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Generalized Whittaker functions on $GSp(2, \boldsymbol{R})$ associated with indefinite quadratic forms

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Abstract. We study the generalized Whittaker models for $G = GSp(2, \boldsymbol{R})$ associated with indefinite binary quadratic forms when they arise from two standard representations of G: (i) a generalized principal series representation induced from the non-Siegel maximal parabolic subgroup and (ii) a (limit of) large discrete series representation. We prove the uniqueness of such models with moderate growth property. Moreover we express the values of the corresponding generalized Whittaker functions on a one-parameter subgroup of G in terms of the Meijer G-functions.

0. Introduction.

Let $\mathbf{G} = GSp(2)$ be the symplectic group with similitude defined over the field $\boldsymbol{Q}$ of rational numbers. When we wish to write down the Fourier expansion of automorphic forms on $\mathbf{G}_{\boldsymbol{A}}$ along its Siegel parabolic subgroup, we necessitate not only the Whittaker models but also the *generalized* Whittaker models (see Section 1 for details). Our concern in this paper is the local theory of generalized Whittaker models at the real place, which still lies in an intermediate state. For example, in the paper [**PS**], which had circulated since the late 1970s, I. I. Piatetski-Shapiro stated the multiplicity free theorem of such models for $G := \mathbf{G}_{\boldsymbol{R}} = GSp(2, \boldsymbol{R})$ without a proof ([**PS**, Theorem 3.1]). However nobody seems to establish it up to today. For the generalized Whittaker model for G associated with a *definite* binary quadratic form, there are some results supporting Piatetski-Shapiro's assertion. Besides H. Yamashita's result [**Y**] in a general setting, there are several detailed studies for specific kinds of representations of G ([**Ni**], [**Mi-1**], [**Mi-2**], [**Is**]), where the multiplicity free results as well as explicit formulae of generalized Whittaker functions are obtained (see Subsection 8.3). On the other hand, little is known about the generalized Whittaker models associated

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with *indefinite* binary quadratic forms, although they are equally important in the study of automorphic forms on $\mathbf{G}_{\boldsymbol{A}}$. The purpose of this paper is to study them for two kinds of standard representations of G.

To be more precise, let $\mathbf{P} = \mathbf{M} \ltimes \mathbf{N}$ be the (standard) Siegel parabolic subgroup of $\mathbf{G}$. For each symmetric matrix $\beta = {}^t\beta \in \mathrm{Mat}(2)_{\boldsymbol{R}}$, we define a character $\psi_\beta : \mathbf{N}_{\boldsymbol{R}} \to \boldsymbol{C}^{(1)}$ by

$$\psi_\beta\left(\left(\begin{array}{c|c} I_2 & x \\ \hline 0_2 & I_2 \end{array}\right)\right) = \exp\left(2\pi\sqrt{-1}\,\mathrm{tr}(\beta x)\right).$$

Suppose that the character ψ_β is non-degenerate (i.e. $\det(\beta) \neq 0$). We denote by $\mathbf{M}_\beta$ the identity component (as an algebraic group) of the stabilizer of ψ_β in $\mathbf{M}$ and set $R_\beta := \mathbf{M}_{\beta,\boldsymbol{R}} \ltimes \mathbf{N}_{\boldsymbol{R}}$. For each quasi-character χ of $\mathbf{M}_{\beta,\boldsymbol{R}}$, we have a quasi-character $\chi \cdot \psi_\beta$ of R_β and an induced representation

$$\begin{aligned} &C^\infty(R_\beta\backslash G; \chi \cdot \psi_\beta) \\ &\quad := \left\{W : G \xrightarrow{C^\infty} \boldsymbol{C} \mid W(rg) = (\chi \cdot \psi_\beta)(r)W(g),\ \forall(r,g) \in R_\beta \times G\right\}, \end{aligned}$$

on which G acts by right translation. The totality of functions in $C^\infty(R_\beta\backslash G; \chi\cdot\psi_\beta)$ having moderate growth property is denoted by $C^\infty_{mg}(R_\beta\backslash G; \chi \cdot \psi_\beta)$. We denote the Lie algebra of G by $\mathfrak{g}$ and take a (standard) maximal compact subgroup K of G. For a quasi-simple $(\mathfrak{g}, K)$-module $(\pi, \mathscr{H}_\pi)$, we set

$$\mathbf{GW}_G(\pi, \chi \cdot \psi_\beta) := \mathrm{Hom}_{\mathfrak{g},K}(\mathscr{H}_\pi, C^\infty(R_\beta\backslash G; \chi \cdot \psi_\beta)),$$

which is called the space of generalized Whittaker functionals for π. By a generalized Whittaker function belonging to π, we understand the image $\Phi(v)$ of a vector $v \in \mathscr{H}_\pi$ under some $\Phi \in \mathbf{GW}_G(\pi, \chi \cdot \psi_\beta)$. If $\Phi \in \mathbf{GW}_G(\pi, \chi \cdot \psi_\beta)$ belongs to the subspace

$$\mathbf{GW}^{mg}_G(\pi, \chi \cdot \psi_\beta) := \mathrm{Hom}_{\mathfrak{g},K}\left(\mathscr{H}_\pi, C^\infty_{mg}(R_\beta\backslash G; \chi \cdot \psi_\beta)\right),$$

then we say Φ and $\Phi(v)$ have the moderate growth property. From the view point of automorphic forms, the following two problems are fundamental:

Problem (A). Estimate the dimensions of $\mathbf{GW}_G(\pi, \chi \cdot \psi_\beta)$ and $\mathbf{GW}^{mg}_G(\pi, \chi \cdot \psi_\beta)$. In particular, determine whether the multiplicity free property $\dim_{\boldsymbol{C}} \mathbf{GW}^{mg}_G(\pi, \chi \cdot \psi_\beta) \leq 1$ holds or not.

Problem (B). Find tractable formulae of the generalized Whittaker functions $\Phi(v)$ for $\Phi \in \mathbf{GW}_G^{mg}(\pi, \chi \cdot \psi_\beta)$ and appropriate vectors $v \in \mathscr{H}_\pi$.

In connection with Problem (B), the following problem is also important.

Problem (C). Construct a non-zero element in $\mathbf{GW}_G^{mg}(\pi, \chi \cdot \psi_\beta)$, especially when the estimate $\dim_{\boldsymbol{C}} \mathbf{GW}_G^{mg}(\pi, \chi \cdot \psi_\beta) \leq 1$ holds.

Let us summarize the current status of these problems according as the signature of $\det(\beta)$.

The case of $\det(\beta) > 0$. As we mentioned above, there are some positive results on the Problem (A) in this case. In the first place, H. Yamashita [**Y**] obtained a result closely related to the Problem (A), which we now explain. For an irreducible unitary representation $(\pi, \mathscr{H}_\pi)$ of G, we consider the space of continuous intertwining operators

$$\mathbf{GW}_G^{\infty}(\pi, \chi \cdot \psi_\beta) := \mathrm{Hom}_G(\mathscr{H}_\pi^{\infty}, C^{\infty}\big(R_\beta \backslash G; \chi \cdot \psi_\beta)\big).$$

Here $\mathscr{H}_\pi^{\infty}$ stands for the smooth vectors in $\mathscr{H}_\pi$ equipped with the usual C^{∞}-topology. Note that there is a natural inclusion $\mathbf{GW}_G^{\infty}(\pi, \chi \cdot \psi_\beta) \subset \mathbf{GW}_G^{mg}(\pi, \chi \cdot \psi_\beta)$ (cf. [**Wal**, Section 5.1]). Then a result of H. Yamashita ([**Y**, Theorem 6.9 (3)]), who works for general connected semisimple groups of hermitian type, implies that $\dim_{\boldsymbol{C}} \mathbf{GW}_G^{\infty}(\pi, \chi \cdot \psi_\beta) \leq 1$ under the following three conditions: $\det(\beta) > 0$, the restriction of π to $G_0 := Sp(2, \boldsymbol{R})$ remains irreducible, and the restriction of χ to $\mathbf{M}_{\beta, \boldsymbol{R}} \cap G_0 (\cong O(2))$ is real-valued. Meanwhile, in the early 1990s, S. Niwa [**Ni**] constructed the generalized Whittaker functions on G associated with definite quadratic forms by using theta liftings. He gives an integral expression of the generalized Whittaker function on G belonging to the spherical principal series representation. T. Miyazaki [**Mi-1**], [**Mi-2**] and T. Ishii [**Is**] extended Niwa's study to several standard representations and some derived functor modules by constructing differential equations satisfied by the generalized Whittaker functions. More recently, D. Prasad and R. Takloo-Bighash [**Pr-TB**], [**TB**] determine the dimension of $\mathbf{GW}_G^{\infty}(\pi, \chi \cdot \psi_\beta)$ for the discrete series representations π when $\det(\beta) > 0$ by using theta liftings. In a paper of A. Pitale and R. Schmidt [**Pi-Sch**], the case of holomorphic discrete series representations is also studied.

The case of $\det(\beta) < 0$. First we note that $\mathbf{GW}_G^{mg}(\pi, \chi \cdot \psi_\beta) = \{0\}$ for a holomorphic discrete series representation π of G (cf. [**Pi-Sch**]), which can be seen as a paraphrase of Koecher's principle for holomorphic automorphic forms. In [**Mi-2**], the case of some derived functor modules with small Gel'fand-Kirillov dimensions are studied. However there seems to be no detailed investigation for standard representations. In fact, a recent investigation with T. Ishii tells us that Problem (A) becomes more subtle when $\det(\beta) < 0$. To explain it, suppose that π is an irreducible principal series representation of G induced from the Borel subgroup. Then it seems possible to construct two linearly independent functionals in

$$\mathrm{Hom}_{\mathfrak{g}_0,K_0}\left(\mathscr{H}_\pi, C^\infty_{mg}(R_\beta\backslash G;\chi\cdot\psi_\beta)\right),\quad \mathfrak{g}_0 := \mathfrak{sp}(2,\boldsymbol{R}),\ K_0 := G_0\cap O(4)$$

by using the Novodvorsky local zeta integrals (cf. Section 9). Nevertheless, we can still prove that $\dim_{\boldsymbol{C}} \mathbf{GW}^{mg}_G(\pi,\chi\cdot\psi_\beta) \leq 1$ thanks to the non-trivial outer automorphism of $Sp(2,\boldsymbol{R})$. This phenomenon, which we do not encounter when $\det(\beta) > 0$, suggest that the appropriate group for multiplicity free theorem should be the disconnected group $G = GSp(2,\boldsymbol{R})$, not the connected group $Sp(2,\boldsymbol{R})$.

Now we are in the position to state our main results of this paper:

Main Results (see Theorems 5.1, 6.1, 7.1, and 8.1 for precise statements). Suppose that $\det(\beta) < 0$ and take an arbitrary quasi-character χ of $\mathbf{M}_{\beta,\boldsymbol{R}}$. Let $(\pi,\mathscr{H}_\pi)$ be either (i) an irreducible generalized principal series representation of G induced from the non-Siegel maximal parabolic subgroup P_1 (an irreducible P_1-principal series representation) or (ii) a (limit of) large discrete series representation. Then we have the following assertions:

(1) The space $\mathbf{GW}^{mg}_G(\pi,\chi\cdot\psi_\beta)$ is at most one dimensional.
(2) If π is equivalent to an irreducible P_1-principal series representation, then we have $\dim_{\boldsymbol{C}} \mathbf{GW}_G(\pi,\chi\cdot\psi_\beta) \leq 4$.
(3) For an element $\Phi \in \mathbf{GW}^{mg}_G(\pi,\chi\cdot\psi_\beta)$, the values of the generalized Whittaker function $\Phi(v)$ corresponding to some specific vector $v \in \mathscr{H}_\pi$ on a one-parameter subgroup of G can be expressed by the Meijer G-function $G^{4,0}_{2,4}(z)$.

The main results should play the following important roles in automorphic forms. In the first place, the multiplicity free theorem (1) allows us to express the global generalized Whittaker function arising from a Hecke eigen form on $\mathbf{G}_{\boldsymbol{A}}$ as a product of the generalized Whittaker function on $\mathbf{G}_{\boldsymbol{R}}$ and a function on $\mathbf{G}_{\boldsymbol{A}_\mathrm{f}}$. Besides, there are several automorphic L-functions whose integral representations involve generalized Whittaker functions on $\mathbf{G}_{\boldsymbol{A}}$ (e.g. [**An**], [**PS**], [**An-Ka**], [**PS-Ra**], [**F**]). In fact, the direct motivation of our investigation is to extend our earlier results [**Mo**] on the spinor L-function to a cusp form not having global Whittaker models by evaluating the real local zeta integrals of Andrianov ([**An**], [**PS**]). Note that our explicit formulae in (3) are quite suitable for this purpose, because they are given by inverse Mellin transforms, the Meijer G-functions. We hope to discuss this issue in a future paper, which is also an indispensable step to study arithmetic properties of the spinor L-function (e.g. [**Ha**], [**Le**]).

Our proof of the main results is done by analyzing the differential equations satisfied by the generalized Whittaker functions, which is basically parallel to [**Mi-1**], [**Mi-2**], and [**Is**]. Besides, there are two crucial points in our proof. One is the fact that the generalized Whittaker function with moderate growth property decreases rapidly in a certain direction (Lemma 3.3). The other is the determination of rapidly decreasing solutions of a fourth order generalized hypergeometric equation (Proposition 9.2). Thanks to Proposition 9.2, we can prove our main results without assuming any conditions on χ. This is quite satisfactory when we apply our results to automorphic forms.

The organization of this paper is as follows. In Section 1, we introduce a Fourier expansion of automorphic forms on $\mathbf{G}_{\boldsymbol{A}}$ involving the usual and generalized Whittaker functions, which motivates us to study the generalized Whittaker models. In particular, we prove the absolute convergence of the integral defining the global generalized Whittaker function arising from a cusp form, which is a delicate problem when $-\det(\beta) \in (\boldsymbol{Q}^{\times})^2$. Moreover we give several equivalent conditions for a cusp form on $\mathbf{G}_{\boldsymbol{A}}$ to have a non-zero global Whittaker function (Proposition 1.2). In Section 2 we collect some basic notation concerning the Lie group G_0 and introduce two kinds of standard representations of G in the main results. In Section 3, we define the generalized Whittaker functions on G_0. We discuss the restriction of generalized Whittaker functions to a two-dimensional split torus S of G_0 satisfying $G_0 = (R_\beta \cap G_0)SK_0$ for $\det(\beta) < 0$. We also prove the key observation (Lemma 3.3) mentioned above. Our proof of the main results occupies Section 4–Section 8. In Section 4, we introduce two kinds of differential operators acting on the generalized Whittaker functions, the shift operators and the Casimir operators. We compute the S-radial part of these two differential operators. In Section 5 and Section 6 (resp. Section 7) we treat P_1-principal series representations (resp. (limits of) large discrete series representations). In each section, we derive a system of differential equations for the generalized Whittaker function by using differential operators constructed in Section 4. By a careful analysis of these differential equations, we obtain our explicit formulae. In Section 8, after clarifying the relation between the generalized Whittaker functions on G_0 and those on G, we prove the multiplicity free theorem (Theorem 8.2) for $\det(\beta) < 0$. We also prove the multiplicity free theorem for $\det(\beta) > 0$ (Theorem 8.3), which is a reformulation of the results of T. Miyazaki [**Mi-1**]. In Section 9, we state and prove the above-mentioned key result (Proposition 9.2) on generalized hypergeometric differential equations. In the final section, we propose an approach to Problem (C) via Novodvorsky's local zeta integrals ([**No**]) when $\det(\beta) < 0$.

Notation and Conventions.

(i) For each place v of the field $\boldsymbol{Q}$ of rational numbers, we denote by $\boldsymbol{Q}_v$ the completion of $\boldsymbol{Q}$ at v. The module of an element $x \in \boldsymbol{Q}_v$ is denoted by $|x|_v$. For a finite place p of $\boldsymbol{Q}$, $\boldsymbol{Z}_p$ stands for the ring of integers in $\boldsymbol{Q}_p$. The adele ring (resp. the ring of finite adeles, the idele group) is denoted by $\boldsymbol{A}$ (resp. $\boldsymbol{A}_{\mathrm{f}}$, $\boldsymbol{A}^\times$). The module of an element $x \in \boldsymbol{A}^\times$ is denoted by $|x|_{\boldsymbol{A}}$ or simply by $|x|$. Unless otherwise stated, we understand that all the measures on locally compact unimodular groups are the Haar measures.

(ii) Let $\mathbf{G} = GSp(2)$ be the symplectic group with similitude of degree two, which is defined by

$$\mathbf{G} := \left\{ g \in GL(4) \mid {}^t g J_4 g = \nu(g) J_4 \text{ for some } \nu(g) \in \mathbf{G}_m \right\}, \quad J_4 := \left(\begin{array}{c|c} 0_2 & I_2 \\ \hline -I_2 & 0_2 \end{array} \right).$$

We regard $\mathbf{G}$ as an algebraic group defined over $\boldsymbol{Q}$. For any $\boldsymbol{Q}$-algebra R, the group of R-valued points of $\mathbf{G}$ is denoted by $\mathbf{G}_R$. We adopt the same convention for other algebraic groups. For each element $g = (g_v) = (g_{v,i,j})_{1 \le i,j \le 4} \in \mathbf{G}_{\boldsymbol{A}}$, we define its norm $\|g\|$ by

$$\|g\| := \prod_v \|g_v\|_v \quad \text{with} \quad \|g_v\|_v := \max\left\{ |g_{v,i,j}|_v, |(g_v^{-1})_{i,j}|_v \mid 1 \le i,j \le 4 \right\}.$$

As a maximal compact subgroup $K_{\boldsymbol{A}}$ of $\mathbf{G}_{\boldsymbol{A}}$, we take

$$K_{\boldsymbol{A}} := \prod_v K_v \quad \text{with } K_\infty := O(4) \cap \mathbf{G}_{\boldsymbol{R}} \text{ and } K_p := GL(4, \boldsymbol{Z}_p) \cap \mathbf{G}_{\boldsymbol{Q}_p} \ (\forall p < \infty).$$

We also use the following notation:

$$\begin{aligned} G &:= \mathbf{G}_{\boldsymbol{R}} = GSp(2, \boldsymbol{R}), & G_0 &\equiv Sp(2, \boldsymbol{R}) = \{ g \in G \mid \nu(g) = 1 \}, \\ K &= K_\infty, & K_0 &:= K \cap G_0. \end{aligned}$$

We write the Lie algebras of G, G_0, and K_0 by $\mathfrak{g}$, $\mathfrak{g}_0$, and $\mathfrak{k}$, respectively. If there is no fear of confusion, we do not distinguish a smooth representation of G (resp. G_0) from its underlying $(\mathfrak{g}, K)$-module (resp. $(\mathfrak{g}_0, K_0)$-module). For an arbitrary Lie subalgebra $\mathfrak{l}$ of $\mathfrak{g}$, its complexification $\mathfrak{l} \otimes \boldsymbol{C}$ is denoted by $\mathfrak{l}_{\boldsymbol{C}}$. We denote the dual space $\mathrm{Hom}_{\boldsymbol{C}}(\mathfrak{l}_{\boldsymbol{C}}, \boldsymbol{C})$ of $\mathfrak{l}_{\boldsymbol{C}}$ by $\mathfrak{l}_{\boldsymbol{C}}^*$.

(iii) Let L be a Lie group with Lie algebra $\mathfrak{l} = \mathrm{Lie}(L)$. For a C^∞-function f on L, we set

$$[R_X f](x) := \left. \frac{d}{dt} \right|_{t=0} f(x \exp(tX)), \quad X \in \mathfrak{l}, \ x \in L.$$

This action of $\mathfrak{l}$ can be extended to that of the universal enveloping algebra $U(\mathfrak{l})$ of $\mathfrak{l}$. We also write $f(x;X)$ for $[R_X f](x)$ $(X \in U(\mathfrak{l}))$.

(iv) For a positive integer n, $\mathrm{Mat}(n)$ stands for the algebraic group of $n \times n$ matrices defined over $\boldsymbol{Q}$.

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1. Fourier expansions of automorphic forms on $GSp(2)$.

In this section, we formulate a Fourier expansion of automorphic forms on $GSp(2)_{\boldsymbol{A}}$ along the Siegel parabolic subgroup (cf. [**PS**], [**Su**]) in order to motivate our study of generalized Whittaker functions on $GSp(2,\boldsymbol{R})$. The expansions are expressed in terms of the global Whittaker functions and the global *generalized* Whittaker functions. Although almost all materials here might be found in the literature, we put them together for the sake of convenience. Moreover we give interesting equivalent conditions for a cusp form to have a non-zero global Whittaker function (Proposition 1.2).

1.1. The first step of the Fourier expansion.

The center of $\mathbf{G}$ is given by $\mathbf{Z} := \{z 1_4 \in G \mid z \in \mathbf{G}_m\}$. We denote the space of automorphic forms on $\mathbf{G}_{\boldsymbol{A}}$ (resp. the space of cusp forms on $\mathbf{G}_{\boldsymbol{A}}$) with central character $\omega : \boldsymbol{Q}^\times \backslash \boldsymbol{A}^\times \to \boldsymbol{C}^{(1)}$ by $\mathscr{A}(\mathbf{G}_{\boldsymbol{Q}}\mathbf{Z}_{\boldsymbol{A}}\backslash \mathbf{G}_{\boldsymbol{A}};\omega)$ (resp. $\mathscr{A}^{cusp}(\mathbf{G}_{\boldsymbol{Q}}\mathbf{Z}_{\boldsymbol{A}}\backslash \mathbf{G}_{\boldsymbol{A}};\omega)$). We fix a maximal parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ corresponding to the short root (the so-called Siegel parabolic subgroup) and its Levi decomposition $\mathbf{P} = \mathbf{M}\mathbf{N}$ as follows:

$$\mathbf{P} := \left\{ \left(\begin{array}{c|c} * & * \\ \hline 0_2 & * \end{array} \right) \in \mathbf{G} \right\},$$

$$\mathbf{M} := \left\{ m(h,\lambda) := \left(\begin{array}{c|c} h & 0_2 \\ \hline 0_2 & \lambda {}^t h^{-1} \end{array} \right) \middle|\, h \in GL(2),\ \lambda \in \mathbf{G}_m \right\},$$

$$\mathbf{N} := \left\{ \left(\begin{array}{c|c} I_2 & x \\ \hline 0_2 & I_2 \end{array} \right) \middle|\, x \in \mathrm{Sym}(2) \right\},$$

where we set $\mathrm{Sym}(2) := \{x \in \mathrm{Mat}(2) \mid {}^t x = x\}$. Let $\psi : \boldsymbol{Q}\backslash\boldsymbol{A} \to \boldsymbol{C}^{(1)}$ be a non-trivial character of $\boldsymbol{Q}\backslash\boldsymbol{A}$ characterized by $\psi(t_\infty) = \exp(2\pi\sqrt{-1}t_\infty)$ $(t_\infty \in \boldsymbol{R})$. For each $\beta \in \mathrm{Sym}(2)_{\boldsymbol{Q}}$, we define a character ψ_β of $\mathbf{N}_{\boldsymbol{A}}$ by

$$\psi_\beta\left(\left(\begin{array}{c|c} I_2 & x \\ \hline 0_2 & I_2 \end{array}\right)\right) = \psi(\mathrm{tr}(\beta x)).$$

An automorphic form $F \in \mathscr{A}(\mathbf{G}_{\boldsymbol{Q}}\mathbf{Z}_{\boldsymbol{A}}\backslash\mathbf{G}_{\boldsymbol{A}};\omega)$ have the following Fourier expansion

$$F(g) = \sum_{\beta\in\mathrm{Sym}(2)_{\boldsymbol{Q}}} F_\beta(g), \quad F_\beta(g) := \int_{\mathbf{N}_{\boldsymbol{Q}}\backslash\mathbf{N}_{\boldsymbol{A}}} dn\, F(ng)\psi_\beta(n)^{-1}. \tag{1.1}$$

For each $\beta \in \mathrm{Sym}(2)_{\boldsymbol{Q}}$, the β-th coefficient function $F_\beta(g)$ satisfies the relation

$$F_\beta(ng) = \psi_\beta(n)F_\beta(g), \quad \forall(n,g) \in \mathbf{N}_{\boldsymbol{A}} \times \mathbf{G}_{\boldsymbol{A}}.$$

In order to describe relations between the coefficient functions $\{F_\beta | \beta \in \mathrm{Sym}(2)_{\boldsymbol{Q}}\}$, we introduce the right action of $\mathbf{M}$ on $\mathrm{Sym}(2)$ by

$$\beta\cdot m(h,\lambda) := \lambda^{-1}\,{}^t h\beta h, \quad (\beta \in \mathrm{Sym}(2),\ m(h,\lambda) \in \mathbf{M}). \tag{1.2}$$

Lemma 1.1. *Take two symmetric matrices $\beta, \beta' \in \mathrm{Sym}(2)_{\boldsymbol{Q}}$. Suppose that there exists an element $m \in \mathbf{M}_{\boldsymbol{Q}}$ such that $\beta' = \beta\cdot m$. Then we have*

$$F_{\beta'}(g) = F_\beta(mg), \quad \forall g \in \mathbf{G}_{\boldsymbol{A}}.$$

For $\beta, \beta' \in \mathrm{Sym}(2)_{\boldsymbol{Q}}$, we write $\beta \sim_{\mathbf{M}_{\boldsymbol{Q}}} \beta'$ if there exists an element $m \in \mathbf{M}_{\boldsymbol{Q}}$ such that $\beta' = \beta\cdot m$. In view of Lemma 1.1, an automorphic form $F \in \mathscr{A}(\mathbf{Z}_{\boldsymbol{A}}\mathbf{G}_{\boldsymbol{Q}}\backslash\mathbf{G}_{\boldsymbol{A}};\omega)$ is determined by the coefficient functions $\{F_\beta(g) \mid \beta \in \mathrm{Sym}(2)_{\boldsymbol{Q}}/\sim_{\mathbf{M}_{\boldsymbol{Q}}}\}$. A complete set of representatives for the coset space $\mathrm{Sym}(2)_{\boldsymbol{Q}}/\sim_{\mathbf{M}_{\boldsymbol{Q}}}$ is given by

$$\{0_2\} \cup \left\{\beta^{(1)} := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\} \cup \left\{\beta^{(2)}_d := \begin{pmatrix} 1 & 0 \\ 0 & -d \end{pmatrix} \,\middle|\, d \in \boldsymbol{Q}^\times/(\boldsymbol{Q}^\times)^2\right\}. \tag{1.3}$$

Suppose that F is a cusp form. Then the function $F_{0_2}(g)$ vanishes identically, because it is nothing but the constant term of F along $\mathbf{P}$. Hence the equation (1.1) tells us that at least one of the following three conditions holds:

(Rk1): there exists a non-zero symmetric matrix $\beta \in \mathrm{Sym}(2)_{\boldsymbol{Q}}$ such that $\det(\beta) =$

0 and $F_\beta(g) \not\equiv 0$;
(Rk2-D): there exists a symmetric matrix $\beta \in \mathrm{Sym}(2)_{\boldsymbol{Q}}$ such that $\det(\beta) > 0$ and $F_\beta(g) \not\equiv 0$;
(Rk2-ID): there exists a symmetric matrix $\beta \in \mathrm{Sym}(2)_{\boldsymbol{Q}}$ such that $\det(\beta) < 0$ and $F_\beta(g) \not\equiv 0$.

