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Small subdivisions of simplicial complexes with the metric topology

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Abstract. D. W. Henderson established the metric topology vertion of J. H. C. Whitehead's Theorem on small subdivisions of simplicial complexes. However, his proof is valid only for locally finite-dimensional simplicial complexes. In this note, we give a complete proof of Henderson's Theorem for arbitrary simplicial complexes.

1. Introduction.

For a simplicial complex K, the polyhedron $|K|$ has two topologies, the Whitehead (weak) topology and the metric topology. By $|K|_{\mathrm{w}}$ and $|K|_{\mathrm{m}}$, we denote $|K|$ with the Whitehead (weak) topology and the metric topology, respectively. Unless K is locally finite, $|K|_{\mathrm{w}} \neq |K|_{\mathrm{m}}$ as spaces. For a simplicial subdivision K' of K, $|K'|_{\mathrm{w}} = |K|_{\mathrm{w}}$ but $|K'|_{\mathrm{m}} \neq |K|_{\mathrm{m}}$ as spaces. We call a simplicial subdivision K' of K an *admissible subdivision* if $|K'|_{\mathrm{m}} = |K|_{\mathrm{m}}$ as spaces.[1] The barycentric subdivision $\operatorname{Sd} K$ of K is admissible. Recall that the star $\operatorname{St}(\sigma, K)$ at $\sigma \in K$ is the subcomplex of K consisting of all faces of simplexes having σ as a face. Let $\mathscr{S}_K = \{|\operatorname{St}(v, K)| \mid v \in K^{(0)}\}$, where $K^{(0)}$ is the set of all vertices of K.

The following theorem is due to J. H. C. Whitehead [**3**], which is very important because one can use this theorem to prove the paracompactness of $|K|_{\mathrm{w}}$, the simplicial approximation theorem, etc.

Theorem 1 (J. H. C. Whitehead). *Let K be an arbitrary simplicial complex. For any open cover $\mathscr{U}$ of $|K|_{\mathrm{w}}$, there exists a simplicial subdivision K' of K such that $\mathscr{S}_{K'}$ refines $\mathscr{U}$.*

In [**1**, Lemma V.7], D. W. Henderson established the following metric topology version of Whitehead's Theorem above, which is a key lemma to prove basic

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[1] D. W. Henderson [**1**] called this a *proper subdivision*. In [**2**], the suitable word "admissible" is adopted rather than "proper" because the metric defined by such a subdivision is admissible.

theorems on non-separable infinite-dimensional manifolds.

Theorem 2 (D. W. Henderson). *Let K be an arbitrary simplicial complex. For any open cover $\mathscr{U}$ of $|K|_{\mathrm{m}}$, there exists an admissible subdivision K' of K such that $\mathscr{S}_{K'}$ refines $\mathscr{U}$.*

Although his proof is valid for a locally finite-dimensional simplicial complex, it is not valid in general. The problem is the existence of the integer $n(s)$ for a simplex s in the proof. The n-th barycentric subdivision $\mathrm{Sd}^n K$ of K is inductively defined by $\mathrm{Sd}^n K = \mathrm{Sd}(\mathrm{Sd}^{n-1} K)$, where $\mathrm{Sd}^0 K = K$. As well known, when $\dim K < \infty$,

$$\mathrm{mesh}_{\rho_K} \mathrm{Sd}^n K = 2\left(\frac{\dim K}{\dim K + 1}\right)^n \quad \text{for each } n \in \boldsymbol{N},$$

where ρ_K is the barycentric metric (the definition is given in Preliminaries). Hence, if the star at a simplex s in the complex is finite-dimensional then such an $n(s)$ exists. However, when the star at s is infinite-dimensional, such an $n(s)$ does not exist even locally, that is, no matter how large n is, the size of simplexes of $N_n(s)$ is not small anywhere in s. This follows from the proposition below:

Proposition 3. *Let K be a simplicial complex and $x \in |K|$. Suppose that the star of the carrier $\sigma \in K$ of x contains an infinite full complex.*[2] *For each $n \in \boldsymbol{N}$ and $\varepsilon > 0$, there are infinitely many vertices $u_i \in (\mathrm{Sd}^n K)^{(0)}$, $i \in \boldsymbol{N}$, such that $\rho_K(x, u_i) > 2 - \varepsilon$ and every finite set of u_i's, togather with the vertices of the carrier of x in $\mathrm{Sd}^n K$, spans a simplex of $\mathrm{Sd}^n K$.*

In this note, we shall show Proposition 3 and give a complete proof of Theorem 2 without local finite-dimensionality.

