma; \boldsymbol{Z})$ is endowed with a non-trivial polarized Hodge structure of weight 2, which we regard as the period of (S, σ). The lattice $H^-(S, \boldsymbol{Z})$ contains the orthogonal direct sum $H(S, \sigma; \boldsymbol{Z}) \perp H_-(S, \sigma; \boldsymbol{Z})$ as a sublattice of finite index. More precisely, the quotient group

$$D_\sigma := \frac{H^-(S, \boldsymbol{Z})}{[H_-(S, \sigma; \boldsymbol{Z}) \oplus H(S, \sigma; \boldsymbol{Z})]} \tag{3}$$

is 2-elementary. We call this quotient D_σ *the patching group* of σ.

The global Torelli theorem for K3 surfaces (resp. Enriques surfaces) is generalized to that for pairs of K3 surfaces (resp. Enriques surfaces) and involutions.

THEOREM 2.2. *Let X and X' be two K3 surfaces and let τ and τ' be involutions of X and X', respectively. If there exists an orientation preserving Hodge isometry $\alpha : H^2(X', \boldsymbol{Z}) \to H^2(X, \boldsymbol{Z})$ such that the diagram*

$$\begin{tikzcd} H^2(X', \boldsymbol{Z}) \arrow[r, "\alpha"] \arrow[d, "\tau'^*"'] & H^2(X, \boldsymbol{Z}) \arrow[d, "\tau^*"] \\ H^2(X', \boldsymbol{Z}) \arrow[r, "\alpha"] & H^2(X, \boldsymbol{Z}) \end{tikzcd}$$

commutes, then there exists an isomorphism $\varphi : X \to X'$ such that $\varphi \circ \tau = \tau' \circ \varphi$.

PROOF. If neither τ nor τ' has a fixed point, this is the global Torelli theorem for Enriques surfaces. The proof in [**1**, Chapter VIII, Section 21], especially its key Proposition (21.1), works also in our general case as follows.

Let h' be a τ'-invariant ample divisor class of X' and put $h = \alpha(h')$. By our assumption, h is τ-invariant and belongs to the positive cone of $H^{1,1}(X, \boldsymbol{Z})$. If h is ample, we are done by the global Torelli theorem for K3 surfaces. If not, there exists a (-2) curve $D \simeq \boldsymbol{P}^1$ with $(h.D) \leq 0$. Since $(h.D + \tau(D)) = 2(h.D) \leq 0$,

$D+\tau(D)$ is not nef. Hence we have $(D.\tau(D)) = 1, 0$ or -2. Replace α with $r_{D+\tau(D)} \circ \alpha$ if $(D, \tau(D)) = 1$, with $r_D \circ r_{\tau(D)} \circ \alpha$ if $(D, \tau(D)) = 0$ and with $r_D \circ \alpha$ if $(D, \tau(D)) = -2$, where r_D is the reflection with respect to a (-2) divisor class D. Then we have $(\alpha(h').D) > 0$. Repeating this process, $\alpha(h')$ becomes ample after a finitely many steps. □

THEOREM 2.3. *Let S and S' be two Enriques surfaces and let σ and σ' be involutions of S and S', respectively. If there exists an orientation preserving Hodge isometry $\alpha : H^-(S', \boldsymbol{Z}) \to H^-(S, \boldsymbol{Z})$ such that the diagram*

$$\begin{array}{ccc} H^-(S', \boldsymbol{Z}) & \xrightarrow{\alpha} & H^-(S, \boldsymbol{Z}) \\ \Big\downarrow{\scriptstyle \sigma'^*} & & \Big\downarrow{\scriptstyle \sigma^*} \\ H^-(S', \boldsymbol{Z}) & \xrightarrow{\alpha} & H^-(S, \boldsymbol{Z}) \end{array}$$

commutes, then there exists an isomorphism $\varphi : S \to S'$ such that $\varphi \circ \sigma = \sigma' \circ \varphi$.