REMARK. A result of J. S. Li [**Li**] tells us that an arbitrary non-zero cusp form $F \in \mathscr{A}^{cusp}(\mathbf{G}_{\boldsymbol{Q}}\mathbf{Z}_{\boldsymbol{A}}\backslash\mathbf{G}_{\boldsymbol{A}};\omega)$ satisfies either (Rk2-D) or (Rk2-ID). This can be seen from Proposition 1.2 below, too. It is well known that a holomorphic cusp form F on $\mathbf{G}_{\boldsymbol{A}}$ satisfies neither (Rk1) nor (Rk2-ID).

In the next two subsections, we shall express the functions $F_\beta(g)$ $(\beta \neq 0_2)$ in terms of the global Whittaker functions and the global generalized Whittaker functions to get a finer expansion of $F(g)$.

1.2. The second step of the Fourier expansion, the case of det(β) = 0.

In this subsection, we consider the case where $\beta \in \mathrm{Sym}(2)_{\boldsymbol{Q}}$ satisfies $\det(\beta) = 0$ and $\beta \neq 0_2$. By Lemma 1.1, we may assume that $\beta = \beta^{(1)} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We fix a maximal unipotent subgroup of $\mathbf{G}$ defined over $\boldsymbol{Q}$ as follows:

$$\mathbf{N}_0 := \left\{ n(x_0,x_1,x_2,x_3) := \left(\begin{array}{cc|cc} 1 & & x_1 & x_2 \\ & 1 & x_2 & x_3 \\ \hline & & 1 & \\ & & & 1 \end{array}\right) \left(\begin{array}{cc|cc} 1 & x_0 & & \\ & 1 & & \\ \hline & & 1 & \\ & & -x_0 & 1 \end{array}\right) \in \mathbf{G} \right\}.$$

For each $g \in \mathbf{G}_{\boldsymbol{A}}$, we put $h_F(x_0;g) = F_{\beta^{(1)}}(n(x_0,0,0,0)g)$, $(x_0 \in \boldsymbol{A}/\boldsymbol{Q})$. Applying the Fourier inversion formula to $h_F(x_0;g)$, we have

$$F_\beta(g) = \int_{\boldsymbol{Q}\backslash\boldsymbol{A}} h_F(x_0;g)dx_0 + \sum_{\alpha\in\boldsymbol{Q}^\times} \boldsymbol{W}_F(\mathrm{diag}(\alpha,1,\alpha^{-1},1)g). \tag{1.4}$$

Here $\boldsymbol{W}_F(g)$ is the *global Whittaker function* attached to F defined by

$$\begin{aligned}\boldsymbol{W}_F(g) &:= \int_{\boldsymbol{Q}\backslash\boldsymbol{A}} h_F(x_0;g)\psi(x_0)^{-1}dx_0 \\ &= \int_{(\boldsymbol{Q}\backslash\boldsymbol{A})^4} F(n(x_0,x_1,x_2,x_3)g)\psi(x_0+x_3)^{-1}dx_1dx_2dx_3dx_0. \end{aligned} \tag{1.5}$$

An automorphic form F on $\mathbf{G}_{\boldsymbol{A}}$ is said to be globally generic if the global Whittaker function $\boldsymbol{W}_F(g)$ attached to F does not vanish identically. We give several characterizations of globally generic cusp forms:

Proposition 1.2. *For a cusp form $F(g)$ on $\mathbf{G}_{\boldsymbol{A}}$, the following four conditions are equivalent*:

(i) *F is globally generic*;
(ii) *the condition* (Rk1) *holds*;
(iii) *for each $\beta \in \mathrm{Sym}(2)_{\boldsymbol{Q}}$ satisfying $-\det(\beta) \in (\boldsymbol{Q}^\times)^2$, we have $F_\beta(g) \not\equiv 0$*;
(iv) *the integral $F_Z(g) := \int_{\boldsymbol{Q}\backslash\boldsymbol{A}} F(n(0,x_1,0,0)g)dx_1$ $(g \in G_{\boldsymbol{A}})$ does not vanish identically.*

Proof. (i) ⇒ (ii): This is immediate from the definition (1.5) of the global Whittaker function. (ii) ⇒ (i): Since F is assumed to be a cusp form, the first term of the right hand side of (1.4) vanishes. This implies that the global Whittaker function $\boldsymbol{W}_F(g)$ does not vanish identically. (i) ⇒ (iii): Define two elements w_1 and w_2 of $\mathbf{G}_{\boldsymbol{Q}}$ by

$$w_1 := \left(\begin{array}{cc|cc} & 1 & & \\ 1 & & & \\ \hline & & & 1 \\ & & 1 & \end{array}\right) \quad \text{and} \quad w_2 := \left(\begin{array}{cc|cc} 1 & & & \\ & & & -1 \\ \hline & & 1 & \\ & 1 & & \end{array}\right). \tag{1.6}$$

Then we have

$$\begin{aligned} &\boldsymbol{W}_F(g) \\ &= \int_{(\boldsymbol{Q}\backslash\boldsymbol{A})^4} F\big(n(x_0,x_1,0,0)n(0,0,x_2,x_3)g\big)\psi(-x_0-x_3)dx_1dx_0dx_2dx_3 \\ &= \int_{(\boldsymbol{Q}\backslash\boldsymbol{A})^4} F\big(w_2^{-1}n(x_0,x_1,0,0)n(0,0,x_2,x_3)g\big)\psi(-x_0-x_3)dx_1dx_0dx_2dx_3 \\ &= \int_{(\boldsymbol{Q}\backslash\boldsymbol{A})^4} F\big(n(0,x_1,-x_0,0)w_2^{-1}n(0,0,x_2,x_3)g\big)\psi(-x_0-x_3)dx_1dx_0dx_2dx_3. \end{aligned}$$

Therefore we know that

$$\int_{(\boldsymbol{Q}\backslash\boldsymbol{A})^2} F\big(n(0,x_1,x_0,0)g\big)\psi(x_0)\, dx_0dx_1 \not\equiv 0.$$

Hence it follows from the equality

$$\int_{(\boldsymbol{Q}\backslash\boldsymbol{A})^2} F\bigl(n(0,x_1,x_2,0)g\bigr)\psi(x_2)\,dx_1dx_2$$
$$= \sum_{a\in\boldsymbol{Q}}\int_{(\boldsymbol{Q}\backslash\boldsymbol{A})^3} F\bigl(n(0,x_1,x_2,x_3)g\bigr)\psi(x_2+ax_3)\,dx_1dx_2dx_3$$

that there exists $a \in \boldsymbol{Q}$ such that $F_\beta(g) \neq 0$ for $\beta = \left(\begin{smallmatrix} 0 & -1/2 \\ -1/2 & -a\end{smallmatrix}\right)$. By Lemma 1.1, this implies (iii). Finally the implications (iii) ⇒ (iv) ⇒ (i) are proved in [**K-R-S**, Lemma 8.2], for example. □

Remark. The above proof shows that the implications (i) ⇒ (ii) and (i) ⇒ (iii) hold even if F is not a cusp form. The converse implications (ii) ⇒ (i) and (iii) ⇒ (i) are not valid if F is not a cusp form (cf. [**Ma**, p. 306]).

1.3. The second step of the Fourier expansion, the case of $\det(\beta) \neq 0$.

Next we consider the case where $\det(\beta) \neq 0$. For each $\beta \in \mathrm{Sym}(2)_{\boldsymbol{Q}}$ with $\det(\beta) \neq 0$, the identity component (as an algebraic group) of the stabilizer of β in $\mathbf{M}$ is given by $\mathbf{M}_\beta = \{m(h,\det(h)) \mid {}^th\beta h = \det(h)\beta\}$. Let

$$k_\beta := \boldsymbol{Q}[t]/(t^2+\det\beta) \cong \begin{cases} \boldsymbol{Q}(\sqrt{-\det\beta}) & -\det\beta \notin (\boldsymbol{Q}^\times)^2; \\ \boldsymbol{Q}\oplus\boldsymbol{Q} & -\det\beta \in (\boldsymbol{Q}^\times)^2, \end{cases}$$

be a quadratic separable algebra over $\boldsymbol{Q}$. We set $\boldsymbol{A}_{k_\beta} := \boldsymbol{A}\otimes_{\boldsymbol{Q}} k_\beta$. Note that there is an isomorphism $\mathbf{M}_\beta \cong \mathrm{Res}_{k_\beta/\boldsymbol{Q}}\,GL(1)$ or $\mathbf{M}_\beta \cong GL(1)\times GL(1)$ according as $-\det\beta \notin (\boldsymbol{Q}^\times)^2$ or $-\det\beta \in (\boldsymbol{Q}^\times)^2$. By Lemma 1.1, we have $F_\beta(\gamma g) = F_\beta(g)$ $(\gamma \in \mathbf{M}_{\beta,\boldsymbol{Q}})$. In order to get an expansion of $F_\beta(g)$, we set

$$\Xi_\omega := \bigl\{\chi : \mathbf{M}_{\beta,\boldsymbol{Q}}\backslash\mathbf{M}_{\beta,\boldsymbol{A}} \to \boldsymbol{C}^{(1)} \mid \text{character}, \ \chi(z) = \omega(z)\ (\forall z \in \mathbf{Z}_{\boldsymbol{A}})\bigr\},$$
$$\Xi_0 := \bigl\{\chi : \mathbf{M}_{\beta,\boldsymbol{Q}}\backslash\mathbf{M}_{\beta,\boldsymbol{A}} \to \boldsymbol{C}^{(1)} \mid \text{character}, \ \chi(z) = 1\ (\forall z \in \mathbf{Z}_{\boldsymbol{A}})\bigr\}.$$

We define the subgroup $\mathbf{R}_\beta$ of $\mathbf{G}$ by $\mathbf{R}_\beta = \mathbf{M}_\beta \ltimes \mathbf{N}$. For each character $\chi \in \Xi_\omega$, we define a character $\chi\cdot\psi_\beta$ of $\mathbf{R}_{\beta,\boldsymbol{A}}$ by

$$(\chi\cdot\psi_\beta)(mn) = \chi(m)\psi_\beta(n), \quad (m,n) \in \mathbf{M}_{\beta,\boldsymbol{A}}\times\mathbf{N}_{\boldsymbol{A}}.$$

We consider the following integral

$$W_F^{\chi\cdot\psi_\beta}(g) := \int_{\mathbf{Z}_{\boldsymbol{A}}\mathbf{M}_{\beta,\boldsymbol{Q}}\backslash\mathbf{M}_{\beta,\boldsymbol{A}}} F_\beta(mg)\chi(m)^{-1}dm, \quad g \in \mathbf{G}_{\boldsymbol{A}}. \tag{1.7}$$

If the integral (1.7) converges absolutely, then we call $W_F^{\chi\cdot\psi_\beta}(g)$ the *global generalized Whittaker function* attached to F with respect to the character $\chi\cdot\psi_\beta$. The global generalized Whittaker function $W_F^{\chi\cdot\psi_\beta}(g)$ belongs to the following space

$$C^\infty(\mathbf{R}_{\beta,\boldsymbol{A}}\backslash\mathbf{G}_{\boldsymbol{A}};\chi\cdot\psi_\beta) \\ := \Big\{W:\mathbf{G}_{\boldsymbol{A}}\overset{C^\infty}{\to}\boldsymbol{C}\mid W(rg)=(\chi\cdot\psi_\beta)(r)W(g),\quad \forall(r,g)\in\mathbf{R}_{\beta,\boldsymbol{A}}\times\mathbf{G}_{\boldsymbol{A}}\Big\}. \tag{1.8}$$

For a fixed Haar measure on $\mathbf{Z}_{\boldsymbol{A}}\mathbf{M}_{\beta,\boldsymbol{Q}}\backslash\mathbf{M}_{\beta,\boldsymbol{A}}$, we denote by $d\chi$ the Haar measure on Ξ_0 dual to it. Fix an arbitrary element χ_1 of Ξ_ω. Through the bijection $\Xi_0\ni\chi\mapsto\chi\cdot\chi_1\in\Xi_\omega$, we have a measure on Ξ_ω, which is independent of the choice of the base point χ_1. The measure on Ξ_ω obtained in this manner is also denoted by $d\chi$. Then we have the following expansion of F_β when $\det(\beta)\neq 0$.

Proposition 1.3. *Let $F\in\mathscr{A}(\mathbf{G}_{\boldsymbol{Q}}\mathbf{Z}_{\boldsymbol{A}}\backslash\mathbf{G}_{\boldsymbol{A}};\omega)$ be an automorphic form on $\mathbf{G}_{\boldsymbol{A}}$ with central character ω.*

(i) *Suppose that $-\det(\beta)\notin(\boldsymbol{Q}^\times)^2$. Then $\mathbf{Z}_{\boldsymbol{A}}\mathbf{M}_{\beta,\boldsymbol{Q}}\backslash\mathbf{M}_{\beta,\boldsymbol{A}}$ is a compact abelian group and the integral* (1.7) *converges absolutely. Moreover if we normalize the Haar measure on $\mathbf{Z}_{\boldsymbol{A}}\mathbf{M}_{\beta,\boldsymbol{Q}}\backslash\mathbf{M}_{\beta,\boldsymbol{A}}$ so that the total volume is one, then we have the following inversion formula*:

$$F_\beta(g)=\int_{\Xi_\omega}W_F^{\chi\cdot\psi_\beta}(g)d\chi=\sum_{\chi\in\Xi_\omega}W_F^{\chi\cdot\psi_\beta}(g). \tag{1.9}$$

(ii) *Suppose that $-\det(\beta)\in(\boldsymbol{Q}^\times)^2$. If F is a cusp form, then the integral* (1.7) *converges absolutely. Moreover we have*

$$F_\beta(g)=\int_{\Xi_\omega}W_F^{\chi\cdot\psi_\beta}(g)d\chi. \tag{1.10}$$

Proof.

(i) Suppose that $-\det(\beta)\notin(\boldsymbol{Q}^\times)^2$. Then the abelian group $\mathbf{Z}_{\boldsymbol{A}}\mathbf{M}_{\beta,\boldsymbol{Q}}\backslash\mathbf{M}_{\beta,\boldsymbol{A}}$ is compact, for it is isomorphic to $\boldsymbol{A}^\times k_\beta^\times\backslash\boldsymbol{A}_{k_\beta}^\times$. Hence the measure $d\chi$ of Ξ_ω is the pointing measure. This proves the inversion formula (1.9).

(ii) By Lemma 1.1, we may assume that $\beta=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. Then we have

$$\mathbf{M}_\beta=\{z\,\mathrm{diag}(y,1,1,y)\mid y,z\in\boldsymbol{G}_m\}.$$

We shall prove the convergence of the integral

$$\int_{\boldsymbol{Q}^\times\backslash\boldsymbol{A}^\times} d^\times y \int_{\mathbf{N}_{\boldsymbol{Q}}\backslash\mathbf{N}_{\boldsymbol{A}}} dn|F(n\,\mathrm{diag}(y,1,1,y)g)| \tag{1.11}$$

for each $g \in \mathbf{G}_{\boldsymbol{A}}$. Since $\mathrm{diag}(y,1,1,y)$ commutes with $n(0,0,x_2,0)$, it suffices to show that the integral

$$\int_{\boldsymbol{Q}^\times\backslash\boldsymbol{A}^\times} d^\times y \int_{(\boldsymbol{Q}\backslash\boldsymbol{A})^2} dx_1 dx_3 \big|F(n(0,x_1,0,x_3)\,\mathrm{diag}(y,1,1,y)g)\big| \tag{1.12}$$

converges absolutely for all $g \in \mathbf{G}_{\boldsymbol{A}}$ and defines a bounded function on each compact subset Ω of $\mathbf{G}_{\boldsymbol{A}}$. Moreover, it is easy to see that

$$\begin{aligned}&\int_{\substack{\boldsymbol{Q}^\times\backslash\boldsymbol{A}^\times\\|y|\leq 1}} d^\times y \int_{(\boldsymbol{Q}\backslash\boldsymbol{A})^2} dx_1 dx_3 \big|F(n(0,x_1,0,x_3)\,\mathrm{diag}(y,1,1,y)g)\big|\\ &\quad= \int_{\substack{\boldsymbol{Q}^\times\backslash\boldsymbol{A}^\times\\|y|\geq 1}} d^\times y \int_{(\boldsymbol{Q}\backslash\boldsymbol{A})^2} dx_1 dx_3 \big|F(n(0,x_3,0,x_1)\,\mathrm{diag}(y,1,1,y)w_1 g)\big|.\end{aligned}$$

Hence our assertion follows from Lemma 1.4 below. □

LEMMA 1.4. *Let $F \in \mathscr{A}^{cusp}(\mathbf{G}_{\boldsymbol{Q}}\mathbf{Z}_{\boldsymbol{A}}\backslash\mathbf{G}_{\boldsymbol{A}};\omega)$ be a cusp form on $\mathbf{G}_{\boldsymbol{A}}$ with central character ω. Fix $r \geq 1$ and a compact subset Ω of $\mathbf{G}_{\boldsymbol{A}}$. Then there exists a constant $C > 0$ such that*

$$|F(n(0,x_1,0,x_3)\,\mathrm{diag}(y,1,1,y)u)| \leq C \times |y|_{\boldsymbol{A}}^{-r},$$
$$\forall (x_1,x_3) \in \boldsymbol{A}^2,\ \forall y \in \boldsymbol{A}^\times \text{ with } |y|_{\boldsymbol{A}} \geq 1, \text{ and } \forall u \in \Omega.$$

PROOF (similar to [**J-S**, Lemma 3.4 (i)]). Let $\mathbf{A}_0 := \{\mathrm{diag}(a_0a_1, a_0a_2, a_1^{-1}, a_2^{-1}) \mid a_i \in \boldsymbol{G}_m\}$ be the maximal $\boldsymbol{Q}$-split torus in $\mathbf{G}$. For $\infty \geq t' > t > 0$, we set

$$\mathbf{A}_{0,\boldsymbol{A}}(t';t) := \left\{ \mathrm{diag}(a_0a_1, a_0a_2, a_1^{-1}, a_2^{-1}) \in \mathbf{A}_{0,\boldsymbol{A}} \,\middle|\, t' > \left|\frac{a_1}{a_2}\right| > t \text{ and } t' > |a_0a_2^2| > t \right\}.$$

By the compactness of $K_{\boldsymbol{A}} \cdot \Omega$, we have

$$K_{\boldsymbol{A}} \cdot \Omega \subset \mathbf{N}_{0,\boldsymbol{A}} \cdot \mathbf{A}_{0,\boldsymbol{A}}(t_2;t_1) \cdot K_{\boldsymbol{A}}$$

for some $\infty > t_2 > t_1 > 0$. Since F is a cusp form with central character, it is rapidly decreasing on $\mathbf{N}_{0,\boldsymbol{A}}\mathbf{A}_{0,\boldsymbol{A}}(\infty;t)K_{\boldsymbol{A}}$ for every $t>0$. That is, for each $(r_1,r_2)\in\boldsymbol{R}^2$ there exists a constant $C>0$ such that

$$\begin{aligned}&\left|F\left(n\operatorname{diag}(a_0a_1,a_0a_2,a_1^{-1},a_2^{-1})k\right)\right|\le C\left|\frac{a_1}{a_2}\right|^{-r_1}\left|a_0a_2^2\right|^{-r_2}\\ &\forall n\in\mathbf{N}_{0,\boldsymbol{A}},\ \forall\operatorname{diag}\left(a_0a_1,a_0a_2,a_1^{-1},a_2^{-1}\right)\in\mathbf{A}_{0,\boldsymbol{A}}(\infty;t),\ \forall k\in K_{\boldsymbol{A}}.\end{aligned}\tag{1.13}$$

We shall rewrite the element $n(0,x_1,0,x_3)\operatorname{diag}(y,1,1,y)u$ in order that we can apply the estimate (1.13) to it. By reduction theory for $GL(2)$, there exists a constant $t_0>0$ independent of x_3 and y such that

$$\begin{pmatrix}1 & x_3y\\ 0 & y\end{pmatrix}=\gamma_2\begin{pmatrix}1 & x_3'\\ 0 & 1\end{pmatrix}\begin{pmatrix}b_1 & 0\\ 0 & b_2\end{pmatrix}k_2$$

for some $\gamma_2\in GL(2)_{\boldsymbol{Q}}$, $b_1,b_2\in\boldsymbol{A}^\times$ with $|b_1/b_2|\ge t_0$, $x_3'\in\boldsymbol{A}$, and $k_2\in O(2)\cdot GL(2,\widehat{\boldsymbol{Z}})$. We also have

$$\begin{pmatrix}y & x_1\\ 0 & 1\end{pmatrix}=\begin{pmatrix}\det(\gamma_2) & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & x_1'\\ 0 & 1\end{pmatrix}\begin{pmatrix}b_1b_2 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\det(k_2) & 0\\ 0 & 1\end{pmatrix}$$

with $x_1'=\det(\gamma_2{}^{-1})x_1$. Hence we have

$$n(0,x_1,0,x_3)\operatorname{diag}(y,1,1,y)=\gamma n\left(0,x_1',0,x_3'\right)\operatorname{diag}(b_1b_2,b_1,1,b_2)k\tag{1.14}$$

for some $\gamma\in\mathbf{G}_{\boldsymbol{Q}}$ and $k\in K_{\boldsymbol{A}}$.

