J. Math. Soc. Japan
Vol. 63, No. 3 (2011) pp. 1027–1037
doi: 10.2969/jmsj/06331027

# A non-autonomous model problem for the Oseen-Navier-Stokes flow with rotating effects

By Matthias Geissert and Tobias Hansel

(Received Apr. 27, 2010)

**Abstract.** Consider the Navier-Stokes flow past a rotating obstacle with a general time-dependent angular velocity and a time-dependent outflow condition at infinity. After rewriting the problem on a fixed domain, one obtains a non-autonomous system of equations with unbounded drift terms. It is shown that the solution to a model problem in the whole space case $\boldsymbol{R}^d$ is governed by a strongly continuous evolution system on $L^p_\sigma(\boldsymbol{R}^d)$ for $1 < p < \infty$. The strategy is to derive a representation formula, similar to the one known in the case of non-autonomous Ornstein-Uhlenbeck equations. This explicit formula allows to prove $L^p$-$L^q$ estimates and gradient estimates for the evolution system. These results are key ingredients to obtain (local) mild solutions to the full nonlinear problem by a version of Kato's iteration scheme.

## 1. Introduction and main result.

In this paper we consider a model problem in $\boldsymbol{R}^d$ for the flow of an incompressible, viscous fluid past a rotating obstacle with an additional time-dependent outflow condition at infinity. The equations describing this problem are the Navier-Stokes equations in an exterior domain varying in time with an additional condition for the velocity field at infinity.

In order to motivate our model problem, let $\mathscr{O} \subset \boldsymbol{R}^d$ be a compact obstacle with smooth boundary, let $\Omega := \boldsymbol{R}^d \setminus \mathscr{O}$ be the exterior of the obstacle and let $m \in C([0,\infty); \boldsymbol{R}^{d\times d})$ be a continuous matrix-valued function. Then, the exterior of the rotated obstacle at time $t > 0$ is represented by $\Omega(t) := Q(t)\Omega$ where $Q(t)$ solves the ordinary differential equation

$$\begin{cases} \partial_t Q(t) = m(t)Q(t), & t > 0, \\ Q(0) = \mathrm{Id}. \end{cases} \tag{1.1}$$

---

2000 *Mathematics Subject Classification.* Primary 35Q30; Secondary 76D03, 76D05.
*Key Words and Phrases.* Navier-Stokes flow, Oseen flow, rotating obstacle, non-autonomous PDE.
The second author was supported by the DFG International Research Training Group 1529 on *Mathematical Fluid Dynamics* at TU Darmstadt.

With a prescribed velocity field $v_\infty \in C^1([0,\infty);\boldsymbol{R}^d)$ at infinity, the equations for the fluid on the time-dependent domain $\Omega(t)$ with no-slip boundary condition take the form

$$
\begin{aligned}
v_t - \Delta v + v\cdot\nabla v + \nabla \mathrm{q} &= 0 && \text{in } \Omega(t)\times(0,\infty),\\
\operatorname{div} v &= 0 && \text{in } \Omega(t)\times(0,\infty),\\
v(t,y) &= m(t)y && \text{on } \partial\Omega(t)\times(0,\infty),\\
\lim_{|y|\to\infty} v(t,y) &= v_\infty(t) && \text{for } t\in(0,\infty),\\
v(0,y) &= u_0(y) && \text{in } \Omega,
\end{aligned}
\tag{1.2}
$$

where $v$ and $\mathrm{q}$ are the unknown velocity field and the pressure of the fluid, respectively.

The disadvantage of this description is the variability of the domain $\Omega(t)$, and the fact that the equations do not fit into the $L^p$-setting, due the velocity condition at infinity. Assume for the time being that $m(t)$ is skew symmetric for $t>0$; this implies that for all $t>0$ the matrix $Q(t)$ is orthogonal. Then, by setting

$$
x = Q(t)^{\mathrm{T}}y,\quad u(t,x) = Q(t)^{\mathrm{T}}(v(t,y)-v_\infty(t)),\quad \mathrm{p}(t,x)=\mathrm{q}(t,y), \tag{1.3}
$$

the above equations can be transformed to the reference domain $\Omega$ and the new velocity field $u$ vanishes at infinity. Then (1.2) is equivalent to the following system of equations