PROOF. Let $\tilde{S}$ and $\tilde{S}'$ be the covering K3 surfaces. Each has an action of $G := \boldsymbol{Z}/2\boldsymbol{Z} \times \boldsymbol{Z}/2\boldsymbol{Z}$. It suffices to show a G-equivariant Torelli theorem for K3 surfaces $\tilde{S}$ and $\tilde{S}'$. (The remaining part is the same as the usual global Torelli theorem for Enriques surfaces.) The proof goes as the preceding theorem if the G-orbit of D consists of one or two irreducible components. Assume that the G-orbit of D has four irreducible components and let L be the sublattice spanned by them. If L is negative definite, then L is of type $4A_1$ or $2A_2$. Hence the same argument as the preceding theorem works. Otherwise $(h.D)$ is positive by the Hodge index theorem. □

3. Period of a numerically reflective involution.

In this section and the next we assume that σ is numerically reflective and study the patching group D_σ (defined by the formula (3)) in detail.

Let $H^2(S, \boldsymbol{Z})_f$ be the torsion free part of $H^2(S, \boldsymbol{Z})$. σ acts on $H^2(S, \boldsymbol{Z})_f$ as a reflection with respect to a class $e = e_\sigma$. Since $H^2(S, \boldsymbol{Z})_f$ is an even unimodular lattice with respect to the intersection form, we have $(e^2) = -2$. Let N_R and N_K be the anti-invariant part of the action of σ_R and σ_K, respectively. Both N_R and N_K contains the pull-back $\tilde{e} \in H^2(\tilde{S}, \boldsymbol{Z})$ of e. The orthogonal complement of $\tilde{e}$ in N_R is $H(S, \sigma; \boldsymbol{Z})$, and that in N_K is $H_-(S, \sigma; \boldsymbol{Z})$. Since N_K is isomorphic to $E_8(2)$ ([**9**, Section 5], [**11**, Lemma 2.1]), we have

LEMMA 3.1. *$H_-(S, \sigma; \boldsymbol{Z}) \simeq E_7(2)$.*

In particular, the discriminant group A_- of $H_-(S,\sigma;\boldsymbol{Z})$ is $u(2)^{\perp 3} \perp (4)$, whose underlying group is $(\boldsymbol{Z}/2\boldsymbol{Z})^{\oplus 6} \oplus (\boldsymbol{Z}/4\boldsymbol{Z})$, in the notation of [**12**, Section 1].

There are two lattice-types of numerically reflective involutions:

Proposition 3.2. *The patching group D_σ is of order 2^a and $H(S,\sigma;\boldsymbol{Z})$ is isomorphic to $\langle -4\rangle \perp U(2) \perp U(a)$ for $a = 1$ or 2.*

Proof. The lattice $H_-(S,\sigma;\boldsymbol{Z})$ is not 2-elementary by the above lemma. Since $H^-(S,\boldsymbol{Z})$ is 2-elementary, D_σ is not trivial. Let $a \geq 1$ be the length of the patching group D_σ. Then we have

$$[\operatorname{disc} H^-(S,\boldsymbol{Z})] \cdot 2^{2a} = [\operatorname{disc} H_-(S,\sigma;\boldsymbol{Z})] \cdot [\operatorname{disc} H(S,\sigma;\boldsymbol{Z})].$$

The discriminant group of $H^-(S,\boldsymbol{Z})$ is an abelian groups of type (2^{10}). By the above lemma, the discriminant of $H(S,\sigma;\boldsymbol{Z})$ equals $-2^{(2+2a)}$. More precisely, the discriminant group A_+ of $H(S,\sigma;\boldsymbol{Z})$ is an abelian group of type $(2^{2a},4)$. Since $H(S,\sigma;\boldsymbol{Z})$ is of rank 5, we have $a \leq 2$.

If $a = 2$, then $H(S,\sigma;\boldsymbol{Z})(1/2)$ is an even (integral) lattice with discriminant -2. Hence $H(S,\sigma;\boldsymbol{Z}))(1/2)$ is isomorphic to $\langle -2\rangle \perp U \perp U$ by Kneser's uniqueness theorem for indefinite lattices ([**12**, Section 1]). If $a = 1$, then we have $H(S,\sigma;\boldsymbol{Z}) \simeq \langle -4\rangle \perp U(2) \perp U$ by the uniqueness theorem again. □

The lattice $H^-(S,\boldsymbol{Z})$ is a $\boldsymbol{Z}$-submodule of the direct sum $H_-(S,\sigma;\boldsymbol{Q}) \oplus H(S,\sigma;\boldsymbol{Q})$. Hence the patching group D_σ is a subgroup of the discriminant group $A_- \perp A_+$ of the lattice $H_-(S,\sigma;\boldsymbol{Z}) \perp H(S,\sigma;\boldsymbol{Z})$. The discriminant group A_+ is either $u(2)^{\perp 2} \perp (4)$ or $u(2) \perp (4)$.