We first consider the case where $b_2\in\boldsymbol{A}^\times$ in (1.14) can be taken so that $|b_2|\ge 1$. Then we have

$$n(0,x_1,0,x_3)\operatorname{diag}(y,1,1,y)u\in\mathbf{G}_{\boldsymbol{Q}}\mathbf{N}_{0,\boldsymbol{A}}\mathbf{A}_{0,\boldsymbol{A}}(\infty;t_3)K_{\boldsymbol{A}}$$

with $t_3:=t_1\min\{1,t_0\}$. Hence for each $(r_1,r_2)\in\boldsymbol{R}^2$, there exists a constant $C_1>0$ such that

$$F\left(n(0,x_1,0,x_3)\operatorname{diag}(y,1,1,y)u\right)\le C_1\times|b_2|^{-r_1}\left|\frac{b_1}{b_2}\right|^{-r_2}.$$

We set $r_1=2r$ and $r_2=r$. By noting $|b_1b_2|=|y|$, we have the estimate of the lemma in this case. Next we suppose that $|b_2|\le 1$. In the expression

$$n(0,x_1,0,x_3)\,\mathrm{diag}(y,1,1,y) = \gamma w_1 n\bigl(0,x_3',0,x_1'\bigr)\,\mathrm{diag}(b_1,b_1b_2,b_2,1)w_1^{-1}k,$$

we note that $\gamma w_1 \in \mathbf{G}_{\boldsymbol{Q}}$, $\mathrm{diag}(b_1,b_1b_2,b_2,1) \in \mathbf{A}_{0,\boldsymbol{A}}(\infty;1)$ and $w_1^{-1}k \in K_{\boldsymbol{A}}$. Hence for each $(r_1,r_2) \in \boldsymbol{R}^2$ we can take a constant $C_2 > 0$ such that

$$F\bigl(n(0,x_1,0,x_3)\,\mathrm{diag}(y,1,1,y)u\bigr) \le C_2 \times |b_2|^{r_1}|b_1b_2|^{-r_2}.$$

By putting $r_1 = 0$ and $r_2 = r$, we have the estimate of the lemma for $|b_2| \le 1$, too. □

1.4. The case of cusp forms.

Suppose that $F \in \mathscr{A}^{cusp}(\mathbf{G}_{\boldsymbol{Q}}\mathbf{Z}_{\boldsymbol{A}}\backslash\mathbf{G}_{\boldsymbol{A}};\omega)$ is a cusp form on $GSp(2)_{\boldsymbol{A}}$. Then, combining the results of the previous two subsections, we have the following Fourier expansion of F:

$$\begin{aligned} F(g) &= \sum_{\gamma\in\mathbf{B}'_{\boldsymbol{Q}}\backslash\mathbf{M}_{\boldsymbol{Q}}} F_{\beta^{(1)}}(\gamma g) + \frac{1}{2}\sum_{d\in(\boldsymbol{Q}^\times)^2\backslash\boldsymbol{Q}^\times}\ \sum_{\gamma\in\mathbf{M}_{\beta_d^{(2)},\boldsymbol{Q}}\backslash\mathbf{M}_{\boldsymbol{Q}}} F_{\beta_d^{(2)}}(\gamma g) \\ &= \sum_{\gamma\in\mathbf{B}'_{\boldsymbol{Q}}\backslash\mathbf{M}_{\boldsymbol{Q}}}\ \sum_{\alpha\in\boldsymbol{Q}^\times} \boldsymbol{W}_F(\mathrm{diag}(\alpha,1,\alpha^{-1},1)\gamma g) \\ &\quad + \frac{1}{2}\sum_{d\in(\boldsymbol{Q}^\times)^2\backslash\boldsymbol{Q}^\times}\ \sum_{\gamma\in\mathbf{M}_{\beta_d^{(2)},\boldsymbol{Q}}\backslash\mathbf{M}_{\boldsymbol{Q}}} \int_{\Xi_\omega} W_F^{\chi\cdot\psi_{\beta_d^{(2)}}}(\gamma g)d\chi. \end{aligned}$$

Here $\mathbf{B}' := \bigl\{m\bigl(\bigl(\begin{smallmatrix} a_{11} & a_{12} \\ 0 & a_{22}\end{smallmatrix}\bigr), a_{22}^2\bigr) \in \mathbf{M}\bigr\}$ is the stabilizer of $\beta^{(1)}$ in $\mathbf{M}$. Note that the factor $1/2$ comes from the fact that $\mathbf{M}_{\beta,\boldsymbol{Q}}$ is of index two in the stabilizer of β in $\mathbf{M}_{\boldsymbol{Q}}$.

Remark. The above expansion plus Proposition 1.2 implies that for a non-zero cusp form F we can find β and χ such that the associated global generalized Whittaker function $W_F^{\chi\cdot\psi_\beta}(\gamma g)$ does not vanish. This is quite satisfactory, because we can study the spinor L-function by the method of Andrianov [**An**], [**PS**] without assuming any global conditions on F.

1.5. Local generalized Whittaker functions.

We introduce the local generalized Whittaker functions, which are local counterpart of the global generalized Whittaker functions $W_F^{\chi\cdot\psi_\beta}(g)$. For a place v of $\boldsymbol{Q}$, we denote by ψ_v the restriction of the character ψ to $\boldsymbol{Q}_v$. For each $\beta \in \mathrm{Sym}(2)_{\boldsymbol{Q}_v}$, we define a character $\psi_{v,\beta}$ of $\mathbf{N}_{\boldsymbol{Q}_v}$ by

$$\psi_{v,\beta}\left(\left(\begin{array}{c|c} I_2 & x \\ \hline 0_2 & I_2 \end{array}\right)\right) = \psi_v(\mathrm{tr}(\beta x)).$$

As before, we define $\mathbf{M}_\beta$ to be the identity component of the stabilizer of β in $\mathbf{M}$ and set $\mathbf{R}_\beta := \mathbf{M}_\beta \ltimes \mathbf{N}$, which are algebraic subgroups of $\mathbf{G}$ defined over $\boldsymbol{Q}_v$. For each quasi-character $\chi : \mathbf{M}_{\beta,\boldsymbol{Q}_v} \to \boldsymbol{C}^\times$, we set

$$\begin{aligned} &C^\infty\big(\mathbf{R}_{\beta,\boldsymbol{Q}_v}\backslash\mathbf{G}_{\boldsymbol{Q}_v}; \chi\cdot\psi_{v,\beta}\big) \\ &\quad := \Big\{W : \mathbf{G}_{\boldsymbol{Q}_v} \xrightarrow{C^\infty} \boldsymbol{C} \mid W(rg) = (\chi\cdot\psi_{v,\beta})(r)W(g),\ \forall(r,g)\in\mathbf{R}_{\beta,\boldsymbol{Q}_v}\times\mathbf{G}_{\boldsymbol{Q}_v}\Big\}, \end{aligned}$$

on which $\mathbf{G}_{\boldsymbol{Q}_v}$ acts by right translation.

We now suppose that $v=\infty$ is the real place and set $R_\beta := \mathbf{R}_{\beta,\boldsymbol{R}}$. Recall that a C^∞-function $f : G \to \boldsymbol{C}$ is said to be *of moderate growth* if there exist constants $C>0$ and $M>0$ such that $|f(g)| < C\|g\|_\infty^M$ holds for all $g\in G$. If f and its all derivatives $f(g;X)$ $(X\in U(\mathfrak{g}))$ are of moderate growth with an exponent $M>0$ common to all $X\in U(\mathfrak{g})$, then the function f is said to be *of uniformly moderate growth.* We denote by $C^\infty_{mg}(R_\beta\backslash G;\chi\cdot\psi_\beta)$ (resp. $C^\infty_{umg}(R_\beta\backslash G;\chi\cdot\psi_\beta)$) the totality of functions in $C^\infty(R_\beta\backslash G;\chi\cdot\psi_\beta)$ of moderate growth (resp. of uniformly moderate growth). Note that an automorphic form $F\in\mathscr{A}(\mathbf{Z}_{\boldsymbol{A}}\mathbf{G}_{\boldsymbol{Q}}\backslash\mathbf{G}_{\boldsymbol{A}};\omega)$ is of moderate growth in the sense that there exist $C>0$ and $M>0$ such that

$$|F(g)| \le C\times\|g\|^M, \quad \forall g\in\mathbf{G}_{\boldsymbol{A}}.$$

Hence, for each fixed $g_{\mathrm{f}}\in\mathbf{G}_{\boldsymbol{A}_{\mathrm{f}}}$, the function $W_F^{\chi\cdot\psi_\beta}(g_{\mathrm{f}}g_\infty)$ in $g_\infty\in\mathbf{G}_{\boldsymbol{R}}$ belongs to the space $C^\infty_{mg}(R_\beta\backslash G;\chi\cdot\psi_\beta)$. Hence we are led to the following definition.

Definition 1.5. Let $(\pi,\mathscr{H}_\pi)$ be a quasi-simple $(\mathfrak{g},K)$-module.

(i) By a *generalized Whittaker functional* belonging to π, we understand an element of the intertwining space

$$\mathbf{GW}_G(\pi,\chi\cdot\psi_\beta) := \mathrm{Hom}_{\mathfrak{g},K}\big(\mathscr{H}_\pi, C^\infty(R_\beta\backslash G;\chi\cdot\psi_\beta)\big).$$

(ii) If $\Phi\in\mathbf{GW}_G(\pi,\chi\cdot\psi_\beta)$ belongs to the subspace

$$\mathbf{GW}_G^{mg}(\pi,\chi\cdot\psi_\beta) := \mathrm{Hom}_{\mathfrak{g},K}\big(\mathscr{H}_\pi, C^\infty_{mg}(R_\beta\backslash G;\chi\cdot\psi_\beta)\big),$$

then we say that it has the *moderate growth property.*

It is preferable that the multiplicity free property $\dim_{\boldsymbol{C}}\mathbf{GW}_G^{mg}(\pi,\chi\cdot\psi_\beta)\le 1$

holds for an arbitrary irreducible $(\mathfrak{g}, K)$-module $(\pi, \mathscr{H}_\pi)$ and an arbitrary quasi-character $\chi \cdot \psi_\beta$. One of the main purposes of this paper is to prove that this expectation is true when $\det(\beta) < 0$ and $(\pi, \mathscr{H}_\pi)$ is equivalent to one of two kinds of irreducible $(\mathfrak{g}, K)$-modules introduced in the next section.

Before closing this section, we mention some results on the generalized Whittaker models in the non-archimedean case. In the first place, Bump, Friedberg, and Furusawa [**Bu-Fr-Fu**] obtained an explicit formula of the unramified generalized Whittaker function by the method of [**C-S**]. This is the ultimate result extending an earlier result of Andrianov [**An**, Theorem 1] (cf. [**Su**, Proposition 2-5]). Some results related to the multiplicity freeness are proved in [**No**], [**No-PS**], where the authors consider the stabilizer group of ψ_β instead of its connected component $\mathbf{M}_\beta$. Moreover F. Rodier [**Ro**] proved the following multiplicity free theorem for the cuspidal representations of $\mathbf{G}_{\boldsymbol{Q}_p}$:

Proposition 1.6 ([**Ro**, Theorem, p. 126]). *Suppose that $v = p < \infty$ is a finite place of $\boldsymbol{Q}$. Let $(\pi, \mathscr{H}_\pi)$ be a cuspidal irreducible admissible representation of $\mathbf{G}_{\boldsymbol{Q}_p}$ with trivial central character. Then the intertwining space* $\mathrm{Hom}_{\mathbf{G}_{\boldsymbol{Q}_p}}(\pi, C^\infty(\mathbf{R}_{\beta,\boldsymbol{Q}_p}\backslash\mathbf{G}_{\boldsymbol{Q}_p}; \chi \cdot \psi_\beta))$ *is at most one dimensional.*

2. The group $Sp(2, \boldsymbol{R})$ and its representations.

In this section, we fix some notation concerning the Lie group $Sp(2, \boldsymbol{R})$ and introduce two kinds of standard representations of it.

2.1. Root systems and the irreducible K_0-modules.

The maximal compact subgroup K_0 of $G_0 = Sp(2, \boldsymbol{R})$ is isomorphic to the unitary group $U(2) := \{g \in GL(2, \boldsymbol{C}) \mid {}^t\overline{g}g = I_2\}$ of degree two. Fix an isomorphism $\kappa : U(2) \cong K_0$ by

$$\kappa : U(2) \ni A + \sqrt{-1}B \mapsto k_{A,B} := \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in K_0, \quad (A, B \in \mathrm{Mat}(2)_{\boldsymbol{R}}).$$

The differential κ_* of κ defines an isomorphism of Lie algebras: $\kappa_* : \mathfrak{gl}(2, \boldsymbol{C}) \cong \mathfrak{k}_{\boldsymbol{C}}$. A compact Cartan subalgebra of $\mathfrak{g}_0$ is given by $\mathfrak{h} := \boldsymbol{R}T_1 \oplus \boldsymbol{R}T_2$, where

$$T_1 := \kappa_*\left(\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & 0 \end{pmatrix}\right), \quad T_2 := \kappa_*\left(\begin{pmatrix} 0 & 0 \\ 0 & \sqrt{-1} \end{pmatrix}\right).$$

Define a $\boldsymbol{C}$-basis $\{e_1, e_2\}$ of $\mathfrak{h}^*_{\boldsymbol{C}}$ by $e_i(T_j) = \sqrt{-1}\delta_{ij}$ $(1 \le i, j \le 2)$. Then the root system $\Delta = \Delta(\mathfrak{g}_{0,\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}})$ for the pair $(\mathfrak{g}_{0,\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}})$ is given by $\Delta(\mathfrak{g}_{0,\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}}) = \{\pm 2e_1, \pm 2e_2, \pm(e_1 \pm e_2)\}$. We denote by Δ_c (resp. Δ_{nc}) the set of compact roots (resp. the set of non-compact roots) in Δ: $\Delta_c = \{\pm(e_1 - e_2)\}$ (resp. $\Delta_{nc} = \Delta \backslash \Delta_c$). We take a positive system Δ^+ of Δ as $\Delta^+ := \{2e_1, e_1+e_2, 2e_2, e_1-e_2\}$. Then the sets of compact and non-compact positive roots in Δ are given by $\Delta_c^+ = \Delta_c \cap \Delta^+$ and $\Delta_{nc}^+ = \Delta_{nc} \cap \Delta^+$, respectively. For each symmetric matrix $A \in \mathrm{Sym}(2)_{\boldsymbol{C}}$, we define elements $p_\pm(A)$ of $\mathfrak{g}_{0,\boldsymbol{C}}$ by

$$p_\pm(A) := \left(\begin{array}{c|c} A & \pm\sqrt{-1}A \\ \hline \pm\sqrt{-1}A & -A \end{array}\right) \in \mathfrak{g}_{0,\boldsymbol{C}}.$$

Then we can take the root vectors $X_{(\alpha_1,\alpha_2)} \in \mathfrak{g}_{0,\boldsymbol{C}}$ corresponding to the non-compact roots $\alpha_1 e_1 + \alpha_2 e_2 = (\alpha_1, \alpha_2) \in \Delta_{nc}$ as follows:

$$X_{\pm(2,0)} := p_\pm\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right), \quad X_{\pm(1,1)} := p_\pm\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right), \quad X_{\pm(0,2)} := p_\pm\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right).$$

We set $\mathfrak{p}_\pm := \bigoplus_{\alpha \in \Delta_{nc}^+} \boldsymbol{C} X_{\pm\alpha}$.

The set of all the irreducible finite-dimensional representations of K_0 is parameterized by their highest weights relative to Δ_c^+. For each dominant integral weight $q = (q_1, q_2) = q_1 e_1 + q_2 e_2 \in \mathfrak{h}_{\boldsymbol{C}}^*$ ($q_i \in \boldsymbol{Z}, q_1 \geq q_2$), we denote the corresponding irreducible finite-dimensional representation by $(\tau_{(q_1,q_2)}, V_{(q_1,q_2)})$. The dimension of the representation space $V_{(q_1,q_2)}$ is given by $d+1$, where we set $d = d_q = q_1 - q_2$. There is a basis $\{v_k \mid 0 \leq k \leq d\}$ of $(\tau_{(q_1,q_2)}, V_{(q_1,q_2)})$ satisfying

$$\tau_q\left(\kappa_*\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)\right) v_k = (k+q_2)v_k, \quad \tau_q\left(\kappa_*\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)\right) v_k = (-k+q_1)v_k,$$

$$\tau_q\left(\kappa_*\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)\right) v_k = (k+1)v_{k+1}, \quad \tau_q\left(\kappa_*\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)\right) v_k = (d-k+1)v_{k-1},$$

which we call the *standard basis* of $(\tau_{(q_1,q_2)}, V_{(q_1,q_2)})$. Here we understand that $v_{-1} = v_{d+1} = 0$. Note that $(\mathrm{Ad}, \mathfrak{p}_+)$ and $(\mathrm{Ad}, \mathfrak{p}_-)$ are equivalent to $\tau_{(2,0)}$ and $\tau_{(0,-2)}$, respectively. The correspondence of the bases are given by

$$(X_{(2,0)}, X_{(1,1)}, X_{(0,2)}) \mapsto (v_2, v_1, v_0), \quad (X_{(0,-2)}, X_{(-1,-1)}, X_{(-2,0)}) \mapsto (v_2, -v_1, v_0).$$

The simple Lie algebra $\mathfrak{g}_0$ has a $\boldsymbol{R}$-split Cartan subalgebra $\mathfrak{a} := \boldsymbol{R}H_1 \oplus \boldsymbol{R}H_2$, where we set

$$H_1 := \mathrm{diag}(1, 0, -1, 0), \quad H_2 := \mathrm{diag}(0, 1, 0, -1).$$

We denote the basis of $\mathfrak{a}_{\boldsymbol{C}}^*$ dual to $\{H_1, H_2\}$ by $\{e_1', e_2'\}$. Then the root system $\Sigma = \Sigma(\mathfrak{g}_0, \mathfrak{a})$ for $(\mathfrak{g}_0, \mathfrak{a})$ is given by $\{\pm 2e_1', \pm 2e_2', \pm(e_1' \pm e_2')\}$. As a positive system of Σ, we take $\Sigma_+ := \{2e_1', 2e_2', e_1'+e_2', e_1'-e_2'\}$. For each $\alpha = (\alpha_1, \alpha_2) = \alpha_1 e_1' + \alpha_2 e_2' \in \Sigma_+$, we fix a root vector E_α as

$$E_{(2,0)} := n_*\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right), \quad E_{(1,1)} := n_*\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right),$$

$$E_{(0,2)} := n_*\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right), \quad E_{(1,-1)} := \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right).$$

Here we set $n_*(A) := \left(\begin{array}{c|c} 0_2 & A \\ \hline 0_2 & 0_2 \end{array}\right) \in \mathfrak{g}_0$ for $A \in \mathrm{Sym}(2)_{\boldsymbol{R}}$. For a negative root $-\alpha$ ($\alpha \in \Sigma_+$), we fix a root vector $E_{-\alpha}$ as $E_{-\alpha} := {}^t E_\alpha$.

2.2. Standard representations of $Sp(2, \boldsymbol{R})$.

In this subsection, we introduce two kinds of quasi-simple admissible representations of $Sp(2, \boldsymbol{R})$.

2.2.1. P_1-principal series representations.

We define the non-Siegel maximal parabolic subgroup P_1 of $Sp(2, \boldsymbol{R})$ to be the stabilizer of the line $\boldsymbol{R}\cdot{}^t(1,0,0,0)$ in $Sp(2, \boldsymbol{R})$. We fix the Langlands decomposition $P_1 = M_1 A_1 N_1$ of P_1 as follows:

$$M_1 := \left\{ \left(\begin{array}{cc|cc} \epsilon_1 & & & \\ & a & & b \\ \hline & & \epsilon_1 & \\ & c & & d \end{array}\right) \,\middle|\, \epsilon_1 = \pm 1, \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \boldsymbol{R}) \right\},$$

$$A_1 := \left\{ \mathrm{diag}(a_1, 1, a_1^{-1}, 1) \mid a_1 > 0 \right\}, \quad N_1 := \{ n(x_0, x_1, x_2, 0) \mid x_i \in \boldsymbol{R} \}.$$

Let $D_n^{\pm}$ ($n \geq 1$) be the (limit of) discrete series representation of $SL(2, \boldsymbol{R})$ with Blattner parameter $\pm n$. Then we have an irreducible unitary representation (σ, V_σ) of M_1 characterized by $\sigma(\mathrm{diag}(-1,1,-1,1)) = \epsilon$ ($\epsilon = \pm 1$) and $\sigma|_{SL(2,\boldsymbol{R})} \cong D_n^{\pm}$, which we denote by $\sigma = \epsilon \otimes D_n^{\pm}$. For $\nu_1 \in \boldsymbol{C}$, we define a quasi-character $\exp(\nu_1) : A_1 \to \boldsymbol{C}^\times$ by

$$\exp(\nu_1)\big(\mathrm{diag}(a_1, 1, a_1^{-1}, 1)\big) = a_1^{\nu_1}.$$

Then the P_1-principal series representation $I(P_1; \sigma, \nu_1)$ of G_0 is realized on the space of all C^∞-functions $f : G_0 \longrightarrow V_\sigma$ satisfying

$$f(mang) = \sigma(m)\exp(\nu_1 + 2)(a)f(g),$$
$$\forall(m, a, n, g) \in M_1 \times A_1 \times N_1 \times G_0,$$

on which G_0 acts by right translation. The infinitesimal character of π is given by $\nu_1 e'_1 + (n-1)e'_2 \in \mathfrak{a}^*_{\boldsymbol{C}}/\sim_W$, where $W = W(\mathfrak{g}_0, \mathfrak{a})$ is the Weyl group for the pair $(\mathfrak{g}_0, \mathfrak{a})$. For an admissible representation $(\pi, \mathscr{H}_\pi)$ of G_0 and an irreducible finite-dimensional representation τ of K_0, we set $[\pi; \tau] := \dim_{\boldsymbol{C}} \mathrm{Hom}_{K_0}(\tau, \pi)$, the multiplicity of τ in π. From [**Mi-1**, Proposition 6.3], we quote the following:

Proposition 2.1. *Let $\pi = I(P_1; \sigma, \nu_1)$ be a P_1-principal series representation of G_0. Then we have the following assertions*:

(i) *if $\sigma = (-1)^n \otimes D_n^+$, then we have $[\pi; \tau_{(n,n)}] = 1$ and $[\pi; \tau_{(n-2,n-2)}] = 0$;*
(ii) *if $\sigma = (-1)^{n+1} \otimes D_n^+$, then we have $[\pi; \tau_{(n,n-1)}] = 1$ and $[\pi; \tau_{(n-1,n-2)}] = 0$;*
(iii) *if $\sigma = (-1)^n \otimes D_n^-$, then we have $[\pi; \tau_{(-n,-n)}] = 1$ and $[\pi; \tau_{(2-n,2-n)}] = 0$;*
(iv) *if $\sigma = (-1)^{n+1} \otimes D_n^-$, then we have $[\pi; \tau_{(1-n,-n)}] = 1$ and $[\pi; \tau_{(2-n,1-n)}] = 0$.*

We say that a P_1-principal series representation $\pi = I(P_1; \sigma, \nu_1)$ of G_0 is *even* (resp. *odd*) if σ is of the form $\sigma = (-1)^n \otimes D_n^\pm$ (resp. $\sigma = (-1)^{n+1} \otimes D_n^\pm$). We denote by $I(P_1; \sigma, \nu_1)[c]$ ($c \in \boldsymbol{C}$) the representation π of $GSp(2, \boldsymbol{R})$ characterized by

$$\pi|_{G_0} \cong I(P_1; \sigma, \nu_1) \oplus I(P_1; \sigma^\vee, -\nu_1) \quad \text{and} \quad \pi(zI_4) = z^c \ (\forall z > 0).$$

Here $\sigma^\vee$ stands for the contragradient representation of σ.

2.2.2. (Limits of) large discrete series representations.

Let $D_{(\lambda_1,\lambda_2)}$ be the (limit of) large discrete series representation of $Sp(2, \boldsymbol{R})$ with minimal K_0-type $\tau_{(\lambda_1,\lambda_2)}$, where $(\lambda_1, \lambda_2) \in \boldsymbol{Z}^{\oplus 2}$ satisfies $1 - \lambda_1 \le \lambda_2 \le 0$ or $1 + \lambda_2 \le -\lambda_1 \le 0$ (cf. [**Kn**, Theorem 9.20, Theorem 12.26, and p. 626–627], [**Mo**, Subsection (1.2)]). The infinitesimal character of π is given by $(\lambda_1 - 1)e_1 + \lambda_2 e_2 \in \mathfrak{a}^*_{\boldsymbol{C}}/\sim_W$. We also note the following:

Proposition 2.2. *Let $\pi = D_{(\lambda_1,\lambda_2)}$ be a (limit of) large discrete series representation of G_0 with $1 - \lambda_1 \le \lambda_2 \le 0$. Then the minimal K_0-type $\tau_{(\lambda_1,\lambda_2)}$ of π occurs in π with multiplicity one. Moreover, if an irreducible finite-dimensional representation τ of K_0 occurs in π, then τ is equivalent to $\tau_{(q_1,q_2)}$ with*

$$(q_1, q_2) = (\lambda_1, \lambda_2) + k(1, 1) + l(0, -2) \quad \textit{for some } k, l \in \boldsymbol{Z}_{\ge 0}.$$

For each $(\lambda_1, \lambda_2) \in \boldsymbol{Z}^{\oplus 2}$ satisfying $1-\lambda_1 \leq \lambda_2 \leq 0$ and $c \in \boldsymbol{C}$, there exists an irreducible admissible representation π of $GSp(2, \boldsymbol{R})$ characterized by

$$\pi\,|_{Sp(2,\boldsymbol{R})} = D_{(\lambda_1,\lambda_2)} \oplus D_{(-\lambda_2,-\lambda_1)} \quad \text{and} \quad \pi(zI_4) = z^c, \quad (\forall z > 0),$$

which we denote by $\pi = D_{(\lambda_1,\lambda_2)}[c]$.