$$
\begin{aligned}
&\left.\begin{aligned}
&u_t - \Delta u - \mathscr{M}(t)x\cdot\nabla u + \mathscr{M}(t)u + Q(t)^{\mathrm{T}}v_\infty(t)\cdot\nabla u\\
&\quad - Q(t)^{\mathrm{T}}\partial_t v_\infty(t) + u\cdot\nabla u + \nabla\mathrm{p}
\end{aligned}\right\} = 0 && \text{in } \Omega\times(0,\infty),\\
&\operatorname{div} u = 0 && \text{in } \Omega\times(0,\infty),\\
&u(t,x) = \mathscr{M}(t)x - Q(t)^{\mathrm{T}}v_\infty(t) && \text{on } \partial\Omega\times(0,\infty),\\
&\lim_{|x|\to\infty} u(t,x) = 0 && \text{for } t\in(0,\infty),\\
&u(0,x) = u_0(x) && \text{in } \Omega,
\end{aligned}
\tag{1.4}
$$

where $\mathscr{M}(t) := Q(t)^{\mathrm{T}}m(t)Q(t)$. The main difficulty in dealing with this problem arises since the term $\mathscr{M}(t)x\cdot\nabla$ has unbounded coefficients. In particular, the lower order terms cannot be treated by classical perturbation theory for the Stokes operator.

Note that even if we assume that $m(t)\equiv m$ is independent of time (this implies that also $\mathscr{M}(t)\equiv\mathscr{M}$ is independent of time), equation (1.4) is still non-autonomous due to the time-dependent first order term $Q(t)^{\mathrm{T}}v_\infty \cdot \nabla$ (except in some special cases discussed below).

However, by using localization techniques similar to [**GHH06**], this problem is finally reduced to a model problem in $\boldsymbol{R}^d$ and a model problem in a bounded domain. Since $Q(t)\partial_t v_\infty(t) \equiv F(t)$, $t > 0$, i.e. it is constant in space, we may put this term in the pressure p. Hence, in this paper we discuss the following linearized model problem in $\boldsymbol{R}^d$

$$\begin{aligned} u_t - \Delta u - (M(t)x + f(t)) \cdot \nabla u + M(t)u + \nabla \mathrm{p} &= 0 \qquad \text{in } \boldsymbol{R}^d \times (0,\infty), \\ \operatorname{div} u &= 0 \qquad \text{in } \boldsymbol{R}^d \times (0,\infty), \\ u(0) &= u_0 \qquad \text{in } \boldsymbol{R}^d, \end{aligned} \tag{1.5}$$

where we allow general coefficients $M \in C([0,\infty); \boldsymbol{R}^{d\times d})$ and $f \in C([0,\infty); \boldsymbol{R}^d)$. If we set $M(t) := Q(t)^{\mathrm{T}}m(t)Q(t)$ and $f(t) := -Q(t)^{\mathrm{T}}v_\infty(t)$ then we obtain the linearization of equation (1.4) with $\Omega = \boldsymbol{R}^d$. Such a model problem also arises in the analysis of a rotating body with translational velocity $-v_\infty(t)$, see [**Far05**].

Existence and uniqueness of a mild solution of an autonomous variant of problem (1.2) *without* an outflow condition, i.e. $v_\infty \equiv 0$, and $m(t) \equiv m$, was investigated in quite a few papers, see [**His99a**], [**His99b**], [**GHH06**] and [**HS05**]. Hishida was even able to deal with a time dependent rotation in [**His01**], however only for angular velocities of a special form.

For the problem including an additional outflow condition at infinity, there are only a few results. Indeed, in the special case, where $m(t)x = \omega(t) \times x$ and $\omega : [0,\infty) \to \boldsymbol{R}^3$ is the angular velocity of the obstacle and $v_\infty : [0,\infty) \to \boldsymbol{R}^3$ a time-dependent outflow velocity, Borchers [**Bor92**] constructed weak non-stationary solutions for the equations (1.4). Moreover, Shibata [**Shi08**] studied the special case where $m(t) \equiv m$, $v_\infty(t) = v_\infty$ and $mv_\infty = 0$. The condition $mv_\infty = 0$, i.e. $Q(t)^{\mathrm{T}}v_\infty = kv_\infty$ for $k \in \{-1, 1\}$, ensures that (1.4) is still an autonomous equation and the solution of (1.4) is governed by a $C_0$-semigroup which is *not* analytic. The physical meaning of the additional condition $mv_\infty = 0$ is that the outflow direction of the fluid is parallel to the axis of rotation of the obstacle. The stationary problem of this latter situation was analysed in [**Far05**].