Both A_- and A_+ contains exactly one copy of $\boldsymbol{Z}/4\boldsymbol{Z}$ as their direct summand. Let $\zeta_\pm \in A_\pm$ be the unique element which is twice an element $\eta_\pm$ of order four. We call $(\zeta_-,\zeta_+) \in A_- \perp A_+$ *the canonical element*.

Lemma 3.3. *D_σ contains the canonical element (ζ_-,ζ_+).*

Proof. Both $H_-(S,\sigma;\boldsymbol{Q})$ and $H(S,\sigma;\boldsymbol{Q})$ are primitive in $H^-(S,\boldsymbol{Q})$. Hence D_σ does not contain $(0,\zeta_+)$ or $(\zeta_-,0)$. Hence the intersection $D_\sigma \cap (2A_- \oplus 2A_+)$ is either 0 or generated by (ζ_-,ζ_+). We consider the intersection number of an element of D_σ and (η_-,η_+). Since the intersection number of (ζ_-,ζ_+) and (η_-,η_+) is zero (in $\boldsymbol{Q}/\boldsymbol{Z}$), the intersection number with (η_-,η_+) is a linear form on $\bar{D}_\sigma$, the image of D_σ in $A := (A_-)_{(2)}/\{0,\zeta_-\} \oplus (A_+)_{(2)}/\{0,\zeta_+\}$. Since the induced bilinear form on the group A is non-degenerate, there exists an element $(\beta_-,\beta_+) \in A_- \oplus A_+$ whose intersection number with D_σ is the same as (η_-,η_+). It follows that $(\eta_- + \beta_-, \eta_+ + \beta_+)$ is perpendicular to D_σ. Since $D_\sigma^\perp/D_\sigma$ is 2-

elementary, $2 \times (\eta_- + \beta_-, \eta_+ + \beta_+) = (\zeta_-, \zeta_+)$ is contained in D_σ. □

The patching group D_σ is generated by the canonical element (ζ_-, ζ_+) when it is of order two.

LEMMA 3.4. *If D_σ is of order four, then D_σ is generated by the canonical element and an element $(\pi_-, \pi_+) \in A_- \oplus A_+$ of order two such that $q_-(\pi_-) = q_+(\pi_+) = 0 \in \boldsymbol{Q}/2\boldsymbol{Z}$, where $q_\pm$ are the quadratic forms on $A_\pm$.*

PROOF. $D_\sigma \simeq \boldsymbol{Z}/2\boldsymbol{Z} \oplus \boldsymbol{Z}/2\boldsymbol{Z}$ is generated by (ζ_-, ζ_+) and an element (π_-, π_+). Since D_σ is totally isotropic, we have $q_-(\pi_-) = q_+(\pi_+)$. This common value belongs to $\boldsymbol{Z}/2\boldsymbol{Z}$. If it is non-zero, replace $\pi_\pm$ with $\zeta_\pm + \pi_\pm$. Then we have $q_-(\pi_-) = q_+(\pi_+) = 0$. □

4. Numerically reflective involution with ord $D_\sigma = 4$.

Let σ be a numerically reflective involution of an Enriques surface S and assume that the patching group D_σ is of order four. By Propositions 1.3 and 3.2, we have

PROPOSITION 4.1. *There exists a principally polarized abelian surface (A, Θ) such that $H(S, \sigma; \boldsymbol{Z})$ is isomorphic to $H^2(A, \Theta; \boldsymbol{Z})(2)$ as a polarized Hodge structure.*

Let $\pi_+ \in (A_+)_{(2)}$ be as in Lemma 3.4. Since $q_+(\pi_+) = 0$, π_+ is the Plücker coordinate of a subgroup $G_\sigma \subset A_{(2)}$ of order four. Since $(A_+)_{(2)}$ is the orthogonal complement of $[\Theta/2]$ in $H^2(A, ((1/2)\boldsymbol{Z})/\boldsymbol{Z})$, G_σ is Göpel (Definition 1.4 and Lemma 1.5).