3. Generalized Whittaker functions on $Sp(2, R)$.

We introduce the generalized Whittaker functions on $G_0 = Sp(2, \boldsymbol{R})$ and prove their elementary properties. The relation with the generalized Whittaker functions on $G = GSp(2, \boldsymbol{R})$ will be discussed in Subsection 8.1.

3.1. Definition of local generalized Whittaker functions on $Sp(2, R)$.

We fix a symmetric matrix $\beta \in \mathrm{Sym}(2)_{\boldsymbol{R}}$ satisfying $\det(\beta) \neq 0$. Let

$$T_\beta := \{t = m(h,1) \in \mathbf{M}_{\boldsymbol{R}} \mid {}^t h \beta h = \beta\}$$

be the stabilizer of β in $\mathbf{M}_{\boldsymbol{R}} \cap G_0$. Then the identity component (with respect to the Euclidean topology) T_β° of T_β is isomorphic to $SO(2)$ or $\boldsymbol{R}_{>0}^\times$ according as $\det(\beta) > 0$ or $\det(\beta) < 0$. We set $R_\beta^1 := T_\beta^\circ \ltimes \mathbf{N}_{\boldsymbol{R}}$. For a quasi-character $\chi : T_\beta^\circ \to \boldsymbol{C}^\times$, we define a quasi-character $\chi\cdot\psi_\beta : R_\beta^1 \to \boldsymbol{C}^\times$ of R_β^1 by $(\chi\cdot\psi_\beta)(tn) = \chi(t)\psi_\beta(n)$ $(t \in T_\beta^\circ, n \in \mathbf{N}_{\boldsymbol{R}})$. Then we have a smooth representation induced from $\chi \cdot \psi_\beta$:

$$\begin{aligned} &C^\infty\big(R_\beta^1\backslash G_0; \chi\cdot\psi_\beta\big) \\ &\quad := \big\{W : G_0 \xrightarrow{C^\infty} \boldsymbol{C} \mid W(rg) = (\chi\cdot\psi_\beta)(r)W(g), \quad \forall (r,g) \in R_\beta^1 \times G_0\big\}, \end{aligned}$$

on which G_0 acts by right translation. As in Subsection 1.5, a function $f : G_0 \to \boldsymbol{C}$ is said to be of moderate growth if there exists a constant $C > 0$ and $M > 0$ such that $|f(g)| \leq C\|g\|^M$ for all $g \in G_0$. If f and its all derivatives $f(g; X)$ $(X \in U(\mathfrak{g}_0))$ are of moderate growth with exponent $M > 0$ common to all $X \in U(\mathfrak{g}_0)$, then we say that the function f is *of uniformly moderate growth*. We denote by $C_{mg}^\infty(R_\beta^1\backslash G_0; \chi\cdot\psi_\beta)$ (resp. $C_{umg}^\infty(R_\beta^1\backslash G_0; \chi\cdot\psi_\beta)$) the space of functions $f \in C^\infty(R_\beta^1\backslash G_0; \chi\cdot\psi_\beta)$ of moderate growth (resp. of uniformly moderate growth).

Definition 3.1. Let $(\pi, \mathscr{H}_\pi)$ be a quasi-simple $(\mathfrak{g}_0, K_0)$-module.

(i) By a *generalized Whittaker functional*, we understand an element of the intertwining space

$$\mathbf{GW}_{G_0}(\pi, \chi \cdot \psi_\beta) := \mathrm{Hom}_{\mathfrak{g}_0, K_0}\left(\mathscr{H}_\pi, C^\infty(R^1_\beta \backslash G_0; \chi \cdot \psi_\beta)\right).$$

For a vector $v \in \mathscr{H}_\pi$, we call its image $\Phi(v)$ under some $\Phi \in \mathbf{GW}_{G_0}(\pi, \chi \cdot \psi_\beta)$ a *generalized Whittaker function* on G_0 belonging to π.

(ii) If $\Phi \in \mathbf{GW}_{G_0}(\pi, \chi \cdot \psi_\beta)$ belongs to the subspace

$$\mathbf{GW}^{mg}_{G_0}(\pi, \chi \cdot \psi_\beta) := \mathrm{Hom}_{\mathfrak{g}_0, K_0}\left(\mathscr{H}_\pi, C^\infty_{mg}(R^1_\beta \backslash G_0; \chi \cdot \psi_\beta)\right),$$

then we say that Φ and $\Phi(v)$ have the *moderate growth property.*

(iii) For a non-zero element $\Phi \in \mathbf{GW}_{G_0}(\pi, \chi \cdot \psi_\beta)$, we call the whole image $\Phi(\mathscr{H}_\pi)$ a *generalized Whittaker model of* π. If $\Phi \in \mathbf{GW}^{mg}_{G_0}(\pi, \chi \cdot \psi_\beta)$, then we say $\Phi(\mathscr{H}_\pi)$ has the *moderate growth property.*

Remark.

(i) In the literature, generalized Whittaker functions are sometimes called Bessel functions (e.g. [**F**], [**TB**], [**Pr-TB**], [**Pi-Sch**]), Siegel-Whittaker functions (e.g. [**Is**]), or generalized Bessel functions (e.g. [**No**], [**No-PS**]).

(ii) Since the generalized Whittaker function $\Phi(v)$ is right K_0-finite and $Z(\mathfrak{g}_0)$-finite, it is a real analytic function on G_0 by the elliptic regularity theorem.

Let $\beta, \beta' \in \mathrm{Sym}(2)_{\boldsymbol{R}}$ be two symmetric matrices with $\det(\beta) \neq 0$ and $\det(\beta') \neq 0$. Suppose that there exists an element $m_0 = m(h, 1) \in \mathbf{M}_{\boldsymbol{R}} \cap G_0$ such that $\beta \cdot m_0 = {}^t h \beta h = \beta'$. Define a quasi-character χ' of $T^\circ_{\beta'} = m_0^{-1} T^\circ_\beta m_0$ by $\chi'(t') := \chi(m_0 t' m_0^{-1})$ $(t' \in T^\circ_{\beta'})$. Then we have the following isomorphism of G_0-modules

$$C^\infty\left(R^1_\beta \backslash G_0; \chi \cdot \psi_\beta\right) \cong C^\infty\left(R^1_{\beta'} \backslash G_0; \chi' \cdot \psi_{\beta'}\right), \tag{3.1}$$

which assigns $W(g)$ to $W(m_0 g)$. For a finite-dimensional K_0-module (τ, V_τ), we set

$$\begin{aligned} &C^\infty\left(R^1_\beta \backslash G_0 / K_0; \chi \cdot \psi_\beta; \tau\right) \\ &\quad := \left\{W : G_0 \xrightarrow{C^\infty} V_\tau^\vee \mid W(rgk) = (\chi \cdot \psi_\beta)(r) \tau^\vee(k)^{-1} f(g), \right. \\ &\qquad\quad \left. \forall (r, g, k) \in R^1_\beta \times G_0 \times K_0\right\}, \end{aligned}$$

where $(\tau^\vee, V_\tau^\vee)$ is the contragradient representation of (τ, V_τ). Fix a K_0-equivariant map $\iota_\tau : V_\tau \to \mathscr{H}_\pi$. For a generalized Whittaker functional $\Phi \in \mathbf{GW}_{G_0}(\pi, \chi \cdot \psi_\beta)$, we have a $V_\tau^\vee$-valued C^∞-function $W \in C^\infty(R^1_\beta \backslash G_0 / K_0; \chi \cdot \psi_\beta; \tau)$ characterized by

$$\Phi(\iota_\tau(v))(g) = \langle W(g), v\rangle, \quad \forall g \in G_0,\ \forall v \in V_\tau.$$

Here $\langle\cdot\,,\cdot\rangle$ stands for the canonical pairing between $V_\tau^\vee$ and V_τ. We call the function $W \in C^\infty(R_\beta^1\backslash G_0/K_0; \chi\cdot\psi_\beta;\tau)$ defined from Φ in this way the *generalized Whittaker function on* G_0 *of type* $(\pi, \chi\cdot\psi_\beta, \iota_\tau)$. If τ is irreducible and occurs in π with multiplicity one, then we simply say that f is of type $(\pi,\chi\cdot\psi_\beta,\tau)$. If $\Phi \in \mathbf{GW}_{G_0}^{mg}(\pi,\chi\cdot\psi_\beta)$, then we say that the corresponding $W \in C^\infty(R_\beta^1\backslash G_0/K_0;\chi\cdot\psi_\beta;\tau)$ has the moderate growth property.

3.2. Radial parts.

From now on, we shall assume that $\det(\beta) < 0$. In view of (3.1), we may suppose that $\beta = \left(\begin{smallmatrix} 0 & c/2 \\ c/2 & 0 \end{smallmatrix}\right)$ $(c \in \boldsymbol{R}^\times)$. We may further assume that $c = 1$, but we *do not* assume it. It is easy to check that

$$T_\beta^\circ = \left\{ \mathrm{diag}\left(\sqrt{y_1}, \frac{1}{\sqrt{y_1}}, \frac{1}{\sqrt{y_1}}, \sqrt{y_1}\right) \,\middle|\, y_1 > 0 \right\}.$$

For $(t,y) \in \boldsymbol{R}\times\boldsymbol{R}_{>0}$, we set

$$a(t,y) := \exp\left\{ t\cdot\frac{H_0}{2} + \log(y)\cdot\frac{(H_1+H_2)}{2}\right\} \ \text{ with } \ H_0 := E_{(1,-1)} + E_{(-1,1)} \in \mathfrak{g}_0.$$

Then we have

$$a(t,y) = \left(\begin{array}{cc|cc} y^{1/2}\,\mathrm{ch}(t/2) & y^{1/2}\,\mathrm{sh}(t/2) & & \\ y^{1/2}\,\mathrm{sh}(t/2) & y^{1/2}\,\mathrm{ch}(t/2) & & \\ \hline & & y^{-1/2}\,\mathrm{ch}(t/2) & y^{-1/2}\,\mathrm{sh}(-t/2) \\ & & y^{-1/2}\,\mathrm{sh}(-t/2) & y^{-1/2}\,\mathrm{ch}(t/2) \end{array}\right).$$

Here we use the abbreviated expressions

$$\mathrm{ch}(t) := \cosh(t), \quad \mathrm{sh}(t) := \sinh(t), \quad \text{and} \quad \mathrm{th}(t) := \tanh(t).$$

We define a closed abelian subgroup S of G_0 by

$$S := \{a(t,y) \mid t \in \boldsymbol{R}, y > 0\}.$$

Then we have the following:

LEMMA 3.2. *The multiplication map* $R_\beta^1 \times S \times K_0 \ni (r,a,k) \mapsto rak \in G_0$ *gives a diffeomorphism* $R_\beta^1 \times S \times K_0 \cong G_0$.

Proof. First we prove the surjectivity of the multiplication map. Consider the usual action of $SL(2,\boldsymbol{R})$ on the upper half plane $\boldsymbol{H}_1 := \{z \in \boldsymbol{C} \mid \mathrm{Im}(z) > 0\}$: $g \cdot \langle z \rangle := (az+b)/(cz+d)$ $\big(g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right),\ z \in \boldsymbol{H}_1\big)$. From the equality

$$\begin{pmatrix} \sqrt{y_1} & 0 \\ 0 & \dfrac{1}{\sqrt{y_1}} \end{pmatrix} \begin{pmatrix} \mathrm{ch}(t/2) & \mathrm{sh}(t/2) \\ \mathrm{sh}(t/2) & \mathrm{ch}(t/2) \end{pmatrix} \cdot \langle \sqrt{-1} \rangle = y_1 \left\{ \mathrm{th}(t) + \frac{\sqrt{-1}}{\mathrm{ch}(t)} \right\}, \quad (y_1 > 0, t \in \boldsymbol{R}),$$

we know that $\mathbf{M}_{\boldsymbol{R}} \cap G_0 = T^\circ_\beta S(\mathbf{M}_{\boldsymbol{R}} \cap K_0)$. Since $G_0 = \mathbf{N}_{\boldsymbol{R}}(\mathbf{M}_{\boldsymbol{R}} \cap G_0)K_0$, this proves $R^1_\beta S K_0 = G_0$. To prove the injectivity, it is enough to show that $ra_1 = a_2 k$ for some $r \in R^1_\beta$, $a_1, a_2 \in S$, and $k \in K_0$ implies $r = k = I_4$ and $a_1 = a_2$, the proof of which is an easy computation. Finally, Lemma 4.2 in the next section tells us that

$$\mathfrak{g}_0 = \mathrm{Ad}(a^{-1})\,\mathrm{Lie}\left(R^1_\beta\right) \oplus \mathrm{Lie}(S) \oplus \mathfrak{k}, \quad \forall a \in S. \tag{3.2}$$

This decomposition implies that the multiplication map is a diffeomorphism. □

Let $C^\infty(S;\tau)$ be the space of $V_\tau^\vee$-valued C^∞-functions on S. It follows from Lemma 3.2 that the restriction map

$$C^\infty\left(R^1_\beta \backslash G_0 / K_0; \chi \cdot \psi_\beta; \tau\right) \to C^\infty(S;\tau)$$

is an isomorphism. We call the restriction $W_{|S}(a(t,y))$ of $W(g) \in C^\infty(R^1_\beta \backslash G_0/K_0; \chi \cdot \psi_\beta; \tau)$ to S the S-radial part of $W(g)$. A well-known theorem of Harish-Chandra [**HC**, Theorem 1] (see also [**Bo**, 5.6], [**Bu**, Section 2.10]) tells us that a C^∞-function $f : G_0 \to \boldsymbol{C}$ of moderate growth is of uniformly moderate growth if it is right K_0-finite and $Z(\mathfrak{g}_0)$-finite. Hence we have the following isomorphism

$$\mathbf{GW}^{mg}_{G_0}(\pi, \chi \cdot \psi_\beta) \cong \mathrm{Hom}_{\mathfrak{g}_0, K_0}\left(\mathscr{H}_\pi, C^\infty_{umg}\left(R^1_\beta \backslash G_0; \chi \cdot \psi_\beta\right)\right) \tag{3.3}$$

for a quasi-simple $(\mathfrak{g}_0, K_0)$-module $(\pi, \mathscr{H}_\pi)$. The following lemma will play a crucial role in the proof of our main results.

Lemma 3.3. *Suppose that* $W(g) \in C^\infty_{umg}(R^1_\beta \backslash G_0; \chi \cdot \psi_\beta)$. *For each* $N \geq 0$, *there exists a constant* $C > 0$ *such that* $|W(a(0,y))| \leq C y^{-N}$ *for all* $y > 0$.

Proof. Since $W(g)$ is assumed to be of uniformly moderate growth, there exists a constant $N > 0$ such that for each $l \geq 0$ we can find $C_l > 0$ satisfying

$$\left|W(a(0,y);E^l_{(1,1)})\right| \le C_l \times (\max\{y,y^{-1}\})^N, \quad \forall l \ge 0, \forall y > 0.$$

On the other hand, we have

$$\begin{aligned} & W\big(a(0,y);E_{(1,1)}\big) \\ & = \frac{d}{ds}\bigg|_{s=0} W\big(a(0,y)\exp(sE_{(1,1)})\big) \\ & = \frac{d}{ds}\bigg|_{s=0} W\big(\exp(syE_{(1,1)})a(0,y)\big) = 2\pi\sqrt{-1}cyW(a(0,y)). \end{aligned} \tag{3.4}$$

Hence we have the assertion of the lemma. □

4. Differential operators.

In this section we introduce two kinds of differential operators, that is, the shift operators and the Casimir operator, which will be used to construct differential equations satisfied by the generalized Whittaker functions in Section 5–Section 7.

4.1. Schmid operators.

In this subsection, we introduce the Schmid operators, which are used to define shift operators in the next subsection. Let (τ, V_τ) be a finite-dimensional representation of K_0. Recall that $\beta = \left(\begin{smallmatrix} 0 & c/2 \\ c/2 & 0 \end{smallmatrix}\right)$. For $\mu \in \boldsymbol{C}$, we define a quasi-character χ_μ of T°_β by

$$\chi_\mu\left(\operatorname{diag}\left(\sqrt{y_1}, \frac{1}{\sqrt{y_1}}, \frac{1}{\sqrt{y_1}}, \sqrt{y_1}\right)\right) = y_1^\mu, \quad y_1 > 0. \tag{4.1}$$

For a K_0-equivariant map

$$\phi_\tau \in \operatorname{Hom}_{K_0}\left(\tau, C^\infty\big(R^1_\beta\backslash G_0;\chi_\mu\cdot\psi_\beta\big)\right) \cong C^\infty\big(R^1_\beta\backslash G_0/K_0;\chi_\mu\cdot\psi_\beta;\tau\big),$$

we define K_0-equivariant maps $\phi_{\mathfrak{p}_\pm\otimes\tau} : \mathfrak{p}_\pm \otimes V_\tau \to C^\infty(R^1_\beta\backslash G_0;\chi\cdot\psi_\beta)$ by

$$\phi_{\mathfrak{p}_\pm\otimes\tau}(X\otimes v)(g) := \phi_\tau(v)(g;X), \quad (X \in \mathfrak{p}_\pm, v \in V_\tau, g \in G_0). \tag{4.2}$$

The assignment of ϕ_τ to $\phi_{\mathfrak{p}_\pm\otimes\tau}$ defines the gradient type differential operators

$$\nabla^\pm : C^\infty\big(R^1_\beta\backslash G_0/K_0;\chi_\mu\cdot\psi_\beta;\tau\big) \to C^\infty\big(R^1_\beta\backslash G_0/K_0;\chi_\mu\cdot\psi_\beta;\tau\otimes\operatorname{Ad}_{\mathfrak{p}_\pm}\big).$$

These operators $\nabla^{\pm}$ are called the *Schmid operators*. By the identification $(\tau \otimes \mathrm{Ad}_{\mathfrak{p}_{\pm}})^{\vee} \cong \tau^{\vee} \otimes \mathrm{Ad}_{\mathfrak{p}_{\mp}}$, we regard the image $[\nabla^{\pm} W](g)$ of $W \in C^{\infty}(R_{\beta}^{1} \backslash G_0/K_0; \chi_{\mu} \cdot \psi_{\beta}; \tau)$ as a $(V_{\tau}^{\vee} \otimes \mathfrak{p}_{\mp})$-valued C^{∞}-function on G_0. By using the root vectors X_{α} $(\alpha \in \Delta_{nc})$ introduced in Section 2, we can express $[\nabla^{\pm} W](g)$ as follows:

$$[\nabla^{\pm} W](g) = W\big(g; X_{\pm(2,0)}\big) \otimes X_{\mp(2,0)} + \frac{1}{2} W\big(g; X_{\pm(1,1)}\big) \otimes X_{\mp(1,1)} + W\big(g; X_{\pm(0,2)}\big) \otimes X_{\mp(0,2)}.$$

The S-radial parts of the actions of the Schmid operators $\nabla^{\pm}$ can be described as follows:

Proposition 4.1. *Suppose that $W \in C^{\infty}(R_{\beta}^{1} \backslash G_0/K_0; \chi_{\mu} \cdot \psi_{\beta}; \tau)$. Then we have*

$$\begin{aligned}
&[\nabla^{+} W](a(t,y)) \\
&= \left\{ \delta_y - \mathrm{sh}(t) \cdot (2\pi c y) + \frac{\mu}{\mathrm{ch}(t)} + (\tau^{\vee} \otimes \mathrm{Ad}) \left(\kappa_* \left(\begin{pmatrix} -1 & \frac{\mathrm{th}(t)}{2} \\ -\frac{\mathrm{th}(t)}{2} & 0 \end{pmatrix} \right) \right) - 3 \right\} \\
&\quad \cdot W(a(t,y)) \otimes X_{(-2,0)} \\
&\quad + \left\{ \partial_t - \mathrm{ch}(t) \cdot (2\pi c y) + \mathrm{th}(t) - \frac{1}{2} (\tau^{\vee} \otimes \mathrm{Ad}) \left(\kappa_* \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \right) \right\} \\
&\qquad \cdot W(a(t,y)) \otimes X_{(-1,-1)} \\
&\quad + \left\{ \delta_y - \mathrm{sh}(t) \cdot (2\pi c y) - \frac{\mu}{\mathrm{ch}(t)} + (\tau^{\vee} \otimes \mathrm{Ad}) \left(\kappa_* \left(\begin{pmatrix} 0 & -\frac{\mathrm{th}(t)}{2} \\ \frac{\mathrm{th}(t)}{2} & -1 \end{pmatrix} \right) \right) - 3 \right\} \\
&\qquad \cdot W(a(t,y)) \otimes X_{(0,-2)}
\end{aligned}$$

and

$$\begin{aligned}
&[\nabla^{-} W](a(t,y)) \\
&= \left\{ \delta_y + \mathrm{sh}(t) \cdot (2\pi c y) + \frac{\mu}{\mathrm{ch}(t)} + (\tau^{\vee} \otimes \mathrm{Ad}) \left(\kappa_* \left(\begin{pmatrix} 1 & \frac{\mathrm{th}(t)}{2} \\ -\frac{\mathrm{th}(t)}{2} & 0 \end{pmatrix} \right) \right) - 3 \right\} \\
&\quad \cdot W(a(t,y)) \otimes X_{(2,0)}
\end{aligned}$$

$$+\left\{\partial_t + \mathrm{ch}(t)\cdot(2\pi c y) + \mathrm{th}(t) + \frac{1}{2}(\tau^\vee \otimes \mathrm{Ad})\left(\kappa_*\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)\right)\right\}$$

$$\cdot W(a(t,y)) \otimes X_{(1,1)}$$

$$+\left\{\delta_y + \mathrm{sh}(t)\cdot(2\pi c y) - \frac{\mu}{\mathrm{ch}(t)} + (\tau^\vee \otimes \mathrm{Ad})\left(\kappa_*\left(\begin{pmatrix} 0 & -\dfrac{\mathrm{th}(t)}{2} \\ \dfrac{\mathrm{th}(t)}{2} & 1 \end{pmatrix}\right)\right) - 3\right\}$$

$$\cdot W(a(t,y)) \otimes X_{(0,2)}.$$

Here we use the abbreviated expressions:

$$\partial_t := \frac{\partial}{\partial t}, \qquad \delta_y := y\frac{\partial}{\partial y}. \tag{4.3}$$

Proof. For any function $W(g) \in C^\infty(R_\beta^1\backslash G_0/K_0; \chi\cdot\psi_\beta; \tau)$, we have the following equality

$$\begin{aligned} &W\left(a(t,y); \mathrm{Ad}(a(t,y)^{-1})(\xi)\cdot\left(\frac{H_0}{2}\right)^l\cdot\left(\frac{H_1+H_2}{2}\right)^m\cdot\eta\right) \\ &\quad = (\chi\cdot\psi_\beta)(\xi)\tau^\vee(\widehat{\eta})\partial_t^l\delta_y^m W(a(t,y)), \\ &\qquad \forall\xi \in U(\mathrm{Lie}(R_\beta^1)), \quad \forall l, \forall m, \geq 0, \quad \forall\eta \in U(\mathfrak{k}). \end{aligned} \tag{4.4}$$

Here $U(\mathfrak{k}) \ni \eta \mapsto \widehat{\eta} \in U(\mathfrak{k})$ is the anti-automorphism of $U(\mathfrak{k})$ characterized by $\widehat{X} = -X$ for $X \in \mathfrak{k}_{\boldsymbol{C}}$. By using (4.4) and Lemma 4.2 below, we can compute $\nabla^{\pm}W(a(t,y))$. □

Lemma 4.2. *For each $a(t,y) \in S$, the root vectors X_α ($\alpha \in \Delta_{nc}$) are decomposed as follows*:

$$\begin{aligned} &X_{\pm(2,0)} \\ &\quad = \mathrm{Ad}(a(t,y)^{-1})\left\{\pm 2\sqrt{-1}y\cdot n_*\left(\begin{pmatrix} \mathrm{ch}^2(t/2) & \mathrm{sh}(t)/2 \\ \mathrm{sh}(t)/2 & \mathrm{sh}^2(t/2) \end{pmatrix}\right) + \frac{1}{2\,\mathrm{ch}(t)}(H_1 - H_2)\right\} \\ &\qquad + \frac{1}{2}(H_1+H_2) + \kappa_*\left(\begin{pmatrix} \pm 1 & -\mathrm{th}(t)/2 \\ \mathrm{th}(t)/2 & 0 \end{pmatrix}\right), \end{aligned}$$

$$X_{\pm(1,1)}$$
$$= \mathrm{Ad}(a(t,y)^{-1})\Big\{ \pm 2\sqrt{-1}y \cdot n_*\Big(\begin{pmatrix} \mathrm{sh}(t) & \mathrm{ch}(t) \\ \mathrm{ch}(t) & \mathrm{sh}(t)\end{pmatrix}\Big)\Big\} + H_0 \pm \kappa_*\Big(\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\Big),$$

$$X_{\pm(0,2)}$$
$$= \mathrm{Ad}(a(t,y)^{-1})\Big\{ \pm 2\sqrt{-1}y \cdot n_*\Big(\begin{pmatrix} \mathrm{sh}^2(t/2) & \mathrm{sh}(t)/2 \\ \mathrm{sh}(t)/2 & \mathrm{ch}^2(t/2)\end{pmatrix}\Big) - \frac{1}{2\,\mathrm{ch}(t)}(H_1 - H_2)\Big\}$$
$$+ \frac{1}{2}(H_1 + H_2) + \kappa_*\Big(\begin{pmatrix} 0 & \mathrm{th}(t)/2 \\ -\mathrm{th}(t)/2 & \pm 1\end{pmatrix}\Big).$$

Remark. In [**Mi-1**], the corresponding computation is done for $\beta = \left(\begin{smallmatrix} h_1 & 0 \\ 0 & h_2\end{smallmatrix}\right)$, $h_1h_2 \neq 0$. However, our anti-diagonal choice of β makes our final formulae in Theorem 7.1 simple as long as we use the standard basis introduced in Subsection 2.2. This can be foreseen from the computation of local Novodvorsky integrals in [**Mo**, Proposition 8] (see Section 9).

