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Relativistic Hamiltonians with dilation analytic potentials diverging at infinity

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Abstract. We investigate the spectral properties of the Dirac operator with a potential $V(x)$ and two relativistic Schrödinger operators with $V(x)$ and $-V(x)$, respectively. The potential $V(x)$ is assumed to be dilation analytic and diverge at infinity. Our approach is based on an abstract theorem related to dilation analytic methods, and our results on the Dirac operator are obtained by analyzing dilated relativistic Schrödinger operators. Moreover, we explain some relationships of spectra and resonances between Schrödinger operators and the Dirac operator as the nonrelativistic limit.

1. Introduction.

We first consider the Dirac operator

$$H(c) = c\alpha \cdot D + mc^2\beta + V(x) \tag{1}$$

in the Hilbert space $L^2(\boldsymbol{R}^3)^4$, where $c > 0$ is the speed of light, $m > 0$ the rest mass of a relativistic particle moving in an electric potential $V \in C(\boldsymbol{R}^3 \to \boldsymbol{R})$ and $\alpha \cdot D = \sum_{j=1}^{3} \alpha_j D_j$, where $D = -i\nabla_x = (D_1, D_2, D_3)$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. Here each α_j and β are 4×4 Hermitian matrices defined by

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},$$

where

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$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are Pauli matrices, and I_n is the $n \times n$ unit matrix.

It is believed that a Dirac operator converges to the corresponding Schrödinger operator acting in $L^2(\boldsymbol{R}^3)$

$$S = -\frac{1}{2m}\Delta + V(x) \tag{2}$$

in some sense if the speed of light, c, goes to infinity (the nonrelativistic limit) and this expectation has been verified by many authors [**3**], [**4**], [**19**], [**20**], [**22**], [**23**], if the potential $V(x)$ *decays* uniformly at infinity. Indeed, in this case the resolvent $(H(c) - mc^2 - z)^{-1}$, $\operatorname{Im} z \neq 0$, converges to

$$\begin{pmatrix} (S-z)^{-1}I_2 & 0 \\ 0 & 0 \end{pmatrix} \tag{3}$$

as $c \to \infty$ in the operator norm (see, e.g., [**19**]), and the spectrum of the Dirac operator is similar to that of the Schrödinger operator, that is, $\sigma_{\mathrm{ess}}(H(c)) = (-\infty, -mc^2] \cup [mc^2, \infty)$, $\sigma_{\mathrm{d}}(H(c)) \subset (-mc^2, mc^2)$, and $\sigma_{\mathrm{ess}}(S) = [0, \infty)$, $\sigma_{\mathrm{d}}(S) \subset (-\infty, 0)$. In this paper we denote the spectrum of H by $\sigma(H)$, the discrete spectrum $\sigma_{\mathrm{d}}(H)$, the essential spectrum $\sigma_{\mathrm{ess}}(H)$, the point spectrum $\sigma_{\mathrm{p}}(H)$, the continuous spectrum $\sigma_{\mathrm{c}}(H)$, the absolutely continuous spectrum $\sigma_{\mathrm{ac}}(H)$, the singular continuous spectrum $\sigma_{\mathrm{sc}}(H)$ and the resolvent set $\rho(H)$ (cf. [**12**], [**14**]).

On the other hand, if the potential *diverges at infinity*:

$$V(x) \to +\infty \quad \text{as } |x| \to \infty, \tag{4}$$

their spectra are quite different. Indeed, the Schrödinger operator S has a purely discrete spectrum, whereas the Dirac operator $H(c)$ has a purely absolutely continuous spectrum covering the whole real line $(-\infty, +\infty)$ for a wide class of radial potentials [**17**], [**18**], [**19**]. Therefore, in this case, we cannot expect the norm resolvent convergence of $H(c) - mc^2$ to S as in the case of decaying potentials since their spectra are quite different. However, we can consider S as the nonrelativistic limit of $H(c)$ even in this case. In fact, there are two typical approaches to relate them: "spectral concentration" [**11**], [**21**] and "resonances" [**1**], [**20**].

In this paper we study this problem from the standpoint of resonances. Moreover, we consider the nonrelativistic limit of the spectral projection of the Dirac operator, which plays a crucial role in the study of "spectral concentration".

We assume the the following in this paper.

Assumptions.

(V1): $V(x)$ is a real-valued continuous function on $\boldsymbol{R}^3$ and there are constants $M > 0$, $K > 0$, a small constant $a_0 > 0$ and a $C(S^2)$-valued analytic function $V(z,\cdot)$ of z defined on S_{a_0},

$$S_{a_0} := \{re^{i\tau} \in \boldsymbol{C}; r \in (0,\infty),\ -a_0 < \tau < a_0\},$$

such that

$$\sup_{\omega \in S^2} |V(z,\omega)| \le K(1+|z|)^M \tag{5}$$

for all $z \in S_{a_0}$ and $V(r,\omega) = V(r\omega)$ if $r > 0$ and $\omega \in S^2$.

Define a function $V_\theta(x)$ for each $\theta \in \boldsymbol{C}$ with $|\operatorname{Im}\theta| < a_0$ by

$$V_\theta(x) := V\big(e^\theta |x|, \hat{x}\big), \quad \hat{x} = \frac{x}{|x|},\ x \neq 0.$$

(V2): There is a constant $R_0 > 0$ such that for each $\tau \in (-a_0, a_0)$ the function $V_{i\tau}(x)$ is C^∞ for $|x| > R_0$ and satisfies the estimate

$$\big|\partial_x^\alpha V_{i\tau}(x)\big| \le K_\alpha |x|^{M-|\alpha|} \tag{6}$$

for $|\alpha| \ge 0$ and for $|x| > R_0$ uniformly in $|\tau| < a_0$.

(V3): There exists a constant $K_0 > 0$ such that

$$V(x) \ge K_0|x|^M, \quad x \cdot \nabla V(x) \ge K_0 |x|^M \tag{7}$$

for $|x| \ge R_0$.

Here we note that it follows from the estimate (5) and the Cauchy integral formula that the following estimates hold

$$\big\|\partial_z^\alpha V(z,\cdot)\big\|_{L^\infty(S^2)} \le K_\alpha (1+|z|)^{M-|\alpha|}, \quad z \in S_{a_0} \tag{8}$$

for all α.

The *resonances* of $H(c)$ are defined as the eigenvalues of the dilated Dirac operator

$$H(c,\theta) := c\alpha \cdot e^{-\theta} D + mc^2\beta + V_\theta(x) \tag{9}$$

for $\theta \in \boldsymbol{C}$ with $0 < \operatorname{Im}\theta < a_0$ (see [**1**]). Amour, Brummelhuis and Nourrigat [**1**] show that under a similar assumption the family of $H(c,\theta)$ is an analytic family of type (A) [**12**], [**16**] with compact resolvent, and so $H(c,\theta)$ has a purely discrete spectrum. The standard argument of the complex scaling method shows that the resonances are independent of θ with $0 < \operatorname{Im}\theta < a_0$ [**2**], [**6**], [**16**]. In [**1**] they prove that there are resonances of $H(c) - mc^2$ near each eigenvalue of the Schrödinger operator S if c is large enough and the resonances converge to the eigenvalue as $c \to \infty$.

Our main purpose is to clarify these mechanisms by introducing two relativistic Schrödinger operators,

$$L_\pm(c) := \pm\sqrt{-c^2\Delta + m^2c^4} - mc^2 + V(x) \quad \text{in } L^2(\boldsymbol{R}^3), \tag{10}$$

as intermediates between the Dirac operator $H(c)$ and the Schrödinger operator S, though relativistic Schrödinger operators are not considered in [**1**]. Here we note that each operator is a self-adjoint operator with a core $\mathscr{S}$, the Schwartz class on $\boldsymbol{R}^3$, under our assumptions (see [**8**], [**9**]).

Before giving a description of our results, we should remark that we denote by $\|\cdot\|$ (resp. $(\cdot,\cdot)$) the norm (resp. the scalar product) of the Hilbert space $L^2(\boldsymbol{R}^3)^4$ and also use these notations for other Hilbert spaces if they do not cause a confusion. The notation $\|\cdot\|$ is also used for operator norms.

Applying the FWT transformation $U_c(D)$ to $H(c) - mc^2$ (see Section 2 for the definition of $U_c(D)$), we have

$$L(c) := U_c(D)(H(c) - mc^2)U_c(D)^{-1} = L_1(c) + W(c), \tag{11}$$

where

$$L_1(c) := \begin{pmatrix} L_+(c)I_2 & 0 \\ 0 & L_-(c)I_2 \end{pmatrix}, \tag{12}$$

and

$$W(c) = W(c,x,D) := U_c(D)V(x)U_c(D)^{-1} - V(x). \tag{13}$$

If $V(x)$ is *smooth* (i.e. C^∞) on $\boldsymbol{R}^3$, a simple calculus of pseudodifferential operators leads to the following estimate

$$\left\| W(c)\langle x\rangle^{-M+1} \right\| \le Kc^{-1}, \quad c \ge 1$$

for some $K > 0$, where $\langle x\rangle := (1+|x|^2)^{1/2}$. This estimate seems to allow $W(c)$ as a perturbation of $L_1(c)$ for at least large $c > 0$, but the idea cannot be used directly since it is difficult to control $W(c)$ by $L_-(c)$ which has no global ellipticity. Hence, to avoid this problem we introduce the idea of complex scaling as in [**1**]. Let us fix a small constant a with $0 < a < a_0$ and define

$$\Omega := \{\theta \in \boldsymbol{C}; |\operatorname{Im}\theta| < a\},$$
$$\Omega_+ := \{\theta \in \boldsymbol{C}; 0 < \operatorname{Im}\theta < a\}.$$

Under our assumptions we can define the following operators

$$L_\pm(c,\theta) := \pm\sqrt{-c^2 e^{-2\theta}\Delta + m^2c^4} - mc^2 + V_\theta(x), \tag{14}$$

$$L(c,\theta) := \begin{pmatrix} L_+(c,\theta)I_2 & 0 \\ 0 & L_-(c,\theta)I_2 \end{pmatrix} + W(c,\theta), \tag{15}$$

where $W(c,\theta) := W(c, e^\theta x, e^{-\theta}D)$ for $\theta \in \Omega$. Let $\{\mathscr{U}(t)\}_{t\in\boldsymbol{R}}$ be the dilation group on $\boldsymbol{R}^3$,

$$\mathscr{U}(t)f(x) = e^{(3t/2)}f(e^t x).$$

Now we state our results. The details are discussed in the following sections.

We see that $\{L_+(c,\theta)\}$ is an analytic family of type (A) in Ω, with compact resolvent, and satisfies

$$\mathscr{U}(t)L_+(c,\theta)\mathscr{U}(t)^{-1} = L_+(c,\theta+t), \quad t \in \boldsymbol{R},\ \theta \in \Omega. \tag{16}$$

Thus $L_+(c,\theta)$ has a purely discrete spectrum and the standard argument on the dilation analyticity gives the following theorem since $L_+(c) = L_+(c,0)$.

Theorem 1.1.

(a) *The discrete spectrum $\sigma_{\mathrm{d}}(L_+(c,\theta))$ is independent of $\theta \in \Omega$, denoted by $\Sigma_+(c)$, and coincides with $\sigma_{\mathrm{p}}(L_+(c))$.*
(b) *$L_+(c)$ has a purely discrete spectrum.*

However, $\{L_-(c,\theta)\}$ is an analytic family of type (A) *only* in Ω_+ (not in Ω). Each $L_-(c,\theta)$ also has compact resolvent for each $\theta \in \Omega_+$ and satisfies

$$\mathscr{U}(t)L_-(c,\theta)\mathscr{U}(t)^{-1} = L_-(c,\theta+t), \quad t \in \boldsymbol{R},\ \theta \in \Omega_+. \tag{17}$$

Thus, the spectrum of $L_-(c,\theta)$ (also consisting of a discrete spectrum only) is independent of $\theta \in \Omega_+$. Moreover, we can prove that the resolvent $(L_-(c)-z)^{-1}$, $\operatorname{Im} z < 0$, is the strong limit of $(L_-(c,\theta)-z)^{-1}$ as $\Omega_+ \ni \theta \to 0$ (Proposition 2.1). But, the spectra of $L_-(c)$ and $L_-(c,\theta)$ are quite different. Indeed, we have

Theorem 1.2.

(a) *The discrete spectrum* $\sigma_{\mathrm{d}}(L_-(c,\theta))$ *is independent of* $\theta \in \Omega_+$*, denoted by* $\Sigma_-(c)$*, and satisfies*

$$\Sigma_-(c) \subset \overline{\boldsymbol{C}_+}, \quad \Sigma_-(c) \cap \boldsymbol{R} = \sigma_{\mathrm{p}}(L_-(c)),$$

where $\overline{\boldsymbol{C}}_+ := \{z \in \boldsymbol{C}; \operatorname{Im} z \geq 0\}$ *is the closed upper half plane.*

(b) $L_-(c)$ *has at most finitely many eigenvalues, and the multiplicity of each of them is finite.*

(c) $\sigma(L_-(c)) = \boldsymbol{R}$ *and* $\sigma_{\mathrm{sc}}(L_-(c)) = \phi$*. In particular,* $\sigma(L_-(c)) \setminus \sigma_{\mathrm{p}}(L_-(c)) \subset \sigma_{\mathrm{ac}}(L_-(c))$*.*

Remarks.

(i) There are only a few studies about the relativistic Schrödinger operator $\sqrt{-\Delta+1} + v(x)$ with a potential $v(x)$ such that $v(x) \to -\infty$ as $|x| \to \infty$: In [**9**], [**8**] the essential self-adjointness of the operator is investigated.

(ii) Each element of $\Sigma_+(c)$ (resp. $\Sigma_-(c)$) is called a *resonance* of $L_+(c)$ (resp. $L_-(c)$).

The spectra of $L_+(c)$ and $L_-(c)$ are quite different. But, Theorem 3.4 shows that $L_+(c)$ and $L_-(c)$ can be treated in the same framework. They are regarded as boundary values of analytic families $\{L_+(c,\theta)\}$ and $\{L_-(c,\theta)\}$, in Ω_+, respectively (see Section 3 for the definition of a boundary value of an analytic family). We shall prove that self-adjoint operators T defined as a boundary value of a certain analytic family are classified into two categories: type (I) $\sigma(T) = \sigma_{\mathrm{d}}(T)$; type (II) $\sigma(T) = (-\infty,+\infty)$, $\sigma_{\mathrm{sc}}(T) = \emptyset$ (Theorem 3.4). In Section 4, we shall show that $L_+(c)$ is of type (I), and $L_-(c)$ is of type (II).

We also see that $\{L(c,\theta)\}$ is an analytic family of type (A) in Ω_+, with compact resolvent, and satisfies

$$\mathscr{U}_4(t)L(c,\theta)\mathscr{U}_4(t)^{-1} = L(c,\theta+t), \quad t \in \boldsymbol{R},\ \theta \in \Omega_+, \tag{18}$$

where $\mathscr{U}_4(t) := \mathscr{U}(t)I_4$. Thus, the spectrum of $L(c,\theta)$ (also consisting of a discrete spectrum only) is independent of $\theta \in \Omega_+$, and coincides with the set of the resonances of $H(c) - mc^2$ (see the remark after Proposition 4.3). In Section 4 we show that $L(c)$ is of type (II), from which the following theorem follows.

Theorem 1.3.

(a) *The resonances of the Dirac operator $H(c)$ are contained in the closed upper half plane $\overline{\boldsymbol{C}_+}$, and the real resonances coincide with the eigenvalues of $H(c)$. In particular, the set of the eigenvalues* (*if exist*) *is a discrete set. Moreover, the multiplicity of each eigenvalue is finite.*

(b) $\sigma(H(c)) = \boldsymbol{R}$ *and* $\sigma_{\mathrm{sc}}(H(c)) = \phi$. *In particular,* $\sigma(H(c)) \setminus \sigma_{\mathrm{p}}(H(c)) \subset \sigma_{\mathrm{ac}}(H(c))$.

Next we see that there exist resonances of the Dirac operator $H(c) - mc^2$ near each eigenvalue of the Schrödinger operator S if c is sufficiently large and they converge to the eigenvalue as $c \to \infty$.

We fix a constant $L > 0$ and an open interval $I \subset \boldsymbol{R}$ with $I \cap \sigma_d(S) = \{\lambda_j\}_{j=1}^N$ and define a set

$$\mathscr{O} := \{z \in \boldsymbol{C}; \operatorname{Re} z \in I,\ |\operatorname{Im} z| < L\}.$$

For small $\varepsilon > 0$ we also define $B_\varepsilon(\lambda) = \{z \in \boldsymbol{C}; |z - \lambda| \le \varepsilon\}$, $B_\varepsilon^+(\lambda) = B_\varepsilon(\lambda) \cap \boldsymbol{C}_+$ and $\overline{B_\varepsilon^+}(\lambda) = B_\varepsilon(\lambda) \cap \overline{\boldsymbol{C}_+}$, where $\boldsymbol{C}_+ := \{z \in \boldsymbol{C}; \operatorname{Im} z > 0\}$. Let m_j be the multiplicity of the eigenvalue λ_j of S.