LEMMA 4.2. (A, Θ) *in Proposition* 4.1 *is not a product of two elliptic curves. In particular, (A, Θ) is the Jacobian of a curve C_σ of genus two.*

PROOF. Assume that (A, Θ) is the product $E_1 \times E_2$ (as a polarized abelian surface). Then $E_1 \times 0 - 0 \times E_2$, the difference of two fibers, is a (-2)-class in $H^2(A, \Theta; \boldsymbol{Z})$. Let D_+ be its image in $H(S, \sigma; \boldsymbol{Z})$. Then $(D_+^2) = -4$ and $D_+/4$ represents an element $\eta_+ \in A_+$ of order four. Hence $D_+/2$ represents the class ζ_+ in the discriminant group A_+. Let $\tilde{e}$ be the pull-back of $e = e_\sigma \in H^2(S, \boldsymbol{Z})_f$ as in Section 3. Then $\tilde{e} + D_+$ is divisible by two in $H^2(\tilde{S}, \boldsymbol{Z})$ and $(\tilde{e} + D_+)/2$ is an algebraic (-2)-class in N_R. This is a contradiction since N_R is the anti-invariant part of the involution or σ_R. □

LEMMA 4.3. *The pair (C_σ, G_σ) is not bi-elliptic* (*see Definition* 1.6).

Proof. The proof is similar to the preceding lemma. Assume that (C_σ, G_σ) is bi-elliptic. Since $(\Theta.E)=2$, $\Theta-E$ is a (-2)-class in $H^2(A,\Theta;\boldsymbol{Z})$. Let D_+ be its image in $H(S,\sigma;\boldsymbol{Z})$. Then $(D_+^2)=-4$ and $D_+/2$ represents the class $\zeta_+ + \pi_+$ in A_+. $(D_- + D_+)/2$ belongs to $H^-(S,\boldsymbol{Z})$ if D_- belongs to $H_-(S,\sigma;\boldsymbol{Z})$ and $D_-/2$ represents $\zeta_- + \pi_-$. Since $q_-(\zeta_- + \pi_-)=1$ and since $H_-(S,\sigma;\boldsymbol{Z})$ is isomorphic to $E_7(2)$, there is such a D_- with $(D_-^2)=-4$. For this choice, $(D_- + D_+)/2$ is an algebraic (-2)-class. This is a contradiction since $H^-(S,\boldsymbol{Z})$ is the anti-invariant part of the involution ε. □

Summarizing this section, we have

Proposition 4.4. *There exists a unique non-bi-elliptic pair (C_σ, G_σ) of a curve C_σ and a Göpel subgroup G_σ of $J(C_\sigma)$ with the following properties*:

(1) $H(S,\sigma;\boldsymbol{Z}) \simeq H^2(J(C_\sigma),\Theta;\boldsymbol{Z})(2)$ *as a polarized Hodge structure, and*
(2) *the patching subgroup D_σ is generated by the canonical element and an element (π_-, π_+) such that π_+ is the Plücker coordinate of G_σ.*

In the subsequent sections, we conversely construct a numerically reflective involutions σ_G of an Enriques surface from such a pair (C,G) as above (Proposition 6.4).

5. Hutchinson-Göpel involution.

Hutchinson [**7**] discovered an equation which implies (1) by means of theta functions. In this section we describe the automorphism ε_G in a more elementary manner without using the equation (*cf.* Remark 6.3).

Let C be a curve of genus two and $J(C)$ its Jacobian. By the natural morphism $Sym^2 C \to J(C)$ and Abel's theorem, the second symmetric product $Sym^2 C$ of C is the blow-up of $J(C)$ at the origin. Let $\overline{Sym^2}\, C$ be the quotient of $Sym^2 C$ by the involution induced by the hyper-elliptic involution.