2. Preliminaries.

Our notations are different from the paper [**1**]. Here are notations fixed. For a collection $\mathscr{A}$ of subsets of X and $B \subset X$, we use the following notations:

$$\mathscr{A} \mid B = \{A \cap B \mid A \in \mathscr{A}\}, \quad \mathscr{A}[B] = \{A \in \mathscr{A} \mid A \cap B \neq \emptyset\}$$
$$\text{and } \mathrm{st}(B, \mathscr{A}) = \bigcup \mathscr{A}[B].$$

[2] We call $\sigma \in K$ the *carrier* of $x \in |K|$ if x is an interior point of σ, that is, $\sigma \in K$ is the smallest simplex of K containing x. A *full complex* is a simplicial complex such that any finite subset of the vertices spans a simplex.

Given a collection $\mathscr{B}$ of subsets of X, $\mathscr{A}[\bigcup \mathscr{B}]$ is simply denoted by $\mathscr{A}[\mathscr{B}]$. When $\mathscr{B}$ refines $\mathscr{A}$, that is, each $B \in \mathscr{B}$ is contained in some $A \in \mathscr{A}$, we write $\mathscr{B} \prec \mathscr{A}$.

The simplex spanned by vertices $v_0, v_1, \ldots, v_n$ is denoted by $\langle v_0, v_1, \ldots, v_n \rangle$. For simplexes σ and τ, $\sigma \leq \tau$ (or $\sigma < \tau$) means that σ is a face (or a proper face) of τ. The boundary, the interior, the barycenter and the set of vertices of σ are denoted by $\partial \sigma$, $\mathring{\sigma}$, $\hat{\sigma}$ and $\sigma^{(0)}$, respectively.

Let K be a simplicial complex. The n-skeleton of K is denoted by $K^{(n)}$, that is, $K^{(n)} = \{\sigma \in K \mid \dim \sigma \leq n\}$. By $K(n)$, we denote the set of all n-simplexes in K, that is, $K(n) = K^{(n)} \setminus K^{(n-1)}$. For $A \subset |K|$, let

$$
\begin{aligned}
N(A, K) &= \big\{\sigma \in K \mid \exists \tau \in K[A] \ \text{ such that } \ \sigma \leq \tau\big\}, \\
C(A, K) &= K \setminus K[A] = \big\{\sigma \in K \mid \sigma \cap A = \emptyset\big\} \quad \text{and} \\
B(A, K) &= N(A, K) \cap C(A, K).
\end{aligned}
$$

In case $A = |L|$ for a subcomplex $L \subset K$, we simply write $N(L, K)$, $C(L, K)$ and $B(L, K)$ instead of $N(|L|, K)$, $C(|L|, K)$ and $B(|L|, K)$, respectively. Note that $N(\{v\}, K) = \operatorname{St}(v, K)$ for each $v \in K^{(0)}$ but $N(\sigma, K) \supsetneqq \operatorname{St}(\sigma, K)$ for each $\sigma \in K \setminus K^{(0)}$ in general. For each simplex $\sigma \in K$, $|N(\sigma, K)| = \operatorname{st}(\sigma, K)$ and $|\operatorname{St}(\sigma, K)| = \operatorname{st}(\mathring{\sigma}, K) = \operatorname{st}(\hat{\sigma}, K)$.