The assumption $mv_\infty = 0$ was recently relaxed by the second author in [**Han10**]. Indeed, he was able to deal with the model problem in $\boldsymbol{R}^d$ where $m(t)v_\infty \neq 0$ and $v_\infty(t) \equiv v_\infty$. However he assumes that $m(t)$ and $m(s)$ commute for all $t, s > 0$ which can physically be interpreted by the fact that the axis of rotation is fixed.

The aim of this work is to remove the latter additional condition, i.e. $m(t)$ and $m(s)$ need not to commute and $v_\infty$ may be time-dependent.

As usual the Helmholtz projection $\boldsymbol{P}$ allows us to rewrite (1.5) as an abstract Cauchy problem in $L^p_\sigma(\boldsymbol{R}^d)$, where $L^p_\sigma(\boldsymbol{R}^d)$ denotes the space of all solenoidal vector fields in $L^p(\boldsymbol{R})^d$:

$$\begin{aligned} u'(t) - A(t)u(t) &= 0, \qquad t > 0, \\ u(0) &= u_0. \end{aligned} \tag{1.6}$$

Here:

$$\begin{aligned} A(t)u &:= \boldsymbol{P}(\Delta u + (M(t)x + f(t)) \cdot \nabla u + M(t)u), \\ D(A(t)) &:= \left\{ u \in W^{2,p}(\boldsymbol{R}^d)^d \cap L^p_\sigma(\boldsymbol{R}^d) : M(t)x \cdot \nabla u \in L^p(\boldsymbol{R}^d)^d \right\}. \end{aligned}$$

Note that it immediately follows from [**HS05**] that for fixed $t > 0$, the operator $A(t)$ is the generator of a $C_0$-semigroup, which is not analytic. The fact that the semigroup is not analytic prevents us from employing standard generation results for evolution systems, see [**Paz83**, Chapter 5] and references therein. For the same reason, $L^p$-$L^q$ estimates and gradient estimates don't follow from standard arguments.

Therefore, we first derive a representation formula for the solution of (1.5). In order to derive this representation formula we transform (1.5) to a non-autonomous heat equation which can be explicitly solved, see Section 3. It turns out that the transformation to a non-autonomous heat equation is crucial to deal with our problem in this generality since the different transformation used in [**Han10**] caused the additional assumption that $M(t)$ and $M(s)$ commute for all $t, s > 0$.

In the following we denote by $\{U(t,s)\}_{t,s\geq 0}$ the evolution system on $\boldsymbol{R}^d$ generated by the family of matrices $\{-M(t)\}_{t\geq 0}$, i.e.

$$\begin{cases} \partial_t U(t,s) = -M(t)U(t,s), \\ U(s,s) = \mathrm{Id}. \end{cases} \tag{1.7}$$

Note that $\partial_s U(t,s) = U(t,s)M(s)$.

We are now ready to present our main result.

THEOREM 1.1. *Let* $1 < p < \infty$, $M \in C([0,\infty); \boldsymbol{R}^{d\times d})$ *and* $f \in C([0,\infty); \boldsymbol{R}^d)$. *The solution of* (1.6) *is governed by a strongly continuous evolution system* $\{T(t,s)\}_{t\geq s\geq 0} \subset \mathscr{L}(L^p_\sigma(\boldsymbol{R}^d)^d)$. *Moreover, the evolution system* $\{T(t,s)\}_{t\geq s\geq 0}$ *admits the following properties*:

(a) *For* $T_0 > 0$ *set* $M_{T_0} := \sup\{\|U(t,s)\| : t,s \in [0,T_0]\}$. *Then for* $1 < p < \infty$ *and* $p \leq q \leq \infty$ *there exists* $C := C(M_{T_0}, d) > 0$ *such that for* $u \in L^p_\sigma(\boldsymbol{R}^d)$

$$\|T(t,s)u\|_{L^q_\sigma(\boldsymbol{R}^d)} \le C(t-s)^{(-d/2)(1/p-1/q)}\|u\|_{L^p_\sigma(\boldsymbol{R}^d)}, \qquad 0 \le s < t < T_0, \tag{1.8}$$

$$\|\nabla T(t,s)u\|_{L^q(\boldsymbol{R}^d)} \le C(t-s)^{(-d/2)(1/p-1/q)-1/2}\|u\|_{L^p_\sigma(\boldsymbol{R}^d)}, \qquad 0 \le s < t < T_0. \tag{1.9}$$