Since C is a double cover of the projective line $\boldsymbol{P}^1$ with six branch points, $\overline{Sym^2}\, C$ is the double cover of $Sym^2 \boldsymbol{P}^1 \simeq \boldsymbol{P}^2$ with branch six lines $l_1,\dots,l_6$. Moreover, these six lines are tangent lines of the conic Q corresponding to the diagonal $\boldsymbol{P}^1 \hookrightarrow Sym^2 \boldsymbol{P}^1$. Note that the double cover has 15 nodes over 15 intersections $p_{i,j}=l_i\cap l_j$, $1\le i<j\le 6$. These correspond to the 15 non-zero two-torsions of $J(C)$. The minimal resolution of this double cover $\overline{Sym^2}\, C$ is the Jacobian Kummer surface $\mathrm{Km}\, C$.

Three nodes on $\overline{Sym^2}\, C$ are called *Göpel* if they correspond to the three non-zero elements of a Göpel subgroup of $J(C)_{(2)}$. More explicitly, a triple $(p_{ij}, p_{i'j'}, p_{i''j''})$ of nodes is Göpel if and only if all suffixes $i,j,\dots,j''$ are distinct. Hence the Göpel subgroups correspond to the decompositions of the six

Weierstrass points of C into three pairs. Therefore, there are exactly 15 Göpel subgroups.

We now construct an involution of $\mathrm{Km}\, C$ for each Göpel subgroup G. The construction differs a lot according as the Göpel triple is collinear or not. First we consider the non-collinear case, which we are most interested in.

Assume that three points p, q, r on $\boldsymbol{P}^2$ are not collinear. A birational automorphism $\varphi : \boldsymbol{P}^2 \cdots \to \boldsymbol{P}^2$ is called *a Cremona involution with center* p, q, r if there is a linear coordinate $(x : y : z)$ of $\boldsymbol{P}^2$ such that p, q, r is the three vertices of the triangle $xyz = 0$ and that φ is the quadratic Cremona transformation $(x : y : z) \mapsto (x^{-1} : y^{-1} : z^{-1})$. Given a triple $p, q, r \in \boldsymbol{P}^2$, there is a two-parameter family of Cremona involutions with center p, q, r.

Proposition 5.1. *Assume that a Göpel triple* (p_{14}, p_{25}, p_{36}) *of* G *is not collinear. Then there exists a unique quadratic Cremona transformation* φ *with center* p_{14}, p_{25} *and* p_{36} *which maps the line* l_i *onto* l_{i+3} *for* $i = 1, 2, 3$.

Proof. We choose a linear coordinate $(x : y : z)$ of $\boldsymbol{P}^2$ such that p_{14}, p_{25} and p_{36} are the vertices of the triangle $xyz = 0$. Then the six lines are given by

$$l_i : y = \alpha_i x \ (i = 1, 4), \quad l_j : z = \alpha_j y \ (j = 2, 5) \text{ and } l_k : x = \alpha_k z \ (k = 3, 6)$$

for $\alpha_1, \ldots, \alpha_6 \in \boldsymbol{C}^*$. Let

$$\check{Q} : a'x^2 + b'y^2 + c'z^2 + d'yz + e'xz + f'xy = 0$$

be the dual of the conic Q to which the six lines are tangent. Then we have

$$\alpha_1\alpha_4 = \frac{a'}{b'}, \quad \alpha_2\alpha_5 = \frac{b'}{c'}, \quad \alpha_3\alpha_6 = \frac{c'}{a'}$$

and hence $\prod_{i=1}^{6} \alpha_i = 1$. The Cemona involution $(x : y : z) \mapsto (A/x : B/y : 1/z)$ satisfies our requirement if and only if $A = \alpha_3\alpha_6$ and $B = \alpha_2^{-1}\alpha_5^{-1}$. □

The Cremona involution φ in the proposition is lifted to two involutions of $\mathrm{Km}\, C$. One is symplectic and has eight fixed points over the four fixed points of φ. The other has no fixed points (*cf.* (1) of Remark 5.3). We call the latter *the Hutchinson involution associated with the Göpel subgroup* G and denote by ε_G. Since the covering involution β commutes with ε_G, it induces an involution of the Enriques surface $(\mathrm{Km}\, C)/\varepsilon_G$, which we denote by σ_G.