There exist functions $\beta_v^K : |K| \to \boldsymbol{I}$, $v \in K^{(0)}$, such that $\sum_{v \in K^{(0)}} \beta_v^K(x) = 1$ and $x = \sum_{v \in K^{(0)}} \beta_v^K(x) v$ for each $x \in |K|$, where $(\beta_v^K(x))_{v \in K^{(0)}}$ is the barycentric coordinate of $x \in |K|$. It should be noticed that every β_v^K is affine (linear) on each $\sigma \in K$ and $\beta_v^K(\sigma) = 0$ if $v \notin \sigma^{(0)}$. The barycentric metric ρ_K is defined as follows:

$$
\rho_K(x, y) = \sum_{v \in K^{(0)}} \big|\beta_v^K(x) - \beta_v^K(x')\big|.
$$

The metric topology for $|K|$ is induced by this metric.

The open star $O_K(v)$ at $v \in K^{(0)}$ is defined by

$$
O_K(v) = \big(\beta_v^K\big)^{-1}((0, 1]) = |\operatorname{St}(v, K)| \setminus |\operatorname{Lk}(v, K)|.
$$

For each point $x \in |K|$, we denote by $c_K(x)$ the carrier of x in K, that is, $c_K(x) \in K$ is the smallest simplex containing x. Then, $c_K(x)^{(0)} = \{v \in K^{(0)} \mid \beta_v^K(x) > 0\}$. The open star at $x \in |K|$ can be defined as follows:

$$
O_K(x) = \bigcup_{\sigma \in K[x]} \mathring{\sigma} = \bigcap_{v \in c_K(x)^{(0)}} O_K(v).
$$

For each $0 < r \leq 1$, we define

$$O_K(x,r) = (1-r)x + rO_K(x) = \{(1-r)x + ry \mid y \in O_K(x)\},$$

which is an open neighborhood of x in $|K|_{\mathrm{m}}$ contained in the open ball $B_{\rho_K}(x, 2r)$ with center x and radius $2r$. Indeed, for each $y \in O_K(x)$,

$$\begin{aligned}\rho_K((1-r)x + ry, x) &= \sum_{v \in K^{(0)}} \left|\beta_v^K((1-r)x + ry) - \beta_v^K(x)\right| \\ &= \sum_{v \in K^{(0)}} r\left|\beta_v^K(y) - \beta_v^K(x)\right| = r\rho_K(y,x) < 2r.\end{aligned}$$

For a vertex $v \in K^{(0)}$, we have $O_K(v,r) = (\beta_v^K)^{-1}((1-r,1]) = B_{\rho_K}(v,2r)$. The following fact is used in the proof of [**1**, Lemma V.5]:

Lemma 4. *$\{O_K(x,r) \mid 0 < r \leq 1\}$ is an open neighborhood basis at x in $|K|_{\mathrm{m}}$.*

For $A \subset |K|$, let $\beta_A^K = \sum_{v \in K^{(0)} \cap A} \beta_v^K : |K| \to \boldsymbol{I}$. In case A is a simplex $\sigma \in K$, $\sigma = (\beta_\sigma^K)^{-1}(1)$ and $(\beta_\sigma^K)^{-1}((0,1]) = \bigcup_{v \in \sigma^{(0)}} O_K(v)$. The following will be used in the proof of Theorem 2:

Lemma 5. *$(\beta_\sigma^K)^{-1}((1-r,1]) \subset \{y \in |K| \mid \mathrm{dist}_{\rho_K}(y,\sigma) < 2r\}$ for each $\sigma \in K$.*

Proof. For each $y \in (\beta_\sigma^K)^{-1}((1-r,1])$, we have $x \in \sigma$ defined by

$$x = \sum_{v \in \sigma^{(0)}} \frac{\beta_v^K(y)}{\beta_\sigma^K(y)} v \quad \left(\text{i.e., } \beta_v^K(x) = \frac{\beta_v^K(y)}{\beta_\sigma^K(y)} \text{ for each } v \in \sigma^{(0)}\right).$$

Then, it follows that

$$\begin{aligned}\rho_K(x,y) &= \sum_{v \in K^{(0)}} \left|\beta_v^K(x) - \beta_v^K(y)\right| \\ &= \sum_{v \in \sigma^{(0)}} \left(\beta_v^K(x) - \beta_v^K(y)\right) + \sum_{v \in K^{(0)} \setminus \sigma^{(0)}} \beta_v^K(y) \\ &= 2\left(1 - \beta_\sigma^K(y)\right) < 2r.\end{aligned}$$