Theorem 1.4. *For any small $\varepsilon > 0$ there exists a constant $c_\varepsilon > 0$ such that there is no resonance of $H(c) - mc^2$ in $\mathscr{O} \setminus \cup_{j=1}^N \overline{B_\varepsilon^+}(\lambda_j)$ and there are $2m_j$ resonances of $H(c) - mc^2$ in $\overline{B_\varepsilon^+}(\lambda_j)$ for each $j = 1, \ldots, N$, if $c > c_\varepsilon$.*

By Theorems 1.3 and 1.4, we have the following corollary, which guarantees the existence of nonreal resonances of $H(c)$ if $H(c)$ has no eigenvalue.

Corollary 1.5. *Suppose $\sigma_{\mathrm{p}}(H(c)) = \phi$ for all large c. Then for any small $\varepsilon > 0$ there exists a constant $c_\varepsilon > 0$ such that there is no resonance of $H(c) - mc^2$ in $\mathscr{O} \setminus \cup_{j=1}^N B_\varepsilon^+(\lambda_j)$ and there are $2m_j$ resonances of $H(c) - mc^2$ in $B_\varepsilon^+(\lambda_j)$ for each $j = 1, \ldots, N$, if $c > c_\varepsilon$.*

Remarks.

(i) It is known that the Dirac operator $H(c)$ has no eigenvalue if the potential $V(x)$ asymptotically equals to a radial function $q(|x|)$ as $|x| \to \infty$, where $q(r)$ goes to $+\infty$ as $r \to +\infty$ and satisfies some mild condition [**13**], [**25**]. For example, the Dirac operator has no eigenvalue if $V(x) = A|x|^M + O(|x|^{M-1})$, $A > 0$, as $|x| \to \infty$. Moreover, if $V(x)$ is a radial potential with some growth condition, then $H(c)$ has a purely absolutely continuous spectrum (cf. [**17**], [**18**], [**19**]). In particular, if a *radial* potential satisfies our assumptions, then the Dirac operator has a purely absolutely continuous spectrum. Our result gives a condition for the absolute continuity even if we do not assume the radial symmetry of $V(x)$.

(ii) We can also consider the problem on the limits of resonances of $H(c) + mc^2$ as $c \to \infty$. In Section 6, we state a result on this problem (Theorem 6.5), in which eigenvalues of S in Theorem 1.4 are replaced by "resonances" of $(2m)^{-1}\Delta + V(x)$ (see the remark after Proposition 6.1).

(iii) It is a natural question whether or not there exists a resonance of $H(c)$ in a bounded set in $\boldsymbol{C}$. We shall show that there is neither eigenvalue nor resonance in any fixed bounded set if $c > 0$ is sufficiently large (Theorem 6.6).

A result similar to Theorem 1.4 has already been proved by Amour, Brummelhuis and Nourrigat in [**1**], in which they investigate the nonrelativistic limits of resonances in more detail: They obtain an asymptotic expansion of the real part of each resonance and an estimate of its imaginary part. In this work we do not intend to study the nonrelativistic limits of resonances in detail. However, we investigate not only the nonrelativistic limit but also the resonances and the spectra of various operators (Dirac operators, relativistic Schrödinger operators and Schrödinger operators) for fixed $c \geq 1$. Combining these results, we can show that there are nonreal resonances of Dirac operators in $\boldsymbol{C}_+$ under some condition (Corollary 1.5). Here we should note that the results on resonances and spectra are obtained in an abstract framework (Theorem 3.4).

We introduce two relativistic Schrödinger operators $L_\pm(c)$ to explain the structure of the spectrum of $H(c)$, the appearance of resonances of $H(c)$ and their convergence to eigenvalues of the Schrödinger operator S in the nonrelativistic limit. Proposition 5.1 shows that there is no resonance of $L_-(c)$ in any bounded set in $\boldsymbol{C}$ if $c > 0$ is large enough. However, since $L_+(c)$ converges to S in the norm resolvent sense as $c \to \infty$ (Proposition 5.5), for each eigenvalue of S we can find eigenvalues of $L_+(c)$ converging to it as $c \to \infty$. If I is a small open interval containing only an eigenvalue λ of S with multiplicity n, then I is contained in the spectrum of $L_1(c) = L_+(c)I_2 \oplus L_-(c)I_2$ since $\sigma(L_1(c)) = \boldsymbol{R}$, and there exist $2n$ eigenvalues of $L_1(c)$ (counting multiplicity) in I for c large enough. Now suppose there is no eigenvalue of $H(c)$, i.e., $\sigma_{\mathrm{p}}(H(c) - mc^2) = \sigma_{\mathrm{p}}(L(c)) = \phi$. Then the embedded eigenvalues of $L_1(c)$ in I disappear by adding the perturbation $W(c)$ to $L_1(c)$. But, Corollary 1.5 shows that they survive as resonances in the upper half plane $\boldsymbol{C}_+$ near λ and converge to λ as $c \to \infty$.

This is our explanation for the appearance of the resonances of $H(c) - mc^2$ and for the convergence of the resonances to eigenvalues of S in the nonrelativistic limit.

The plan of this paper is as follows. In Section 2 we study $L_\pm(c,\theta)$ in detail. Proposition 2.1 is the main result in the section. In Section 3 we study an abstract theory on self-adjoint operators defined as a *boundary value* of a certain analytic family. Theorem 3.4 is the key to our study on spectral properties and resonances. Theorems 1.1, 1.2 and 1.3 are proved in Section 4 with the help of Theorem 3.4. In Section 5 the nonrelativistic limits of $L_\pm(c,\theta)$ and $L(c,\theta)$ are considered and Theorem 1.4 is proved. Moreover, we obtain a result on the nonrelativistic limit of the spectral projection of the Dirac operator. In Section 6 we state our results on the nonrelativistic limits of resonances for $H(c) + mc^2$ and $H(c)$, respectively, and study a typical example in detail.

2. Relativistic Schrödinger Operators.

Let $\sigma(\xi) := \sqrt{|\xi|^2 + m^2}$, $\xi \in \boldsymbol{R}^3$, and

$$A(\xi) := \left(\frac{\sigma(\xi) + m}{\sigma(\xi)}\right)^{1/2}.$$

Define a 4×4 matrix $U_c(\xi)$ by

$$U_c(\xi) := U\left(\frac{\xi}{c}\right),$$

where

$$U(\xi) := \frac{1}{\sqrt{2}}\left(A(\xi)I_4 + A(\xi)^{-1}\beta\alpha \cdot \frac{\xi}{\sigma(\xi)}\right).$$

The matrix $U_c(\xi)$ is unitary with inverse $U_c(\xi)^{-1} = U_c(-\xi)$ and satisfies

$$\left|\partial_\xi^\alpha U_c(\xi)\right| \le K_\alpha c^{-|\alpha|}\sigma(\xi/c)^{-|\alpha|}, \quad c \ge 1,\, \xi \in \boldsymbol{R}^3 \tag{19}$$

for all multi-index α. Furthermore, by a simple calculation we see that the matrix $U_c(\xi)$ diagonalizes the symbol of the Dirac operator $H(c)$:

$$U_c(\xi)(c\alpha\cdot\xi+mc^2\beta+V(x))U_c(\xi)^{-1}$$
$$=\begin{pmatrix}\big(\sqrt{c^2|\xi|^2+m^2c^4}+V(x)\big)I_2 & 0\\ 0 & \big(-\sqrt{c^2|\xi|^2+m^2c^4}+V(x)\big)I_2\end{pmatrix}.$$

In particular, the free Dirac operator ($V(x)=0$) is diagonalized by the unitary operator $U_c(D)$:

$$U_c(D)H_0(c)U_c(D)^{-1}=\begin{pmatrix}\sqrt{-c^2\Delta+m^2c^4}I_2 & 0\\ 0 & -\sqrt{-c^2\Delta+m^2c^4}I_2\end{pmatrix}.$$

This transformation by $U_c(D)$ is called the FWT transformation (the Foldy-Wouthuysen-Tani transformation). The estimate (19) guarantees that the FWT transformation maps $\mathscr{S}^4$ onto itself, where $\mathscr{S}$ denotes the Schwartz space $\mathscr{S}(\boldsymbol{R}^3)$. Applying this transformation to $H(c)-mc^2=H_0(c)-mc^2+V(x)$, we have (11), (12) and (13). Since the potential $V(x)$ is continuous, $H(c)$ defined on $\mathscr{S}^4$ is essentially self-adjoint (see e.g. [**19**]), and so $L(c)$ defined on $\mathscr{S}^4$ is also essentially self-adjoint since $U_c(D)$ maps $\mathscr{S}^4$ onto itself. Hereafter we denote by $L(c)$ and $H(c)$ the unique self-adjoint extensions of them, respectively.

First we study the relativistic Schrödinger operators $L_\pm(c)$. Let us remark that $L_\pm(c)$ defined on $\mathscr{S}$ are essentially self-adjoint operators (see [**8**], [**9**]) under our assumptions, though the Schrödinger operator $-(2m)^{-1}\Delta-|x|^M$ defined on $\mathscr{S}$ is essentially self-adjoint if and only if $M\le 2$ (cf. [**15**]). We denote again by the same notation $L_\pm(c)$ their self-adjoint extensions, respectively.

Recall that we fix a constant a with $0<a<a_0$ in this paper and define $\Omega=\{\theta\in\boldsymbol{C};|\operatorname{Im}\theta|<a\}$ and $\Omega_+=\{\theta\in\boldsymbol{C};0<\operatorname{Im}\theta<a\}$. Our assumptions on V make it possible to define the following operators on $\mathscr{S}$

$$L_\pm(c,\theta):=\pm\sqrt{-c^2e^{-2\theta}\Delta+m^2c^4}-mc^2+V_\theta(x) \tag{20}$$

for $\theta\in\Omega$, where $\sqrt{-c^2e^{-2\theta}\Delta+m^2c^4}$ is considered as the pseudodifferential operator with symbol $\sqrt{c^2e^{-2\theta}|\xi|^2+m^2c^4}$. Here $\sqrt{z}$ is defined to have the branch on the negative real line. Note that if t is a real number, they are written as

$$L_\pm(c,t)=\mathscr{U}(t)L_\pm(c)\mathscr{U}(t)^{-1} \tag{21}$$

on $\mathscr{S}$, where $\mathscr{U}(t)$ is the dilation group defined by

$$\mathscr{U}(t)f(x)=e^{(3t/2)}f(e^tx).$$

Let us define the weighted L^2-space $L^2_M(\boldsymbol{R}^3)$ by $L^2_M(\boldsymbol{R}^3) = L^2(\boldsymbol{R}^3; \langle x\rangle^{2M}dx)$ and set $D_M := H^1(\boldsymbol{R}^3) \cap L^2_M(\boldsymbol{R}^3)$, where $H^1(\boldsymbol{R}^3)$ is the Sobolev space of order one. Here we note that according to Rellich's criterion any closed operator with domain D_M has compact resolvent if the resolvent set is not empty.

Hereafter we suppose $a > 0$ is sufficiently small and $c \geq 1$. The following proposition is the main result in this section.

PROPOSITION 2.1.

(a) *For each $\theta \in \Omega$ and $c \geq 1$, $L_+(c,\theta)$ defined on $\mathscr{S}$ is closable, and its closure* (*denoted by the same notation $L_+(c,\theta)$*) *has domain D_M. Moreover, its resolvent set is nonempty and, in particular, $L_+(c,\theta)$ has compact resolvent.*

(b) *For each $c \geq 1$ the family of closed operators $\{L_+(c,\theta)\}_{\theta\in\Omega}$ is an analytic family of type* (A) (*e.g.* [**12**], [**16**]) *with the following property*:

$$L_+(c, t+\theta) = \mathscr{U}(t)L_+(c,\theta)\mathscr{U}(t)^{-1}, \quad t \in \boldsymbol{R},\, \theta \in \Omega. \tag{22}$$

(c) *For each $\theta \in \Omega_+$ and $c \geq 1$, $L_-(c,\theta)$ defined on $\mathscr{S}$ is closable and its closure* (*denoted by the same notation $L_-(c,\theta)$*) *has domain D_M. Moreover, its resolvent set is nonempty and, in particular, $L_-(c,\theta)$ has compact resolvent.*

(d) *For each $c \geq 1$, the family of closed operators $\{L_-(c,\theta)\}_{\theta\in\Omega_+}$ is an analytic family of type* (A) *with the following property*:

$$L_-(c, t+\theta) = \mathscr{U}(t)L_-(c,\theta)\mathscr{U}(t)^{-1}, \quad t \in \boldsymbol{R},\, \theta \in \Omega_+. \tag{23}$$

(e) *There is a constant $r_0 > 0$ independent of $c \geq 1$ and $\theta \in \Omega_+$ such that $\{z \in \boldsymbol{C}; \operatorname{Im} z < -r_0\} \subset \rho(L_-(c,\theta))$, the resolvent set of $L_-(c,\theta)$.*

(f) *Let $c \geq 1$ and $\operatorname{Im} z < -r_0$. Then the resolvent $(L_-(c,\theta) - z)^{-1}$ converges to $(L_-(c) - z)^{-1}$ strongly as $\theta \to 0$:*

$$s - \lim_{\Omega_+ \ni \theta \to 0} (L_-(c,\theta) - z)^{-1} = (L_-(c) - z)^{-1}. \tag{24}$$

REMARKS.

(i) Since $L_+(c,\theta)$ has compact resolvent, it has a purely discrete spectrum. Moreover, according to (b), with the help of the standard argument by Aguilar and Combes [**2**] we see that the discrete spectrum is independent of $\theta \in \Omega$ for each $c \geq 1$. In particular, it coincides with that of $L_+(c)$. However, the above argument is valid for $L_-(c,\theta)$ with only $\theta \in \Omega_+$. Actually, the structure of spectrum of $L_-(c,\theta)$ for $\operatorname{Im}\theta > 0$ and that of $L_-(c)$ are quite different (see Theorem 1.2). Thus it seems that the analysis of $L_-(c,\theta)$ for

$\operatorname{Im}\theta > 0$ does not contribute to that of $L_-(c)$. But, as shown in Section 3, the spectral property of $L_-(c,\theta)$ for $\operatorname{Im}\theta > 0$ helps us to determine that of $L_-(c)$ through the relation (24).

(ii) The result of (e) is not optimal. Indeed, combining this proposition with Proposition 3.2 in the next section, we can prove that the lower half plane $\{z \in \boldsymbol{C}; \operatorname{Im} z < 0\}$ is contained in the resolvent set of $L_-(c,\theta)$ for all $\theta \in \Omega_+$.

We have to prepare several lemmas to prove the above proposition.

Lemma 2.2. *Suppose $a > 0$ is small enough. Then for any $\rho_0 > 0$ there exist constants $R_1 > 0$ and $K > 0$ such that*

$$K^{-1}\langle x\rangle^M \le \operatorname{Re} V_{\rho+i\tau}(x) \le K\langle x\rangle^M, \quad |x| \ge R_1 \tag{25}$$

for $\tau \in (-a,a)$ and $\rho \in [-\rho_0,\rho_0]$ and that

$$K^{-1}|\tau|\langle x\rangle^M \le \pm \operatorname{Im} V_{\rho+i\tau}(x) \le K|\tau|\langle x\rangle^M, \quad |x| \ge R_1 \tag{26}$$

for $\pm\tau \in (0,a)$, respectively, and $\rho \in [-\rho_0,\rho_0]$.

Proof. It suffices to prove for $\rho = 0$. Indeed we may replace x by $e^\rho x$ in the obtained result for $\rho = 0$. Let $V_1(x) = x \cdot \nabla V(x)$. Then we can write

$$V_{i\tau}(x) = V(x) + iV_1(x)\tau + V_2(\tau,x)\tau^2. \tag{27}$$

By the estimate (8) there is a constant $K_1 > 0$ such that

$$|V_1(x)|, \quad |V_2(\tau,x)| \le K_1\langle x\rangle^M$$

for $|x| \ge R_0$ uniformly in $\tau \in (-a,a)$. Thus the desired results follow immediately from our assumptions since $a > 0$ is sufficiently small. $\square$

Let $\sigma_\theta(c,\xi) := \sqrt{c^2e^{-2\theta}|\xi|^2 + m^2c^4} - mc^2$. Then we can write

$$L_+(c,\theta) = \sigma_\theta(c,D) + V_\theta(x), \quad L_-(c,\theta) = -\sigma_\theta(c,D) - 2mc^2 + V_\theta(x)$$

on $\mathscr{S}$.

Lemma 2.3.