*In particular, if the evolution system $\{U(t,s)\}_{s,t\ge 0}$ is uniformly bounded, i.e. $M_{T_0} \le M$, for some $M > 0$ and all $T_0 > 0$, we may set $T_0 = \infty$.*

(b) *For $1 < p < q < \infty$, $s \ge 0$ and $u \in L^p_\sigma(\boldsymbol{R}^d)$ we have*

$$\lim_{t\to s,\ t>s}(t-s)^{(d/2)(1/p-1/q)}\|T(t,s)u\|_{L^q_\sigma(\boldsymbol{R}^d)} = 0 \quad \text{and}$$

$$\lim_{t\to s,\ t>s}(t-s)^{1/2}\|\nabla T(t,s)u\|_{L^p(\boldsymbol{R}^d)} = 0.$$

Next we consider the nonlinear problem

$$\begin{aligned} u'(t) - A(t)u(t) + \boldsymbol{P}((u(t)\cdot\nabla)u(t)) &= 0, \qquad t > 0, \\ u(0) &= u_0, \end{aligned} \tag{1.10}$$

with initial value $u_0 \in L^p_\sigma(\boldsymbol{R}^d)$.

For given $0 < T_0 \le \infty$, we call a function $u \in C([0,T_0); L^p_\sigma(\boldsymbol{R}^d))$ a *mild solution* of (1.10) if $u$ satisfies the integral equation

$$u(t) = T(t,0)u_0 - \int_0^t T(t,s)\boldsymbol{P}((u(s)\cdot\nabla)u(s))\mathrm{d}s, \quad t > 0, \tag{1.11}$$

in $L^p_\sigma(\boldsymbol{R}^d)$. By adjusting Kato's iteration scheme (see [**Kat84**]) to our situation the existence of a unique (local) mild solution follows, cf. [**Han10**] for details.

Corollary 1.2. *Let $2 \le d \le p \le q < \infty$, $M \in C([0,\infty); \boldsymbol{R}^{d\times d})$, $f \in C([0,\infty); \boldsymbol{R}^d)$ and $u_0 \in L^p_\sigma(\boldsymbol{R}^d)$. Then there exists $T_0 > 0$ and a unique mild solution $u \in C([0,T_0); L^p_\sigma(\boldsymbol{R}^d))$ of (1.10), which has the properties*

$$t^{(d/2)(1/p-1/q)}u(t) \in C([0,T_0); L^q_\sigma(\boldsymbol{R}^d)), \tag{1.12}$$

$$t^{(d/2)(1/p-1/q)+1/2}\nabla u(t) \in C([0,T_0); L^q(\boldsymbol{R}^d)^{d\times d}). \tag{1.13}$$

*If $p < q$, then in addition*

$$t^{(d/2)(1/p-1/q)}\|u(t)\|_{L^q(\boldsymbol{R}^d)} + t^{1/2}\|\nabla u(t)\|_{L^p(\boldsymbol{R}^d)} \to 0 \qquad \textit{as } t \to 0. \tag{1.14}$$

*Moreover, in the case* $d = p$ *we may set* $T_0 = +\infty$ *provided* $\|u_0\|_{L^d(\boldsymbol{R}^d)}$ *is small enough and* $\{U(t,s)\}_{s,t\geq 0}$ *is uniformly bounded.*

REMARK 1.3. In particular, $\{U(t,s)\}_{s,t\geq 0}$ is uniformly bounded if $M(t)$ is skew symmetric for all $t > 0$.