Thus, we have $\mathrm{dist}_{\rho_K}(y,\sigma) < 2r$. □

For a simplicial subdivision K' of K, $\rho_K \leq \rho_{K'}$ but the topology induced by $\rho_{K'}$ is different from the one induced by ρ_K in general. A simplicial subdivision K' of K is admissible if and only if $\rho_{K'}$ is admissible for the space $|K|_{\mathrm{m}}$. Admissible subdivisions are characterized in [**1**, Lemma V.5] and [**2**, Theorem 2] as follows:

THEOREM 6. *For a simplicial subdivision K' of a simplicial complex K, the following are equivalent*:

(a) *K' is admissible*;
(b) *$O_{K'}(v)$ is open in $|K|_{\mathrm{m}}$ for each $v \in K'^{(0)}$*;
(c) *$K'^{(0)}$ is discrete in $|K|_{\mathrm{m}}$*.

Let K be a simplicial complex and L a subcomplex of K. For each subdivision K' of K, L is subdivided by the subcomplex $K'||L| = \{\tau \in K' \mid \tau \subset |L|\}$ of K'. In particular, every simplex $\sigma \in K$ is triangulated by the subcomplex $K'|\sigma = \{\tau \in K' \mid \tau \subset \sigma\}$ of K'. The barycentric subdivision $\operatorname{Sd}_L K$ of K relative to L is defined as follows:

$$\begin{aligned}\operatorname{Sd}_L K = L &\cup \big\{\langle\hat{\sigma}_1, \dots, \hat{\sigma}_n\rangle \mid \sigma_1 < \cdots < \sigma_n \in K \setminus L\big\} \\ &\cup \big\{\langle v_1, \dots, v_m, \hat{\sigma}_1, \dots, \hat{\sigma}_n\rangle \mid \langle v_1, \dots, v_m\rangle \in L, \\ &\qquad \sigma_1 < \cdots < \sigma_n \in K \setminus L,\ \langle v_1, \dots, v_m\rangle < \sigma_1\big\}.\end{aligned}$$

Then, L is a subcomplex of $\operatorname{Sd}_L K$ and $(\operatorname{Sd}_L K)||C(L,K)| = (\operatorname{Sd} K)||C(L,K)|$. The n-th barycentric subdivision $\operatorname{Sd}_L^n K$ of K relative to L is inductively defined by $\operatorname{Sd}_L^n K = \operatorname{Sd}_L(\operatorname{Sd}_L^{n-1} K)$, where $\operatorname{Sd}_L^0 K = K$. Since $(\operatorname{Sd}_L K)^{(0)} \subset (\operatorname{Sd} K)^{(0)}$, the subdivision $\operatorname{Sd}_L K$ is also admissible by Theorem 6 above, hence so is every $\operatorname{Sd}_L^n K$.

3. Proofs of Proposition 3 and Theorem 2.

PROOF OF PROPOSITION 3. Let $\sigma^{(0)} = \{v_0, v_1, \dots, v_k\}$, that is, $\sigma = \langle v_0, v_1, \dots, v_k\rangle$. By induction on $n \in \boldsymbol{N}$, we shall show the following:

$(\star)_n$ for each $\varepsilon > 0$, there are infinitely many vertices $u_i \in (\operatorname{Sd}^n K)^{(0)}$, $i \in \boldsymbol{N}$, such that $\sum_{j=0}^{k} \beta_{v_j}^K(u_i) < \varepsilon$ and every finite set of u_i's, togather with the vertices of the carrier of x in $\operatorname{Sd}^n K$, spans a simplex of $\operatorname{Sd}^n K$.