(a) *Fix $\rho_0 > 0$. Then there exists a constant $K > 0$, independent of $\tau \in (-a,a)$,*

$\rho \in [-\rho_0, \rho_0]$, $c \geq 1$ and $\xi \in \boldsymbol{R}^3$, such that

$$K^{-1}\frac{c|\xi|^2}{|\xi|+c} \leq \operatorname{Re}\sigma_{\rho+i\tau}(c,\xi) \leq K\frac{c|\xi|^2}{|\xi|+c}. \tag{28}$$

(b) *Fix $\rho_0 > 0$. Then there exists a constant $K' > 0$, independent of $\tau \in (0, a)$, $\rho \in [-\rho_0, \rho_0]$, $c \geq 1$ and $\xi \in \boldsymbol{R}^3$, such that*

$$K'^{-1}\frac{c|\xi|^2}{|\xi|+c}\tau \leq -\operatorname{Im}\sigma_{\rho+i\tau}(c,\xi) \leq K'\frac{c|\xi|^2}{|\xi|+c}\tau. \tag{29}$$

Proof. Note that $a > 0$ is sufficiently small. Due to the relation $\sigma_{\rho+i\tau}(c,\xi) = \sigma_{i\tau}(c, e^{-\rho}\xi)$, it is sufficient to prove the estimates for $\rho = 0$. The first estimate (28) follows immediately from the formula

$$\sigma_{i\tau}(c,\xi) = \frac{ce^{-2i\tau}|\xi|^2}{\sqrt{e^{-2i\tau}|\xi|^2 + c^2m^2} + mc}.$$

Next we write $Re^{-i\phi/2} = \sigma_{i\tau}(c,\xi) + mc^2$ with $R > 0$ and $0 < \phi < 2\tau < 2a$, i.e.,

$$R^2e^{-i\phi} = c^2e^{-2i\tau}|\xi|^2 + m^2c^4.$$

Then

$$\begin{aligned} R &= c\big(|\xi|^4 + m^4c^4 + 2|\xi|^2m^2c^2\cos 2\tau\big)^{1/4}, \\ \tan\phi &= \frac{|\xi|^2\sin 2\tau}{|\xi|^2\cos 2\tau + m^2c^2}. \end{aligned} \tag{30}$$

The first equality implies that $R \sim c(|\xi| + c)$ and the second $\phi \sim 2\tau|\xi|^2(|\xi|^2 + c^2)^{-1}$ uniformly in ξ, since $\tau > 0$ is small enough. Combining these results with $-\operatorname{Im}\sigma_{i\tau}(c,\xi) = R\sin(\phi/2)$, we obtain the estimate (29). □

Combining the last two lemmas, we have

Lemma 2.4. *Define*

$$h(c,x,\xi) := \frac{c|\xi|^2}{|\xi|+c} + \langle x\rangle^M$$

and fix $\rho_0 > 0$. Then there are positive constants K and L independent of $x \in \boldsymbol{R}^3$,

$\xi \in \boldsymbol{R}^3$, $c \geq 1$ and θ such that

$$K^{-1}h(c,x,\xi) - L \leq \mathrm{Re}(\sigma_\theta(c,\xi) + V_\theta(x)) \leq Kh(c,x,\xi) + L \tag{31}$$

for $|\operatorname{Im}\theta| < a$ and $|\operatorname{Re}\theta| < \rho_0$ and that

$$\begin{aligned} K^{-1}(\operatorname{Im}\theta)h(c,x,\xi) - L &\leq \operatorname{Im}(-\sigma_\theta(c,\xi) - 2mc^2 + V_\theta(x)) \\ &\leq K(\operatorname{Im}\theta)h(c,x,\xi) + L \end{aligned} \tag{32}$$

for $0 < \operatorname{Im}\theta < a$ and $|\operatorname{Re}\theta| < \rho_0$.

Lemma 2.5. *Fix $\rho_0 > 0$. Then*

$$\left|\partial_{\xi_k}\sigma_\theta(c,\xi)\right| \leq K|\xi_k|, \qquad k = 1,2,3, \tag{33}$$

$$\left|\partial_\xi^\alpha\sigma_\theta(c,\xi)\right| \leq K_\alpha c(|\xi| + c)^{1-|\alpha|}, \quad |\alpha| \geq 2 \tag{34}$$

for some constants $K > 0$ and $K_\alpha > 0$, uniformly in $c \geq 1$ and $\theta \in \Omega$ with $|\operatorname{Re}\theta| < \rho_0$.

Proof. We only prove the lemma in the case $\operatorname{Im}\theta = 0$ for simplicity. The desired result follows immediately from the relation $\sigma_\theta(c,\xi) + mc^2 = c^2\sigma(e^{-\theta}\xi/c)$ and the estimates

$$\begin{aligned} \left|\partial_{\xi_k}\sigma(\xi)\right| &\leq |\xi_k|\sigma(\xi)^{-1}, \qquad k = 1,2,3, \\ \left|\partial_\xi^\alpha\sigma(\xi)\right| &\leq K_\alpha\sigma(\xi)^{1-|\alpha|}, \quad |\alpha| \geq 2. \end{aligned} \qquad \square$$

In the proof of the next two lemmas we need a class of pseudodifferential operators. Let m, $s \in \boldsymbol{R}$ and denote by $S^{m,s}$ the space of functions $p(x,\xi) \in C^\infty(\boldsymbol{R}^3 \times \boldsymbol{R}^3)$ satisfying

$$\left|\partial_\xi^\alpha\partial_x^\beta p(x,\xi)\right| \leq K_{\alpha\beta}\langle x\rangle^{m-|\beta|}\langle\xi\rangle^{s-|\alpha|} \quad \text{on } \boldsymbol{R}^3 \times \boldsymbol{R}^3$$

for all α and β. We denote by $\Sigma^{m,s}$ the set of pseudodifferential operators $p(x,D)$, $p \in S^{m,s}$, defined by

$$p(x,D)u(x) = \frac{1}{(2\pi)^3}\iint e^{i(x-y)\cdot\xi}p(x,\xi)u(y)d\xi dy.$$

Let

$$s(c,\xi) := \frac{c|\xi|^2}{|\xi|+c}$$

and write

$$V_{\theta,\infty}(x) := \chi_\infty\left(\frac{e^{\operatorname{Re}\theta}|x|}{R_0}\right)V_\theta(x),$$
$$V_{\theta,0}(x) := V_\theta(x) - V_{\theta,\infty}(x),$$

where $\chi_\infty(r)$ is a smooth function on $(0,\infty)$ with $\chi_\infty(r) = 1$ for $r > 2$ and $\chi_\infty(r) = 0$ for $r < 1$. By (V1) and (V2) $V_{\theta,\infty}(x)$ is smooth on $\boldsymbol{R}^3$ and $V_{\theta,0}(x)$ is a bounded function with compact support. We define

$$L_{+,\infty}(c,\theta) := \sigma_\theta(c,D) + V_{\theta,\infty}(x), \quad L_{-,\infty}(c,\theta) := -\sigma_\theta(c,D) - 2mc^2 + V_{\theta,\infty}(x).$$

LEMMA 2.6. *Fix $\theta \in \Omega_+$ and a constant $L > 0$. Then there are positive constants K_1, K_2 and K_3 independent of $c \geq 1$ such that*

$$\|(L_-(c,\theta) - z)f\|^2 + K_1\|f\|^2 \geq K_2\|s(c,D)f\|^2 + K_3\big\|\langle x\rangle^M f\big\|^2 \tag{35}$$

for all $z \in \boldsymbol{C}$ with $|\operatorname{Im} z| \leq L$ and all $f \in \mathscr{S}$.

PROOF. Fix a constant $M_0 > 0$ such that $\operatorname{Im} V_{\theta,\infty}(x) + M_0 \geq 1$ on $\boldsymbol{R}^3$ (see (26)). We have only to give the proof for $z = r - i(M_0+1)$, $r \in \boldsymbol{R}$, and $L_{-,\infty}(c,\theta)$ in place of $L_-(c,\theta)$ because

$$\|(L_-(c,\theta) - z)f\| \geq \|(L_{-,\infty}(c,\theta) - z_0)f\| - (|z - z_0| + \|V_{\theta,0}\|_\infty)\|f\|$$

and $|z - z_0| \leq (L + M_0)$ for z, $z_0 \in \boldsymbol{C}$ with $\operatorname{Re} z = \operatorname{Re} z_0$, $|\operatorname{Im} z| \leq L$ and $\operatorname{Im} z_0 = M_0 + 1$. Define

$$\sigma_{1,\theta}(c,\xi) := \operatorname{Re}\sigma_\theta(c,\xi), \qquad \sigma_{2,\theta}(c,\xi) := \operatorname{Im}\sigma_\theta(c,\xi) - 1,$$
$$V_{1,\theta}(x) := \operatorname{Re} V_{\theta,\infty}(x) - r, \qquad V_{2,\theta}(x) := \operatorname{Im} V_{\theta,\infty}(x) + M_0.$$

Writing $\sigma_\theta(\xi) = \sigma_\theta(c,\xi)$, etc. for simplicity, we have

$$
\begin{aligned}
&\big\|(-\sigma_\theta(D) - 2mc^2 + V_{\theta,\infty}(x) - r + i(M_0+1))f\big\|^2 \\
&\quad = \big((\sigma_{1,\theta}(D) + 2mc^2 - V_{1,\theta})^2 f, f\big) + \big((\sigma_{2,\theta}(D) - V_{2,\theta})^2 f, f\big) \\
&\qquad + \big(\{i[\sigma_{2,\theta}(D), V_{1,\theta}(x)] - i[\sigma_{1,\theta}(D), V_{2,\theta}(x)]\}f, f\big) \\
&\quad \geq \big((\sigma_{2,\theta}(D) - V_{2,\theta})^2 f, f\big) + \big(\{i[\sigma_{2,\theta}(D), V_{1,\theta}(x)] - i[\sigma_{1,\theta}(D), V_{2,\theta}(x)]\}f, f\big).
\end{aligned}
$$

Here taking account of $-\sigma_{2,\theta}(\xi, c) \geq 1$ and $V_{2,\theta}(x) \geq 1$, we write

$$
\begin{aligned}
&(\sigma_{2,\theta}(D) - V_{2,\theta}(x))^2 \\
&\quad = \sigma_{2,\theta}(D)^2 + V_{2,\theta}(x)^2 + (-\sigma_{2,\theta}(D))V_{2,\theta}(x) + V_{2,\theta}(x)(-\sigma_{2,\theta}(D)) \\
&\quad = \sigma_{2,\theta}(D)^2 + V_{2,\theta}(x)^2 + 2V_{2,\theta}(x)^{1/2}(-\sigma_{2,\theta}(D))V_{2,\theta}(x)^{1/2} \\
&\qquad - V_{2,\theta}(x)^{1/2}\big[V_{2,\theta}(x)^{1/2}, \sigma_{2,\theta}(D)\big] - [\sigma_{2,\theta}(D), V_{2,\theta}(x)^{1/2}]V_{2,\theta}(x)^{1/2} \\
&\quad \geq \sigma_{2,\theta}(D)^2 + V_{2,\theta}(x)^2 - V_{2,\theta}(x)^{1/2}\big[V_{2,\theta}(x)^{1/2}, \sigma_{2,\theta}(D)\big] \\
&\qquad - \big[\sigma_{2,\theta}(D), V_{2,\theta}(x)^{1/2}\big]V_{2,\theta}(x)^{1/2} \qquad (36)
\end{aligned}
$$

in the form sense on $\mathscr{S}$. Using Lemma 2.5 and our assumptions on V we can verify that

$$
\begin{aligned}
T_1 &:= -V_{2,\theta}(x)^{1/2}\big[V_{2,\theta}(x)^{1/2}, \sigma_{2,\theta}(D)\big] \in \Sigma^{1,M-1}, \\
T_2 &:= -\big[\sigma_{2,\theta}(D), V_{2,\theta}(x)^{1/2}\big]V_{2,\theta}(x)^{1/2} \in \Sigma^{1,M-1}
\end{aligned}
$$

and that for $j = 1, 2$

$$
K_j' := \sup_{c \geq 1} \big\|\langle D\rangle^{-1} T_j \langle x\rangle^{-(M-1)}\big\| < \infty, \qquad (37)
$$

where $\langle D\rangle := (1-\Delta)^{1/2}$. Therefore we have for any $\varepsilon > 0$

$$
\begin{aligned}
|(T_j f, f)| &\leq K_j' \big\|\langle x\rangle^{M-1} f\big\| \big\|\langle D\rangle f\big\| \\
&\leq K_j'(2\varepsilon)^{-1}\big(\langle x\rangle^{2(M-1)} f, f\big) + K_j'\left(\frac{\varepsilon}{2}\right)(\langle D\rangle^2 f, f). \qquad (38)
\end{aligned}
$$

This implies that

$$T_1 + T_2 \geq -\big(K_1' + K_2'\big)\left[(2\varepsilon)^{-1}\langle x\rangle^{2(M-1)} + \left(\frac{\varepsilon}{2}\right)\langle D\rangle^2\right]. \tag{39}$$

Let $T_3 := i[\sigma_{2,\theta}(D), V_{1,\theta}(x)] - i[\sigma_{1,\theta}(D), V_{2,\theta}(x)]$. Then, since

$$[\sigma_{2,\theta}(D), V_{1,\theta}(x)], \quad [\sigma_{1,\theta}(D), V_{2,\theta}(x)] \in \Sigma^{1,M-1},$$

we can also show that for any $\varepsilon > 0$

$$T_3 \geq -K_3'\left[(2\varepsilon)^{-1}\langle x\rangle^{2(M-1)} + \left(\frac{\varepsilon}{2}\right)\langle D\rangle^2\right],$$

where K_3' is defined by (37) for $j = 3$. Note that for any $K > 0$ there exists $L = L(K) > 0$ such that

$$K\langle x\rangle^{2(M-1)} \leq \langle x\rangle^{2M} + L$$

for all $x \in \boldsymbol{R}^3$, and that $s(c,\xi)^2 \geq |\xi|^2/4$ for $|\xi| \geq 1$ and $c \geq 1$, since $ab/(a+b) \geq 1/2$ for real numbers $a \geq 1$, $b \geq 1$. This implies that $s(c,D)^2 + (1/4) \geq |\xi|^2/4$ on $\boldsymbol{R}^3$. Consequently, by (28) and (36) we obtain

$$\begin{aligned}\big\|(-\sigma_\theta(D) - 2mc^2 + V_{\theta,\infty}(x) - r + i(M_0+1))f\big\|^2 \\ \geq K_2(s(c,D)^2 f, f) + K_3(\langle x\rangle^{2M} f, f) - K_1\|f\|^2\end{aligned}$$

for some positive constants K_1, K_2 and K_3 independent of $c \geq 1$. □

LEMMA 2.7. *Fix $\theta \in \Omega$ and a constant $L \in \boldsymbol{R}$. Then there are positive constants $\tilde{K}_1, \tilde{K}_2$ and $\tilde{K}_3$ independent of $c \geq 1$ such that*

$$\|(L_+(c,\theta) - z)f\|^2 + \tilde{K}_1\|f\|^2 \geq \tilde{K}_2\|s(c,D)f\|^2 + \tilde{K}_3\|\langle x\rangle^M f\|^2 \tag{40}$$

for all $z \in \boldsymbol{C}$ with $\operatorname{Re} z \leq L$ and $f \in \mathscr{S}$.

PROOF. Fix a positive constant $\tilde{M}_0$ such that $\operatorname{Re} V_{\theta,\infty}(x) + \tilde{M}_0 \geq 1$ on $\boldsymbol{R}^3$. As in the proof of Lemma 2.6, we have only to prove for $z = -t - (\tilde{M}_0 + 1) - ir$, $t \geq 0$, $r \in \boldsymbol{R}$, and $L_{-,\infty}(c,\theta)$ in place of $L_-(c,\theta)$. Let

$$\begin{aligned}\tilde{\sigma}_{1,\theta}(c,\xi) &:= \operatorname{Re}\sigma_\theta(c,\xi) + 1, & \tilde{\sigma}_{2,\theta}(c,\xi) &:= \operatorname{Im}\sigma_\theta(c,\xi), \\ \tilde{V}_{1,\theta}(x) &:= \operatorname{Re} V_{\theta,\infty}(x) + \tilde{M}_0, & \tilde{V}_{2,\theta}(x) &:= \operatorname{Im} V_{\theta,\infty}(x) + r.\end{aligned}$$

Then using the following inequality

$$
\begin{aligned}
&\big\|(\sigma_\theta(c,D)+V_{\theta,\infty}(x)+t+\tilde{M}_0+1+ir)f\big\|^2\\
&\quad=\big((\tilde\sigma_{1,\theta}(c,D)+\tilde V_{1,\theta}+t)^2f,f\big)+\big((\tilde\sigma_{2,\theta}(c,D)+\tilde V_{2,\theta})^2f,f\big)\\
&\qquad+\big(\big\{i[\tilde\sigma_{1,\theta}(c,D),\tilde V_{2,\theta}(x)]-i[\tilde\sigma_{2,\theta}(c,D),\tilde V_{1,\theta}(x)]\big\}f,f\big)\\
&\quad\geq\big((\tilde\sigma_{1,\theta}(c,D)+\tilde V_{1,\theta})^2f,f\big)\\
&\qquad+\big(\big\{i[\tilde\sigma_{1,\theta}(c,D),\tilde V_{2,\theta}(x)]-i[\tilde\sigma_{2,\theta}(c,D),\tilde V_{1,\theta}(x)]\big\}f,f\big),
\end{aligned}
$$

we can prove the desired result (40) in a manner similar to that in the proof of Lemma 2.6. □

Proof of Proposition 2.1.

(a) Fix $\theta\in\Omega$. Since $s(c,D)$ and $\langle x\rangle^M$ are closed operators with domains $H^1(\boldsymbol{R}^3)$ and $L^2_M(\boldsymbol{R}^3)$, respectively, and $\mathscr{S}$ is a common core for them, it follows from Lemma 2.7 that $L_+(c,\theta)$ defined on $\mathscr{S}$ is closable and its closure has domain $D(L_+(c,\theta))=D_M$. Apparently, the same results hold for $L_{+,\infty}(c,\theta)$.