4.2. Shift operators.

Suppose that (τ, V_τ) is equivalent to an irreducible finite-dimensional representation $(\tau_{(\lambda_1,\lambda_2)}, V_{(\lambda_1,\lambda_2)})$ of K_0. The tensor product representation $V_\tau^\vee \otimes \mathfrak{p}_\pm$ has the decomposition into irreducible factors:

$$V_\tau^\vee \otimes \mathfrak{p}_+ \cong \begin{cases} V_{(-\lambda_2+2,-\lambda_1)} \oplus V_{(-\lambda_2+1,-\lambda_1+1)} \oplus V_{(-\lambda_2,-\lambda_1+2)} & \text{if } \lambda_1 > \lambda_2; \\ V_{(-\lambda_2+2,-\lambda_1)} & \text{if } \lambda_1 = \lambda_2; \end{cases}$$
$$V_\tau^\vee \otimes \mathfrak{p}_- \cong \begin{cases} V_{(-\lambda_2,-\lambda_1-2)} \oplus V_{(-\lambda_2-1,-\lambda_1-1)} \oplus V_{(-\lambda_2-2,-\lambda_1)} & \text{if } \lambda_1 > \lambda_2; \\ V_{(-\lambda_2,-\lambda_1-2)} & \text{if } \lambda_1 = \lambda_2. \end{cases}$$

Here we understand that $V_{(q_1,q_2)} = \{0\}$ if $q_1 < q_2$. Let P^{up}, P^{ev}, and P^{dn} be the projectors from $V_\tau^\vee \otimes \mathfrak{p}_+$ (resp. $V_\tau^\vee \otimes \mathfrak{p}_-$) to $V_{(-\lambda_2+2,-\lambda_1)}$ (resp. $V_{(-\lambda_2,-\lambda_1-2)}$), $V_{(-\lambda_2+1,-\lambda_1+1)}$ (resp. $V_{(-\lambda_2-1,-\lambda_1-1)}$), and $V_{(-\lambda_2,-\lambda_1+2)}$ (resp. $V_{(-\lambda_2-2,-\lambda_1)}$), which are determined up to constant multiples. Then we have the linear maps

$$P^\bullet \cdot \nabla^+ : C^\infty\big(R^1_\beta\backslash G_0/K_0; \chi\cdot\psi_\beta; \tau\big) \to C^\infty\big(R^1_\beta\backslash G_0/K_0; \chi\cdot\psi_\beta; \tau'\big),$$

where τ' is $\tau_{(\lambda_1+2,\lambda_2)}$, $\tau_{(\lambda_1+1,\lambda_2+1)}$, or $\tau_{(\lambda_1,\lambda_2+2)}$ according as $\bullet = up$, ev, or dn. Similarly we have $P^\bullet \cdot \nabla^-$ ($\bullet = up$, ev, or dn). We call these six differential operators $P^\bullet\cdot\nabla^\pm$ the *shift operators*. We write $W(g) \in C^\infty(R^1_\beta\backslash G_0; \chi\cdot\psi_\beta; \tau_{(\lambda_1,\lambda_2)})$ in the form

$$W(g) = \sum_{k=0}^{d} \phi_k(g) v_k$$

with the standard basis $\{v_k \mid 0 \le k \le d = \lambda_1 - \lambda_2\}$ of $\tau_{(-\lambda_2,-\lambda_1)}$. We frequently write $\phi_k(t,y)$ in place of $\phi_k(a(t,y))$. In terms of the coefficient functions $\phi_k(t,y)$, the S-radial parts of the shift operators are described as follows:

Proposition 4.3.

(i) *We define* C^∞*-functions* $\phi_k^{(2,0)}(g)$ $(0 \le k \le d{+}2)$ *on* G_0 *by* $[P^{up}\cdot\nabla^+ W](g) = \sum_{k=0}^{d+2} \phi_k^{(2,0)}(g) v_k$. *Then we have*

$$\begin{aligned}\phi_k^{(2,0)}(t,y) = &-\frac{1}{4}(d+2-k)(d+1-k)(d-k)\,\mathrm{th}(t)\phi_{k+1}(t,y)\\ &+ \binom{d+2-k}{2}\Big(\delta_y - 2\pi c y\cdot \mathrm{sh}(t) + \frac{\mu}{\mathrm{ch}(t)} + \lambda_1\Big)\phi_k(t,y)\\ &+ k(d+2-k)\Big(- \partial_t + 2\pi c y \cdot \mathrm{ch}(t) + \frac{d}{4}\cdot \mathrm{th}(t)\Big)\phi_{k-1}(t,y)\\ &+ \binom{k}{2}\Big(\delta_y - 2\pi c y\cdot \mathrm{sh}(t) - \frac{\mu}{\mathrm{ch}(t)} + \lambda_1\Big)\phi_{k-2}(t,y)\\ &- \frac{1}{4}k(k-1)(k-2)\,\mathrm{th}(t)\phi_{k-3}(t,y), \qquad (0\le k\le d+2).\end{aligned}$$

(ii) *We define* C^∞*-functions* $\phi_k^{(1,1)}(g)$ $(0 \le k \le d)$ *on* G_0 *by* $[P^{ev}\cdot\nabla^+ W](g) = \sum_{k=0}^{d} \phi_k^{(1,1)}(g) v_k$. *Then we have*

$$\begin{aligned}\phi_k^{(1,1)}(t,y) = &\frac{1}{2}(d-k)(d-k-1)\,\mathrm{th}(t)\phi_{k+2}(t,y)\\ &+ (k-d)\Big(\delta_y - 2\pi c y\cdot \mathrm{sh}(t) + \frac{\mu}{\mathrm{ch}(t)} + \frac{\lambda_1+\lambda_2}{2} - 1\Big)\phi_{k+1}(t,y)\\ &+ (2k-d)\Big(\partial_t - 2\pi c y\cdot \mathrm{ch}(t) + \frac{1}{2}\cdot \mathrm{th}(t)\Big)\phi_k(t,y)\\ &+ k\Big(\delta_y - 2\pi c y\cdot \mathrm{sh}(t) - \frac{\mu}{\mathrm{ch}(t)} + \frac{\lambda_1+\lambda_2}{2} - 1\Big)\phi_{k-1}(t,y)\\ &- \frac{1}{2}k(k-1)\,\mathrm{th}(t)\phi_{k-2}(t,y), \qquad (0\le k\le d).\end{aligned}$$

(iii) *We define C^∞-functions $\phi_k^{(0,2)}(g)$ $(0 \le k \le d-2)$ on G_0 by $[P^{dn} \cdot \nabla^+ W](g) = \sum_{k=0}^{d-2} \phi_k^{(0,2)}(g) v_k$. Then we have*

$$\begin{aligned}
\phi_k^{(0,2)}(t,y) = &-\frac{1}{2}(d-k-2)\,\mathrm{th}(t)\phi_{k+3}(t,y) \\
&+ \left(\delta_y - 2\pi c y \cdot \mathrm{sh}(t) + \frac{\mu}{\mathrm{ch}(t)} + \lambda_2 - 1\right)\phi_{k+2}(t,y) \\
&+ 2\left(\partial_t - 2\pi c y \cdot \mathrm{ch}(t) + \frac{d+2}{4} \cdot \mathrm{th}(t)\right)\phi_{k+1}(t,y) \\
&+ \left(\delta_y - 2\pi c y \cdot \mathrm{sh}(t) - \frac{\mu}{\mathrm{ch}(t)} + \lambda_2 - 1\right)\phi_k(t,y) \\
&- \frac{k}{2}\,\mathrm{th}(t)\phi_{k-1}(t,y), \qquad (0 \le k \le d-2).
\end{aligned}$$

(iv) *We define C^∞-functions $\phi_k^{(0,-2)}(g)$ $(0 \le k \le d+2)$ on G_0 by $[P^{up} \cdot \nabla^- W](g) = \sum_{k=0}^{d+2} \phi_k^{(0,-2)}(g) v_k$. Then we have*

$$\begin{aligned}
\phi_k^{(0,-2)}(t,y) = &\frac{1}{4}(d+2-k)(d+1-k)(d-k)\,\mathrm{th}(t)\phi_{k+1}(t,y) \\
&+ \binom{d+2-k}{2}\left(\delta_y + 2\pi c y \cdot \mathrm{sh}(t) - \frac{\mu}{\mathrm{ch}(t)} - \lambda_2\right)\phi_k(t,y) \\
&+ k(d+2-k)\left(\partial_t + 2\pi c y \cdot \mathrm{ch}(t) - \frac{d}{4} \cdot \mathrm{th}(t)\right)\phi_{k-1}(t,y) \\
&+ \binom{k}{2}\left(\delta_y + 2\pi c y \cdot \mathrm{sh}(t) + \frac{\mu}{\mathrm{ch}(t)} - \lambda_2\right)\phi_{k-2}(t,y) \\
&+ \frac{1}{4}k(k-1)(k-2)\,\mathrm{th}(t)\phi_{k-3}(t,y), \qquad (0 \le k \le d+2).
\end{aligned}$$

(v) *We define C^∞-functions $\phi_k^{(-1,-1)}(g)$ $(0 \le k \le d)$ on G_0 by $[P^{ev} \cdot \nabla^- W](g) = \sum_{k=0}^{d} \phi_k^{(-1,-1)}(g) v_k$. Then we have*

$$\begin{aligned}
\phi_k^{(-1,-1)}(t,y) = &-\frac{1}{2}(d-k)(d-k-1)\,\mathrm{th}(t)\phi_{k+2}(t,y) \\
&+ (k-d)\left(\delta_y + 2\pi c y \cdot \mathrm{sh}(t) - \frac{\mu}{\mathrm{ch}(t)} - \frac{\lambda_1 + \lambda_2}{2} - 1\right)\phi_{k+1}(t,y)
\end{aligned}$$

$$+ (d-2k)\Big(\partial_t + 2\pi cy\cdot \mathrm{ch}(t) + \frac{1}{2}\cdot \mathrm{th}(t)\Big)\phi_k(t,y)$$
$$+ k\Big(\delta_y + 2\pi cy\cdot \mathrm{sh}(t) + \frac{\mu}{\mathrm{ch}(t)} - \frac{\lambda_1+\lambda_2}{2} - 1\Big)\phi_{k-1}(t,y)$$
$$+ \frac{1}{2}k(k-1)\,\mathrm{th}(t)\phi_{k-2}(t,y), \qquad (0\le k\le d).$$

(vi) *We define C^∞-functions $\phi_k^{(-2,0)}(g)$ $(0\le k\le d-2)$ on G_0 by $[P^{dn}\cdot \nabla^- W](g) = \sum_{k=0}^{d-2}\phi_k^{(-2,0)}(g)v_k$. Then we have*

$$\phi_k^{(-2,0)}(t,y) = \frac{1}{2}(d-k-2)\,\mathrm{th}(t)\phi_{k+3}(t,y)$$
$$+ \Big(\delta_y + 2\pi cy\cdot \mathrm{sh}(t) - \frac{\mu}{\mathrm{ch}(t)} - \lambda_1 - 1\Big)\phi_{k+2}(t,y)$$
$$+ (-2)\Big(\partial_t + 2\pi cy\cdot \mathrm{ch}(t) + \frac{d+2}{4}\cdot \mathrm{th}(t)\Big)\phi_{k+1}(t,y)$$
$$+ \Big(\delta_y + 2\pi cy\cdot \mathrm{sh}(t) + \frac{\mu}{\mathrm{ch}(t)} - \lambda_1 - 1\Big)\phi_k(t,y)$$
$$+ \frac{k}{2}\,\mathrm{th}(t)\phi_{k-1}(t,y), \qquad (0\le k\le d-2).$$

Proof. We can prove these formulae in the same manner as [**Mi-1**, Proposition 10.2]. That is, we combine Proposition 4.2 with the formulae of projectors $P^\bullet$ given in [**Mi-1**, Lemmas 3.3, 3.4, and 3.5]. Details are left to the reader. □

4.3. The Casimir operator.

Up to a constant multiple, the Casimir element $\Omega\in U(\mathfrak{g}_0)$ of $\mathfrak{g}_0$ is given by

$$\Omega = H_1^2 + H_2^2 - 4H_1 - 2H_2 + 4E_{(2,0)}\cdot E_{(-2,0)} + 2E_{(1,1)}\cdot E_{(-1,-1)} + 4E_{(0,2)}\cdot E_{(0,-2)} + 2E_{(1,-1)}\cdot E_{(-1,1)}.$$

For each finite-dimensional representation (τ, V_τ) of K_0, we call the differential operator

$$R_\Omega : C^\infty\big(R_\beta^1\backslash G_0/K_0;\chi\cdot\psi_\beta;\tau\big)\to C^\infty\big(R_\beta^1\backslash G_0/K_0;\chi\cdot\psi_\beta;\tau\big)$$

the *Casimir operator*. Then the S-radial part of the Casimir operator R_Ω is given by the following proposition:

Proposition 4.4. *For each* $W \in C^\infty(R^1_\beta \backslash G_0/K_0; \chi\cdot\psi_\beta;\tau)$, *we have*

$$
\begin{aligned}
&[R_\Omega W](a(t,y)) \\
&\quad = \Big\{ 2\partial_t^2 + 2\,\mathrm{th}(t)\partial_t + 2\delta_y^2 - 6\delta_y + \frac{2\mu^2}{\mathrm{ch}^2(t)} - 2\,\mathrm{ch}(2t)(2\pi c y)^2 \\
&\qquad - 2\,\mathrm{sh}(t)(2\pi c y)\tau^\vee\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right) + \frac{2\mu\,\mathrm{th}(t)}{\mathrm{ch}(t)}\tau^\vee\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right) \\
&\qquad - 2\,\mathrm{ch}(t)(2\pi c y)\tau^\vee\left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right) - \frac{1}{2\,\mathrm{ch}^2(t)}\tau^\vee\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}^2\right)\Big\} W(a(t,y)).
\end{aligned}
$$

Proof. We can prove this by using (4.4) and Lemma 4.5 below. □

Lemma 4.5. *For any* $a = a(t,y) \in S$, *the Casimir element* Ω *can be rewritten as*

$$
\begin{aligned}
\Omega &= \frac{1}{2\cdot \mathrm{ch}^2(t)}\,\mathrm{Ad}(a^{-1})((H_1 - H_2)^2) \\
&\quad + \frac{\mathrm{th}(t)}{\mathrm{ch}(t)}\big\{\mathrm{Ad}(a^{-1})(H_1 - H_2)\big\}\cdot\kappa_*\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right) \\
&\quad + \frac{1}{2}(H_1 + H_2)^2 - 3(H_1 + H_2) + \frac{1}{2}H_0^2 + \mathrm{th}(t) H_0 \\
&\quad + \frac{1}{2}(\mathrm{th}^2(t) - 1)\kappa_*\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right)^2 \\
&\quad + y^2\,\mathrm{Ad}(a^{-1})\Big\{4n_*\left(\begin{pmatrix}\mathrm{ch}^2(t/2) & \mathrm{sh}(t)/2\\ \mathrm{sh}(t)/2 & \mathrm{sh}^2(t/2)\end{pmatrix}\right)^2 + 2n_*\left(\begin{pmatrix}\mathrm{sh}(t) & \mathrm{ch}(t)\\ \mathrm{ch}(t) & \mathrm{sh}(t)\end{pmatrix}\right)^2 \\
&\qquad\qquad + 4n_*\left(\begin{pmatrix}\mathrm{sh}^2(t/2) & \mathrm{sh}(t)/2\\ \mathrm{sh}(t)/2 & \mathrm{ch}^2(t/2)\end{pmatrix}\right)^2\Big\} \\
&\quad - 4\sqrt{-1}y\Big\{\mathrm{Ad}(a^{-1})n_*\left(\begin{pmatrix}\mathrm{ch}^2(t/2) & \mathrm{sh}(t)/2\\ \mathrm{sh}(t)/2 & \mathrm{sh}^2(t/2)\end{pmatrix}\right)\Big\}\cdot\kappa_*\left(\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right) \\
&\quad - 2\sqrt{-1}y\Big\{\mathrm{Ad}(a^{-1})n_*\left(\begin{pmatrix}\mathrm{sh}(t) & \mathrm{ch}(t)\\ \mathrm{ch}(t) & \mathrm{sh}(t)\end{pmatrix}\right)\Big\}\cdot\kappa_*\left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right) \\
&\quad - 4\sqrt{-1}y\Big\{\mathrm{Ad}(a^{-1})n_*\left(\begin{pmatrix}\mathrm{sh}^2(t/2) & \mathrm{sh}(t)/2\\ \mathrm{sh}(t)/2 & \mathrm{ch}^2(t/2)\end{pmatrix}\right)\Big\}\cdot\kappa_*\left(\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right).
\end{aligned}
$$

Proof. By a direct computation, we have

$$
\begin{aligned}
\Omega = {} & \frac{1}{2}(H_1 - H_2)^2 + \frac{1}{2}(H_1 + H_2)^2 - (H_1 - H_2) - 3(H_1 + H_2) \\
& + 2E_{(1,-1)} \cdot E_{(-1,1)} + 4E_{(2,0)} \cdot \left\{ E_{(2,0)} - \kappa_* \left(\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & 0 \end{pmatrix} \right) \right\} \\
& + 2E_{(1,1)} \cdot \left\{ E_{(1,1)} - \kappa_* \left(\begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix} \right) \right\} \\
& + 4E_{(0,2)} \cdot \left\{ E_{(0,2)} - \kappa_* \left(\begin{pmatrix} 0 & 0 \\ 0 & \sqrt{-1} \end{pmatrix} \right) \right\}.
\end{aligned}
$$

It is easy to check that

$$
H_1 - H_2 = \mathrm{ch}(t)^{-1} \operatorname{Ad}(a^{-1})(H_1 - H_2) + \mathrm{th}(t) \kappa_* \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right).
$$

Hence we have

$$
\begin{aligned}
(H_1 - H_2)^2 = {} & \left\{ \mathrm{ch}(t)^{-1} \operatorname{Ad}(a^{-1})(H_1 - H_2) + \mathrm{th}(t) \kappa_* \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \right\} \cdot (H_1 - H_2) \\
= {} & \mathrm{ch}(t)^{-1} \operatorname{Ad}(a^{-1})(H_1 - H_2) \\
& \cdot \left\{ \mathrm{ch}(t)^{-1} \operatorname{Ad}(a^{-1})(H_1 - H_2) + \mathrm{th}(t) \kappa_* \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \right\} \\
& + \mathrm{th}(t) \left\{ (H_1 - H_2) \cdot \kappa_* \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) + 2H_0 \right\} \\
= {} & \frac{1}{\mathrm{ch}(t)^2} \operatorname{Ad}(a^{-1})((H_1 - H_2)^2) \\
& + \frac{2\,\mathrm{th}(t)}{\mathrm{ch}(t)} \{ \operatorname{Ad}(a^{-1})(H_1 - H_2) \} \cdot \kappa_* \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \\
& + 2\,\mathrm{th}(t) H_0 + \mathrm{th}(t)^2 \kappa_* \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right)^2.
\end{aligned}
$$

The term $2E_{(1,-1)} \cdot E_{(-1,1)}$ can be rewritten as follows:

$$2E_{(1,-1)} \cdot E_{(-1,1)} = \frac{1}{2}\left\{H_0 - \kappa_*\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)\right\} \cdot \left\{H_0 + \kappa_*\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)\right\}$$
$$= \frac{1}{2}H_0^2 - \frac{1}{2}\kappa_*\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)^2 + H_1 - H_2.$$

The rest of the computation is easy if we notice that $\mathrm{Lie}(\mathbf{N}_{\boldsymbol{R}})$ is stable under $\mathrm{Ad}(S)$. □

5. Generalized Whittaker functions belonging to even P_1-principal series representations.

In this section, we suppose that π is equivalent to an even P_1-principal series representation $I(P_1; (-1)^n \otimes D_n^+, \nu_1)$ $(n \geq 1, \nu_1 \in \boldsymbol{C})$ of G_0. By Proposition 2.1, $\tau = \tau_{(n,n)}$ is a multiplicity one K_0-type of π. By using the differential operators introduced in the previous section, we derive a system of partial differential equations satisfied by the generalized Whittaker function of type $(\pi, \chi \cdot \psi_\beta, \tau)$. From these partial differential equations plus a result on the generalized hypergeometric differential equations (Proposition 9.2), we obtain an explicit formula of $W(g)$ on a one-parameter subgroup $\{a(0,y) \mid y > 0\}$ of G_0 and prove that $\dim_{\boldsymbol{C}} \mathbf{GW}_{G_0}^{mg}(\pi, \chi \cdot \psi_\beta) \leq 1$.

5.1. The main results for even P_1-principal series representations.

We state our main results for even P_1-principal series representations.

Theorem 5.1. *Suppose that π is equivalent to an even irreducible P_1-principal series representation $I(P_1; (-1)^n \otimes D_n^+, \nu_1)$ of G_0. We set $\beta = \left(\begin{smallmatrix} 0 & c/2 \\ c/2 & 0 \end{smallmatrix}\right)$ $(c \in \boldsymbol{R}^\times)$ and fix a quasi-character $\chi = \chi_\mu$ of T_β° as in (4.1). Then we have the following assertions*:

- (i) $\dim_{\boldsymbol{C}} \mathbf{GW}_{G_0}(\pi, \chi \cdot \psi_\beta) \leq 4$.
- (ii) $\dim_{\boldsymbol{C}} \mathbf{GW}_{G_0}^{mg}(\pi, \chi \cdot \psi_\beta) \leq 1$.
- (iii) *Let $\tau = \tau_{(n,n)}$ be a multiplicity one K_0-type of π. For a generalized Whittaker function $W(g)$ of type $(\pi, \chi \cdot \psi_\beta, \tau)$, we define a C^∞-function $\varphi(a)$ on S by*

$$W(a(t,y)) = e^{-2\pi c y \cdot \mathrm{sh}(t)} y^n \varphi(a(t,y)) v_0, \quad \forall a(t,y) \in S,$$

where v_0 is a fixed non-zero vector in $V_\tau^\vee$. If $W(g)$ has the moderate growth property, then we have

$$\varphi(a(0,y)) = C \times G^{4,0}_{2,4}\left((\pi c y)^2 \,\middle|\, \begin{matrix} \alpha_1, \ \alpha_2 \\ \gamma_1, \ \gamma_2, \ \gamma_3, \ \gamma_4 \end{matrix}\right),$$

where $C \in \boldsymbol{C}$ is a constant and

$$\begin{aligned} \alpha_1 &= \frac{\mu+2}{2}, \quad \alpha_2 = \frac{-\mu+2}{2}, \\ \gamma_1 &= \frac{-n+4+\nu_1}{4}, \quad \gamma_2 = \frac{-n+4-\nu_1}{4}, \\ \gamma_3 &= \frac{-n+2+\nu_1}{4}, \quad \gamma_4 = \frac{-n+2-\nu_1}{4}. \end{aligned} \tag{5.1}$$

Here $G^{4,0}_{2,4}\left(z \,\middle|\, \begin{smallmatrix} \alpha_1, \ \alpha_2 \\ \gamma_1, \ \gamma_2, \ \gamma_3, \ \gamma_4 \end{smallmatrix}\right)$ stands for the Meijer G-function, whose definition is recalled in Section 8.

Remark. In view of (3.4), the assertions (i) and (ii) imply that $\dim_{\boldsymbol{C}} \mathbf{GW}_G(\pi, \chi \cdot \psi_\beta) \leq 4$ and $\dim_{\boldsymbol{C}} \mathbf{GW}_G^{mg}(\pi, \chi \cdot \psi_\beta) \leq 1$ when π is an irreducible $(\mathfrak{g}, K)$-module $I(P_1; \epsilon \otimes D_n^+, \nu_1)[c]$. Similar remarks are valid for Theorems 6.1 and 7.1 below.