Then, the result follows because

$$\begin{aligned}\rho_K(x,u_i) &= \sum_{v\in K^{(0)}} \left|\beta^K_v(x)-\beta^K_v(u_i)\right| \\ &\geq \sum_{j=0}^{k}\left(\beta^K_{v_j}(x)-\beta^K_{v_j}(u_i)\right) + \sum_{v\in K^{(0)}\setminus\sigma^{(0)}} \beta^K_v(u_i) \\ &= 2\Bigg(1-\sum_{j=0}^{k}\beta^K_{v_j}(u_i)\Bigg) > 2-2\varepsilon.\end{aligned}$$

To see $(\star)_1$, for each $\varepsilon > 0$, choose $m \in \boldsymbol{N}$ so that $(k+1)/(k+m+2) < \varepsilon$. By the assumption, there are simplexes $\sigma < \sigma_1 < \sigma_2 < \cdots$ with $\dim\sigma_i = k+m+i$. Choose $\tau_0 < \tau_1 < \cdots < \tau_{k_0} = \sigma$ so that $\langle\hat{\tau}_0,\hat{\tau}_1,\ldots,\hat{\tau}_{k_0}\rangle \in \operatorname{Sd} K$ is the carrier of x. Then, $\hat{\tau}_0,\hat{\tau}_1,\ldots,\hat{\tau}_{k_0},\hat{\sigma}_1,\ldots,\hat{\sigma}_l$ span a simplex of $\operatorname{Sd} K$ for each $l \in \boldsymbol{N}$ and

$$\sum_{j=0}^{k}\beta^K_{v_j}(\hat{\sigma}_i) = \sum_{j=0}^{k}\frac{1}{k+m+i+1} \leq \frac{k+1}{k+m+2} < \varepsilon.$$

Now, we prove the implication $(\star)_n \Rightarrow (\star)_{n+1}$. Let $\sigma_0 = c_{\operatorname{Sd}^n K}(x)$ be the carrier of x in $\operatorname{Sd}^n K$. We have $\tau_0 < \tau_1 < \cdots < \tau_{k_0} = \sigma_0$ such that $\langle\hat{\tau}_0,\hat{\tau}_1,\ldots,\hat{\tau}_{k_0}\rangle$ is the carrier of x in $\operatorname{Sd}^{n+1} K$. Then, $k_0 \leq \dim\sigma_0 \leq \dim\sigma = k$. For each $\varepsilon > 0$, choose $m \in \boldsymbol{N}$ so that

$$\frac{\dim\sigma_0+1}{\dim\sigma_0+m+2} < \frac{\varepsilon}{2}.$$

By $(\star)_n$, we have infinitely many vertices $u_i \in (\operatorname{Sd}^n K)^{(0)}$, $i \in \boldsymbol{N}$, such that $\sum_{j=0}^{k}\beta^K_{v_j}(u_i) < \varepsilon/2$ and every finite set of u_i's, togather with the vertices of σ_0, spans a simplex of $\operatorname{Sd}^n K$. For each $i \in \boldsymbol{N}$, let $\sigma_i \in \operatorname{Sd}^n K$ be the simplex spanned by the vertices of σ_0 and $u_1,\ldots,u_{m+i}$. Then, $\dim\sigma_i = \dim\sigma_0 + m + i$. Thus, we have infinitely many vertices $\hat{\sigma}_i \in (\operatorname{Sd}^{n+1} K)^{(0)}$, $i \in \boldsymbol{N}$, such that $\hat{\tau}_0,\hat{\tau}_1,\ldots,\hat{\tau}_{k_0},\hat{\sigma}_1,\ldots,\hat{\sigma}_l$ span a simplex of $\operatorname{Sd}^{n+1} K$ for each $l \in \boldsymbol{N}$. Since $\sigma_0^{(0)} \subset \sigma$ and

$$\hat{\sigma}_l = \sum_{w\in\sigma_0^{(0)}}\frac{1}{\dim\sigma_0+m+l+1}w + \sum_{i=1}^{l}\frac{1}{\dim\sigma_0+m+l+1}u_i,$$