Next we prove that z belongs to the resolvent set of $L_+(c,\theta)$ if $(-\operatorname{Re}z)>0$ is large enough. If the resolvent set of $L_+(c,\theta)$ is proved to be nonempty, then the compactness of the resolvent follows immediately from the fact that the domain of $L_+(c,\theta)$ is D_M and Rellich's criterion. We shall begin by giving the proof for $L_{+,\infty}(c,\theta)$ by constructing a parametrix for $L_{+,\infty}(c,\theta)$.

Let

$$N_1(c,\theta):=\big\{\sigma_\theta(c,\xi)+V_{\theta,\infty}(x)\in\boldsymbol{C};x\in\boldsymbol{R}^3,\ \xi\in\boldsymbol{R}^3\big\}.$$

We first show that it is contained in some cone:

$$N_1(c,\theta)\subset C_1:=\{w\in\boldsymbol{C};|\operatorname{Im}w|<B\operatorname{Re}(w+b)\}\tag{41}$$

for some $b>0$ and $B>0$ independent of $c\geq 1$ and $\theta\in\Omega$ with $|\operatorname{Re}\theta|\leq\rho_0$ for fixed $\rho_0>0$. Indeed, by Lemma 2.2 there are positive constants K and R_1 such that

$$|\operatorname{Im}V_{\theta,\infty}(x)|\leq K\operatorname{Re}V_{\theta,\infty}(x)\tag{42}$$

for $|x|>R_1$. Thus, since $\{V_{\theta,\infty}(x)\in\boldsymbol{C};|x|\leq R_1\}$ is a bounded set, we see that there are constants $B_1>0$ and $b_1>0$ such that

$$\{V_{\theta,\infty}(x) \in \boldsymbol{C}; x \in \boldsymbol{R}^3\} \subset C_{1,V} := \{w \in \boldsymbol{C}; |\operatorname{Im} w| < B_1 \operatorname{Re}(w + b_1)\}. \tag{43}$$

Moreover, it is easy to see that

$$\{\sigma_\theta(c, \xi) \in \boldsymbol{C}; \xi \in \boldsymbol{R}^3\} \subset C_{1,\sigma} := \{w \in \boldsymbol{C}; |\operatorname{Im} w| \leq B_2 \operatorname{Re} w\} \tag{44}$$

for some $B_2 > 0$. Hence (41) follows immediately from (43) and (44).

Suppose $(-\operatorname{Re} z) > 0$ is sufficiently large. Then by (6), (31) and (41) we prove that

$$p_+(z, x, \xi) := \sigma_\theta(c, \xi) + V_{\theta,\infty}(x) - z$$

has the inverse

$$q_+(z, x, \xi) = \frac{1}{p_+(z, x, \xi)}$$

and

$$p_+(z, x, D) q_+(z, x, D) = I + r_0(z, x, D),$$

where $r_0(z, x, \xi)$ satisfies

$$\left|\partial_\xi^\alpha \partial_x^\beta r_0(z, x, \xi)\right| \leq K_{\alpha\beta} |p_+(z, x, \xi)|^{-1}. \tag{45}$$

Thus we see that

$$\|r_0(z, x, D)\| \leq K_0 |\operatorname{Re} z + b|^{-1}$$

for some $K_0 > 0$ and $b > 0$ (see (41)) by the Calderon-Vaillancourt theorem, and that the inverse $(r_0(z, x, D) + I)^{-1}$ exists as a bounded operator $L^2 \to L^2$ if $(-\operatorname{Re} z) > 0$ is large enough. Hence we have

$$p_+(z, x, D) q_+(z, x, D)(I + r_0(z, x, D))^{-1} = I \tag{46}$$

in L^2. In the same way as above we have $q_+(z, x, D) p_+(z, x, D) = I + r_1(z, x, D)$, where r_1 satisfies the same properties as r_0, and

$$(I + r_1(z, x, D))^{-1} q_+(z, x, D) p_+(z, x, D) = I \tag{47}$$

if $(-\operatorname{Re} z) > 0$ is large enough. Consequently, it follows from (46) and (47) that z belongs to $\rho(L_{+,\infty}(c,\theta))$, the resolvent set of $L_{+,\infty}(c,\theta)$, if $(-\operatorname{Re} z) > 0$ is large enough. By (43) the set $\{V_{\theta,\infty}(x) \in \boldsymbol{C}; x \in \boldsymbol{R}^3\}$ is contained in the cone $C_{1,V}$, and thus it is easy to see that the numerical range of the multiplication operator $V_{\theta,\infty}(x)$ is contained in the cone. Similarly, the numerical range of the operator $\sigma_\theta(c,D)$ is contained in the cone $C_{1,\sigma}$ by (44). Thus, the numerical range of $L_{+,\infty}(c,\theta)$ is contained in the cone C_1. Hence, if $(-\operatorname{Re} z) > 0$ is large, then the estimate

$$\left\|(L_{+,\infty}(c,\theta) - z)^{-1}\right\| \leq \frac{1}{-b - \operatorname{Re} z} \tag{48}$$

holds, and so z belongs to $\rho(L_+(c,\theta))$ and

$$(L_+(c,\theta) - z)^{-1} = (L_{+,\infty}(c,\theta) - z)^{-1}\bigl(I + V_{\theta,0}(L_{+,\infty}(c,\theta) - z)^{-1}\bigr)^{-1}$$

if $\|V_{\theta,0}\|(-b - \operatorname{Re} z)^{-1} < 1$.

(b) Let $f \in D_M$. Then both $\|V_\theta f\|$ and $\|\sigma_\theta(c,D)f\|$ are uniformly bounded in θ in any compact set in Ω, and both $(V_\theta f, u)$ and $(\sigma_\theta(c,D)f, u)$ are analytic functions of θ for each $u \in \mathscr{S}$. Thus $L_+(c,\theta)f$ is an L^2-valued analytic function of $\theta \in \Omega$ for each $f \in D_M$. Hence we have concluded that $\{L_+(c,\theta)\}_{\theta\in\Omega}$ is an analytic family of type (A). Since $\mathscr{U}(t)$ maps D_M onto D_M, the relation (22) is valid by the definition of $L_+(c,\theta)$.

(c) Suppose θ belongs to the set $\Omega_+(0) := \{\theta \in \Omega_+; |\operatorname{Re}\theta| < \rho_0\}$ for a positive constant ρ_0. Using Lemma 2.6, we can show that $L_-(c,\theta)$ defined on $\mathscr{S}$ is closable and its closure has domain D_M as in the case of $L_+(c,\theta)$.

Next we shall prove that for each $c \geq 1$ and each $\theta \in \Omega_+$ there is a large constant $r(c,\theta) > 0$ such that $\{z \in \boldsymbol{C}; \operatorname{Im} z < -r(c,\theta)\}$ belongs to the resolvent set of $L_-(c,\theta)$. As in the case $L_+(c,\theta)$ we shall first give the proof for $L_{-,\infty}(c,\theta)$. Let

$$\begin{aligned} N_{2v}(\theta) &:= \bigl\{V_{\theta,\infty}(x) \in \boldsymbol{C}; x \in \boldsymbol{R}^3\bigr\}, \\ N_{2s}(c,\theta) &:= \bigl\{-\sigma_\theta(c,\xi) - 2mc^2 \in \boldsymbol{C}; \xi \in \boldsymbol{R}^3\bigr\}. \end{aligned}$$

It follows from Lemma 2.2 that there are positive constants K and R_1 such that

$$K^{-1}(\operatorname{Im}\theta)\operatorname{Re} V_{\theta,\infty}(x) \leq \operatorname{Im} V_{\theta,\infty}(x) \leq K(\operatorname{Im}\theta)\operatorname{Re} V_{\theta,\infty}(x) \tag{49}$$

for $|x| > R_1$. Thus taking account of the fact that $\{V_{\theta,\infty}(x) \in \boldsymbol{C}; |x| \leq R_1\}$ is bounded, we see that

$$N_{2v}(\theta) \subset \widetilde{N}_{2v}(\theta) := \{w \in \boldsymbol{C}; A_1 \operatorname{Im}\theta < \arg(w - w_1) < A_2\} \tag{50}$$

for some $w_1 \in \boldsymbol{C}$ and positive constants A_1 and A_2 with $A_2 < \pi/2$, where A_1, A_2 and w_1 are independent of $\theta \in \Omega_+(0)$. Moreover, it is easy to verify that (see Lemma 2.3)

$$N_{2s}(c,\theta) \subset \widetilde{N}_{2s}(\theta) := \{w \in \boldsymbol{C}; A_3 \le \arg(w + 2mc^2) \le (\pi - A_4 \operatorname{Im}\theta)\} \tag{51}$$

for some constant A_3 and A_4 with $\pi/2 < A_3 < \pi$, where A_3 and A_4 are independent of $c \ge 1$ and $\theta \in \Omega_+(0)$. Let

$$N_2(c,\theta) := \{-\sigma_\theta(c,\xi) - 2mc^2 + V_{\theta,\infty}(x) \in \boldsymbol{C}; x \in \boldsymbol{R}^3,\ \xi \in \boldsymbol{R}^3\}.$$

Since it is contained in the set

$$N_{2s}(c,\theta) + N_{2v}(\theta) := \{w_s + w_v \in \boldsymbol{C}; w_s \in N_{2s}(c,\theta),\ w_v \in N_{2v}(\theta)\},$$

it follows from (50) and (51) that there are positive constants A_5, A_6 and $w_0 \in \boldsymbol{C}$ independent of $c \ge 1$ and $\theta \in \Omega_+(0)$ such that

$$N_2(c,\theta) \subset \widetilde{N}_{2s}(c,\theta) + \widetilde{N}_{2v}(\theta) \subset C_2(c,\theta), \tag{52}$$

where

$$C_2(c,\theta) := \{w \in \boldsymbol{C}; A_5 \operatorname{Im}\theta < \arg(w - w_0 + 2mc^2) < \pi - A_6 \operatorname{Im}\theta\}.$$

Let

$$p_-(z,x,\xi) := -\sigma_\theta(c,\xi) - 2mc^2 + V_{\theta,\infty}(x) - z$$

for $z \in \boldsymbol{C} \setminus C_2(c,\theta)$. Then by (6), (32), (52) and Lemma 2.5 we see that its inverse

$$q_-(z,x,\xi) = \frac{1}{p_-(z,x,\xi)}$$

exists and

$$p_-(z,x,D)q_-(z,x,D) = I + r_2(z,x,D),$$

for some $r_2(z,x,\xi)$ satisfying

$$\left|\partial_\xi^\alpha \partial_x^\beta r_2(z,x,\xi)\right| \le K_{\alpha\beta} |p_-(z,x,\xi)|^{-1}.$$

Here each constant $K_{\alpha\beta}$ depends on c and θ (see (32)).

Thus as in the case of $p_+(z,x,D)$ we have

$$\|r_2(z,x,D)\| \le K_\theta (\operatorname{Im} w_0 - \operatorname{Im} z)^{-1} < 1$$

for some constant $K_\theta > 0$ if $(-\operatorname{Im} z) > 0$ is sufficiently large. Therefore the inverse $(I + r_2(x,D))^{-1} : L^2 \to L^2$ exists and

$$p_-(z,x,D) q_-(z,x,D)(I + r_2(z,x,D))^{-1} = I$$

in L^2. In the same way as above we have $q_-(z,x,D)p_-(z,x,D) = I + r_3(z,x,D)$, where r_3 satisfies the same properties as r_2, and

$$(I + r_3(z,x,D))^{-1} q_-(z,x,D) p_-(z,x,D) = I. \tag{53}$$

Hence we conclude that $z \in \rho(L_{-,\infty}(c,\theta))$ if $\operatorname{Im} z < -r(c,\theta)$ for some constant $r(c,\theta) > 0$. Since the numerical range of the operator $L_{-,\infty}(c,\theta)$ is contained in the cone $C_2(c,\theta)$, we prove that, if $(-\operatorname{Im} z) > 0$ is large, then

$$\left\|(L_{-,\infty}(c,\theta) - z)^{-1}\right\| \le \frac{1}{\operatorname{Im} w_0 - \operatorname{Im} z} \tag{54}$$

and z belongs to the resolvent set of $L_-(c,\theta)$ if $\|V_{\theta,0}\|(\operatorname{Im} w_0 - \operatorname{Im} z)^{-1} < 1$ in the same way as in the proof of (a) above. Moreover, since $D(L_-(c,\theta)) = D_M$, the resolvent $(L_-(c,\theta) - z)^{-1}$ is a compact operator.

(d) In the same way as in the case $L_+(c,\theta)$, we can prove that $L_-(c,\theta)$ is an analytic family of type (A) and satisfies

$$L_-(c,\theta + t) = \mathscr{U}(t) L_-(c,\theta) \mathscr{U}(t)^{-1}$$

for each $t \in \boldsymbol{R}$ and $\theta \in \Omega_+$.

(e) We may assume that the numerical range of $L_-(c,\theta)$ is contained in the cone $C_2(c,\theta)$ if $(-\operatorname{Im} w_0) > 0$ is large since the difference $V_\theta(x) - V_{\theta,\infty}(x) = V_{\theta,0}(x)$ is bounded, and so its spectrum is contained in $C_2(c,\theta)$,

$$\sigma(L_-(c,\theta)) \subset C_2(c,\theta), \tag{55}$$

since the spectrum consists of eigenvalues only. Moreover, we see that $\{z \in \boldsymbol{C};\ \operatorname{Im} z < \operatorname{Im} w_0\} \subset \rho(L_-(c,\theta))$ and

$$\left\|(L_-(c,\theta) - z)^{-1}\right\| \le \frac{1}{\operatorname{Im} w_0 - \operatorname{Im} z},$$

for z with $\operatorname{Im} z < -r_0$, where $r_0 := -\operatorname{Im} w_0$.

(f) It is easy to see that $L_-(c,\theta)f \to L_-(c)f$ strongly as $\Omega_+ \ni \theta \to 0$ for any $f \in \mathscr{S}$. Moreover, $(L_-(c) - z)\mathscr{S}$ is dense in L^2 since $\mathscr{S}$ is a core of $L_-(c)$, and the above argument shows that $\|(L_-(c,\theta) - z)^{-1}\|$ is uniformly bounded in θ for each c. Thus, using the resolvent equation

$$\begin{aligned}&(L_-(c,\theta) - z)^{-1} - (L_-(c) - z)^{-1}\\ &\quad = -(L_-(c,\theta) - z)^{-1}(L_-(c,\theta) - L_-(c))(L_-(c) - z)^{-1},\end{aligned}$$

we have the desired result. □

3. Analytic Family.

In this section we will show that self-adjoint operators defined as a boundary value of some type of analytic family of closed operators can be classified into two types by following the idea of Aguilar and Combes [**2**] (see also [**5**], [**16**]).

Let T be a self-adjoint operator and $\{T(\theta)\}_{\theta\in\boldsymbol{\Omega}_+}$ a family of closed operators in a Hilbert space $\mathscr{H}$, where $\boldsymbol{\Omega}_+ = \{\theta \in \boldsymbol{C}; 0 < \operatorname{Im}\theta < a\}$ for some $a > 0$. We assume the following:

(A1): $\{T(\theta)\}_{\theta\in\boldsymbol{\Omega}_+}$ is an analytic family in the sense of Kato (see [**12**], [**16**]).

(A2): Each $T(\theta)$ has compact resolvent.

(A3): There is a strongly continuous one-parameter unitary group $\{U(t)\}_{t\in\boldsymbol{R}}$ such that

$$U(t)T(\theta)U(t)^* = T(\theta + t) \tag{56}$$

for $t \in \boldsymbol{R}$ and $\theta \in \boldsymbol{\Omega}_+$.

By (A1) and (A2) each $T(\theta)$ has a purely discrete spectrum and the eigenvalues are analytic functions or branches of one or several analytic functions, and (A3) implies that the eigenvalues of $T(\theta)$ is invariant when θ is changed to $\theta + t$ if t is real. Thus, each eigenvalue is a constant function of $\theta \in \boldsymbol{\Omega}_+$ (see e.g. [**2**], [**16**]). Therefore we obtain

Proposition 3.1. *Suppose* (A1)$\sim$(A3). *Then there is a discrete set* Σ *in* $\boldsymbol{C}$ *such that* $\sigma(T(\theta)) = \sigma_{\mathrm{d}}(T(\theta)) = \Sigma$ *for all* $\theta \in \boldsymbol{\Omega}_+$.

Let $\boldsymbol{C}_\pm = \{z \in \boldsymbol{C}; \pm \operatorname{Im} z > 0\}$. A self-adjoint operator T is related to the analytic family $\{T(\theta)\}_{\theta \in \boldsymbol{\Omega}_+}$ in the following sense.

(A4): There is a nonempty open set $\mathscr{O}_0 \subset \boldsymbol{C}_- \setminus \Sigma$ such that

$$w - \lim_{t \to +0} (T(it) - z)^{-1} = (T - z)^{-1} \quad \text{(weakly)}$$

for each $z \in \mathscr{O}_0$.

For each $s \in \boldsymbol{R}$ define a self-adjoint operator $T(s)$ by $T(s) := U(s) T U(s)^*$. Then $T(0) = T$ and

$$\begin{aligned} w - \lim_{t \to +0} (T(s + it) - z)^{-1} &= w - \lim_{t \to +0} U(s)(T(it) - z)^{-1} U(s)^* \\ &= U(s)(T - z)^{-1} U(s)^* = (T(s) - z)^{-1} \end{aligned}$$

by (A3). Thus the self-adjoint operators $T(s)$, $s \in \boldsymbol{R}$, are regarded as boundary values of the operator-valued function $T(\theta)$ defined on $\boldsymbol{\Omega}_+$. The following proposition shows that the eigenvalues of $T(\theta)$ are located in the closed upper half plane $\overline{\boldsymbol{C}}_+ = \{z \in \boldsymbol{C} : \operatorname{Im} z \geq 0\}$.