Now we assume that a Göpel triple, say (p_{14}, p_{25}, p_{36}), lies on a line l. Let p be the point whose polar with respect to the conic Q is l and $\tilde{\gamma}$ be the involution of $\boldsymbol{P}^2$ whose fixed locus is the union of l and p. Then $\tilde{\gamma}$ maps Q onto itself and interchanges p_i and p_{i+3} for $i = 1, 2$ and 3. $\tilde{\gamma}$ induces involutions of $\mathrm{Km}\, C$ and C. The following is easily verified:

PROPOSITION 5.2. *A Göpel triple of nodes is collinear if and only if* (C, G) *is bi-elliptic. Furthermore, the involution of* $\mathrm{Km}\, C$ *constructed above is the same as* $\mathrm{Km}\, \gamma$ *in Lemma* 1.7.

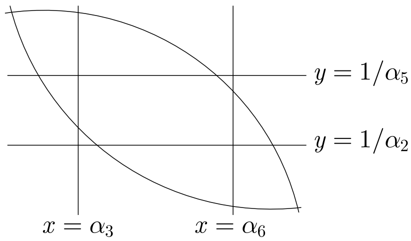
Hence we have constructed an Enriques surface $(\mathrm{Km}\, C)/\varepsilon_G$ with an involution σ_G for every non-bi-elliptic pair (C, G).

REMARK 5.3. Let $\alpha_1, \dots, \alpha_6$ be as in the proof of Proposition 5.1.

(1) The Kummer surface $\mathrm{Km}\, C$ is the minimal resolution of the double cover

$$\bar{S} : \tau^2 = (y - \alpha_1 x)(y - \alpha_4 x)(\alpha_2 y - 1)(\alpha_5 y - 1)(x - \alpha_3)(x - \alpha_6)$$

of $\boldsymbol{P}^1 \times \boldsymbol{P}^1$, where (x, y) is an inhomogeneous coordinate of $\boldsymbol{P}^1 \times \boldsymbol{P}^1$.



The involution

$$\bar{\varepsilon}_G : (\tau, x, y) \mapsto \left(-\frac{AB\tau}{x^2 y^2}, \frac{A}{x}, \frac{B}{y} \right), \quad A = \alpha_3 \alpha_6,\ B = \alpha_2^{-1} \alpha_5^{-1}$$

of $\bar{S}$ has no fixed points. The K3 surface $\bar{S}$ has fourteen nodes and $(\mathrm{Km}\, C)/\varepsilon_G$ is the minimal resolution of the Enriques surface $\bar{S}/\bar{\varepsilon}_G$ with seven nodes.

(2) The Enriques surface $(\mathrm{Km}\, C)/\varepsilon_G$ is the minimal model of the double plane with branch the plane curve

$$x_1 x_2 x_3 x_4 (x_1 x_4 - c_1 x_2 x_3)(x_2 x_4 - c_2 x_1 x_3)(x_3 x_4 - c_3 x_1 x_2) = 0$$

of degree 10 and σ_G is induced by the covering involution, where $(x_1 : x_2 : x_3)$ is a coordinate of $\boldsymbol{P}^2$, $x_4 = -x_1 - x_2 - x_3$ and we put $c_i = (\sqrt{\alpha_i} - \sqrt{\alpha_{i+3}})^2 / (\sqrt{\alpha_i} + \sqrt{\alpha_{i+3}})^2$ for $i = 1, 2, 3$. In the case $\alpha_1 + \alpha_4 = \alpha_2 + \alpha_5 = \alpha_3 + \alpha_6 = 0$, $\mathrm{Km}\, C$ is the minimal model of the Hilbert modular surface $H^2/\Gamma(2)$ associated with the principal congruence subgroup $\Gamma(2)$ of $SL_2(\mathscr{O}_{\boldsymbol{Q}(\sqrt{2})})$ for the ideal (2), and ε_G is induced by the matrix $\left(\begin{smallmatrix} 1+\sqrt{2} & 0 \\ 0 & -1+\sqrt{2} \end{smallmatrix}\right)$ (Hirzebruch [**5**, Section 4], [**4**, Chapter 8]).