## 2. Proof of Theorem 1.1.

Let $M$ be as in Theorem 1.1, and let $\{U(t,s)\}_{s,t\geq 0}$ be the evolution system on $\boldsymbol{R}^d$ that satisfies (1.7). We consider the system of parabolic equations of the form

$$\begin{cases} \partial_t u(t,x) - \mathscr{A}(t)u(t,x) = 0, & t > s,\ x \in \boldsymbol{R}^d, \\ u(s,x) = \varphi(x), & x \in \boldsymbol{R}^d, \end{cases} \tag{2.1}$$

for $s \geq 0$ fixed, initial value $\varphi \in L^p(\boldsymbol{R}^d)^d$ and some $p \in (1,\infty)$. Here the family of operators $\mathscr{A}(t)$ is of the form

$$\mathscr{A}(t)u(x) := \big(\Delta u_i(t,x) + \langle M(t)x + f(t), \nabla u_i(t,x)\rangle\big)_{i=1}^d - M(t)u(t,x), \qquad t > 0,\ x \in \boldsymbol{R}^d.$$

As in [**GL08**, Lemma 3.2] or [**Han10**], we first develop an explicit representation formula. To be more precise, we show in Section 3 that for $p \in (1,\infty)$ and $\varphi \in L^p(\boldsymbol{R}^d)^d$ the solution $u$ to (2.1) is governed by a strongly continuous evolution system $\{\tilde{T}(t,s)\}_{t\geq s} \subset \mathscr{L}(L^p(\boldsymbol{R}^d)^d)$ which is explicitly given by

$$u(t,x) := (\tilde{T}(t,s)\varphi)(x) := (k(t,s,\cdot) * \varphi)(U(s,t)x + g(t,s)), \quad t > s,\ x \in \boldsymbol{R}^d, \tag{2.2}$$

where

$$k(t,s,x) := \frac{1}{(4\pi)^{d/2}(\det Q_{t,s})^{1/2}} U(t,s)\mathrm{e}^{(-1/4)\langle Q_{t,s}^{-1}x,x\rangle}, \qquad t > s \geq 0,\ x \in \boldsymbol{R}^d, \tag{2.3}$$

$$g(t,s) := \int_s^t U(s,r)f(r)\mathrm{d}r, \quad Q_{t,s} := \int_s^t U(s,r)U^*(s,r)\mathrm{d}r, \quad t \geq s \geq 0.$$

Similar to [**DPL07**] one can show that for $\varphi \in C_c^\infty(\boldsymbol{R}^d)^d$ the solution $u$ of (2.1) given by (2.2) is a classical solution.

A simple calculation shows that $\operatorname{div}\tilde{T}(t,s)\varphi = 0$ for $\varphi \in C_{c,\sigma}^{\infty}(\boldsymbol{R}^d)$ and $t \geq s \geq 0$. Hence, the restriction $T(t,s) := \tilde{T}(t,s)|_{L^p_\sigma(\boldsymbol{R}^d)}$ is an evolution system on $L^p_\sigma(\boldsymbol{R}^d)$. In particular, $u(t) := T(t,0)u_0$ is a solution to (1.6).

By similar arguments as in the proofs of [**GL08**, Lemma 3.2] or [**Han10**, Lemma 2.4], for $T_0 > 0$ there exists $C := C(d, M_{T_0}) > 0$ (see Theorem 1.1 for the definition of $M_{T_0}$) such that

$$\begin{aligned} \big\|Q_{t,s}^{-1/2}\big\| &\leq C(t-s)^{-1/2}, \quad 0 \leq s < t < T_0, \\ (\det Q_{t,s})^{1/2} &\geq C(t-s)^{d/2}, \quad 0 \leq s < t < T_0. \end{aligned} \tag{2.4}$$

Moreover, if $M_{T_0}$ is uniformly bounded in $T_0$ we may write $T_0 = \infty$ in (2.4).

Proof of Theorem 1.1. We start by showing the estimate (1.8). Let $T_0 > 0$. By the change of variables $\xi = U(s,t)x$ and by Young's inequality we obtain

$$\|T(t,s)u\|_{L^q_\sigma(\boldsymbol{R}^d)} \leq |\det U(s,t)|^{1/q} \|k(t,s,\cdot)\|_{L^r(\boldsymbol{R}^d)} \|u\|_{L^p_\sigma(\boldsymbol{R}^d)}, \quad t > s \geq 0,$$

where $1 < r < \infty$ with $1/p + 1/r = 1 + 1/q$. Further, by the change of variable $y = Q_{t,s}^{1/2} z$ we obtain

$$\begin{aligned} \|k(t,s,\cdot)\|^r_{L^r(\boldsymbol{R}^d)} &= \|U(t,s)\| \int_{\boldsymbol{R}^d} \left(\frac{1}{(4\pi)^{d/2}} \mathrm{e}^{-|z|^2/4}\right)^r (\det Q_{t,s})^{(1-r)/2} \mathrm{d}z \\ &\leq C \|U(t,s)\| (\det Q_{t,s})^{(1-r)/2}, \quad t \geq s \geq 0, \end{aligned}$$

for some $C > 0$. Now (2.4) yields (1.8).