Proposition 3.2. *Suppose* (A1)$\sim$(A4). *Then* $\Sigma \subset \overline{\boldsymbol{C}_+}$.

Proof. Let A be the generator of $U(t)$, i.e. $U(t) = e^{-itA}$, and let $\mathbf{P}(\cdot)$ be the spectral projection for A. Then $\mathscr{D} := \{u \in \mathscr{H}; \mathbf{P}([-M, M])u = u \text{ for some } M\}$ is dense in $\mathscr{H}$, and $e^{-iwA} u$ is an entire function of w for each $u \in \mathscr{D}$. Moreover, $e^{-iwA}\mathscr{D} = \mathscr{D}$ for each $w \in \boldsymbol{C}$. We fix $z \in \mathscr{O}_0$ and f, g in $\mathscr{D}$ and write $f_\theta = U(-\theta) f$ for simplicity. Then we have the identity by (A3):

$$\big((T(\theta + t) - z)^{-1} f, g\big) = \big((T(\theta) - z)^{-1} f_t, g_t\big) \tag{57}$$

for all $\theta \in \boldsymbol{\Omega}_+$ and all $t \in \boldsymbol{R}$, and by the use of analyticity of both sides in t we get

$$\big((T(\theta + \eta) - z)^{-1} f, g\big) = \big((T(\theta) - z)^{-1} f_\eta, g_{\overline{\eta}}\big) \tag{58}$$

if $\theta \in \boldsymbol{\Omega}_+$, $\theta + \eta \in \boldsymbol{\Omega}_+$. Therefore, by (A4) we have

$$\begin{aligned}\big((T-z)^{-1}f,g\big) &= \lim_{t\to+0}\big((T(it)-z)^{-1}f,g\big)\\ &= \lim_{t\to+0}\big((T(\theta)-z)^{-1}f_{it-\theta},g_{\overline{it-\theta}}\big)\\ &= \big((T(\theta)-z)^{-1}f_{-\theta},g_{-\overline{\theta}}\big). \end{aligned}\tag{59}$$

Since $(T(\theta)-z)^{-1}$ and $(T-z)^{-1}$ are analytic in $\boldsymbol{C}_-\setminus\Sigma$ and in $\boldsymbol{C}_-$, respectively, the above equality holds for all $z\in\boldsymbol{C}_-\setminus\Sigma$. Since $\{f_{-\theta};f\in\mathscr{D}\}=\{g_{-\overline{\theta}};g\in\mathscr{D}\}=\mathscr{D}$ is dense in $\mathscr{H}$, we see that $(T(\theta)-z)^{-1}$ is analytic in $z\in\boldsymbol{C}_-$, and so $\boldsymbol{C}_-\cap\Sigma=\phi$. □

From the key relation (59), which is valid for each $z\in\boldsymbol{C}_-$, we get

$$\big((T-z)^{-1}f_\theta,g_{\overline{\theta}}\big)=\big((T(\theta)-z)^{-1}f,g\big)\tag{60}$$

for $\theta\in\boldsymbol{\Omega}_+$, $z\in\boldsymbol{C}_-$ and $f,\ g\in\mathscr{D}$.

Let $\mathbf{B}(\mathscr{H})$ be the set of bounded operators on $\mathscr{H}$.

LEMMA 3.3. *Let $Q(\theta)$ be a $\mathbf{B}(\mathscr{H})$-valued function defined on the strip $\Gamma:=\{\theta\in\boldsymbol{C};|\operatorname{Im}\theta|\le b\}$ for some $b>0$ such that*

(a) *$Q_{f,g}(\theta):=(Q(\theta)f,g)$ is bounded and continuous on Γ and is analytic in the interior $\Gamma^o:=\{\theta\in\boldsymbol{C},|\operatorname{Im}\theta|<b\}$ for each f,g in a dense set D_0 in $\mathscr{H}$ and that*

(b)
$$\|Q(\theta+t)\|=\|Q(\theta)\|$$

for all $\theta\in\Gamma$ and all $t\in\boldsymbol{R}$.
Then $Q(\theta)$ is a $\mathbf{B}(\mathscr{H})$-valued bounded analytic function of θ in Γ^o.

PROOF. Since

$$\sup_{t\in\boldsymbol{R}}|Q_{f,g}(t\pm ib)|\le\|Q(\pm ib)\|\|f\|\|g\|$$

by (b), we use Hadamard's three line theorem (see [**15**]) to have

$$|Q_{f,g}(\theta)|\le\|Q(-ib)\|^{(1/2)(1-(\operatorname{Im}\theta/b))}\|Q(ib)\|^{(1/2)(1+(\operatorname{Im}\theta/b))}\|f\|\|g\|,\quad \theta\in\Gamma$$

for $f,g\in D_0$. This implies that $\|Q(\theta)\|$ is bounded on Γ since D_0 is dense. Therefore it follows that $(Q(\theta)f,g)$ is analytic in Γ^o for all $f,g\in\mathscr{H}$, and so $Q(\theta)$

is a $\mathbf{B}(\mathscr{H})$-valued bounded analytic function of $\theta \in \Gamma^o$. □

For $E \in \boldsymbol{R}$, let γ be a positively-oriented small circle $|z-E| = \varepsilon$ enclosing E with $\{z \in \boldsymbol{C}; 0 < |z-E| \le \varepsilon\} \cap \Sigma = \phi$ and let

$$P_\theta(E) = -\frac{1}{2\pi i}\int_\gamma (T(\theta) - z)^{-1} dz.$$

Then this operator is the eigenprojection associated with $E \in \sigma_d(T(\theta)) = \Sigma$ if $E \in \Sigma$ and $P_\theta(E) = 0$ otherwise. Moreover, for each $E \in \Sigma$ the projection-valued function $P_\theta(E)$ is analytic in $\theta \in \boldsymbol{\Omega}_+$. In particular, the dimension of the range of $P_\theta(E)$ is independent of θ for each E.

The following is our main result in this section. Let $\mathbf{P}_s(\cdot)$ be the spectral projection of $T(s)$ for $s \in \boldsymbol{R}$.

Theorem 3.4. *Suppose* (A1)$\sim$(A4). *Then*

(a) $\sigma_{\mathrm{d}}(T(\theta)) \cap \boldsymbol{R} = \sigma_{\mathrm{p}}(T)$ *for all* $\theta \in \boldsymbol{\Omega}_+$. *Moreover, for each* $E \in \sigma_{\mathrm{p}}(T)$ *and* $s \in \boldsymbol{R}$, *we have*

$$\lim_{\boldsymbol{\Omega}_+ \ni \theta \to s} \|P_\theta(E) - \mathbf{P}_s(\{E\})\| = 0. \tag{61}$$

In particular, the eigenvalues of T *are discrete and each eigenvalue has finite multiplicity.*

(b) *Either*

(I) T *has a purely discrete spectrum, i.e.* $\sigma(T) = \sigma_{\mathrm{d}}(T)$

or

(II) $\sigma(T) = \boldsymbol{R}$, $\sigma_{\mathrm{sc}}(T) = \phi$

holds. In particular, we have $\sigma(T) \setminus \sigma_{\mathrm{p}}(T) \subset \sigma_{\mathrm{ac}}(T)$ *in the case of* (II).

(c) *If* $\Sigma \cap (\boldsymbol{C} \setminus \boldsymbol{R}) \ne \phi$ *or* $\Sigma = \phi$, *then the case* (II) *holds. Thus,* $\Sigma = \sigma_{\mathrm{p}}(T)$ *in the case of* (I).

(d) *Suppose the case* (I) *above holds and fix* $z \notin \sigma_{\mathrm{d}}(T)$. *Then the resolvent* $(T(\theta) - z)^{-1}$ *has an analytic continuation of* θ *from* $\boldsymbol{\Omega}_+$ *to* $\boldsymbol{\Omega} := \{\theta \in \boldsymbol{C}; |\operatorname{Im}\theta| < a\}$.

Hereafter we call T a *boundary value* of the analytic family $\{T(\theta)\}_{\theta \in \boldsymbol{\Omega}_+}$ and each element of Σ a *resonance* of T, when $\{T(\theta)\}_{\theta \in \boldsymbol{\Omega}_+}$ is given.

Now we discuss two simple and typical examples of T from the point of view of this theorem, though the detail is omitted. Let us consider two Schrödinger operators

$$H_+ := -\Delta + |x|^2, \quad H_- := \Delta + |x|^2 \quad \text{in } L^2(\boldsymbol{R}^3).$$

It is known that they are essentially self-adjoint on $\mathscr{S}$ (see e.g. [**10**], [**15**]) and H_+ is known as the harmonic oscillator.

We denote by the same notation $H_\pm$ their self-adjoint extensions. Define

$$H_+(\theta) := -e^{-2\theta}\Delta + e^{2\theta}|x|^2, \quad |\operatorname{Im}\theta| < \frac{\pi}{4}.$$

Then we can prove that each $H_+(\theta)$ is a closed operator with domain $D(H_+(\theta)) = D(\Delta) \cap D(|x|^2)$ and has compact resolvent and that the family $\{H_+(\theta)\}$ forms an analytic family of type (A). Moreover, we can see that H_+ is a boundary value of the analytic family restricted to $0 < \operatorname{Im}\theta < \pi/4$ and that H_- is a boundary value of the family of operators

$$\begin{aligned} H_-(\theta) &:= e^{-2\theta}\Delta + e^{2\theta}|x|^2 = e^{(\pi/2)i}\big(- e^{-2(\theta-(\pi/4)i)}\Delta + e^{2(\theta-(\pi/4)i)}|x|^2\big) \\ &= iH_+\left(\theta - \frac{\pi}{4}i\right) \end{aligned}$$

for $0 < \operatorname{Im}\theta < \pi/4$. Since $H_+(\theta)$ is an analytic family of type (A) for $-\pi/4 < \operatorname{Im}\theta < \pi/4$, the theorem implies that H_+ is of type (I) in (b). In particular, $\sigma(H_+(\theta)) = \sigma_{\mathrm{d}}(H_+(\theta)) = \sigma_{\mathrm{d}}(H_+) = \{\lambda_{lmn}; l,\, m,\, n = 0, 1, 2, \dots\}$, where

$$\lambda_{lmn} = (2l+1) + (2m+1) + (2n+1) = 2(l+m+n) + 3. \tag{62}$$

Furthermore, by virtue of this fact we know that $\sigma(H_-(\theta)) = \sigma_{\mathrm{d}}(H_-(\theta)) = \{i\lambda_{lmn}; l,\, m,\, n = 0, 1, 2, \dots\}$, i.e., H_- has nonreal resonances. Thus, it follows by (c) that H_- is of type (II) and has a purely absolutely continuous spectrum with $\sigma(H_-) = \boldsymbol{R}$.

Before giving the proof of the theorem we define $T(\theta) := T(\overline{\theta})^*$ for $\theta \in \boldsymbol{\Omega}_-$, where $\boldsymbol{\Omega}_- := \{\theta \in \boldsymbol{C}; -a < \operatorname{Im}\theta < 0\}$. Then $\sigma(T(\theta)) = \sigma_d(T(\theta)) = \overline{\Sigma} := \{z \in \boldsymbol{C}; \overline{z} \in \Sigma\}$ and $(T(\theta) - z)^{-1} = ((T(\overline{\theta}) - \overline{z})^{-1})^*$, $\theta \in \boldsymbol{\Omega}_-$, $z \in \overline{\mathscr{O}_0}$, where $\overline{\mathscr{O}_0} := \{z \in \boldsymbol{C}; \overline{z} \in \mathscr{O}_0\}$. Moreover, the projection $P_\theta(E)$ is defined also for $\theta \in \boldsymbol{\Omega}_-$. The assumptions (A1), (A2) and (A3) are satisfied even if $\boldsymbol{\Omega}_+$ is replaced by $\boldsymbol{\Omega}_-$, and moreover,

$$\text{(A4)}' \qquad w - \lim_{t\to -0}(T(it) - z)^{-1} = (T - z)^{-1}, \quad z \in \overline{\mathscr{O}_0}$$

follows from (A4) immediately.

Proof of Theorem 3.4. We use the idea of the dilation analytic method by Aguilar and Combes [**2**] (see also [**16**]).

(a) Let $E \in \boldsymbol{R}$ and $\theta \in \boldsymbol{\Omega}_+$. We know that the resolvent $(T(\theta)-z)^{-1}$ has the form (see e.g. [**12**], [**16**])

$$(T(\theta)-z)^{-1} = -\frac{P_\theta(E)}{z-E} - \sum_{k=2}^{L} \frac{B_{-k}}{(z-E)^k} + B(z), \tag{63}$$

for z near E, where each B_{-k} is a finite rank operator and $B(z)$ is analytic near E. We first note that a functional calculus for the self-adjoint operator T and (59) give

$$\begin{aligned}(\mathbf{P}_0(\{E\})f,g) &= \lim_{\varepsilon\to+0}(-i\varepsilon)\big((T-(E-i\varepsilon))^{-1}f,g\big)\\ &= \lim_{\varepsilon\to+0}(-i\varepsilon)\big((T(\theta)-(E-i\varepsilon))^{-1}f_{-\theta},g_{-\overline{\theta}}\big)\end{aligned}$$

for f, $g \in \mathscr{D}$. Therefore, noting that $\{f_\theta; f \in \mathscr{D}\} = \mathscr{D}$ is dense for each $\theta \in \boldsymbol{C}$, we see that $B_j = 0$ for all $j = 2, \dots, L$ and

$$\big(\mathbf{P}_0(\{E\})f_\theta, g_{\overline{\theta}}\big) = (P_\theta(E)f,g) \tag{64}$$

for $\theta \in \boldsymbol{\Omega}_+$ if f, $g \in \mathscr{D}$ since $(f_{-\theta})_\theta = f$. Similarly, (64) is also true for $\theta \in \boldsymbol{\Omega}_-$ because $P_\theta(E)^* = P_{\overline{\theta}}(E)$ for $E \in \boldsymbol{R}$. Now we define $P_\theta(E) := \mathbf{P}_\theta(\{E\})$ for $\theta \in \boldsymbol{R}$. Consequently, we see that (64) is true for all $\theta \in \boldsymbol{\Omega}$ and that

$$\|P_{\theta+t}(E)\| = \|P_\theta(E)\|$$

for $\theta \in \boldsymbol{\Omega}$ and $t \in \boldsymbol{R}$. Let

$$P_{f,g}(\theta) := (P_\theta(E)f,g)$$

for $\theta \in \boldsymbol{\Omega}$. If $f, g \in \mathscr{D}$, this is equal to the entire function $(\mathbf{P}_0(\{E\})f_\theta, g_{\overline{\theta}})$ of θ. Fix $b > 0$ with $b < a$. Then

$$\big|P_{f,g}(\theta)\big| \le \|f_\theta\|\|g_{\overline{\theta}}\| \le \Big(\max_{|t|\le b}\|f_{it}\|\Big)\Big(\max_{|t|\le b}\|g_{it}\|\Big)$$

if $|\operatorname{Im}\theta| \le b$. Thus we can apply Lemma 3.3 to conclude that $P_\theta(E)$ is analytic in $\boldsymbol{\Omega}$ because $b > 0$ is arbitrary if $b < a$. In particular, we have

$$P_\theta(E) \to P_{\theta_0}(E) \tag{65}$$

as $\boldsymbol{\Omega}_+ \ni \theta \to \theta_0 \in \boldsymbol{R}$ in the operator norm and so

$$\big\|P_\theta(E) - P_{\theta_0}(E)\big\| < 1$$

if $|\theta - \theta_0|$ is small enough. Since dim $P_\theta(E)$ is independent of $\theta \in \boldsymbol{\Omega}_+$, the above inequality implies that $\dim P_\theta(E)$ is actually independent of $\theta \in \boldsymbol{\Omega}$. Therefore we have $\sigma_{\mathrm{p}}(T) = \Sigma \cap \boldsymbol{R}$.

(b) We first note that (59) holds for any $z \in \boldsymbol{C}_-$ and $\theta \in \boldsymbol{\Omega}_+$. Thus, for any compact interval $J \subset \boldsymbol{R} \setminus \sigma_{\mathrm{p}}(T)$ the limit

$$\begin{aligned}\lim_{\varepsilon \to +0} \big((T - (\lambda - i\varepsilon))^{-1} f, g\big) &= \lim_{\varepsilon \to +0} \big((T(\theta) - (\lambda - i\varepsilon))^{-1} f_{-\theta}, g_{-\overline{\theta}}\big) \\ &= \big((T(\theta) - \lambda)^{-1} f_{-\theta}, g_{-\overline{\theta}}\big)\end{aligned}$$

exists uniformly in $\lambda \in J$ for any $f, g \in \mathscr{D}$ and $\theta \in \boldsymbol{\Omega}_+$ since J is contained in $\boldsymbol{C} \setminus \Sigma$, which is the resolvent set of $T(\theta)$ for $\theta \in \boldsymbol{\Omega}_+$. This means that $J \subset \rho(T)$ or $J \subset \sigma_{\mathrm{ac}}(T)$ since $\mathscr{D}$ is dense. Suppose $J \subset \rho(T)$. Then, $(T - z)^{-1}$ has the analytic continuation from $\boldsymbol{C}_-$ to $\boldsymbol{C}_+$ across the interval J. Thus, (59) holds for $z \in \boldsymbol{C}_+ \setminus \Sigma$ and so $((T - z)^{-1} f, g)$ is a meromorphic function with the poles $\Sigma \cap \boldsymbol{R} = \sigma_{\mathrm{p}}(T)$. Since $\mathscr{D}$ is dense this means $\sigma(T) = \sigma_{\mathrm{d}}(T)$.