it follows that

$$\begin{aligned}\sum_{j=0}^{k}\beta^K_{v_j}(\hat\sigma_l) &= \sum_{j=0}^{k}\frac{1}{\dim\sigma_0+m+l+1}\bigg(\sum_{w\in\sigma_0^{(0)}}\beta^K_{v_j}(w)+\sum_{i=1}^{l}\beta^K_{v_j}(u_i)\bigg)\\ &= \frac{1}{\dim\sigma_0+m+l+1}\bigg(\sum_{w\in\sigma_0^{(0)}}\sum_{j=0}^{k}\beta^K_{v_j}(w)+\sum_{i=1}^{l}\sum_{j=0}^{k}\beta^K_{v_j}(u_i)\bigg)\\ &< \frac{1}{\dim\sigma_0+m+l+1}\bigg(\dim\sigma_0+1+\frac{l\varepsilon}{2}\bigg) < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

This completes the proof. □

For simplexes $\sigma,\tau\in K$, when $\sigma^{(0)}\cup\tau^{(0)}$ spans a simplex, such a simplex is denoted by $\sigma\tau$. Recall that a subcomplex L of a simplicial complex K is full (in K) if each simplex $\sigma\in K[L]$ meets $|L|$ at a face, that is, $\sigma\cap|L|$ is a face of σ. For any subcomplex L of K, $\operatorname{Sd}L$ is a full subcomplex of $\operatorname{Sd}K$.

LEMMA 7. *Let K be a simplicial complex and L a finite-dimensional full subcomplex of K. Every simplicial subdivision B' of $B(L,K)$ extends to a simplicial subdivision N' of $N(L,K)$ such that $L\cup B'\subset N'$ and $N'^{(0)}=L^{(0)}\cup B'^{(0)}$.*

PROOF. For each $\tau\in B'$, $c_K(\hat\tau)\in B(L,K)$ and $\operatorname{Lk}(c_K(\hat\tau),K)\cap L\neq\emptyset$, where $c_K(\hat\tau)$ is the carrier of the barycenter of τ in K. For each $\sigma\in\operatorname{Lk}(c_K(\hat\tau),K)\cap L$, we have $\sigma\tau\subset\sigma c_K(\hat\tau)\in K$. Then, we define

$$N'=L\cup B'\cup\{\sigma\tau\mid \sigma\in\operatorname{Lk}(c_K(\hat\tau),K)\cap L,\ \tau\in B'\}.$$

Obviously, $N'^{(0)}=L^{(0)}\cup B'^{(0)}$. For each $x\in|N(L,K)|\setminus|L\cup B'|$, since L is full in K, we have $\sigma=c_K(x)\cap|L|\in L$. Let σ' be the opposite face of $c_K(x)$ from σ. Then, $\sigma'\in B(L,K)$. Since B' is a subdivision of $B(L,K)$, we have $\tau\in B'$ such that $c_K(\hat\tau)=\sigma'$ and $x\in\sigma\tau$. Thus, N' is a subdivision of $N(L,K)$. □

PROOF OF THEOREM 2. First of all, note that if a subdivision K' of K refines $\mathscr{U}$ then $\mathscr{S}_{K'}$ refines $\operatorname{st}\mathscr{U}=\{\operatorname{st}(U,\mathscr{U})\mid U\in\mathscr{U}\}$. Since every open cover of $|K|_{\mathrm{m}}$ has the open star-refinement, it suffices to construct an admissible subdivision K' of K which refines $\mathscr{U}$. We shall inductively construct admissible subdivisions K_n of K, $n\geqslant 0$, so as to satisfy the following conditions:

(1) K_n is a subdivision of K_{n-1};
(2) $K_n||K^{(n-1)}|=K_{n-1}||K^{(n-1)}|$;
(3) $K_n[K^{(n)}]$ refines $\mathscr{U}$;
(4) $|C(K^{(n-1)},K_n)|=|C(K^{(n-1)},K_{n-1})|$,