To prove the gradient estimate (1.9), we first observe that

$$\nabla T(t,s)u(x) = \int_{\boldsymbol{R}^d} u(U(s,t)x + g(t,s))k(t,s,y)\big(U^T(s,t)Q_{s,t}^{-1}y\big)^T \mathrm{d}y, \qquad t > s \geq 0,\ x \in \boldsymbol{R}^d.$$

Now, (1.9) follows similarly as above.

Since (2.1) is uniquely solvable for $\varphi \in C_c^\infty(\boldsymbol{R}^d)^d$, see Section 3, the law of evolution is valid, i.e.

$$\tilde{T}(t,s)\varphi = \tilde{T}(t,r)\tilde{T}(r,s)\varphi, \tag{2.5}$$

holds for $0 \le s \le r \le t$ and every $\varphi \in C_c^\infty(\boldsymbol{R}^d)^d$. The density of $C_c^\infty(\boldsymbol{R}^d)^d$ in $L^p(\boldsymbol{R}^d)^d$ yields that (2.5) even holds for all $\varphi \in L^p(\boldsymbol{R}^d)^d$.

In order to prove the strong continuity of the map $(t,s) \mapsto \tilde{T}(t,s)$ on $0 \le s \le t$ we apply the change of the variables $y = Q_{t,s}^{1/2} z$, to see that

$$\tilde{T}(t,s)\varphi(x) = \frac{1}{(4\pi)^{d/2}} U(t,s) \cdot \int_{\boldsymbol{R}^d} \varphi\big(U(s,t)x + g(t,s) - Q_{t,s}^{1/2} z\big) \mathrm{e}^{-|z|^2/4} \mathrm{d}z$$

holds. For $t > s$ fixed, we pick two sequences $(t_n)_{n\in\boldsymbol{N}}$ and $(s_n)_{n\in\boldsymbol{N}}$ such that $t_n \ge s_n$ holds for every $n \in \boldsymbol{N}$ and $(t_n, s_n) \to (t,s)$ as $n \to \infty$. For every $\varphi \in C_c^\infty(\boldsymbol{R}^d)^d$ and every $x \in \boldsymbol{R}^d$ we now obtain

$$\varphi\big(U(s_n,t_n)x + g(t_n,s_n) - Q_{t_n,s_n}^{1/2} z\big) \to \varphi\big(U(s,t)x + g(t,s) - Q_{t,s}^{1/2} z\big)$$

as $n \to \infty$. Lebesgue's theorem now yields $\tilde{T}(t_n,s_n)\varphi \to \tilde{T}(t,s)\varphi$ as $n \to \infty$ for every $\varphi \in C_c^\infty(\boldsymbol{R}^d)^d$. The density of $C_c^\infty(\boldsymbol{R}^d)^d$ in $L^p(\boldsymbol{R}^d)^d$ implies the strong continuity.

In order to prove Theorem 1.1(b) let $u \in L_\sigma^p(\boldsymbol{R}^d)$, $t - s \le 1$ and choose $(u_n)_{n\in\boldsymbol{N}} \subset C_{c,\sigma}^\infty(\boldsymbol{R}^d) \subset L_\sigma^p(\boldsymbol{R}^d)$, such that $\lim_{n\to\infty} \|u - u_n\|_{L^p(\boldsymbol{R}^d)} = 0$. The triangle inequality together with the $L^p$-$L^q$ estimates (1.8) imply that there exist constants $C_1, C_2 > 0$ such that