(c) Suppose $\Sigma \cap (\boldsymbol{C} \setminus \boldsymbol{R}) \neq \phi$ and there would be an interval $J \subset \rho(T) \cap \boldsymbol{R}$, then (59) holds for $z \in \boldsymbol{C}_+ \setminus \Sigma$. The right-hand side in (59) has poles at $\Sigma \cap (\boldsymbol{C} \setminus \boldsymbol{R})$ for some $f, g \in \mathscr{D}$, but the left-hand side is analytic in $\boldsymbol{C}_+$. This is a contradiction. Suppose $\Sigma = \phi$. Then $\sigma_{\mathrm{p}}(T) = \phi$ by (a) and the case (II) should hold.

(d) Suppose $\sigma(T) = \sigma_d(T) = \Sigma$, then since both $(T(\theta) - z)^{-1}$ and $(T - z)^{-1}$ are analytic in $z \in \boldsymbol{C} \setminus \sigma_d(T)$, (60) holds for all $z \notin \sigma_d(T)$ and $\theta \in \boldsymbol{\Omega}_+$ if $f, g \in \mathscr{D}$. Recall that $T(\theta) = U(\theta) T U(\theta)^*$ for $\theta \in \boldsymbol{R}$ and $T(\theta) = T(\overline{\theta})^*$ for $\theta \in \boldsymbol{\Omega}_-$. Then we can easily see that (60) holds for all $\theta \in \boldsymbol{\Omega}$ if $z \notin \sigma_d(T)$ and $f, g \in \mathscr{D}$ since $z \notin \overline{\Sigma} = \sigma_d(T)$ and

$$\begin{aligned}\big((T(\theta) - z)^{-1} f, g\big) &= \overline{\big((T(\overline{\theta}) - \overline{z})^{-1} g, f\big)} \\ &= \overline{\big((T - \overline{z})^{-1} g_{\overline{\theta}}, f_\theta\big)} \\ &= \big((T - z)^{-1} f_\theta, g_{\overline{\theta}}\big)\end{aligned}$$

for $\theta \in \boldsymbol{\Omega}_-$. Fix $z \notin \sigma_d(T)$ and define

$$R_{f,g}(\theta) := \big((T(\theta) - z)^{-1} f, g\big)$$

for $f, g \in \mathscr{H}$. Since

$$\|(T(\theta + t) - z)^{-1}\| = \|(T(\theta) - z)^{-1}\|$$

for $\theta \in \boldsymbol{\Omega}$ and $t \in \boldsymbol{R}$, and since $R_{f,g}(\theta) = ((T - z)^{-1} f_\theta, g_{\overline{\theta}})$ for $f, g \in D$, we can prove that $(T(\theta) - z)^{-1}$ is analytic in $\boldsymbol{\Omega}$, as (a) is proved by using Lemma 3.3. □

4. Resonances.

In this section we give the proofs of Theorems 1.1, 1.2 and 1.3. We apply Theorem 3.4 to the relativistic Schrödinger operators $L_\pm(c)$ as follows; $\mathscr{H} = L^2(\boldsymbol{R}^3)$, $\boldsymbol{\Omega}_+ = \Omega_+$, $T = L_\pm(c)$, $T(\theta) = L_\pm(c, \theta)$, $U(t) = \mathscr{U}(t)$. Indeed, Proposition 2.1 guarantees this application, and it will be shown that $L_+(c)$ is of type (I) and $L_-(c)$ is of type (II) in (b) of Theorem 3.4.

PROOF OF THEOREM 1.1. Taking account of Proposition 2.1, we have the theorem as an immediate consequence of Proposition 3.1. □

PROOF OF THEOREM 1.2. As stated in the beginning of this section, Theorem 3.4 can be applied to $L_-(c)$, and then it follows that $\sigma(L_-(c,\theta)) = \sigma_{\mathrm{d}}(L_-(c,\theta)) = \Sigma_-(c)$ is independent of $\theta \in \Omega_+$ and $\sigma(L_-(c,\theta)) \cap \boldsymbol{R} = \sigma_{\mathrm{p}}(L_-(c))$ is a discrete set. Moreover, the multiplicity of each eigenvalue of $L_-(c)$ is finite. We have only to prove that $L_-(c)$ is not of type I. We fix $\theta \in \Omega_+$. Then it follows from (55) that $\sigma(L_-(c,\theta)) \cap \boldsymbol{R}$ is contained in a bounded interval. Since $\sigma(L_-(c,\theta)) = \Sigma_-(c)$, there exists a constant $K > 0$ independent of θ such that $\sigma_{\mathrm{p}}(L_-(c)) \cap ((-\infty, -K] \cup [K, \infty)) = \phi$. Hence the number of eigenvalues (if exist) is finite and each multiplicity is finite. Since the dimension of $L^2(\boldsymbol{R}^3)$ is infinite, this implies that $L_-(c)$ should not be of type (I). Consequently, we have proved the theorem. □

The rest of this section is devoted to the proof of Theorem 1.3. We write

$$L(c,\theta) = L_1(c,\theta) + W(c,\theta), \tag{66}$$

where

$$L_1(c,\theta) := \begin{pmatrix} L_+(c,\theta) I_2 & 0 \\ 0 & L_-(c,\theta) I_2 \end{pmatrix}.$$

LEMMA 4.1. *Fix $\theta \in \Omega$ and let*

$$W_0(c,\theta) := U_c(e^{-\theta}D)V_{\theta,0}(x)U_c(e^{-\theta}D)^{-1} - V_{\theta,0}(x),$$
$$W_\infty(c,\theta) := U_c(e^{-\theta}D)V_{\theta,\infty}(x)U_c(e^{-\theta}D)^{-1} - V_{\theta,\infty}(x).$$

Then, there exists a constant $K > 0$ such that

$$\left\|\langle x\rangle\langle D\rangle W_\infty(c,\theta)\langle x\rangle^{-M}\right\| \le K, \quad \left\|W_\infty(c,\theta)\langle x\rangle^{-M+1}\right\| \le Kc^{-1} \tag{67}$$

for all $c \ge 1$, and

$$\lim_{c\to\infty} W_0(c,\theta)f = 0 \tag{68}$$

strongly for each $f \in L^2(\boldsymbol{R}^3)^4$. In particular, we have

$$\lim_{c\to\infty} W_0(c,\theta)\left(\sqrt{-\Delta+1} + \langle x\rangle^M\right)^{-1} = 0 \tag{69}$$

in the operator norm.

PROOF. Using (19) and the assumption (V2) we can prove (67) with the help of the calculus of pseudodifferential operators. Since $U(e^{-\theta}D/c)$ converges strongly to I as $c \to \infty$, we have (68). (69) follows immediately from (68) since $(\sqrt{-\Delta+1} + \langle x\rangle^M)^{-1}$ is compact. $\square$

PROPOSITION 4.2.

(a) *$L(c,\theta)$ defined on $\mathscr{S}^4$ is closable and its closure (denoted by the same notation $L(c,\theta)$) has domain $D(L(c,\theta)) = (D_M)^4$ for $\theta \in \Omega_+$.*
(b) *The resolvent set of $L(c,\theta)$, $\theta \in \Omega_+$, is not empty and its resolvent $(L(c,\theta) - z)^{-1}$ is compact. Moreover, $L(c,\theta)$ is an analytic family of type* (A) *in $\theta \in \Omega_+$.*
(c) *There is a large constant $T > 0$ independent of c such that*

$$s - \lim_{\Omega_+ \ni \theta \to 0} (L(c,\theta) - z)^{-1} = (L(c) - z)^{-1} \tag{70}$$

for all $z \in \boldsymbol{C}$ with $\operatorname{Re} z < -T$ and $\operatorname{Im} z < -T$.

PROOF.

(a) Since both $L_+(c,\theta)$ and $L_-(c,\theta)$ are closed operators with domain D_M and $\mathscr{S}$ is a common core for them, $L_1(c,\theta)$ is a closed operator with domain $(D_M)^4$

and $\mathscr{S}^4$ is a core. Thus (67) shows that

$$\begin{aligned} &W_\infty(c,\theta)(L_1(c,\theta)-z)^{-1} \\ &\quad = \langle x\rangle^{-1}\langle D\rangle^{-1}\cdot\langle D\rangle\langle x\rangle W_\infty(c,\theta)\langle x\rangle^{-M}\cdot\langle x\rangle^{M}(L_1(c,\theta)-z)^{-1} \end{aligned}$$

is compact since $\langle x\rangle^{-1}\langle D\rangle^{-1}$ is compact. Moreover, $W_0(c,\theta)(L_1(c,\theta)-z)^{-1}$ is also compact because $(L_1(c,\theta)-z)^{-1}$ is compact. Hence $W(c,\theta)$ is $L_1(c,\theta)$-compact and thus $L(c,\theta)$ is a closed operator with domain $(D_M)^4$ and $\mathscr{S}^4$ is a core.

(b) We first prove the estimate

$$\big\|\langle x\rangle^{M-1}(L_{+,\infty}(c,\theta)-z)^{-1}\big\| \le K|\operatorname{Re} z|^{-\min(1/M,1)} \tag{71}$$

for all z such that $(-\operatorname{Re} z)>0$ is sufficiently large. When $0<M\le 1$, this follows immediately from (48), and thus we shall consider the case $M>1$. If $(-\operatorname{Re} z)>0$ is sufficiently large, then (46) implies

$$\big(L_{+,\infty}(c,\theta)-z\big)^{-1} = q_+(z,x,D)(1+r_0(z,x,D))^{-1} \tag{72}$$

and $\|r_0(z,x,D)\|<1/2$. The symbol $q_+(z,x,\xi)$ satisfies

$$\begin{aligned} \big|\partial_\xi^\alpha\partial_x^\beta(\langle x\rangle^{M-1}q_+(z,x,\xi))\big| &\le K_{\alpha\beta}\frac{\langle x\rangle^{M-1}}{|\operatorname{Re} p_+(z,x,\xi)|} \le K'_{\alpha\beta}\frac{\langle x\rangle^{M-1}}{\operatorname{Re} V_{\theta,\infty}(x)-\operatorname{Re} z} \\ &\le K''_{\alpha\beta}\frac{1}{(\langle x\rangle^{M}-\operatorname{Re} z)^{1/M}} \le K'''_{\alpha\beta}\frac{1}{(-\operatorname{Re} z)^{1/M}}, \end{aligned}$$

from which we have

$$\big\|\langle x\rangle^{M-1}q_+(z,x,D)\big\| \le K(-\operatorname{Re} z)^{-1/M},$$

and so

$$\big\|\langle x\rangle^{M-1}(L_{+,\infty}(c,\theta)-z)^{-1}\big\| \le K'(-\operatorname{Re} z)^{-1/M}.$$

Thus we have (71). In a similar way we have

$$\big\|\langle x\rangle^{M-1}(L_{-,\infty}(c,\theta)-z)^{-1}\big\| \le K|\operatorname{Im} z|^{-\min(1/M,1)} \tag{73}$$

for all z such that $(-\operatorname{Im} z)>0$ is sufficiently large. Therefore, by (67) we have shown that

$$\left\| W_\infty(c,\theta)(L_{1,\infty}(c,\theta)-z)^{-1} \right\| \le K|z|^{-\min(1/M,1)}$$

for $z \in \mathscr{Z}_T := \{z \in \boldsymbol{C}; (9/8)\pi < \arg z < (11/8)\pi,\ |z| > T\}$ if T is large enough, where $L_{1,\infty}(c,\theta)$ is defined by replacing $L_\pm(c,\theta)$ by $L_{\pm,\infty}(c,\theta)$, respectively, in the definition of $L_1(c,\theta)$. If T is large, then we also see that

$$\left\| W_0(c,\theta)(L_{1,\infty}(c,\theta)-z)^{-1} \right\| \le K|z|^{-1}, \quad z \in \mathscr{Z}_T$$

by (48) and (54). Hence we have obtained the estimate

$$\left\| W(c,\theta)(L_{1,\infty}(c,\theta)-z)^{-1} \right\| \le K|z|^{-\min(1/M,1)}, \quad z \in \mathscr{Z}_T$$

for large T. Using (48), (54) and the resolvent equation, we have

$$(L_1(c,\theta)-z)^{-1} = (L_{1,\infty}(c,\theta)-z)^{-1}\big(I + V_{\theta,0}(c,\theta)(L_{1,\infty}(c,\theta)-z)^{-1}\big)^{-1},$$

if $z \in \mathscr{Z}_T$ with large T and

$$\left\| W(c,\theta)(L_1(c,\theta)-z)^{-1} \right\| < \frac{1}{2}.$$

Hence such z belongs to the resolvent set of $L(c,\theta)$ and

$$(L(c,\theta)-z)^{-1} = (L_1(c,\theta)-z)^{-1}\big(I + W(c,\theta)(L_1(c,\theta)-z)^{-1}\big)^{-1}.$$

Since $L_1(c,\theta)$ has compact resolvent, the above formula shows that $L(c,\theta)$ also has compact resolvent. Furthermore, it is not difficult to verify that $L(c,\theta)f$ ($f \in (D_M)^4$) is analytic in θ. Therefore $\{L(c,\theta)\}_{\theta\in\Omega_+}$ is an analytic family of type (A). Let $\mathscr{U}_4(t) := \mathscr{U}(t)I_4$, where $\mathscr{U}(t)$ is the dilation group. Then by definition we see that

$$\mathscr{U}_4(t)L(c,\theta)\mathscr{U}_4(t)^{-1} = L(c,\theta+t)$$

for all $\theta \in \Omega_+$ and all $t \in \boldsymbol{R}$. Thus the spectrum of $L(c,\theta)$ is independent of $\theta \in \Omega_+$.

(c) Let us introduce a family of operators

$$\tilde{H}(c,\theta) := ce^{-\theta}\alpha\cdot D + \beta mc^2 - mc^2 + V_\theta(x)$$

for $\theta \in \Omega_+$. Then, noting that

$$L(c,\theta) = U_c(e^\theta D)\tilde{H}(c,\theta)U_c(e^\theta D)^{-1} \tag{74}$$

on $\mathscr{S}^4$ and that both $U_c(e^\theta D)$ and $U_c(e^\theta D)^{-1}$ map $\mathscr{S}^4$ onto $\mathscr{S}^4$ and $(D_M)^4$ onto $(D_M)^4$, respectively, we know that $\tilde{H}(c,\theta)$, $\theta \in \Omega_+$ is a closed operator with domain $(D_M)^4$ and so has compact resolvent. Moreover, $\sigma(\tilde{H}(c,\theta)) = \sigma(L(c,\theta))$ is independent of $\theta \in \Omega_+$. Using (50) we easily see that the numerical range $Nu(\tilde{H}(c,\theta))$ of $\tilde{H}(c,\theta)$ is contained in the half plane

$$\operatorname{Im} z \geq -\tan(\operatorname{Im}\theta)\operatorname{Re} z + b_1 \tag{75}$$

for some b_1 independent of $\theta \in \Omega_+$. Since the spectrum of $\tilde{H}(c,\theta)$ consists of eigenvalues only, it is contained in the half plane (75), and there exist constants $K > 0$ and $T > 0$ such that $\|(\tilde{H}(c,\theta) - z)^{-1}\| \leq 1/(|z| - K)$ for all z with $\operatorname{Re} z < -T$ and $\operatorname{Im} z < -T$. Thus the spectrum of $L(c,\theta)$ is contained in the half plane (75) and $\|(L(c,\theta) - z)^{-1}\|$ is uniformly bounded in $\theta \in \Omega_+$ for each z as above. Noting that $\mathscr{S}^4$ is a common core of $L(c)$ and $L(c,\theta)$, we have

$$\begin{aligned}(L(c,\theta) - z)^{-1}u& \\ = (L(c) - z)^{-1}u &- (L(c,\theta) - z)^{-1}(L(c,\theta) - L(c))(L(c) - z)^{-1}u\end{aligned} \tag{76}$$

for $u \in (L(c) - z)\mathscr{S}^4$. We can easily prove that

$$\|(L(c,\theta) - L(c))f\| \to 0$$

as $\Omega_+ \ni \theta \to 0$ for each $f \in \mathscr{S}^4$. Thus it follows from the uniform boundedness of $\|(L(c,\theta) - z)^{-1}\|$ that

$$s - \lim_{\Omega_+ \ni \theta \to 0}(L(c,\theta) - z)^{-1}f = (L(c) - z)^{-1}f \tag{77}$$

for each $f \in (L(c) - z)\mathscr{S}^4$ and z with $\operatorname{Re} z < -T$ and $\operatorname{Im} z < -T$. Since $(L(c) - z)\mathscr{S}^4$ is dense, we obtain (70). $\square$

This proposition shows that $L(c)$ is a boundary value of the analytic family $\{L(c,\theta)\}_{\theta\in\Omega_+}$.