6. Period of $(\mathrm{Km}\, C)/\varepsilon_G$.

Returning to the case where (C, G) is not bi-elliptic, we compute the periods of the Enriques surface $(\mathrm{Km}\, C)/\varepsilon_G$ and the involution σ_G. The Jacobian Kummer surface $\mathrm{Km}\, C$ is a double cover of the blow-up R of $\boldsymbol{P}^2$ at the 15 points p_{ij}, $1 \le i < j \le 6$. The pull-back of $H^2(R, \boldsymbol{Q})$ has $\{h, N_{ij},\ 1 \le i < j \le 6\}$ as a $\boldsymbol{Q}$-basis, where h is the pull-back of a line and N_{ij} is the (-2) curves over p_{ij}.

We assume for simplicity that the Göpel triple is (p_{14}, p_{25}, p_{36}). Let $\bar{R}$ be the blow-up of $\boldsymbol{P}^2$ at p_{14}, p_{25} and p_{36}. The Cremona involution φ in Proposition 5.1 acts on the Picard group of $\bar{R}$ as the reflection with respect to the (-2)-class $l - E_{14} - E_{25} - E_{36}$, where E_{14}, E_{25} and E_{36} are the exceptional curves. φ interchanges $p_{i,j}$ with $p_{i+3,j+3}$, and $p_{i,j+3}$ with $p_{j,i+3}$ for $1 \le i < j \le 3$. Hence we have

Proposition 6.1. *The action of ε_G on the pull-back of $H^2(R, \boldsymbol{Q})$ is the composite of the permutation*

$$N_{i,j} \leftrightarrow N_{i+3,j+3}, \quad N_{i,j+3} \leftrightarrow N_{j,i+3} \quad (1 \le i < j \le 3)$$

of type $(2)^6$ and the reflection with respect to the (-4)-class $h - N_{14} - N_{25} - N_{36}$.

By the proposition,

$$\{h - N_{14} - N_{25} - N_{36},\ N_{ij} - N_{i+3,j+3},\ N_{i,j+3} - N_{j,i+3}\}, \tag{4}$$

with $1 \le i < j \le 3$, is a $\boldsymbol{Q}$-basis of $H_-(\mathrm{Km}\, C/\varepsilon_G, \sigma_G; \boldsymbol{Q})$. N_0, the $(-2)\boldsymbol{P}^1$ over the origin, maps onto the conic Q.

Proposition 6.2. *$h - N_0$ is invariant by ε_G and anti-invariant by β.*

Proof. There exists a cubic curve $D : r(x, y, z) = 0$ such that $D \cap C$ consists of the 6 tangent points $l_i \cap Q$, $1 \le i \le 6$. The union of 6 lines is defined by $r(x, y, z)^2 - q(x, y, z)s(x, y, z)$ for a suitable quartic form $s(x, y, z)$. Choose a cubic curve D such that it passes the Göpel triple. Then the quartic curve $s(x, y, z) = 0$ is singular at the Göpel triple. By the Cremona symmetry, $s(x, y, z)$ is a constant multiple of $q(yz, xz, xy)$. Hence the double cover $\overline{Sym^2}\, C$ is defined by

$$\tau^2 = r(x, y, z)^2 - cq(x, y, z)q(yz, xz, xy) \tag{5}$$

for a constant $c \in \boldsymbol{C}^*$. The rational function $\{r(x,y,z)+\tau\}/\{r(x,y,z)-\tau\}$ on $\mathrm{Km}\, C$ gives a rational equivalence between two divisors $N_0 + \beta\varepsilon_G(N_0)$ and $\varepsilon_G(N_0) + \beta(N_0)$. Hence $\beta(N_0) - N_0$ is ε_G-invariant. Since $\beta(N_0) + N_0$ is linearly equivalent to $2h$, we have our proposition. □

REMARK 6.3. By (5) the linear system $|h + N_0|$ gives a birational morphism from the double cover $\overline{Sym^2\, C}$ to the quartic $cq(x,y,z)t^2 + 2r(x,y,z)t + q(yz,xz,xy) = 0$ in $\boldsymbol{P}^3$, which is essentially the equation (1).