$$\begin{aligned}
&(t-s)^{(d/2)(1/p-1/q)} \|T(t,s)u\|_{L_\sigma^q(\boldsymbol{R}^d)} \\
&\quad \le (t-s)^{(d/2)(1/p-1/q)} \|T(t,s)(u-u_n)\|_{L_\sigma^q(\boldsymbol{R}^d)} \\
&\qquad + (t-s)^{(d/2)(1/p-1/q)} \|T(t,s)u_n\|_{L_\sigma^q(\boldsymbol{R}^d)} \\
&\quad \le C_1 \|u-u_n\|_{L_\sigma^p(\boldsymbol{R}^d)} + C_2 (t-s)^{(d/2)(1/p-1/q)} \|u_n\|_{L_\sigma^q(\boldsymbol{R}^d)}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le t-s \le 1,\ n \in \boldsymbol{N}.
\end{aligned}$$

Hence, $\lim_{t\to s}(t-s)^{(d/2)(1/p-1/q)} \|T(t,s)u\|_{L_\sigma^q(\boldsymbol{R}^d)} = 0$ by letting first $t \to s$ and then $n \to \infty$. The second assertion in Theorem 1.1(b) is proved in a similar way. □

## 3. Representation formula.

In this section the representation formula (2.2) is derived. The general idea is to do a coordinate transformation in order to eliminate the unbounded drift and the zero order term of the operator $\mathscr{A}(t)$. For this purpose we set

$$z := U(s,t)x + g(t,s),$$

where

$$g(t,s) := \int_s^t U(s,r)f(r)\mathrm{d}r,$$

and we look for a solution $u$ of (2.1) with initial value $\varphi \in C_c^\infty(\boldsymbol{R}^d)^d$ in the form

$$u(t,x) = U(t,s)w(t, U(s,t)x + g(t,s)). \tag{3.1}$$

By recalling (1.7) we obtain from a straightforward computation that

$$\begin{aligned}\partial_t u(t,x) = & -M(t)U(t,s)w(t,z)\\ & + U(t,s)\big(\langle U(s,t)M(t)x + U(s,t)f(t), \nabla w_i(t,z)\rangle\big)_{i=1}^d\\ & + U(t,s)\partial_t w(t,z),\end{aligned}$$

holds. Moreover, we can write equation (3.1) component-wise as

$$u_i(t,x) = \sum_{j=1}^d U_{ij}(t,s)w_j(t, U(s,t)x + g(t,s)), \quad \text{for } i = 1, \dots, d,$$

and thus for the spatial derivatives of $u$ we obtain

$$\begin{aligned}\nabla u_i(t,x) &= \sum_{j=1}^d U_{ij}(t,s)U^*(s,t)\nabla w_j(t,z),\\ \nabla^2 u_i(t,x) &= \sum_{j=1}^d U_{ij}(t,s)U^*(s,t)\nabla^2 w_j(t,z)U(s,t).\end{aligned}$$

In particular, the drift term can be written as

$$\langle M(t)x + f(t), \nabla u_i(t,x)\rangle = \sum_{j=1}^d U_{ij}(t,s)\langle U(s,t)M(t)x + U(s,t)f(t), \nabla w_j(t,z)\rangle.$$

Thus, the function $u$ solves problem (2.1) if and only if for every $i = 1, \dots, d$, the function $w_i : \boldsymbol{R}^d \to \boldsymbol{R}$ is a solution to

$$\begin{cases} \partial_t w_i(t,z) = \operatorname{Tr}[U(s,t)U^*(s,t)\nabla^2 w_i(t,z)], & t > s, z \in \boldsymbol{R}^d, \\ w_i(s,z) = \varphi_i(z), & z \in \boldsymbol{R}^d. \end{cases} \tag{3.2}$$

By our transformation we now obtained an uncoupled system of parabolic equations with coefficients only depending on $t$. More precisely, for $i = 1, \dots, d$, the equation (3.2) is a non-autonomous heat equation. It is well known that such a problem can be uniquely solved (cf. [**DPL07**, Proposition 2.1]) and that for every $\varphi_i \in C_c^\infty(\boldsymbol{R}^d)$ its unique solution is explicitly given by the formula

$$w_i(t,z) = \frac{1}{(4\pi)^{d/2}(\det Q_{t,s})^{1/2}} \int_{\boldsymbol{R}^d} \varphi_i(z-y) \mathrm{e}^{(-1/4)\langle Q_{t,s}^{-1} y, y\rangle} \mathrm{d}y, \tag{3.3}$$

where

$$Q_{t,s} = \int_s^t U(s,r)U^*(s,r)\mathrm{d}r. \tag{3.4}$$

Now, via (3.1), the unique solution to our original problem (2.1) is given by the representation formula

$$u(t,x) = (k(t,s,\cdot) * \varphi)(U(s,t)x + g(t,s)), \tag{3.5}$$

where the kernel $k(t,s,x)$ is defined in (2.3).