PROPOSITION 4.3.

(a) *The set* $\Sigma(c) := \sigma_{\mathrm{d}}(L(c,\theta))$ *is independent of* $\theta \in \Omega_+$ *and satisfies*

$$\Sigma(c) \subset \overline{\boldsymbol{C}_+}, \quad \Sigma(c) \cap \boldsymbol{R} = \sigma_{\mathrm{p}}(L(c)).$$

Moreover, the multiplicity of each eigenvalue (*if exists*) *of* $L(c)$ *is finite.*

(b) $\sigma(L(c)) = \boldsymbol{R}$ *and* $\sigma_{\mathrm{sc}}(L(c)) = \phi$. *In particular,* $\sigma(L(c)) \setminus \sigma_{\mathrm{p}}(L(c)) \subset \sigma_{\mathrm{ac}}(L(c))$.

PROOF. Since $L(c)$ is a boundary value of the analytic family $\{L(c,\theta)\}_{\theta\in\Omega_+}$, we have only to prove that $L(c)$ is of type (II) in (b) of Theorem 3.4. To do so we first show that $L(c)$ is not bounded below. We take a function $\phi \in C_0^\infty(\boldsymbol{R}^3)^4$ such that $\|\phi\| = 1$ and $(\alpha \cdot D\phi, \phi) \neq 0$, and define $\phi_\lambda(x) := |\lambda|^{3/2}\phi(\lambda x)$ for $\lambda \in \boldsymbol{R}$, $\lambda \neq 0$. Then $\|\phi_\lambda\| = 1$ and $(c\alpha \cdot D\phi_\lambda, \phi_\lambda) = \lambda(c\alpha \cdot D\phi, \phi)$, and the support of $\phi_\lambda(x)$ is contained in some bounded set for all λ with $|\lambda| \geq 1$, and hence $((\beta mc^2 + V(x))\phi_\lambda, \phi_\lambda)$ is uniformly bounded for $|\lambda| \geq 1$. Thus $(H(c)\phi_\lambda, \phi_\lambda)$ is not bounded above nor below in λ, which implies that $H(c)$ is not bounded above nor below, and so is $L(c)$. However, the set of the eigenvalues of $L(c,\theta)$, $\theta \in \Omega_+$, is contained in the half plane (75), and so the set of the eigenvalues of $L(c)$ is bounded from below since $\sigma(L(c,\theta)) \cap \boldsymbol{R} = \sigma_{\mathrm{p}}(L(c))$. Therefore, we see that $L(c)$ should not be of type (I). Consequently, we have proved that $L(c)$ is of type (II). □

REMARK. We call an element of $\Sigma(c)$ a *resonance* of $L(c)$, and in [**1**] eigenvalues of $\tilde{H}(c,\theta)$, which are independent of $\theta \in \Omega_+$, are called resonances of $H(c) - mc^2$. The above equality (74) shows that these two sets of resonances coincide.

PROOF OF THEOREM 1.3. Theorem 1.3 follows from this proposition and the above remark. □

REMARKS.

(i) It is a natural question whether or not there is a resonance. In the next section we will show that there are resonances near each eigenvalue of the Schrödinger operator $S = -(2m)^{-1}\Delta + V(x)$ if c is large enough.

(ii) As shown in the proof of Proposition 4.2(c), the resonances of $L(c)$ are contained in the half-plane (75). Hence, the set of the eigenvalues of $H(c)$ is bounded from below.

5. Nonrelativistic Limit for $H(c) - mc^2$.

In this section we study the operators $L_\pm(c,\theta)$ and $L(c,\theta)$ in the nonrelativistic limit and give the proof of Theorem 1.4. Furthermore, we study the nonrelativistic limit of the spectral projection of $H(c) - mc^2$ at the end of this section.

Recall that we fix a constant $L > 0$ and an open interval $I \subset \boldsymbol{R}$ with $I \cap \sigma_d(S) = \{\lambda_j\}_{j=1}^N$ and define a set $\mathscr{O} := \{z \in \boldsymbol{C}; \operatorname{Re} z \in I, |\operatorname{Im} z| < L\}$.

Proposition 5.1. *Fix $\theta \in \Omega_+$. There are constants $c_0 > 0$ and $K > 0$ such that $\mathscr{O} \subset \rho(L_-(c,\theta))$ and*

$$\left\|(L_-(c,\theta) - z)^{-1}\right\| \le Kc^{-2} \tag{78}$$

for all $c > c_0$ and all $z \in \mathscr{O}$.

Proof. Combining (52) and the boundedness of $V_{\theta,0}$, we may assume that the numerical range of $L_-(c,\theta)$ is contained in the cone $C_2(c,\theta)$ with some new w_0, and so the spectrum of $L_-(c,\theta)$ is contained in the cone since the spectrum consists of only eigenvalues and each eigenvalue is in the numerical range. Hence, if c is sufficiently large, the set $\mathscr{O}$ and the cone have no intersection and the distance between them is larger than *const.*c^2. Hence we obtain the estimate (78). □

Let $S(\theta) := -(2m)^{-1}e^{-2\theta}\Delta + V_\theta(x)$, $\theta \in \Omega$. The following proposition shows that the Schrödinger operator S is of type (I) in Theorem 3.4.

Proposition 5.2.

(a) *$S(\theta)$ defined on $\mathscr{S}$ is closable and its closure (denoted by the same notation $S(\theta)$) has the domain $D(S(\theta)) = D(-\Delta) \cap L^2_M(\boldsymbol{R}^3)$.*
(b) *The resolvent set of $S(\theta)$ is not empty and its resolvent is compact. In particular, $S(\theta)$ has a purely discrete spectrum.*
(c) *$\{S(\theta)\}_{\theta\in\Omega}$ is an analytic family of type (A), and*

$$\mathscr{U}(t)S(\theta)\mathscr{U}(t)^{-1} = S(\theta + t)$$

for all $\theta \in \Omega$ and $t \in \boldsymbol{R}$.
(d) *The spectrum of $S(\theta)$ is independent of θ, and in particular, coincides with that of S.*

Outline of the proof. We see that $L_+(c,\theta)f \to S(\theta)f$ as $c \to \infty$ for each $f \in \mathscr{S}$ and $\theta \in \Omega$. Thus, since $s(c,\xi) \to |\xi|^2$ as $c \to \infty$, we have by Lemma 2.7

$$\|(S(\theta)-z)f\|^2+\|f\|^2 \geq K_1\|\Delta f\|^2+K_2\big\|\langle x\rangle^M f\big\|^2, \quad f \in \mathscr{S} \tag{79}$$

for some positive constants K_1 and K_2. Using this estimate and an argument similar to that in the proof of Proposition 2.1, we can obtain the proposition.

We next study the nonrelativistic limit of $L_+(c,\theta)$. First of all we consider the operators $\sigma_\theta(c,D)$ and $-e^{-2\theta}(2m)^{-1}\Delta$ with $\theta \in \Omega$ and $c \geq 1$. It is known that $\mathscr{S}$ is a common core of them and that $D(\sigma_\theta(c,D)) = H^1(\boldsymbol{R}^3)$ and $D(-e^{-2\theta}(2m)^{-1}\Delta) = H^2(\boldsymbol{R}^3)$. Moreover, it is easy to see that all $z \in \boldsymbol{C}$ with $\operatorname{Re} z < 0$ belong to the resolvent sets of them.

Lemma 5.3. *Let G be a compact set in $\{z \in \boldsymbol{C}; \operatorname{Re} z < 0\}$ and fix $\theta \in \Omega$. Then there is a constant $K > 0$ such that*

$$\sup_{z\in G}\left\|(\sigma_\theta(c,D)-z)^{-1}-\left(-\frac{e^{-2\theta}}{2m}\Delta - z\right)^{-1}\right\| \leq Kc^{-2} \tag{80}$$

for $c \geq 1$.

Proof. We estimate the symbol of $(\sigma_\theta(c,D)-z)^{-1}-(-e^{-2\theta}(2m)^{-1}\Delta-z)^{-1}$ for $\theta = 0$ for simplicity. By a simple calculation we have

$$\sqrt{c^2|\xi|^2+m^2c^4}-mc^2-\frac{|\xi|^2}{2m} = \frac{-c^2|\xi|^4}{2m\big(\sqrt{c^2|\xi|^2+m^2c^4}+mc^2\big)^2}$$

and so

$$\big(\sqrt{c^2|\xi|^2+m^2c^4}-mc^2-z\big)^{-1}-\left(\frac{|\xi|^2}{2m}-z\right)^{-1} = F_1F_2F_3,$$

where $F_1 = (2m)^{-1}(\sqrt{c^2|\xi|^2+m^2c^4}-mc^2)(\sqrt{c^2|\xi|^2+m^2c^4}-mc^2-z)^{-1}$, $F_2 = (\sqrt{c^2|\xi|^2+m^2c^4}+mc^2)^{-1}$ and $F_3 = |\xi|^2((2m)^{-1}|\xi|^2-z)^{-1}$. Since $|F_1|$, $c^2|F_2|$ and $|F_3|$ are uniformly bounded in $\xi \in \boldsymbol{R}^3$, $c \geq 1$ and $z \in G$, we have the desired result. $\square$

Using the above lemma we can prove that $L_+(c,\theta)$ converges to $S(\theta)$ in the norm resolvent sense.

Lemma 5.4. *Let G be a compact set in $\{z \in \boldsymbol{C}; \operatorname{Re} z < 0\}$ such that $G \cap (\sigma(S) \cup \sigma(L_+(c))) = \phi$ for all $c \geq 1$ and fix $\theta \in \Omega$. Then there is a constant $K > 0$ such that*

$$\sup_{z\in G}\left\|(L_+(c,\theta)-z)^{-1}-(S(\theta)-z)^{-1}\right\|\le Kc^{-2}$$

for $c\ge 1$.

Proof. Let $z\in G$. Then $z\in\rho(S(\theta))\cap\rho(L_+(c,\theta))$ and the following resolvent equation holds:

$$\begin{aligned}&(L_+(c,\theta)-z)^{-1}-(S(\theta)-z)^{-1}\\&\quad=-(L_+(c,\theta)-z)^{-1}(L_+(c,\theta)-S(\theta))(S(\theta)-z)^{-1}\end{aligned}\tag{81}$$

on $(S(\theta)-z)\mathscr{S}$. Since $\mathscr{S}$ is a core of $S(\theta)$ it holds on the whole space $L^2(\boldsymbol{R}^3)$. Similarly, we have

$$(S(\theta)-z)^{-1}=\left(-\frac{e^{-2\theta}}{2m}\Delta-z\right)^{-1}-\left(-\frac{e^{-2\theta}}{2m}\Delta-z\right)^{-1}V_\theta(S(\theta)-z)^{-1},\tag{82}$$

$$(L_+(c,\theta)-z)^{-1}=(\sigma_\theta(c,D)-z)^{-1}-(L_+(c,\theta)-z)^{-1}V_\theta(\sigma_\theta(c,D)-z)^{-1}.\tag{83}$$

Here we note that $(L_+(c,\theta)-z)^{-1}V_\theta\subset(V_{\overline{\theta}}(L_+(c,\overline{\theta})-\overline{z})^{-1})^*$ and $D(L_+(c,\theta))=D_M\subset D(V_\theta)$ allow us to consider $(L_+(c,\theta)-z)^{-1}V_\theta$ as a bounded operator. We also know from Lemma 2.7 that $\|(L_+(c,\theta)-z)^{-1}V_\theta\|$ is uniformly bounded in $c\ge 1$. Thus substituting (82) and (83) into the right-hand side of (81) and noting that

$$\left\|\left(-(2m)^{-1}e^{-2\theta}\Delta-z\right)^{-1}(L_+(c,\theta)-S(\theta))(\sigma_\theta(c,D)-z)^{-1}\right\|\le Kc^{-2}$$

for some $K>0$ by (80), we arrive at the desired result. □

Proposition 5.5. *Let G be a compact set in $\rho(S)$ and fix $\theta\in\Omega$. Then there are constants $c_0>0$ and $K>0$ such that $G\subset\rho(L_+(c,\theta))$ for $c\ge c_0$ and*

$$\sup_{z\in G}\left\|(L_+(c,\theta)-z)^{-1}-(S(\theta)-z)^{-1}\right\|\le Kc^{-2}$$

for $c\ge c_0$.

Proof. Lemma 5.4 implies that $L_+(c,\theta)$ converges to $S(\theta)$ in the generalized sense and so the proposition follows immediately from Theorem 2.25 and (3.10) in Chapter IV of [**12**], since $\rho(S)=\rho(S(\theta))$. □

This result implies that for each eigenvalue λ (with multiplicity n) of S there exist n eigenvalues (counting multiplicity) $\lambda_j(c)$, $j = 1, \dots, n$, of $L_+(c)$ near λ for large c and $\lambda_j(c) \to \lambda$ as $c \to \infty$.

Define $\mathscr{O}_\varepsilon := \mathscr{O} \setminus \cup_{j=1}^N B_\varepsilon(\lambda_j)$. Then we have the following proposition.

PROPOSITION 5.6. *Fix $\theta \in \Omega_+$. Then for any $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that $\mathscr{O}_\varepsilon \subset \rho(L(c,\theta)) \cap \rho(S)$ for $c > c_\varepsilon$ and*

$$\lim_{c\to\infty} \sup_{z\in\mathscr{O}_\varepsilon} \left\| (L(c,\theta) - z)^{-1} - \begin{pmatrix} (S(\theta)-z)^{-1} I_2 & 0 \\ 0 & 0 \end{pmatrix} \right\| = 0. \tag{84}$$

PROOF. If c is large enough, it follows from Propositions 5.1 and 5.5 that $\mathscr{O}_\varepsilon \subset \rho(L_1(c,\theta))$ and that $\|(L_\pm(c,\theta) - z)^{-1}\|$ is uniformly bounded in c and $z \in \mathscr{O}_\varepsilon$, and thus it follows from Lemmas 2.6 and 2.7 that $\|\langle x\rangle^M (L_\pm(c,\theta) - z)^{-1}\|$ is uniformly bounded in c and $z \in \mathscr{O}_\varepsilon$. Therefore by the use of (67) we have for large c

$$\sup_{z\in\mathscr{O}_\varepsilon} \left\| W_\infty(c,\theta)(L_1(c,\theta) - z)^{-1} \right\| \le K c^{-1}. \tag{85}$$

Since $c \ge 1$, a simple calculation shows that $2s(c,\xi) - |\xi| \ge 0$ for $|\xi| \ge 1$ and $s(c,\xi) + 1 \ge |\xi|$ for $|\xi| \le 1$. Thus we have immediately

$$\sup_{c\ge 1} \left\| \sqrt{-\Delta+1}(s(c,D)+1)^{-1} \right\| < \infty,$$

and so Lemmas 2.6 and 2.7 imply

$$\sup_{c\ge 1, z\in\mathscr{O}_\varepsilon} \left\| \left(\sqrt{-\Delta+1} + \langle x\rangle^M\right)(L_1(c,\theta) - z)^{-1} \right\| < \infty.$$

Therefore, writing

$$\begin{aligned} &W_0(c,\theta)(L_1(c,\theta) - z)^{-1} \\ &\quad = W_0(c,\theta)\left(\sqrt{-\Delta+1} + \langle x\rangle^M\right)^{-1}\left(\sqrt{-\Delta+1} + \langle x\rangle^M\right)(L_1(c,\theta) - z)^{-1} \end{aligned}$$

and using (69), we get

$$\lim_{c\to\infty} \left\| W_0(c,\theta)(L_1(c,\theta) - z)^{-1} \right\| = 0$$

uniformly in $z \in \mathscr{O}_\varepsilon$. Consequently, we obtain

$$\lim_{c\to\infty} \left\| W(c,\theta)(L_1(c,\theta)-z)^{-1} \right\| = 0$$

uniformly in $z \in \mathscr{O}_\varepsilon$. Hence we can find a large c_ε such that the bounded inverse of $I + W(c,\theta)(L_1(c,\theta)-z)^{-1}$ exists, so that $\mathscr{O}_\varepsilon \subset \rho(L(c,\theta))$ for $c \geq c_\varepsilon$ and

$$(L(c,\theta)-z)^{-1} = (L_1(c,\theta)-z)^{-1}(I + W(c,\theta)(L_1(c,\theta)-z)^{-1})^{-1}.$$

Hence

$$\lim_{c\to\infty} \left\| (L(c,\theta)-z)^{-1} - (L_1(c,\theta)-z)^{-1} \right\| = 0 \tag{86}$$

uniformly in $z \in \mathscr{O}_\varepsilon$. Combining Propositions 5.1, 5.5 and this estimate, we get the desired result. □

Now we are in a position to prove Theorem 1.4.

Proof of Theorem 1.4. Let

$$P_j(A) := \frac{-1}{2\pi i}\int_{|z-\lambda_j|=\varepsilon} (A-z)^{-1}dz$$

be the eigenprojection for an operator A associated with the eigenvalues in the open disc $|z-\lambda_j| < \varepsilon$. Then we have

$$\lim_{c\to\infty} \left\| P_j(L(c,\theta)) - \begin{pmatrix} P_j(S(\theta))I_2 & 0 \\ 0 & 0 \end{pmatrix} \right\| = 0 \tag{87}$$

by Proposition 5.6. Thus, since $\dim P_j(S(\theta)) = \dim P_j(S)$, we have $\dim P_j(L(c,\theta)) = 2\dim P_j(S) = 2m_j$ if c is large. Since the resonances of $H(c)-mc^2$ coincide with those of $L(c)$ and since there is no resonance of $H(c)-mc^2$ in $\boldsymbol{C}_-$, we have proved the theorem. □

Using Proposition 5.6, we have a result on the nonrelativistic limit of the spectral projection of the Dirac operator.