By Propositions 6.1 and 6.2,

$$\big\{h - N_0,\ h - N_{14},\ h - N_{25},\ h - N_{36},\ N_{ij} + N_{i+3,j+3},\ N_{i,j+3} + N_{j,i+3}\big\}, \quad (6)$$

with $1 \le i < j \le 3$, is an orthogonal $\boldsymbol{Q}$-basis of $\pi^* H^2(\mathrm{Km}\, C/\varepsilon_G, \boldsymbol{Q})$. In particular, σ_G acts on $\pi^* H^2(\mathrm{Km}\, C/\varepsilon_G, \boldsymbol{Q})$ as the reflection with respect to $h - N_0$. Hence we have

PROPOSITION 6.4. *The involution σ_G of the Enriques surface* $(\mathrm{Km}\, C)/\varepsilon_G$ *is numerically reflective.*

Moreover, the inverse of the correspondence $(S,\sigma) \mapsto (C_\sigma, G_\sigma)$ of Proposition 4.4 is given by this construction $(C,G) \mapsto (\mathrm{Km}\, C/\varepsilon_G, \sigma_G)$:

PROPOSITION 6.5.

(1) *The polarized Hodge structure* $H(\mathrm{Km}\, C/\varepsilon_G, \sigma_G; \boldsymbol{Z})$ *is isomorphic to* $H^2(J(C), \Theta; \boldsymbol{Z})(2)$.
(2) *The patching group of σ_G is of order four, and generated by the canonical element and (π_-, π_G), where π_G is the Plücker coordinate of G.*

PROOF. By (4) and (6), $H(\mathrm{Km}\, C/\varepsilon_G, \sigma_G; \boldsymbol{Z})$ is the orthogonal complement of the lattice generated by the 17 classes h, N_0 and N_{ij}, $1 \le i < j \le 6$, in $H^2(\mathrm{Km}\, C, \boldsymbol{Z})$. Let $H \in H^2(\mathrm{Km}\, C, \boldsymbol{Z})$ be the (4)-class in Λ corresponding to $\Theta \in H^2(J(C), \boldsymbol{Z})$ in the way of Lemma 1.1. It is easily checked that $H = h + N_0$. Hence we have (1).

The patching group is order four by (1) and Proposition 3.2 since $H^2(J(C), \Theta; \boldsymbol{Z})(2) \simeq \langle -4 \rangle \perp U(2) \perp U(2)$. By Proposition 6.1, both $N_{12} - N_{45}$ and $N_{15} - N_{24}$ belong to $H_-(\mathrm{Km}\, C/\varepsilon_G, \sigma_G; \boldsymbol{Z})$. Since the two-torsion points p_{12}, p_{45}, p_{15} and p_{24} form a coset of $G \subset J(C)_{(2)}$, $([(N_{12} - N_{45} + N_{15} - N_{24})/2], \pi_G)$ belongs to the patching group of σ_G by Lemma 1.2. □

PROOF OF THEOREM 2. Let σ be a numerically reflective involution of an Enriques surface S and assume that the patching group D_σ is of order four. Let

(C_σ, G_σ) be as in Proposition 4.4 and σ' be the numerically reflective involution σ_G of the Enriques surface $S' := \mathrm{Km}\, C/\varepsilon_G$ for $C = C_\sigma$ and $G = G_\sigma$. By Proposition 6.5, $H(S, \sigma; \boldsymbol{Z})$ is isomorphic to $H(S', \sigma'; \boldsymbol{Z})$ as a polarized Hodge structure. Moreover, the A_+-components of their patching groups are the same. Both are generated by ζ_+ and the Plücker coordinate π_G of G.

Now we look at the A_--components. Two lattices $H_-(S, \sigma; \boldsymbol{Z})$ and $H_-(S', \sigma'; \boldsymbol{Z})$ are $E_7(2)$ by Lemma 3.1. The A_--components of patching groups are generated by ζ_+ and π_- with $q_-(\pi_-) = 0$. The Weyl group W of E_7 acts on $A_- \simeq u(2)^{\perp 3} \perp (4)$ preserving ζ_-. There are 63 α's with $q_-(\alpha) = 0$ in $(A_-)_{(2)}$ and W acts transitively on them. Hence a Hodge isometry between $H(S, \sigma; \boldsymbol{Z})$ and $H(S', \sigma'; \boldsymbol{Z})$ extends to a $\boldsymbol{Z}/2\boldsymbol{Z}$-equivariant Hodge isometry between $H^-(S, \boldsymbol{Z})$ and $H^-(S', \boldsymbol{Z})$. Now the theorem follows from Theorem 2.3. $\square$

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