Note that the right hand side of (3.5) is even well defined for each $L^p(\boldsymbol{R}^d)^d$-function $\varphi$. Thus, this explicit formula can be used to define an evolution system on $L^p(\boldsymbol{R}^d)^d$ in the following way. For $\varphi \in L^p(\boldsymbol{R}^d)^d$ we set

$$\tilde{T}(t,s)\varphi := \begin{cases} \varphi & \text{for } t = s, \\ (k(t,s,x) * \varphi)(U(s,t)x + g(t,s)) & \text{for } t > s. \end{cases}$$

Since problem (3.2) is uniquely solvable it follows via (3.1) that $\tilde{T}(t,s)\varphi$ is the unique solution of (2.1) for initial value $\varphi \in C_c^\infty(\boldsymbol{R}^d)^d$.

## References

[Bor92] W. Borchers, Zur Stabilität und Faktorisierungsmethode für die Navier-Stokes-Gleichungen inkompressibler viskoser Flüssigkeiten, Habilitation, 1992, Habilitation, Universität Paderborn.

[DPL07] G. Da Prato and A. Lunardi, Ornstein-Uhlenbeck operators with time periodic coefficients, J. Evol. Equ., **7** (2007), 587–614.

[Far05] R. Farwig, An $L^q$-analysis of viscous fluid flow past a rotating obstacle, Tohoku

Math. J. (2), **58** (2005), 129–147.

[GHH06] M. Geissert, H. Heck and M. Hieber, $L^p$-theory of the Navier-Stokes flow in the exterior of a moving or rotating obstacle, J. Reine Angew. Math., **596** (2006), 45–62.

[GL08] M. Geissert and A. Lunardi, Invariant measures and maximal $L^2$ regularity for nonautonomous Ornstein-Uhlenbeck equations, J. Lond. Math. Soc. (2), **77** (2008), 719–740.

[Han10] T. Hansel, On the Navier-Stokes equations with rotating effect and prescribed outflow velocity, J. Math. Fluid Mech. online first, 2010, doi: 10.1007/s00021-010-0026-x.

[His99a] T. Hishida, An existence theorem for the Navier-Stokes flow in the exterior of a rotating obstacle, Arch. Ration. Mech. Anal., **150** (1999), 307–348.

[His99b] T. Hishida, The Stokes operator with rotation effect in exterior domains, Analysis (Munich), **19** (1999), 51–67.

[His01] T. Hishida, On the Navier-Stokes flow around a rigid body with a prescribed rotation, Proceedings of the Third World Congress of Nonlinear Analysts, Part 6 (Catania, 2000), Nonlinear Anal., **47** (2001), 4217–4231.

[HS05] M. Hieber and O. Sawada, The Navier-Stokes equations in $\boldsymbol{R}^n$ with linearly growing initial data, Arch. Ration. Mech. Anal., **175** (2005), 269–285.

[Kat84] T. Kato, Strong $L^p$-solutions of the Navier-Stokes equation in $\boldsymbol{R}^m$, with applications to weak solutions, Math. Z., **187** (1984), 471–480.

[Paz83] A. Pazy, Semigroups of linear operators and applications to partial differential equations, Springer-Verlag, New York, 1983.

[Shi08] Y. Shibata, On the Oseen semigroup with rotating effect, Functional analysis and evolution equations, Birkhäuser, Basel, 2008, pp. 595–611.

Matthias Geissert
Department of Mathematics
Technische Universität Darmstadt
Schlossgartenstr. 7
64289 Darmstadt, Germany
E-mail: geissert@mathematik.tu-darmstadt.de

Tobias Hansel
International Research Training Group 1529
Technische Universität Darmstadt
Schlossgartenstr. 7
64289 Darmstadt, Germany
E-mail: hansel@mathematik.tu-darmstadt.de