Proposition 5.7. *Let $I = [\alpha,\beta]$ be an interval such that $I \cap \sigma(S) = \{\lambda_0\}$ with $\alpha < \lambda_0 < \beta$ or $I \cap \sigma(S) = \phi$. Then we have*

$$s - \lim_{c\to\infty} \mathbf{P}_{H(c)-mc^2}(I)f = \mathbf{P}f, \tag{88}$$

for each $f \in L^2(\boldsymbol{R}^3)^4$, where

$$\mathbf{P} = \begin{pmatrix} \mathbf{P}_S(\{\lambda_0\})I_2 & 0 \\ 0 & 0 \end{pmatrix} \tag{89}$$

if $I \cap \sigma(S) = \{\lambda_0\}$ and $\mathbf{P} = 0$ if $I \cap \sigma(S) = \phi$, where $\mathbf{P}_A(\cdot)$ denotes the spectral projection of a self-adjoint operator A.

Remark. A similar result replaced f by

$$\begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} f$$

has already been proved in more general cases in [**11**].

Proof. Since $U_c(D)\mathbf{P}_{H(c)-mc^2}(I)U_c(D)^{-1} = \mathbf{P}_{L(c)}(I)$ and $U_c(D)$, $U_c(D)^{-1} \to I$ strongly as $c \to \infty$, we have only to prove

$$s - \lim_{c\to\infty} \mathbf{P}_{L(c)}(I)f = \mathbf{P}f. \tag{90}$$

Using Theorem 1.4 we first note that there is a constant $c_0 > 0$ such that α, $\beta \notin \sigma_{\mathrm{p}}(L(c))$ for all $c > c_0$ since the real resonances of $L(c)$ coincide with the eigenvalues of $L(c)$. Let f be in the dense set

$$\bigl\{f \in L^2(\boldsymbol{R}^3)^4; f_\theta := \mathscr{U}_4(-\theta)f \text{ has an analytic continuation from } \boldsymbol{R} \text{ to } \boldsymbol{C}\bigr\}.$$

Then by the relation

$$\bigl((L(c)-z)^{-1}f, f\bigr) = \bigl((L(c,\theta)-z)^{-1}f_{-\theta}, f_{-\overline{\theta}}\bigr)$$

for $\operatorname{Im} z < 0$ (see (59)) we have

$$\begin{aligned}
(\mathbf{P}_{L(c)}(I)f, f) &= \lim_{\varepsilon\to+0} \frac{-1}{\pi} \operatorname{Im}\left(\int_\alpha^\beta \bigl((L(c)-E+i\varepsilon)^{-1}f, f\bigr)dE\right) \\
&= \lim_{\varepsilon\to+0} \frac{-1}{\pi} \operatorname{Im}\left(\int_\alpha^\beta \bigl((L(c,\theta)-E+i\varepsilon)^{-1}f_{-\theta}, f_{-\overline{\theta}}\bigr)dE\right) \\
&= \frac{-1}{\pi} \operatorname{Im}\left(\int_\gamma \bigl((L(c,\theta)-z)^{-1}f_{-\theta}, f_{-\overline{\theta}}\bigr)dz\right),
\end{aligned}$$

where γ is the positively oriented half round from α to β in the lower half-plane (in which there is no resonance of $L(c)$). Hence using Proposition 5.6 and denoting

$$Q_\theta(z) := \begin{pmatrix} (S(\theta)-z)^{-1}I_2 & 0 \\ 0 & 0 \end{pmatrix},$$

we get

$$\begin{aligned}
\lim_{c\to\infty}(\mathbf{P}_{L(c)}(I)f,f) &= \frac{-1}{\pi}\operatorname{Im}\left(\int_\gamma (Q_\theta(z)f_{-\theta}, f_{-\overline{\theta}})dz\right) \\
&= \frac{-1}{\pi}\operatorname{Im}\left(\int_\gamma (Q_0(z)f,f)dz\right) \\
&= \frac{-1}{2\pi i}\int_{\gamma+\widetilde{\gamma}} (Q_0(z)f,f)dz = (\mathbf{P}f,f),
\end{aligned}$$

where we denote by $\widetilde{\gamma}$ the positively oriented half round from β to α in the upper half-plane. Since both $\mathbf{P}_{L(c)}(I)$ and $\mathbf{P}$ are orthogonal projections, we can easily see that this result implies (90). □

6. Nonrelativistic limits for $H(c)+mc^2$ and $H(c)$.

In Section 5 we have considered the resonances of $H(c)-mc^2$ in a bounded domain $\mathscr{O}$ when c is large, and have shown that they are closely related with the eigenvalues of S in the domain. In this section we will consider those of $H(c)+mc^2$ and $H(c)$ in a bounded domain $\mathscr{O}_1$. We show that, if c is sufficiently large, there is no resonance of $H(c)$ in $\mathscr{O}_1$, but there are resonances of $H(c)+mc^2$ near "resonances" (in $\mathscr{O}_1$) of the operator $\widetilde{S} := (2m)^{-1}\Delta + V(x)$ defined on $\mathscr{S}$. Here we should remark that the essential self-adjointness of $\widetilde{S}$ requires some conditions, for example, $M \le 2$ in our assumptions. But, we do not need to assume the essential self-adjointness because, as we see below, the location of resonances of $H(c)+mc^2$ does not depend on the spectrum of a self-adjoint extension of $\widetilde{S}$ but the one of the dilated operator $\widetilde{S}(\theta) := (2m)^{-1}e^{-2\theta}\Delta + V_\theta(x)$, $\theta \in \Omega_+$. Therefore, we do not need any condition on $M > 0$ in the following results.

PROPOSITION 6.1.

(a) *$\widetilde{S}(\theta)$ defined on $\mathscr{S}$ is closable and its closure (denoted by the same notation $\widetilde{S}(\theta)$) has the domain $D(\widetilde{S}(\theta)) = D(-\Delta)\cap L^2_M(\boldsymbol{R}^3)$.*
(b) *The resolvent set of $\widetilde{S}(\theta)$ is not empty and its resolvent is compact. In particular, $\widetilde{S}(\theta)$ has a purely discrete spectrum.*

(c) $\{\widetilde{S}(\theta)\}_{\theta\in\Omega_+}$ *is an analytic family of type* (A), *and*

$$\mathscr{U}(t)\widetilde{S}(\theta)\mathscr{U}(t)^{-1} = \widetilde{S}(\theta+t)$$

for all $\theta \in \Omega_+$ *and* $t \in \boldsymbol{R}$.

(d) *The spectrum of* $\widetilde{S}(\theta)$ *is independent of* θ, *which is denoted by* $\widetilde{\Sigma}$.

Remark. We call an element of $\widetilde{\Sigma}$ a *resonance* of $\widetilde{S}$ even if there are many self-adjoint extensions of $\widetilde{S}$.

Outline of the proof. We see that $(L_-(c,\theta)+2mc^2)f \to \widetilde{S}(\theta)f$ as $c \to \infty$ for each $f \in \mathscr{S}$ and $\theta \in \Omega_+$. Thus, since $s(c,\xi) \to |\xi|^2$ as $c \to \infty$, we have by Lemma 2.6

$$\big\|(\widetilde{S}(\theta)-z)f\big\|^2 + \|f\|^2 \geq K_1\|\Delta f\|^2 + K_2\big\|\langle x\rangle^M f\big\|^2, \quad f \in \mathscr{S} \tag{91}$$

for some positive constants K_1 and K_2. Using the estimate we can obtain the proposition in a manner similar to that in the proof of Proposition 2.1.

Moreover, as in the proof of Proposition 5.5 we can prove

Proposition 6.2. *Let* G *be a compact set in* $\boldsymbol{C}\setminus\widetilde{\Sigma}$ *and fix* $\theta \in \Omega_+$. *Then there are constants* $c_0 > 0$ *and* $K > 0$ *such that* $G \subset \rho(L_-(c,\theta)+2mc^2)$ *for* $c \geq c_0$ *and*

$$\sup_{z\in G}\big\|(L_-(c,\theta)+2mc^2-z)^{-1} - (\widetilde{S}(\theta)-z)^{-1}\big\| \leq Kc^{-2}$$

for $c \geq c_0$.

By Propositions 6.1 and 6.2 we have the following corollary.

Corollary 6.3.

(a) $\widetilde{\Sigma} \subset \overline{\boldsymbol{C}_+}$.

(b) *If* $\widetilde{S} = (2m)^{-1}\Delta + V(x)$ *defined on* $\mathscr{S}$ *is an essentially self-adjoint operator, then the self-adjoint extension* (*denoted by the same notation* $\widetilde{S}$) *is of type* (II) *as a boundary value of the analytic family* $\{\widetilde{S}(\theta)\}_{\theta\in\Omega_+}$.

Remark. The essential self-adjointness and the absolute continuity of the spectrum of $\tilde{S}$ under the condition $M \leq 2$ have been extensively studied by many papers (see, e.g. [**24**] and its references).

Proof.

(a) If there exists an eigenvalue of $\widetilde{\Sigma}$ in $\boldsymbol{C}_-$, then Proposition 6.2 implies that there exist eigenvalues of $L_-(c,\theta)+2mc^2$ near it for large $c>0$. But, this contradicts the fact that the eigenvalues of $L_-(c,\theta)$ are all in $\overline{\boldsymbol{C}_+}$. Hence we have proved (a).

(b) Taking account of the fact that the numerical range of $\widetilde{S}(\theta)$ is contained in the cone $\{w\in\boldsymbol{C}; A_1\operatorname{Im}\theta<\arg(w-w_0)<\pi-A_1\operatorname{Im}\theta\}$ for some $w_0\in\boldsymbol{C}$ and $A_1>0$, we can prove, as in the proof of (f) in Proposition 2.1, that $(\widetilde{S}(\theta)-z)^{-1}f$ converges to $(\widetilde{S}-z)^{-1}f$ strongly as $\Omega_+\ni\theta\to 0$ for all $f\in L^2(\boldsymbol{R}^3)$ and for all z with $(-\operatorname{Im}z)>0$ sufficient large. Hence we can prove (b) as in the proof of Theorem 1.2. □

Let $\mathscr{O}_1$ be a bounded open set in $\boldsymbol{C}$. In the proof of (a) of Proposition 2.1 we see that there are positive constants K_0, b and b_0 such that $z\in\rho(L_+(c,\theta))$ and $\|(L_+(c,\theta)-z)^{-1}\|\le K_0(-b-\operatorname{Re}z)^{-1}$ if $\operatorname{Re}z<-b_0$. Thus the following proposition follows.

Proposition 6.4. *Fix $\theta\in\Omega_+$. There are constants $c_0>0$ and $K>0$ such that $\mathscr{O}_1\subset\rho(L_+(c,\theta)+2mc^2)$ and*

$$\left\|(L_+(c,\theta)+2mc^2-z)^{-1}\right\|\le Kc^{-2} \tag{92}$$

for all $c>c_0$ and all $z\in\mathscr{O}_1$. Moreover, the same results hold with mc^2 in place of $2mc^2$.

Let $\mathscr{O}_1\cap\widetilde{\Sigma}=\{\mu_j\}_{j=1,\dots,N_0}$ and let n_j be the algebraic multiplicity of the eigenvalue μ_j of $\widetilde{S}(\theta)$ (independent of $\theta\in\Omega_+$).

Now we state our main result in this section. Since we can prove this result in almost the same way as in the proof of Theorem 1.4 by considering $L(c)+2mc^2$ in place of $L(c)$ with the help of Propositions 6.2 and 6.4, we omit the proof.

Theorem 6.5. *For any small $\varepsilon>0$ there exists a constant $c_\varepsilon>0$ such that there is no resonance of $H(c)+mc^2$ in $\mathscr{O}_1\setminus\cup_{j=1}^{N_0}\overline{B_\varepsilon^+}(\mu_j)$ and there are $2n_j$ resonances of $H(c)+mc^2$ in $\overline{B_\varepsilon^+}(\mu_j)$ for each $j=1,\dots,N_0$, if $c>c_\varepsilon$.*

We next consider the case $H(c)$ and show that there is neither eigenvalue nor resonance in any bounded set in $\boldsymbol{C}$ if $c>0$ is sufficiently large. We note that the proof of Proposition 5.1 shows that the proposition with $L_-(c,\theta)+mc^2$ in place of $L_-(c,\theta)$ is valid. Thus, combining this with Proposition 6.4 we see that for each $\theta\in\Omega_+$ there is a constant $c_0>0$ such that $\mathscr{O}_1$ is contained in the resolvent set of $L_1(c,\theta)+mc^2$ and the estimate

$$\left\|(L_1(c,\theta)+mc^2-z)^{-1}\right\| \le Kc^{-2} \tag{93}$$

holds for all $c > c_0$ and all $z \in \mathscr{O}_1$. Hence, as in the proof of Proposition 5.6, we can prove $\mathscr{O}_1 \subset \rho(L(c,\theta)+mc^2)$ if $c > 0$ is sufficiently large, and so we have the following.

THEOREM 6.6. *Let $\mathscr{O}_1$ be a bounded open set. Then there is a constant $c_0 > 0$ such that there is neither eigenvalue nor resonance of $H(c)$ in $\mathscr{O}_1$ if $c > c_0$.*

Finally, we discuss a typical example, $V(x) = |x|^\gamma$, $\gamma > 0$, though the detail is omitted. It is easy to see that the potential satisfies our assumptions (V1)∼(V3). The operator $H_+^{(\gamma)} := -\Delta + |x|^\gamma$ in $L^2(\boldsymbol{R}^3)$ is self-adjoint with a core $\mathscr{S}$ and has a purely discrete spectrum $\{\nu_j\}_{j=1}^\infty$ and $H_+^{(\gamma)} = H_+^{(\gamma)}(0)$, where

$$H_+^{(\gamma)}(\theta) := -e^{-2\theta}\Delta + e^{\gamma\theta}|x|^\gamma = e^{-2\theta}\big(-\Delta + e^{(2+\gamma)\theta}|x|^\gamma\big), \quad \theta \in \boldsymbol{C}.$$

Note that, if $\gamma = 2$, each ν_j can be written explicitly (see Section 3). We can see that $H_+^{(\gamma)}(\theta)$ is a closed operator with domain $D(\Delta) \cap D(|x|^\gamma)$ for $-\pi < (2+\gamma)\operatorname{Im}\theta < \pi$, i.e., $\theta \in \Omega^\gamma := \{\theta \in \boldsymbol{C}; -\beta_0 < \operatorname{Im}\theta < \beta_0\}$, where $\beta_0 := \pi(2+\gamma)^{-1}$ and has compact resolvent. Moreover, the family $\{H_+^{(\gamma)}(\theta)\}$ forms an analytic family of type (A) and the set of their eigenvalues (independent of θ) coincides with the set $\{\nu_j\}_{j=1}^\infty$. Next we consider the operator $H_-^{(\gamma)} := \Delta + |x|^\gamma = -(-\Delta - |x|^\gamma)$. It is known that the operator $H_-^{(\gamma)}$ defined on $\mathscr{S}$ is an essentially self-adjoint operator if and only if $0 < \gamma \le 2$ (cf. [**15**]). But, for any $\gamma > 0$ we can see that

$$H_-^{(\gamma)}(\theta) := e^{-2\theta}\Delta + e^{\gamma\theta}|x|^\gamma = -e^{-2\theta}\big(-\Delta - e^{(2+\gamma)\theta}|x|^\gamma\big)$$

is a closed operator with domain $D(\Delta) \cap D(|x|^\gamma)$ and with a core $\mathscr{S}$ for $\theta \in \Omega_+^\gamma := \{\theta \in \boldsymbol{C}; 0 < \operatorname{Im}\theta < \beta_0\}$ and has compact resolvent. Moreover, there is a simple relation between $H_+^{(\gamma)}(\theta)$ and $H_-^{(\gamma)}(\theta)$,

$$H_-^{(\gamma)}(\theta) = -e^{-2i\beta_0}H_+^{(\gamma)}(\theta - i\beta_0),$$

for $\theta \in \Omega_+^\gamma$. Hence the set of the eigenvalues of $H_-^{(\gamma)}(\theta)$, which is independent of θ and is $\widetilde{\Sigma}$ in Proposition 6.1, coincides with the set $\{-e^{-2i\beta_0}\nu_j\}_{j=1}^\infty$. Proposition 6.2 implies that if $\widetilde{\Sigma}$ has a nonreal element λ there exists a nonreal resonance $\lambda(c)$ of $L_-(c)$ for large c such that $\lambda(c) + 2mc^2 \to \lambda$ as $c \to \infty$. Hence if $V(x) = |x|^\gamma$ and $m = 1/2$, then the above argument implies that $L_-(c)$ has nonreal resonances for large c. Furthermore, in this case Theorems 1.4 and 6.5 say that there are resonances of $H(c)$ near each $-e^{-2i\beta_0}\nu_j - mc^2$ and each $\nu_j + mc^2$ if $c > 0$ is large.

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