5.2. Proof of Theorem 5.1.

The starting point of our proof of Theorem 5.1 is the following:

Proposition 5.2. *Let $W(g)$ be a generalized Whittaker function of type $(\pi, \chi_\mu \cdot \psi_\beta, \tau_{(n,n)})$. Then we have*

$$\left[(P^{dn} \cdot \nabla^-) \circ (P^{up} \cdot \nabla^-) W\right](g) = 0, \tag{5.2}$$

$$[R_\Omega W](g) = \left\{\nu_1^2 + (n-1)^2 - 5\right\} W(g). \tag{5.3}$$

Proof. The first equation can be easily obtained from Proposition 2.1. The image of the Casimir element Ω under the Harish-Chandra isomorphism $Z(\mathfrak{g}_0) \to U(\mathfrak{a})^W$ is given by $H_1^2 + H_2^2 - 5$. This combined with the infinitesimal character of π given in Subsection 2.2 proves the second equation. □

The S-radial parts of the system of differential equations in Proposition 5.2 are given as follows:

Proposition 5.3. *Let $\varphi(a(t,y)) \in C^\infty(S)$ be as in Theorem 5.1. Then the equations (5.2) and (5.3) are equivalent to the following differential equations*

$$\left\{ -\partial_t^2 - \mathrm{th}(t)\partial_t + \delta_y^2 - \delta_y - \frac{\mu^2}{\mathrm{ch}^2(t)} \right\} \varphi(a(t,y)) = 0 \tag{5.4}$$

and

$$\left\{ \partial_t^2 + (-4\pi cy \cdot \mathrm{ch}(t) + \mathrm{th}(t))\partial_t + \delta_y^2 + (-4\pi cy \cdot \mathrm{sh}(t) + 2n - 3)\delta_y + \frac{\mu^2}{\mathrm{ch}^2(t)} + \frac{1}{2}(n-2)^2 - \frac{1}{2}\nu_1^2 \right\} \varphi(a(t,y)) = 0, \tag{5.5}$$

respectively.

Proof. This can be easily obtained by using Proposition 4.3 (iv), (vi) and Proposition 4.4. □

We shall derive an ordinary differential equation for $\varphi(a(0,y))$ from the system in Proposition 5.3. First we note that, under (5.4), the equation (5.5) can be replaced by

$$\left\{ -2\pi cy \cdot \mathrm{ch}(t)\partial_t + \left(\delta_y + \frac{n-2+\nu_1}{2}\right)\left(\delta_y + \frac{n-2-\nu_1}{2}\right) - 2\pi cy \cdot \mathrm{sh}(t)\delta_y \right\} \varphi(a(t,y)) = 0. \tag{5.6}$$

Since $\varphi(a(t,y))$ is a real analytic function, we can express the function $\varphi(a(t,y))$ in the form

$$\varphi(a(t,y)) = \sum_{j \geq 0} \varphi^{\langle j \rangle}(y) t^j.$$

Note that $\varphi(a(0,y)) = \varphi^{\langle 0 \rangle}(y)$. It follows from the equation (5.4) that

$$-2\varphi^{\langle 2 \rangle}(y) + \left(\delta_y^2 - \delta_y - \mu^2\right)\varphi^{\langle 0 \rangle}(y) = 0. \tag{5.7}$$

From (5.6), we have

$$-2\pi cy \cdot \varphi^{\langle 1 \rangle}(y) + \left(\delta_y + \frac{n-2+\nu_1}{2}\right)\left(\delta_y + \frac{n-2-\nu_1}{2}\right)\varphi^{\langle 0 \rangle}(y) = 0 \tag{5.8}$$

and

$$- 4\pi cy \cdot \varphi^{\langle 2\rangle}(y) + \left(\delta_y + \frac{n-2+\nu_1}{2}\right)\left(\delta_y + \frac{n-2-\nu_1}{2}\right)\varphi^{\langle 1\rangle}(y) \\ - 2\pi cy \cdot \delta_y \varphi^{\langle 0\rangle}(y) = 0. \quad (5.9)$$

By eliminating $\varphi^{\langle 2\rangle}(y)$ and $\varphi^{\langle 1\rangle}(y)$ from (5.7), (5.8), and (5.9), we have

$$\left\{\left(\delta_y + \frac{n-4+\nu_1}{2}\right)\left(\delta_y + \frac{n-4-\nu_1}{2}\right)\left(\delta_y + \frac{n-2+\nu_1}{2}\right)\left(\delta_y + \frac{n-2-\nu_1}{2}\right) \\ - (2\pi cy)^2 \cdot \left(\delta_y^2 - \mu^2\right)\right\}\varphi^{\langle 0\rangle}(y) = 0. \quad (5.10)$$

We introduce a new variable $z = (\pi cy)^2$. Then the function $\phi(z) = \varphi^{\langle 0\rangle}(y)$ satisfies the following generalized hypergeometric differential equation

$$\left\{z \prod_{j=1}^{2}(\delta_z - \alpha_j + 1) - \prod_{i=1}^{4}(\delta_z - \gamma_i)\right\}\phi(z) = 0, \quad \delta_z = z\frac{d}{dz} \quad (5.11)$$

with the parameters α_i and γ_j in the theorem. Since the functions $\varphi^{\langle j\rangle}(y)$ ($j > 0$) are determined recursively from $\varphi^{\langle 0\rangle}(y)$ by (5.6), we have $\dim_{\boldsymbol{C}} \mathbf{GW}_{G_0}(\pi, \chi \cdot \psi_\beta) \leq 4$. Moreover, it follows from Proposition 9.2 plus Lemma 3.3 that the function $\phi(z)$ coming from an element in $\mathbf{GW}^{mg}_{G_0}(\pi, \chi \cdot \psi_\beta)$ is a constant multiple of $G^{4,0}_{2,4}\left(z \,\middle|\, {}^{\alpha_1,\ \alpha_2}_{\gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4}\right)$. This proves the assertions (ii) and (iii). □

Remark. It seems difficult to determine the coefficient function $\varphi^{\langle j\rangle}(y)$ for all $j \geq 0$. Following a suggestion of the referee, we express the function $\varphi(t, y)$

$$\varphi(t, y) = \mathrm{ch}(t)^{-\mu} \sum_{j \geq 0} \widetilde{\varphi}^{\langle j\rangle}(y) \frac{(2\,\mathrm{sh}(t))^j}{j!}.$$

Then, from the equations (5.4) and (5.5), we have

$$\widetilde{\varphi}^{\langle 2j\rangle}(y) = \int_L \frac{\left(\dfrac{-\mu+1}{2} - s\right)_j \prod_{i=1}^4 \Gamma(\gamma_i - s)}{\Gamma(\alpha_1 - j - s)\Gamma(\alpha_2 - s)} (\pi cy)^{2s} \frac{ds}{2\pi\sqrt{-1}},$$

$$\widetilde{\varphi}^{\langle 2j+1\rangle}(y) = \int_L \frac{\left(\dfrac{-\mu+2}{2} - s\right)_j \prod_{i=1}^4 \Gamma(\gamma_i - s)}{\Gamma(\alpha_1 - 1/2 - j - s)\Gamma(\alpha_2 - 1/2 - s)} (\pi cy)^{2s} \frac{ds}{2\pi\sqrt{-1}},$$

for $j \geq 0$, where the path L of integration is a loop starting and ending at $+\infty$ and encircling all the poles of integrands.

6. Generalized Whittaker functions belonging to odd P_1-principal series representations.

In this section we consider the case where π is equivalent to an odd P_1-principal series representation $I(P_1; (-1)^{n+1} \otimes D_n^+, \nu_1)$ $(n \geq 1, \nu_1 \in \boldsymbol{C})$ of G_0.

6.1. The main results for odd P_1-principal series representations.

We state our main results for odd P_1-principal series representations.

Theorem 6.1. *Let $\pi = I(P_1; (-1)^{n+1} \otimes D_n^+, \nu_1)$ be an odd irreducible P_1-principal series representation of G_0. We set $\beta = \left(\begin{smallmatrix} 0 & c/2 \\ c/2 & 0 \end{smallmatrix}\right)$ $(c \in \boldsymbol{R}^\times)$ and fix a quasi-character $\chi = \chi_\mu$ of T_β° as in* (4.1). *Then we have the following assertions*:

(i) $\dim_{\boldsymbol{C}} \mathbf{GW}_{G_0}(\pi, \chi \cdot \psi_\beta) \leq 4$.

(ii) $\dim_{\boldsymbol{C}} \mathbf{GW}_{G_0}^{mg}(\pi, \chi \cdot \psi_\beta) \leq 1$.

(iii) *Note that $\tau = \tau_{(n,n-1)}$ is a multiplicity one K_0-type of π. For a generalized Whittaker function $W(g)$ of type $(\pi, \chi \cdot \psi_\beta, \tau)$, we define two C^∞-functions $\varphi_k(a)$ $(k = 0, 1)$ on S by*

$$W(a(t,y)) = e^{-2\pi cy \cdot \mathrm{sh}(t)} y^n \big\{\varphi_1(a(t,y))v_1 + \varphi_0(a(t,y))v_0\big\}, \quad \forall a(t,y) \in S,$$

where $\{v_0, v_1\}$ is a standard basis of $V_\tau^\vee$. If $W(g)$ has the moderate growth property, then we have

$$\varphi_k(a(0,y)) = (-1)^k C \times G_{2,4}^{4,0}\left((\pi cy)^2 \,\middle|\, \begin{matrix} \alpha_1^{(k)},\ \alpha_2^{(k)} \\ \gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4 \end{matrix}\right),$$

where $C \in \boldsymbol{C}$ is a constant common to $k = 0, 1$ and the constants $\alpha_i^{(k)}$ and γ_j are given by

$$\alpha_1^{(0)} = \frac{\mu+1}{2}, \quad \alpha_2^{(0)} = \frac{-\mu+2}{2}, \quad \alpha_1^{(1)} = \frac{-\mu+1}{2}, \quad \alpha_2^{(1)} = \frac{\mu+2}{2},$$

$$\gamma_1 = \frac{-n+4+\nu_1}{4}, \quad \gamma_2 = \frac{-n+4-\nu_1}{4}, \tag{6.1}$$

$$\gamma_3 = \frac{-n+2+\nu_1}{4}, \quad \gamma_4 = \frac{-n+2-\nu_1}{4}.$$

6.2. Proof of Theorem 6.1.

The starting point of our proof of Theorem 6.1 is the following:

PROPOSITION 6.2. *Let $W(g)$ be a generalized Whittaker function of type $(\pi, \chi_\mu \cdot \psi_\beta, \tau_{(n,n-1)})$. Then we have*

$$[(P^{ev} \cdot \nabla^-)W](g) = 0, \tag{6.2}$$

$$[R_\Omega W](g) = \left\{\nu_1^2 + (n-1)^2 - 5\right\}W(g). \tag{6.3}$$

PROOF. This can be proved in the same manner as Proposition 5.2. □

The S-radial parts of the system of differential equations in Proposition 6.2 are given as follows:

PROPOSITION 6.3. *Let $\varphi_k(a(t,y)) \in C^\infty(S)$ $(k = 0, 1)$ be as in Theorem* 6.1.

(i) *The equation* (6.2) *is equivalent to the system of the equations*

$$\left(\partial_t + \frac{1}{2}\,\mathrm{th}(t)\right)\varphi_1(a(t,y)) - \left(\delta_y + \frac{\mu}{\mathrm{ch}(t)} - \frac{1}{2}\right)\varphi_0(a(t,y)) = 0 \tag{6.4}$$

and

$$\left(\delta_y - \frac{\mu}{\mathrm{ch}(t)} - \frac{1}{2}\right)\varphi_1(a(t,y)) - \left(\partial_t + \frac{1}{2}\,\mathrm{th}(t)\right)\varphi_0(a(t,y)) = 0. \tag{6.5}$$

(ii) *The equation* (6.3) *is equivalent to the system of the equations*

$$\begin{aligned}\Big\{&\partial_t^2 + (-4\pi cy \cdot \mathrm{ch}(t) + \mathrm{th}(t))\partial_t + \delta_y^2 + (-4\pi cy \cdot \mathrm{sh}(t) + 2n - 3)\delta_y \\ &- 2\pi cy \cdot \mathrm{sh}(t) + \frac{\mu^2}{\mathrm{ch}^2(t)} + \frac{1}{2}(n-2)^2 - \frac{1}{2}\nu_1^2 - \frac{1}{4}(\mathrm{th}^2(t) - 1)\Big\}\varphi_1(a(t,y)) \\ &+ \left\{\mu\frac{\mathrm{th}(t)}{\mathrm{ch}(t)} - 2\pi cy \cdot \mathrm{ch}(t)\right\}\varphi_0(a(t,y)) = 0\end{aligned} \tag{6.6}$$

and

$$\Big\{ -\mu\frac{\mathrm{th}(t)}{\mathrm{ch}(t)} - 2\pi cy\cdot \mathrm{ch}(t)\Big\}\varphi_1(a(t,y))$$
$$+ \Big\{\partial_t^2 + (-4\pi cy\cdot\mathrm{ch}(t) + \mathrm{th}(t))\partial_t + \delta_y^2$$
$$+ (-4\pi cy\cdot\mathrm{sh}(t) + 2n-3)\delta_y - 2\pi cy\cdot\mathrm{sh}(t) + \frac{\mu^2}{\mathrm{ch}^2(t)}$$
$$+ \frac{1}{2}(n-2)^2 - \frac{1}{2}\nu_1^2 - \frac{1}{4}(\mathrm{th}^2(t)-1)\Big\}\varphi_0(a(t,y)) = 0. \tag{6.7}$$

Proof. This can be easily proved by using Proposition 4.3 (v) and Proposition 4.4. □

First we prove the assertion (i) of the theorem. By computing

$$\Big(\partial_t + \frac{1}{2}\,\mathrm{th}(t)\Big)\cdot(6.4) - \Big(\delta_y + \frac{\mu}{\mathrm{ch}(t)} - \frac{1}{2}\Big)\cdot(6.5),$$

we have

$$\Big\{\partial_t^2 + \mathrm{th}(t)\partial_t + \frac{1}{4}\,\mathrm{th}^2(t) + \frac{1}{2\,\mathrm{ch}^2(t)} - \Big(\delta_y - \frac{1}{2}\Big)^2 + \frac{\mu^2}{\mathrm{ch}^2(t)}\Big\}\varphi_1(a(t,y))$$
$$+ \mu\cdot\frac{\mathrm{sh}(t)}{\mathrm{ch}^2(t)}\varphi_0(a(t,y)) = 0. \tag{6.8}$$

Hence, under (6.4) and (6.5), we can replace the equation (6.6) by

$$\Big\{\Big(\delta_y + \frac{n-2+\nu_1}{2}\Big)\Big(\delta_y + \frac{n-2-\nu_1}{2}\Big) - 2\pi cy\cdot\mathrm{sh}(t)\Big(\delta_y - \frac{1}{2}\Big)\Big\}\varphi_1(a(t,y))$$
$$- 2\pi cy\cdot\mathrm{ch}(t)\Big(\delta_y + \frac{\mu}{\mathrm{ch}(t)}\Big)\varphi_0(a(t,y)) = 0. \tag{6.9}$$

Similarly the equation (6.7) can be replaced by

$$- 2\pi cy\cdot\mathrm{ch}(t)\Big(\delta_y - \frac{\mu}{\mathrm{ch}(t)}\Big)\varphi_1(a(t,y))$$
$$+ \Big\{\Big(\delta_y + \frac{n-2+\nu_1}{2}\Big)\Big(\delta_y + \frac{n-2-\nu_1}{2}\Big) - 2\pi cy\cdot\mathrm{sh}(t)\Big(\delta_y - \frac{1}{2}\Big)\Big\}$$
$$\cdot\varphi_0(a(t,y)) = 0. \tag{6.10}$$

We temporarily put $\varphi_2(t,y) := \delta_y\varphi_0(t,y)$ and $\varphi_3(t,y) := \delta_y\varphi_1(t,y)$. It follows from (6.4), (6.5), (6.9), and (6.10) that there exists a set of C^∞-functions $A_{k,l}(t,y)$, $B_{k,l}(t,y) \in C^\infty(A_\beta)$ $(0 \le k,l \le 3)$ such that

$$\partial_t\varphi_k(t,y) = \sum_{l=0}^{3} A_{k,l}(t,y)\varphi_l(t,y) \quad \text{and} \quad \delta_y\varphi_k(t,y) = \sum_{l=0}^{3} B_{k,l}(t,y)\varphi_l(t,y),$$

which proves (i). In order to prove the remaining assertions, we express the functions $\varphi_k(a(t,y))$ in the form

$$\varphi_k(a(t,y)) = \sum_{j\ge 0} \varphi_k^{\langle j\rangle}(y)t^j, \quad k = 0,1.$$

From (6.9) and (6.10), we know that

$$\left(\delta_y + \frac{n-2+\nu_1}{2}\right)\left(\delta_y + \frac{n-2-\nu_1}{2}\right)\varphi_1^{\langle 0\rangle}(y) - 2\pi cy(\delta_y+\mu)\varphi_0^{\langle 0\rangle}(y) = 0 \quad (6.11)$$

and

$$-2\pi cy(\delta_y-\mu)\varphi_1^{\langle 0\rangle}(y) + \left(\delta_y + \frac{n-2+\nu_1}{2}\right)\left(\delta_y + \frac{n-2-\nu_1}{2}\right)\varphi_0^{\langle 0\rangle}(y) = 0, \quad (6.12)$$

respectively. Eliminating $\varphi_1^{\langle 0\rangle}(y)$ or $\varphi_0^{\langle 0\rangle}(y)$ from (6.11) and (6.12), we have

$$\left\{z\prod_{i=1}^{2}\left(\delta_z - \alpha_i^{(k)} + 1\right) - \prod_{j=1}^{4}(\delta_z - \gamma_j)\right\}\varphi_k^{\langle 0\rangle}(z) = 0, \quad k = 0,1, \qquad (6.13)$$

where $z = (\pi cy)^2$ and the constants $\alpha_i^{(k)}$ and γ_j are as in the theorem. It follows from Proposition 9.2 plus Lemma 3.3 that

$$\varphi_k^{\langle 0\rangle}(z) = C_k \times G_{2,4}^{4,0}\left(z \,\middle|\, \begin{matrix} \alpha_1^{(k)},\ \alpha_2^{(k)} \\ \gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4 \end{matrix}\right)$$

with some constants C_k. By (6.11), we know that $C_0 = -C_1$, which proves the assertion (iii). It can be seen from (6.4) and (6.5) that the functions $\varphi_k^{\langle j\rangle}(y)$ $(k = 0,1,\ j > 0)$ are determined recursively from $\varphi_0^{\langle 0\rangle}(y)$ and $\varphi_1^{\langle 0\rangle}(y)$. This proves (ii). □

7. Generalized Whittaker functions belonging to (limits of) large discrete series representations.

In this section we consider the case where π is equivalent to a (limit of) large discrete series representation.

7.1. The main results for (limits of) large discrete series representations.

Our main results for (limits of) large discrete series representations are as follows.

Theorem 7.1. *Suppose that π is equivalent to a* (*limit of*) *large discrete series representation $D_{(\lambda_1,\lambda_2)}$ $(1-\lambda_1 \le \lambda_2 \le 0)$ of G_0. We set $\beta = \left(\begin{smallmatrix} 0 & c/2 \\ c/2 & 0 \end{smallmatrix}\right)$ $(c \in \boldsymbol{R}^\times)$ and take a quasi-character $\chi = \chi_\mu$ of T_β° as in* (4.1). *Then we have the following assertions*:

(i) $\dim_{\boldsymbol{C}} \mathbf{GW}^{mg}_{G_0}(\pi, \chi\cdot\psi_\beta) \le 1$.

(ii) *Let $\tau = \tau_{(\lambda_1,\lambda_2)}$ be the minimal K_0-type of π. Suppose that $W(g)$ is a generalized Whittaker function of type $(\pi, \chi\cdot\psi_\beta, \tau_{(\lambda_1,\lambda_2)})$ with moderate growth property. We define C^∞-functions $\varphi_k(a)$ $(0 \le k \le d = \lambda_1 - \lambda_2)$ on S by*

$$W(a(t,y)) = e^{-2\pi cy\cdot \mathrm{sh}(t)} \sum_{k=0}^{d} \varphi_k(a(t,y))v_k, \quad \forall a(t,y) \in S,$$

where $\{v_k \mid 0 \le k \le d\}$ is the standard basis of $V_\tau^\vee$. Then we have

$$\varphi_k(a(0,y)) = C \times (-1)^k \times G^{4,0}_{2,4}\left((\pi cy)^2 \,\middle|\, \begin{matrix} \alpha_1^{(k)},\ \alpha_2^{(k)} \\ \gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4 \end{matrix}\right), \quad (0 \le k \le d),$$

where $C \in \boldsymbol{C}$ is a constant independent of $0 \le k \le d$ and the constants $\alpha_i^{(k)}$ and γ_j are given by

$$\begin{aligned} \alpha_1^{(k)} &= \frac{-\mu+\lambda_1-k+2}{2}, \quad \alpha_2^{(k)} = \frac{\mu+\lambda_2+k+2}{2}, \\ \gamma_1 &= \frac{\lambda_1+\lambda_2+4}{4}, \quad \gamma_2 = \frac{\lambda_1-\lambda_2+4}{4}, \\ \gamma_3 &= \frac{\lambda_1+\lambda_2+2}{4}, \quad \gamma_4 = \frac{\lambda_1-\lambda_2+2}{4}. \end{aligned} \tag{7.1}$$

Remark. It is likely that the estimate $\dim_{\boldsymbol{C}} \mathbf{GW}_{G_0}(\pi, \chi\cdot\psi_\beta) \le 4$ holds as in the case of P_1-principal series representations. But much more computation seems necessary to confirm it. Since our principal interest lies in the subspace $\mathbf{GW}^{mg}_{G_0}(\pi, \chi \cdot \psi_\beta)$, we do not pursue this issue here.

7.2. Differential equations ($d \geq 2$).

We shall construct a system of partial differential equations satisfied by the functions $\varphi_k(a)$. As we shall see later, the case of $d = 1$ can be reduced to the case of odd P_1-principal series representations. Hence we assume that $d \geq 2$. Then, by the location of K_0-types of π (Proposition 2.2), we know that a generalized Whittaker function $W(g)$ of type $(\pi, \chi_\mu \cdot \psi_\beta, \tau_{(\lambda_1,\lambda_2)})$ satisfies the following three equations:

$$(P^{ev} \cdot \nabla^-)W(g) = 0, \quad (P^{dn} \cdot \nabla^-)W(g) = 0, \quad (P^{dn} \cdot \nabla^+)W(g) = 0. \tag{7.2}$$

We rewrite the system (7.2) in terms of the coefficient functions.

PROPOSITION 7.2.

(i) *The equation* $[P^{ev} \cdot \nabla^- W](g) = 0$ *is equivalent to the system*:

$$\begin{aligned}
&-\frac{1}{2}(d-k)(d-k-1)\,\mathrm{th}(t)\varphi_{k+2}(a) \\
&+ (k-d)\left(\delta_y - \frac{\mu}{\mathrm{ch}(t)} - \frac{\lambda_1+\lambda_2+2}{2}\right)\varphi_{k+1}(a) \\
&+ (d-2k)\left(\partial_t + \frac{1}{2}\,\mathrm{th}(t)\right)\varphi_k(a) + k\left(\delta_y + \frac{\mu}{\mathrm{ch}(t)} - \frac{\lambda_1+\lambda_2+2}{2}\right)\varphi_{k-1}(a) \\
&+ \frac{1}{2}k(k-1)\,\mathrm{th}(t)\varphi_{k-2}(a) = 0, \qquad (0 \leq k \leq d).
\end{aligned} \tag{7.3}$$

(ii) *The equation* $[P^{dn} \cdot \nabla^- W](g) = 0$ *is equivalent to the system*:

$$\begin{aligned}
&\frac{1}{2}(d-k-1)\,\mathrm{th}(t)\varphi_{k+2}(a) + \left(\delta_y - \frac{\mu}{\mathrm{ch}(t)} - \lambda_1 - 1\right)\varphi_{k+1}(a) \\
&\quad + (-2)\left(\partial_t + \frac{d+2}{4}\,\mathrm{th}(t)\right)\varphi_k(a) + \left(\partial_t + \frac{\mu}{\mathrm{ch}(t)} - \lambda_1 - 1\right)\varphi_{k-1}(a) \\
&\quad + \frac{1}{2}(k-1)\,\mathrm{th}(t)\varphi_{k-2}(a) = 0, \qquad (1 \leq k \leq d-1).
\end{aligned} \tag{7.4}$$

(iii) *The equation* $[P^{dn} \cdot \nabla^+ W](g) = 0$ *is equivalent to the system*:

$$
\begin{aligned}
&-\frac{1}{2}(d-k-1)\operatorname{th}(t)\varphi_{k+2}(a)+\left(\delta_y-4\pi cy\cdot\operatorname{sh}(t)+\frac{\mu}{\operatorname{ch}(t)}+\lambda_2-1\right)\varphi_{k+1}(a)\\
&\quad+2\left(\partial_t-4\pi cy\cdot\operatorname{sh}(t)+\frac{d+2}{4}\operatorname{th}(t)\right)\varphi_k(a)\\
&\quad+\left(\delta_y-4\pi cy\cdot\operatorname{sh}(t)-\frac{\mu}{\operatorname{ch}(t)}+\lambda_2-1\right)\varphi_{k-1}(a)\\
&\quad-\frac{1}{2}(k-1)\operatorname{th}(t)\varphi_{k-2}(a)=0, \qquad (1\le k\le d-1). \qquad (7.5)
\end{aligned}
$$

Proof. This can be easily obtained from Proposition 4.3. □

The system in the above proposition can be rewritten as follows:

Lemma 7.3. *The system of differential equations in Proposition* 7.2 *is equivalent to the following system*:

$$
\begin{aligned}
&\frac{1}{2}(k-d)\varphi_{k+1}(a)+\left(-\partial_t+\frac{1}{2}(-d+k-1)\operatorname{th}(t)\right)\varphi_k(a)\\
&\quad+\left(\delta_y+\frac{\mu}{\operatorname{ch}(t)}+\frac{k}{2}-\lambda_1-1\right)\varphi_{k-1}(a)+\frac{1}{2}(k-1)\operatorname{th}(t)\varphi_{k-2}(a)=0,\\
&\qquad (1\le k\le d), \qquad (7.6)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}(d-k-1)\operatorname{th}(t)\varphi_{k+2}(a)+\left(-\delta_y+\frac{\mu}{\operatorname{ch}(t)}+\frac{\lambda_1+\lambda_2+k}{2}+1\right)\varphi_{k+1}(a)\\
&\quad+\left(\partial_t+\frac{1}{2}(k+1)\operatorname{th}(t)\right)\varphi_k(a)+\frac{k}{2}\varphi_{k-1}(a)=0, \qquad (0\le k\le d-1), \quad (7.7)
\end{aligned}
$$

$$
\begin{aligned}
&\left(\delta_y-2\pi cy\cdot\operatorname{sh}(t)-\frac{d+2}{2}\right)\varphi_{k+1}(a)-4\pi cy\cdot\operatorname{ch}(t)\varphi_k(a)\\
&\quad+\left(\delta_y-2\pi cy\cdot\operatorname{sh}(t)-\frac{d+2}{2}\right)\varphi_{k-1}(a)=0, \qquad (1\le k\le d-1). \quad (7.8)
\end{aligned}
$$

Proof. By eliminating the terms involving $\varphi_{k+2}(a)$ and $\varphi_{k-2}(a)$ from (7.3) and (7.4), we have (7.6) and (7.7), respectively. Moreover, by computing (7.5) + (7.6) − (7.7), we have (7.8). □

It is also useful to note that

$$
\begin{aligned}
&-\frac{1}{2}(d-k-1)\,\mathrm{th}(t)\varphi_{k+2}(a)+\Big(-\delta_y+\frac{\mu}{\mathrm{ch}(t)}+\lambda_2+k+1\Big)\varphi_{k+1}(a)\\
&\quad+\frac{1}{2}(2k-d)\,\mathrm{th}(t)\varphi_k(a)+\Big(\delta_y+\frac{\mu}{\mathrm{ch}(t)}-\lambda_1+k-1\Big)\varphi_{k-1}(a)\\
&\quad+\frac{1}{2}(k-1)\,\mathrm{th}(t)\varphi_{k-2}(a)=0, \qquad (1\le k\le d-1), \qquad (7.9)
\end{aligned}
$$

which can be obtained by computing $(7.6)_k+(7.7)_k$.

7.3. Proof of Theorem 7.1 ($d \geq 3$).

In this subsection we prove Theorem 7.1 when $d\geq 3$. As in the case of P_1-principal series representations, we express the functions $\varphi_k(a(t,y))$ in the form

$$
\varphi_k(a(t,y))=\sum_{j\geq 0}\varphi_k^{\langle j\rangle}(y)t^j,\quad 0\leq k\leq d.
$$

From the equations (7.8) and (7.9), we have

$$
\Big(\delta_y-\frac{d+2}{2}\Big)\varphi_{k+1}^{\langle 0\rangle}(y)-4\pi cy\cdot\varphi_k^{\langle 0\rangle}(y)+\Big(\delta_y-\frac{d+2}{2}\Big)\varphi_{k-1}^{\langle 0\rangle}(y)=0, \qquad (1\le k\le d-1), \quad (7.10)
$$

and

$$
(\delta_y-\mu-\lambda_2-k-1)\varphi_{k+1}^{\langle 0\rangle}(y)-(\delta_y+\mu+k-\lambda_1-1)\varphi_{k-1}^{\langle 0\rangle}(y)=0, \qquad (1\le k\le d-1), \quad (7.11)
$$

respectively. By eliminating the terms involving $\varphi_{k+1}^{\langle 0\rangle}(y)$ and $\varphi_{k-1}^{\langle 0\rangle}(y)$, we have

$$
-2\pi cy(\delta_y-\mu-\lambda_2-k)\varphi_k^{\langle 0\rangle}(y)+\Big(\delta_y-\frac{\lambda_1+\lambda_2+2}{2}\Big)\Big(\delta_y-\frac{d+2}{2}\Big)\varphi_{k-1}^{\langle 0\rangle}(y)=0, \qquad (1\le k\le d-1) \quad (7.12)
$$

and

$$
\Big(\delta_y-\frac{\lambda_1+\lambda_2+2}{2}\Big)\Big(\delta_y-\frac{d+2}{2}\Big)\varphi_{k+1}^{\langle 0\rangle}(y)-2\pi cy\Big(\delta_y+\mu+k-\lambda_1\Big)\varphi_k^{\langle 0\rangle}(y)=0, \qquad (1\le k\le d-1). \quad (7.13)
$$

From these two formulae, we know that each of the functions $\varphi_k^{\langle 0\rangle}(y)$ $(1 \le k \le d-1)$ satisfies

$$\Big\{ z \prod_{i=1}^{2} \big(\delta_z - \alpha_i^{(k)} + 1\big) - \prod_{j=1}^{4} (\delta_z - \gamma_j) \Big\} \varphi_k^{\langle 0\rangle}(z) = 0, \quad z = (\pi c y)^2, \tag{7.14}$$

where the constants $\alpha_i^{(k)}$ and γ_j are as in Theorem 7.1. Hence it follows from Proposition 9.2 and Lemma 3.3 that

$$\varphi_k^{\langle 0\rangle}(z) = C_k \times G_{2,4}^{4,0}\left(z \,\middle|\, \begin{matrix} \alpha_1^{(k)},\ \alpha_2^{(k)} \\ \gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4 \end{matrix} \right), \quad 1 \le k \le d-1 \tag{7.15}$$

with some constants C_k. Moreover by using (7.12)

$$\varphi_0^{\langle 0\rangle}(y) = -C_1 \times G_{2,4}^{4,0}\left((\pi c y)^2 \,\middle|\, \begin{matrix} \alpha_1^{(0)},\ \alpha_2^{(0)} \\ \gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4 \end{matrix} \right) + \phi(y)$$

with a function $\phi(y)$ annihilated by $(\delta_y - (\lambda_1 + \lambda_2 + 2)/2)(\delta_y - (d+2)/2)$. Since $\varphi_0^{\langle 0\rangle}(y)$ is rapidly decreasing as $y \to +\infty$, $\phi(y)$ must be identically zero. Similarly we have

$$\varphi_d^{\langle 0\rangle}(y) = -C_{d-1} \times G_{2,4}^{4,0}\left((\pi c y)^2 \,\middle|\, \begin{matrix} \alpha_1^{(d)},\ \alpha_2^{(d)} \\ \gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4 \end{matrix} \right),$$

and $C_k = -C_{k+1}$ $(1 \le k \le d-2)$. This proves the assertion (ii). The functions $\varphi_k^{\langle j\rangle}(y)$ $(j > 0, 0 \le k \le d)$ are determined from $\varphi_k^{\langle 0\rangle}(y)$ $(0 \le k \le d)$ via (7.6) and (7.7). Hence we have $\dim_{\boldsymbol{C}} \mathbf{GW}_{G_0}^{mg}(\pi, \chi \cdot \psi_\beta) \le 1$.

7.4. Proof of Theorem 7.1 ($d = 2$).

Next we suppose that $d = 2$, i.e $(\lambda_1, \lambda_2) = (2, 0)$ or $(1, -1)$. By computing $(7.6)_{k=1} + (7.7)_{k=1}$ and $(7.6)_{k=1} - (7.7)_{k=1}$, we have

$$\left(\delta_y - \frac{\mu}{\mathrm{ch}(t)} - \lambda_1\right)\varphi_2(a) - \left(\delta_y + \frac{\mu}{\mathrm{ch}(t)} - \lambda_1\right)\varphi_0(a) = 0 \tag{7.16}$$

and

$$\left(\delta_y - \frac{\mu}{\mathrm{ch}(t)} - \lambda_1 - 1\right)\varphi_2(a) - 2(\partial_t + \mathrm{th}(t))\varphi_1(a) + \left(\delta_y + \frac{\mu}{\mathrm{ch}(t)} - \lambda_1 - 1\right)\varphi_0(a) = 0, \tag{7.17}$$

respectively. We analyze the system of partial differential equations consisting of (7.16), (7.17), $(7.6)_{k=2}$, $(7.7)_{k=0}$, and $(7.8)_{k=1}$. We eliminate $\varphi_1(a)$ from $(7.6)_{k=2}$ by using $(7.8)_{k=1}$. Then we have

$$\left\{\left(\delta_y + \frac{\mu}{\mathrm{ch}(t)} - \lambda_1 - 1\right)(\delta_y - 2\pi cy\cdot\mathrm{sh}(t) - 2) - 4\pi cy\cdot\mathrm{ch}(t)\left(\partial_t + \frac{1}{2}\,\mathrm{th}(t)\right)\right\}\varphi_2(a) + \left\{2\pi cy\cdot\mathrm{sh}(t) + \left(\delta_y + \frac{\mu}{\mathrm{ch}(t)} - \lambda_1 - 1\right)(\delta_y - 2\pi cy\cdot\mathrm{sh}(t) - 2)\right\}\varphi_0(a) = 0. \tag{7.18}$$

Similarly we eliminate $\varphi_1(a)$ from $(7.7)_{k=0}$ and (7.17) by using $(7.8)_{k=1}$ to get

$$\left\{2\pi cy\cdot\mathrm{sh}(t) + \left(\delta_y - \frac{\mu}{\mathrm{ch}(t)} - \lambda_1 - 1\right)(\delta_y - 2\pi cy\cdot\mathrm{sh}(t) - 2)\right\}\varphi_2(a) + \left\{\left(\delta_y - \frac{\mu}{\mathrm{ch}(t)} - \lambda_1 - 1\right)(\delta_y - 2\pi cy\cdot\mathrm{sh}(t) - 2) - 4\pi cy\cdot\mathrm{ch}(t)\right\}\varphi_0(a) = 0 \tag{7.19}$$

and

$$\left\{2\pi cy\cdot\mathrm{ch}(t)\left(\delta_y - \frac{\mu}{\mathrm{ch}(t)} - \lambda_1\right) - (\delta_y - 2\pi cy\cdot\mathrm{sh}(t) - 2)\partial_t\right\}\varphi_2(a) + \left\{2\pi cy\cdot\mathrm{ch}(t)\left(\delta_y + \frac{\mu}{\mathrm{ch}(t)} - \lambda_1\right) - (\delta_y - 2\pi cy\cdot\mathrm{sh}(t) - 2)\partial_t\right\}\varphi_0(a) = 0, \tag{7.20}$$

respectively. By setting $t = 0$ in (7.18), (7.19), and (7.20), we obtain

$$-4\pi cy\cdot\varphi_2^{\langle 1\rangle}(y) + (\delta_y + \mu - \lambda_1 - 1)(\delta_y - 2)\varphi_2^{\langle 0\rangle}(y) + (\delta_y + \mu - \lambda_1 - 1)(\delta_y - 2)\varphi_0^{\langle 0\rangle}(y) = 0, \tag{7.21}$$

$$(\delta_y - \mu - \lambda_1 - 1)(\delta_y - 2)\varphi_2^{\langle 0\rangle}(y) - 4\pi cy\cdot\varphi_0^{\langle 1\rangle}(y) + (\delta_y - \mu - \lambda_1 - 1)(\delta_y - 2)\varphi_0^{\langle 0\rangle}(y) = 0, \tag{7.22}$$

and

$$2\pi cy(\delta_y - \mu - \lambda_1)\varphi_2^{\langle 0\rangle}(y) - (\delta_y - 2)\varphi_2^{\langle 1\rangle}(y)$$
$$+ 2\pi cy(\delta_y + \mu - \lambda_1)\varphi_0^{\langle 0\rangle}(y) - (\delta_y - 2)\varphi_0^{\langle 1\rangle}(y) = 0, \tag{7.23}$$

respectively. We eliminate the terms involving $\varphi_2^{\langle 1\rangle}(y)$ and $\varphi_0^{\langle 1\rangle}(y)$ in (7.23) by using (7.21) and (7.22). Then we get

$$\big\{(\delta_y - 2)(\delta_y - 3)(\delta_y - \lambda_1 - 1) - (2\pi cy)^2(\delta_y - \mu - \lambda_1)\big\}\varphi_2^{\langle 0\rangle}(y)$$
$$+ \big\{(\delta_y - 2)(\delta_y - 3)(\delta_y - \lambda_1 - 1) - (2\pi cy)^2(\delta_y + \mu - \lambda_1)\big\}\varphi_0^{\langle 0\rangle}(y) = 0. \tag{7.24}$$

If we set $t = 0$ in (7.16), then we have

$$-(\delta_y - \mu - \lambda_1)\varphi_2^{\langle 0\rangle}(y) + (\delta_y + \mu - \lambda_1)\varphi_0^{\langle 0\rangle}(y) = 0. \tag{7.25}$$

It follows from (7.24) and (7.25) that

$$\left\{z\prod_{i=1}^{2}\big(\delta_z - \alpha_i^{(k)} + 1\big) - \prod_{j=1}^{4}(\delta_z - \gamma_j)\right\}\varphi_k^{\langle 0\rangle}(z) = 0, \quad z = (\pi cy)^2, \tag{7.26}$$

for $k = 0, 2$, where the constants $\alpha_i^{(k)}$ and γ_j are as in (7.1). Hence it holds that

$$\varphi_k^{\langle 0\rangle}(z) = C_k \times G_{2,4}^{4,0}\left(z \,\middle|\, \begin{matrix} \alpha_1^{(k)},\ \alpha_2^{(k)} \\ \gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4 \end{matrix}\right), \tag{7.27}$$

for $k = 0, 2$ with some constants C_0 and C_2. By (7.25), we know that $C_0 = C_2$. By setting $t = 0$ in $(7.8)_{k=1}$, we have

$$\begin{aligned}\varphi_1^{\langle 0\rangle}(z) &= \frac{1}{2}z^{-1/2}(\delta_z - 2)\big\{\varphi_2^{\langle 0\rangle}(z) + \varphi_0^{\langle 0\rangle}(z)\big\} \\ &= -C_0 \times G_{2,4}^{4,0}\left((\pi cy)^2 \,\middle|\, \begin{matrix} \alpha_1^{(1)},\ \alpha_2^{(1)} \\ \gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4 \end{matrix}\right),\end{aligned}$$

which proves the assertion (ii) of Theorem 7.1 for the case of $d = 2$. Finally it is easy to see that $\varphi_k^{\langle j\rangle}(y)$ ($j > 0, 0 \le k \le 2$) can be determined recursively from $\varphi_k^{\langle 0\rangle}(y)$ ($0 \le k \le 2$) by using $(7.6)_{k=1,2}$ and $(7.7)_{k=0}$. Hence we have the assertion (i) for the case of $d = 2$.

7.5. Proof of Theorem 7.1 ($d = 1$).

Finally we consider the case where $d = 1$, i.e. $\pi \cong D_{(1,0)}$. By comparing the infinitesimal characters, we know that a generalized Whittaker function $W(g)$ of type $(\pi, \chi \cdot \psi_\beta, \tau_{(1,0)})$ satisfies the equations (6.2) and (6.3) with $\nu_1 = 0$. Hence the assertions in Theorem 7.1 follow from our computation in Section 6.

Remark. It can be easily checked that $D_{(1,0)}$ is equivalent to a $(\mathfrak{g}_0, K_0)$-submodule of $I(P_1; 1 \otimes D_1^+, 0)$. It is likely that $D_{(1,0)}$ is equivalent to $I(P_1; 1 \otimes D_1^+, 0)$ itself. But it is not necessary to verify this for our purpose.

8. Multiplicity free theorems for $GSp(2, \boldsymbol{R})$.

In this section we derive the multiplicity free results for the case of $\det(\beta) < 0$ from Theorems 5.1, 6.1, and 7.1. Also we reformulate the multiplicity free results for the case of $\det(\beta) > 0$ in [**Mi-1**] in the setting of this paper.

8.1. The transition from G to G_0.

First we clarify the relation between the generalized Whittaker functions on G and those on G_0. For a quasi-character χ of M_β, we denote the restriction of χ to T_β° by the same letter. If $\det(\beta) > 0$, then we have $G = R_\beta G_0 \sqcup R_\beta \gamma_0 G_0$ with $\gamma_0 = \mathrm{diag}(-1, -1, 1, 1) \in G$. Therefore, the assignment of $W(g)$ to $(W(g_0), W(\gamma_0 g_0))$, $(g_0 \in G_0)$ gives the following isomorphism

$$C^\infty\big(R_\beta\backslash G; \chi \cdot \psi_\beta\big) \cong C^\infty\big(R_\beta^1\backslash G_0; \chi \cdot \psi_\beta\big) \oplus C^\infty\big(R_\beta^1\backslash G_0; \chi \cdot \psi_{-\beta}\big). \tag{8.1}$$

Next we suppose that $\det(\beta) < 0$. Then we have $G = R_\beta G_0$ and $R_\beta \cap G_0 = \{\pm I_4\} R_\beta^1$. Hence the restriction map gives the following isomorphism

$$\begin{aligned} &C^\infty\big(R_\beta\backslash G; \chi \cdot \psi_\beta\big) \\ &\quad \cong \big\{W \in C^\infty\big(R_\beta^1\backslash G_0; \chi \cdot \psi_\beta\big) \mid W((-I_4)g) = \chi(-I_4)W(g)\big\}. \end{aligned} \tag{8.2}$$

Let $(\pi, \mathscr{H}_\pi)$ be an admissible smooth representation of G whose central character $\omega_\pi : \mathbf{Z}_{\boldsymbol{R}}(\cong \boldsymbol{R}^\times) \to \boldsymbol{C}^\times$ coincides with the restriction of χ to $\mathbf{Z}_{\boldsymbol{R}}$. Suppose that $\mathscr{H}_\pi$ is a direct sum $\mathscr{H}_\pi = \mathscr{H}_{\pi_+} \oplus \mathscr{H}_{\pi_-}$ of two smooth representations of G_0 satisfying $\mathscr{H}_{\pi_-} = \pi(\gamma_0)\mathscr{H}_{\pi_+}$. Then it follows from (8.1), (8.2), and $\chi(-I_4) = \omega_\pi(-1)$ that the following isomorphism holds

$$\mathbf{GW}_G(\pi, \chi \cdot \psi_\beta) \cong \begin{cases} \mathbf{GW}_{G_0}(\pi_+, \chi \cdot \psi_\beta) \oplus \mathbf{GW}_{G_0}(\pi_-, \chi \cdot \psi_\beta) & \text{if } \det(\beta) > 0; \\ \mathbf{GW}_{G_0}(\pi_+, \chi \cdot \psi_\beta) & \text{if } \det(\beta) < 0. \end{cases} \tag{8.3}$$

We also note that

$$\mathbf{GW}_{G_0}(\pi_+, \chi \cdot \psi_{-\beta}) \cong \mathbf{GW}_{G_0}(\pi_-, \chi \cdot \psi_{\beta}). \tag{8.4}$$

8.2. The multiplicity free results for the case of $\det(\beta) < 0$.

Now we can prove the following the multiplicity free results for $G = GSp(2, \boldsymbol{R})$.

THEOREM 8.1. *Let $(\pi; \mathscr{H}_\pi)$ be an irreducible $(\mathfrak{g}, K)$-module which is equivalent to either $I(P_1; D_n \otimes \epsilon, \nu)[c]$ $(n \geq 1, \epsilon = \pm 1, \nu \in \boldsymbol{C})$ or $D_{(\lambda_1, \lambda_2)}[c]$ $(1 - \lambda_1 \leq \lambda_2 \leq 0)$. Fix a symmetric matrix $\beta \in \mathrm{Sym}(2)_{\boldsymbol{R}}$ with $\det(\beta) < 0$ and take an arbitrary quasi-character χ of $\mathbf{M}_{\beta, \boldsymbol{R}}$. Then we have*

(i) $\dim_{\boldsymbol{C}} \mathbf{GW}_G^{mg}(\pi, \chi \cdot \psi_\beta) \leq 1$.
(ii) *If π is equivalent to the representation $I(P_1; \sigma, \nu)[c]$ $(n \geq 1, \epsilon = \pm 1, \nu \in \boldsymbol{C})$, then we have*

$$\dim_{\boldsymbol{C}} \mathbf{GW}_G(\pi, \chi \cdot \psi_\beta) \leq 4.$$

PROOF. We may assume that $\chi(zI_4) = \omega_\pi(z)$ $(\forall z \in \boldsymbol{R}^\times)$, because otherwise we have $\mathbf{GW}_G(\pi, \chi \cdot \psi_\beta) = \{0\}$. Then in view of (8.3), we know that both of our assertions are direct consequences of Theorems 5.1, 6.1, and 7.1. □

8.3. The multiplicity free results for the case of $\det(\beta) > 0$.

The multiplicity free problem (Problem (A) in the introduction) is discussed in [**Mi-1**] for the representations considered in this paper. Although the formulation in [**Mi-1**] is different from ours, we can paraphrase the results of [**Mi-1**] as follows:

THEOREM 8.2. *Let $(\pi, \mathscr{H}_\pi)$ be an irreducible $(\mathfrak{g}, K)$-module which is equivalent to either $I(P_1; D_n \otimes \epsilon, \nu)[c]$ $(n \geq 1, \epsilon = \pm 1, \nu \in \boldsymbol{C})$ or $D_{(\lambda_1, \lambda_2)}[c]$ $(1 - \lambda_1 \leq \lambda_2 \leq 0,\ \lambda_1 - \lambda_2 \geq 4)$. Fix a symmetric matrix $\beta \in \mathrm{Sym}(2)_{\boldsymbol{R}}$ with $\det(\beta) > 0$ and take an arbitrary quasi-character χ of $\mathbf{M}_{\beta, \boldsymbol{R}}$. Then we have*

$$\dim_{\boldsymbol{C}} \mathbf{GW}_G^{mg}(\pi, \chi \cdot \psi_\beta) \leq 1.$$

PROOF. As in the proof of Theorem 8.1, we may suppose the compatibility condition $\chi(z) = \omega_\pi(z)$ $(\forall z \in \boldsymbol{R}^\times)$. Then we have

$$\mathbf{GW}_G^{mg}(\pi, \chi \cdot \psi_\beta) \cong \mathbf{GW}_{G_0}^{mg}(\pi_+, \chi \cdot \psi_\beta) \oplus \mathbf{GW}_{G_0}^{mg}(\pi_+, \chi \cdot \psi_{-\beta}).$$

Here π_+ is equivalent to $I(P_1; D_n \otimes \epsilon, \nu)$ with $n \geq 1$, $\epsilon = \pm 1$, $\nu \in \boldsymbol{C}$ or $D_{(\lambda_1, \lambda_2)}$

with $1-\lambda_1<\lambda_2<0$. From now on, we consider the case where π_+ is equivalent to an even P_1-principal series representation $I(P_1;D_n\otimes(-1)^n,\nu)$. The other cases can be treated in the same way. In view of (8.4), we may suppose that β is positive definite. Further we may suppose that $\beta=I_2$ without any loss of generality. Recall that there exists a unique vector $v_0\in\pi$ such that $\pi(k_{A,B})v_0=\det(A+\sqrt{-1}B)^n v_0$ $(k_{A,B}\in K_0)$ (cf. Proposition 2.1). Take an intertwining operator $\Phi_\pm\in\mathbf{GW}^{mg}_{G_0}(\pi_+,\chi\cdot\psi_{\pm\beta})$ and set $W^{\pm}_{v_0}(g):=\Phi_\pm(v_0)(g)$. By [**Mi-1**, p. 261], we can expand each of the functions $W^{\pm}_{v_0}(\mathrm{diag}(a_1,a_2,a_1^{-1},a_1^{-2}))$ as follows:

$$\begin{aligned}&W^{\pm}_{v_0}\big(\,\mathrm{diag}\,\big(a_1,a_2,a_1^{-1},a_1^{-2}\big)\big)\\&\quad=(a_1a_2)^{n+1}e^{\mp\pi(a_1^2+a_2^2)}\sum_{\ell\geq 0}p^{\pm}_{m_0+2\ell}\big(a_1^2+a_2^2\big)\big(a_1^2-a_2^2\big)^{m_0+2\ell},\quad a_i>0.\end{aligned}$$

Here m_0 is a non-negative integer. It is shown in [**Mi-1**, (7.8), p. 261] that $p^{\pm}_{m_0}(y)$ satisfies a second-order ordinary differential equation. Moreover the other coefficient functions $p^{\pm}_m(y)$ $(m>m_0)$ are determined recursively from $p^{\pm}_{m_0}(y)$. By a simple calculation, we have

$$\begin{aligned}&W^{\pm}_{(H_1-H_2)^{m_0}v_0}\big(\,\mathrm{diag}(a,a,a^{-1},a^{-1})\big)\\&\quad=m_0!(2a^2)^{m_0}\times a^{2(n+1)}e^{\mp 2\pi a^2}p^{\pm}_{m_0}(2a^2).\end{aligned}\tag{8.5}$$

On the other hand, it is easy to check that

$$\begin{aligned}&W^{\pm}_{(H_1-H_2)^{m_0}v_0}\big(\,\mathrm{diag}(a,a,a^{-1},a^{-1});E_{2,0}\big)\\&\quad=\pm 2\pi\sqrt{-1}a^2W^{\pm}_{(H_1-H_2)^{m_0}v_0}\big(\,\mathrm{diag}(a,a,a^{-1},a^{-1})\big).\end{aligned}$$

Hence, as in the proof of Lemma 3.3, we conclude that for each $N\geq 0$ there exists a constant $C>0$ such that

$$\big|W^{\pm}_{(H_1-H_2)^{m_0}v_0}\big(\,\mathrm{diag}(a,a,a^{-1},a^{-1})\big)\big|\leq Ca^{-N},\quad\forall a>0.$$

This combined with the asymptotic behavior of the solutions of the ordinary differential equations for $p^{+}_{m_0}(y)$ (cf. [**Mi-1**, p. 261–262]) implies that $\dim_{\boldsymbol{C}}\mathbf{GW}^{mg}_{G_0}(\pi_+,\chi\cdot\psi_\beta)\leq 1$. Similarly we can conclude that $\dim_{\boldsymbol{C}}\mathbf{GW}^{mg}_{G_0}(\pi_+,\chi\cdot\psi_{-\beta})=0$ due to the factor $e^{+2\pi a^2}$ in the right hand side of (8.5). This proves the theorem. □

Remark. We impose the condition $\lambda_1 - \lambda_2 \geq 4$, because the computation in [**Mi-1**, Sections 10–11] is carried out under this assumption. On the other hand, since the computation in [**Mi-1**, Section 7] for $I(P_1; D_n \otimes \epsilon, \nu)$ $(n \geq 2)$ remains valid for $n = 1$, we do not exclude the case of $n = 1$ from Theorem 8.2.

9. Rapidly decreasing solutions of a certain generalized hypergeometric differential equation.

In this section we prove a result on generalized hypergeometric differential equations. In order to state it, we recall the definition of the Meijer G-functions (cf. [**Er**], [**Me**]).

Definition 9.1. Suppose that m, n, p, and q are integers with $q \geq 1$, $0 \leq n \leq p \leq q$, and $0 \leq m \leq q$; suppose further that the number z satisfies the inequality $0 < |z| < 1$ if $q = p$, $z \neq 0$ if $q > p$, moreover that the numbers a_i $(1 \leq i \leq p)$ and b_j $(1 \leq j \leq q)$ fulfill the condition

$$a_i - b_j \notin \boldsymbol{Z}_{>0} \quad (1 \leq i \leq n;\, 1 \leq j \leq m).$$

Then the function $G^{m,n}_{p,q}(z)$ is defined as follows

$$\begin{aligned} G^{m,n}_{p,q}(z) &\equiv G^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix}\right) \\ &:= \int_L \frac{\prod_{1 \leq j \leq m} \Gamma(b_j - s) \prod_{1 \leq i \leq n} \Gamma(1 - a_i + s)}{\prod_{m+1 \leq j \leq q} \Gamma(1 - b_j + s) \prod_{n+1 \leq i \leq p} \Gamma(a_i - s)}\, z^s \frac{ds}{2\pi\sqrt{-1}}, \end{aligned} \tag{9.1}$$

where the path L of integration is a loop starting and ending at $+\infty$ and encircling all the poles of $\Gamma(b_j - s)$ $(1 \leq j \leq m)$ once in the negative direction, but none of the poles of $\Gamma(1 - a_i + s)$ $(1 \leq i \leq n)$.

What we need in this paper is the following proposition, which characterizes the function $G^{4,0}_{2,4}(z)$ up to a constant multiple.

Proposition 9.2. *Consider the following ordinary differential equation*

$$P(z, \delta_z)\phi(z) \equiv \left\{ z \prod_{i=1}^{2} (\delta_z - a_i + 1) - \prod_{j=1}^{4} (\delta_z - b_j) \right\} \phi(z) = 0, \quad a_i, b_j \in \boldsymbol{C}. \tag{9.2}$$

Let $\phi(z)$ *be a solution of* (9.2) *on* $(0, +\infty)$. *Suppose that* $\phi(z)$ *is rapidly decreasing in the sense that for each* $N > 0$ *there exists a constant* $C > 0$ *such that*

$$|\phi(z)| \le C \times z^{-N}, \quad \forall z \in (0, +\infty). \tag{9.3}$$

Then $\phi(z)$ is a constant multiple of the G-function $G^{4,0}_{2,4}\left(z \,\middle|\, {a_1,\ a_2 \atop b_1,\ b_2,\ b_3,\ b_4}\right)$.

Remark.

(i) As can be seen from the proof below, under the conditions $a_1 - a_2 \notin \boldsymbol{Z}$ and $a_i - b_j \notin \boldsymbol{Z}_{>0}$ ($\forall i, j \in \boldsymbol{Z}$), Proposition 9.2 is an easy consequence of the asymptotic expansion of G-functions due to Barnes.

(ii) If we want to extend Proposition 9.2 to the higher order hypergeometric differential equations, it seems better to use the general theory of asymptotic expansions ([**Was**, Chapters IV–V]).

Proof. We shall prove the proposition by constructing a basis $\{\phi_k(z) \mid 1 \le k \le 4\}$ for the solution space of the differential equation (9.2). We set

$$\phi_3(z) := G^{4,0}_{2,4}\left(z \,\middle|\, {a_1,\ a_2 \atop b_1,\ b_2,\ b_3,\ b_4}\right), \quad \phi_4(z) := G^{4,0}_{2,4}\left(ze^{2\pi\sqrt{-1}} \,\middle|\, {a_1,\ a_2 \atop b_1,\ b_2,\ b_3,\ b_4}\right).$$

Then it is easy to see that $\phi_3(z)$ and $\phi_4(z)$ are solutions of (9.2). Moreover it is known that they satisfy the following estimates due to Barnes ([**Ba**], see also [**Me**, p. 131]):

$$\phi_3(z) = \exp(-2\sqrt{z}) z^{\vartheta} \left(\sqrt{\pi} + O(z^{-1/2})\right), \qquad z \to +\infty, z \in \boldsymbol{R}, \tag{9.4}$$

$$\phi_4(z) = \exp(2\sqrt{z}) \left(ze^{2\pi\sqrt{-1}}\right)^{\vartheta} \left(\sqrt{\pi} + O(z^{-1/2})\right), \quad z \to +\infty, z \in \boldsymbol{R}. \tag{9.5}$$

Here we put $\vartheta := 1/2(-1/2 - \sum_{i=1,2} a_i + \sum_{1 \le j \le 4} b_j)$. We have to find two other solutions of (9.2) to make a basis. First we consider the case where $a_1 - a_2 \notin \boldsymbol{Z}$. To each a_i $(i = 1, 2)$, we shall attach a solution $\phi_i(z)$ of (9.2) having the property

$$\phi_i(z) = C_i \times z^{a_i - 1}(1 + O(z^{-1})), \quad z \in \boldsymbol{R},\ z \to +\infty \tag{9.6}$$

for some constants $C_i \in \boldsymbol{C}^{\times}$ $(i = 1, 2)$. If $a_1 - b_j \notin \boldsymbol{Z}_{>0}$ for every $1 \le j \le 4$, then the function

$$\phi_1(z) := G^{4,1}_{2,4}\left(ze^{\pi\sqrt{-1}} \,\middle|\, {a_1,\ a_2 \atop b_1,\ b_2,\ b_3,\ b_4}\right)$$

is a solution of (9.2) on $(0, \infty)$. By shifting the path L to the left and computing the residue, we know that $\phi_1(z)$ satisfies (9.6). Next suppose that $a_1 - b_j \in \boldsymbol{Z}_{>0}$

for some $1 \le j \le 4$. We may assume that

$$\begin{aligned} & a_1 - b_j \in \boldsymbol{Z}_{>0}\ (1 \le j \le m), \quad a_1 - b_j \notin \boldsymbol{Z}_{>0}\ (m < j \le 4); \\ & \operatorname{Re}(b_1) \ge \operatorname{Re}(b_2) \ge \cdots \ge \operatorname{Re}(b_m). \end{aligned} \tag{9.7}$$

Put $l = a_1 - b_1 - 1 (\ge 0)$. Then there exists a solution of (9.2) of the form $\sum_{k=0}^{l} c_k z^{b_1+k}$ with $c_k \ne 0$ for all $0 \le k \le l$, which we denote by $\phi_1(z)$. Then (9.6) holds for $\phi_1(z)$ in this case, too. Interchanging the roles of a_1 and a_2, we have a solution $\phi_2(z)$ of (9.2) satisfying (9.6). It follows from the asymptotic behavior of $\phi_i(z)$ $(1 \le i \le 4)$ given above that the set $\{\phi_i(z) \mid 1 \le i \le 4\}$ is linearly independent and that any rapidly decreasing solution $\phi(z)$ of (9.2) must be a constant multiple of $\phi_3(z)$. Hence the proposition follows when $a_1 - a_2 \notin \boldsymbol{Z}$.

From now on we consider the case where $a_1 - a_2 \in \boldsymbol{Z}$. Without any loss of generality, we may and do assume that

$$a_1, a_2 \in \boldsymbol{Z} \ \text{ and } \ a_2 \ge a_1. \tag{9.8}$$

We divide our construction of a basis for the solution space of (9.2) into the following four cases:

Case 1: $a_i - b_j \notin \boldsymbol{Z}_{>0}$ $(1 \le \forall i \le 2,\ 1 \le \forall j \le 4)$;
Case 2: $b_j \in [a_1, a_2 - 1] \cap \boldsymbol{Z}$ and $b_k \in (-\infty, a_1 - 1] \cap \boldsymbol{Z}$ for some $1 \le j, k \le 4$;
Case 3: $b_j \in [a_1, a_2 - 1] \cap \boldsymbol{Z}$ for some $1 \le j \le 4$ and $b_j \notin (-\infty, a_1 - 1] \cap \boldsymbol{Z}$ for all $1 \le j \le 4$;
Case 4: $b_j \notin [a_1, a_2 - 1] \cap \boldsymbol{Z}$ for all $1 \le j \le 4$ and $b_j \in (-\infty, a_1 - 1] \cap \boldsymbol{Z}$ for some $1 \le j \le 4$.

Case 1: It is easy to see that

$$\phi_1(z) = G^{4,1}_{2,4}\left(z e^{\pi\sqrt{-1}} \,\middle|\, \begin{matrix} a_1,\ a_2 \\ b_1,\ b_2,\ b_3,\ b_4 \end{matrix} \right), \quad \phi_2(z) = G^{4,2}_{2,4}\left(z \,\middle|\, \begin{matrix} a_1,\ a_2 \\ b_1,\ b_2,\ b_3,\ b_4 \end{matrix} \right)$$

are solutions of (9.2). If $a_2 > a_1$, then $\phi_1(z)$ and $\phi_2(z)$ satisfy the estimates (9.6). This proves the proposition in this case. If $a_1 = a_2$, then the estimate (9.6) is still valid for $\phi_1(z)$ and

$$\phi_2(z) = C_2 \times z^{a_1 - 1}(-2\gamma + \log(z)) + O(z^{a_1 - 2 + \epsilon}), \quad z \in \boldsymbol{R},\ z \to +\infty,\ \epsilon > 0, \tag{9.9}$$

for some constant $C_2 \in \boldsymbol{C}^\times$. Here $\gamma = \lim_{s \to 0}(1/s - \Gamma(s))$ is Euler's constant. Hence the proposition holds for $a_1 = a_2$, too.

Case 2: The condition implies that $a_2 > a_1$. Then there exist two solutions $\phi_1(z)$ and $\phi_2(z)$ of (9.2) of the form

$$\phi_1(z) = \sum_{i=b_k}^{a_1-1} c_i z^i, \text{ with } c_{a_1-1} \neq 0, \quad \phi_2(z) = \sum_{i=b_j}^{a_2-1} c_i' z^i, \text{ with } c_{a_2-1}' \neq 0.$$

The set $\{\phi_i(z) \mid 1 \leq i \leq 4\}$ of functions on $(0, \infty)$ forms a basis for the solution space of (9.2). It is then easy to see that the proposition is valid in this case.

Case 3: The condition implies that $a_2 > a_1$ and allows us to define a solution

$$\phi_1(z) = G_{2,4}^{4,1}\left(z e^{\pi\sqrt{-1}} \,\middle|\, \begin{matrix} a_1, \ a_2 \\ b_1, \ b_2, \ b_3, \ b_4 \end{matrix} \right)$$

of (9.2) on $(0, \infty)$ satisfying (9.6). Moreover, as in Case 2, there exists a solution $\phi_2(z)$ of (9.2) of the form

$$\phi_2(z) = \sum_{i=b_j}^{a_2-1} c_i' z^i \quad \text{with } c_{a_2-1}' \neq 0.$$

The set $\{\phi_i(z) \mid 1 \leq i \leq 4\}$ of functions on $(0, \infty)$ forms a basis for the solution space of (9.2). Now it is easy to see that the proposition is valid in this case, too.

Case 4: We enumerate b_j $(1 \leq j \leq 4)$ so that the condition (9.7) holds. Then, as in Case 2, there exists a solution $\phi_1(z)$ of (9.2) of the form

$$\phi_1(z) = \sum_{i=b_1}^{a_1-1} c_i z^i \quad \text{with } c_i \neq 0 \ (b_1 \leq \forall i \leq a_1 - 1).$$

We seek for another solution $\phi_2(z)$ of (9.2). Put $\phi_0(z) := \phi_2(z) - \phi_1(z) \cdot \log(z)$. We express the differential operator $P(z, \delta_z)$ as $P(z, \delta_z) = \sum_{i=0}^{4} p_i(z)\delta^i$ with some polynomials $p_i(z)$ $(0 \leq i \leq 4)$ in z. We define another differential operator $\widetilde{P}(z, \delta_z)$ by

$$\widetilde{P}(z, \delta_z) = \sum_{i=1}^{4} i p_i(z) \delta_z^{i-1}.$$

Then it is easy to see that

$$P(z, \delta_z)\phi_2(z) = P(z, \delta_z)\phi_0(z) + \widetilde{P}(z, \delta_z)\phi_1(z).$$

First we consider the case of $m \geq 2$. Then $\widetilde{P}(z,\delta_z)\phi_1(z)$ belongs to

$$\boldsymbol{C}z^{a_2-1} \oplus \boldsymbol{C}z^{a_2-2} \oplus \cdots \oplus \boldsymbol{C}z^{b_2+1}.$$

Hence we can find

$$\phi_0(z) \in \boldsymbol{C}z^{a_2-1} \oplus \boldsymbol{C}z^{a_2-2} \oplus \cdots \oplus \boldsymbol{C}z^{b_2}$$

such that $P(z,\delta_z)\phi_0(z)+\widetilde{P}(z,\delta_z)\phi_1(z) = 0$. Hence we have a basis for the solution of (9.2) and the proposition follows in this case.

Next we suppose that $m = 1$. In this case, we have

$$\widetilde{P}(z,\delta_z)\phi_1(z) = \sum_{i=b_1}^{a_2-1} c_i'' z^i \quad \text{for some } c_i'' \in \boldsymbol{C}.$$

By using $m = 1$, we have $c_{b_1}'' \neq 0$. Consider the following function

$$\Phi(z) := \int_{L'} \frac{\Gamma(1-a_1+s)}{\Gamma(a_2-s)} \prod_{1\leq j\leq 4} \Gamma(b_j-s)\big(ze^{\sqrt{-1}\pi}\big)^s \frac{ds}{2\pi\sqrt{-1}},$$

where the path L' of integration is a loop starting and ending at $+\infty$ and encircling all the poles of $\prod_{j=1}^4 \Gamma(b_j-s)$ once in the negative direction, but none of b_1-k ($k \in \boldsymbol{Z}_{>0}$). By shifting the path of integration to the left, we know that

$$\Phi(z) = \frac{\prod_{1\leq j\leq 4}\Gamma(b_j-b_1+1)}{\Gamma(a_2-b_1+1)} \operatorname{Res}_{s=b_1-a_1} \Gamma(s)(ze^{\sqrt{-1}\pi})^{b_1-1} + O(z^{b_1-2}), \quad (9.10)$$

when $z \to \infty, z \in \boldsymbol{R}$. By a simple computation, we have

$$P(z,\delta_z)\Phi(z) = \left(\int_{L''} - \int_{L'}\right) \frac{\Gamma(1-a_1+s)\prod_{j=1}^4\Gamma(b_j-s+1)}{\Gamma(a_2-s)} \big(ze^{\sqrt{-1}\pi}\big)^s \frac{ds}{2\pi\sqrt{-1}}$$

where the path L'' is given by $L'' := \{s+1 \mid s \in L'\}$. Hence we have

$$P(z,\delta_z)\Phi(z) = \alpha z^{b_1}$$

with

$$\alpha = \frac{\prod_{j=1}^{4} \Gamma(b_j - b_1 + 1)}{\Gamma(a_2 - b_1)} \times \mathrm{Res}_{s=b_1} \Gamma(1 - a_1 + s) \times e^{\pi\sqrt{-1}b_1}.$$

Note that $\alpha \neq 0$. If $a_2 - 1 = b_1$, then $P(z, \delta_z)\bigl\{-(c''_{b_1}/\alpha)\Phi(z) + \phi_1(z)\log(z)\bigr\} = 0$. If $a_2 - 1 > b_1$, then

$$P(z, \delta_z)\left\{-\frac{c''_{b_1}}{\alpha}\Phi(z) + \phi_1(z)\log(z)\right\} \in \boldsymbol{C}z^{a_2-1} \oplus \boldsymbol{C}z^{a_2-2} \oplus \cdots \oplus \boldsymbol{C}z^{b_1+1}.$$

Hence we can find a function

$$\widetilde{\phi}(z) \in \boldsymbol{C}z^{a_2-1} \oplus \boldsymbol{C}z^{a_2-2} \oplus \cdots \oplus \boldsymbol{C}z^{b_1+1}$$

such that

$$P(z, \delta_z)\widetilde{\phi}(z) = P(z, \delta_z)\left\{\frac{c''_{b_1}}{\alpha}\Phi(z) - \phi_1(z)\log(z)\right\}.$$

Summing up, we know that the function

$$\phi_2(z) := \begin{cases} -\dfrac{c''_{b_1}}{\alpha}\Phi(z) + \phi_1(z)\log(z) & \text{if } a_2 - 1 = b_1, \\ \widetilde{\phi}(z) - \dfrac{c''_{b_1}}{\alpha}\Phi(z) + \phi_1(z)\log(z) & \text{if } a_2 - 1 > b_1 \end{cases}$$

is a solution of (9.2). Hence we have a basis for the solution space of (9.2). Now our assertion follows from (9.4), (9.5), and (9.10). □

10. A concluding remark.

In this section, we propose a way of constructing a non-zero element $\Phi \in \mathbf{GW}^{mg}_{G_0}(\pi, \chi \cdot \psi_\beta)$, which might be useful for finding the values of the generalized Whittaker function on the whole group S. This is also an interesting problem from the representation theoretic point of view ([**G-P**]). First we recall the Whittaker function on $G_0 = Sp(2, \boldsymbol{R})$. A maximal unipotent subgroup of G_0 is given by $N_0 := \mathbf{N}_{0,\boldsymbol{R}}$. Any character of N_0 can be written as

$$\psi_{c_0,c_3} : N_0 \ni n(x_0, x_1, x_2, x_3) \mapsto \exp\bigl(2\pi\sqrt{-1}(c_0x_0 + c_3x_3)\bigr) \in \boldsymbol{C}^{(1)}$$

with some $c_0, c_3 \in \boldsymbol{R}$. We assume that ψ_{c_0,c_3} is non-degenerate, that is $c_0 c_3 \neq 0$. We denote by $C^\infty_{mg}(N_0\backslash G_0;\psi_{c_0,c_3})$ the space of C^∞-functions $\boldsymbol{W}: G_0 \to \boldsymbol{C}$ satisfying two conditions

- $\boldsymbol{W}(ng) = \psi_{c_0,c_3}(n)\boldsymbol{W}(g), \quad \forall (n,g) \in N_0 \times G_0,$
- $\boldsymbol{W}(g)$ is of moderate growth.

The group G_0 acts on the space $C^\infty_{mg}(N_0\backslash G_0;\psi_{c_0,c_3})$ by right translation. Let $(\pi, \mathscr{H}_\pi)$ be an irreducible $(\mathfrak{g}_0, K_0)$-module. Suppose that there exists a non-zero intertwining operator

$$\Psi : \mathscr{H}_\pi \to C^\infty_{mg}(N_0\backslash G_0;\psi_{c_0,c_3})$$

for some (c_0, c_3). We assume that $c_0 = c_3 = 1$, which is not an essential restriction. It is known that Ψ is unique up to a constant multiple ([**Wal**, Theorem 8.8]). We call the whole image $\Psi(\mathscr{H}_\pi)$ the Whittaker model of π. For a Whittaker function $\boldsymbol{W}(g) \in \Psi(\mathscr{H}_\pi)$ and $\mu \in \boldsymbol{C}$, we consider the following integral

$$Z_N(g;\mu,\boldsymbol{W}) := \int_0^\infty d^\times u \int_{\boldsymbol{R}} dx\ \boldsymbol{W}\left(\frac{1}{\sqrt{u}}\left(\begin{array}{cc|cc} u & & & \\ & u & & \\ \hline & & 1 & \\ & x & & 1 \end{array}\right) w_2 g\right) u^{-\mu-1}, \quad g \in G_0.$$

As we noted in [**Is-Mo**, p. 5706] (see also [**G-P**]), this is a variant of Novodvorsky's local zeta integral for $GSp(2,\boldsymbol{R})$. It seems not difficult to show that the integral $Z_N(g;\mu,\boldsymbol{W})$ converges absolutely and defines a function in $C^\infty_{mg}(R^1_\beta\backslash G_0;\chi_\mu\cdot\psi_\beta)$ for $\beta = \left(\begin{smallmatrix} 0 & 1/2 \\ 1/2 & 0 \end{smallmatrix}\right)$ when $\mathrm{Re}(\mu) \ll 0$. Indeed, it is readily seen that our evaluation [**Mo**, Proposition 8] of Novodvorsky's local zeta integrals is compatible with the explicit formula obtained in Theorem 7.1. On the other hand, in order to get a non-zero element in $\mathbf{GW}^{mg}_{G_0}(\pi,\chi_\mu\cdot\psi_\beta)$ for every $\mu \in \boldsymbol{C}$, we have to prove the meromorphic continuability of $Z_N(g;\mu,\boldsymbol{W})$. We hope to discuss it in a future paper.

Remark. The assignment $W(g) \to W(w_1 g)$ gives an isomorphism

$$C^\infty_{mg}\big(R^1_\beta\backslash G_0;\chi_\mu\cdot\psi_\beta\big) \cong C^\infty_{mg}\big(R^1_\beta\backslash G_0;\chi_{-\mu}\cdot\psi_\beta\big).$$

Hence we have $\mathbf{GW}^{mg}_{G_0}(\pi,\chi_\mu\cdot\psi_\beta) \cong \mathbf{GW}^{mg}_{G_0}(\pi,\chi_{-\mu}\cdot\psi_\beta)$.

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