equivalently $|N(K^{(n-1)}, K_n)| = |N(K^{(n-1)}, K_{n-1})|$,

where $K_{-1} = \operatorname{Sd} K$ and $K^{(-1)} = \emptyset$. Then, (2) guarantees that $K' = \bigcup_{n\in\boldsymbol{N}} K_n||K^{(n)}|$ is a simplicial subdivision of K, where one should note that $K_0||K^{(0)}| = K^{(0)} \subset K_1||K^{(1)}|$. By (3), K' refines $\mathscr{U}$. Since each K_n is admissible, $K'^{(0)}||K^{(n)}| = K_n^{(0)}||K^{(n)}|$ is discrete in $|K|_{\mathrm{m}}$ by (2). Since $|C(K^{(n)}, K')| \subset |C(K^{(n)}, K_n)|$ by (2) and (4), $C(K^{(n)}, K')^{(0)}$ has no accumulation points in $|K^{(n)}|$. Then, it follows that $K'^{(0)}$ is discrete in $|K|_{\mathrm{m}}$, which means that K' is an admissible subdivision of K by Theorem 6.

For each vertex $v \in K^{(0)}$, choose $1/2 < t_v < 1$ so that $(\beta_v^{\operatorname{Sd}^2 K})^{-1}([t_v, 1])$ is contained in some $U_v \in \mathscr{U}$ (Lemma 5 or 4). Dividing each $\sigma \in (\operatorname{Sd}^2 K)[v] \setminus \{v\}$ into two cells by $(\beta_v^{\operatorname{Sd}^2 K})^{-1}(t_v)$, we have a cell complex L subdividing $\operatorname{Sd}^2 K$, that is,

$$
\begin{aligned}
L = K^{(0)} \cup C(K^{(0)}, \operatorname{Sd}^2 K) \\
\cup \big\{\sigma \cap (\beta_v^{\operatorname{Sd}^2 K})^{-1}(t_v),\ \sigma \cap (\beta_v^{\operatorname{Sd}^2 K})^{-1}([0, t_v]), \\
\sigma \cap (\beta_v^{\operatorname{Sd}^2 K})^{-1}([t_v, 1]) \mid \sigma \in (\operatorname{Sd}^2 K)[v] \setminus \{v\},\ v \in K^{(0)}\big\}.
\end{aligned}
$$

Then, $L^{(0)}$ is discrete in $|K|_{\mathrm{m}}$. Indeed, $L^{(0)}$ consists of the vertices $(\operatorname{Sd}^2 K)^{(0)}$ and the points

$$v_w = (1 - t_v)w + t_v v, \quad v \in K^{(0)}, \quad w \in \operatorname{Lk}(v, \operatorname{Sd}^2 K)^{(0)}.$$

Since $\operatorname{Sd}^2 K$ is an admissible subdivision of K, $(\operatorname{Sd}^2 K)^{(0)}$ is discrete in $|K|_{\mathrm{m}}$. On the other hand, $\{(\beta_v^{\operatorname{Sd}^2 K})^{-1}(t_v) \mid v \in K^{(0)}\}$ is discrete in $|K|_{\mathrm{m}}$. Then, it suffices to see that $\{v_w \mid w \in \operatorname{Lk}(v, \operatorname{Sd}^2 K)^{(0)}\}$ is discrete in $(\beta_v^{\operatorname{Sd}^2 K})^{-1}(t_v)$ for each $v \in K^{(0)}$. Note that the metric $\rho_{\operatorname{Sd}^2 K}$ is admissible for $|K|_{\mathrm{m}}$. For each $w, w' \in \operatorname{Lk}(v, \operatorname{Sd}^2 K)^{(0)}$,

$$\rho_{\operatorname{Sd}^2 K}(v_w, v_{w'}) = \beta_w^{\operatorname{Sd}^2 K}(v_w) + \beta_{w'}^{\operatorname{Sd}^2 K}(v_{w'}) = 2(1 - t_v).$$

Now, let K_0 be a simplicial subdivision of L with $K_0^{(0)} = L^{(0)}$. Since $K_0^{(0)} = L^{(0)}$ is discrete in $|K|_{\mathrm{m}}$, K_0 is an admissible subdivision of K by Theorem 6. Observe that

$$|\operatorname{St}(v, K_0)| = \big(\beta_v^{\operatorname{Sd} K}\big)^{-1}([t_v, 1]) \subset U_v \text{ for } v \in K_0^{(0)}.$$

Then, K_0 satisfies (3).

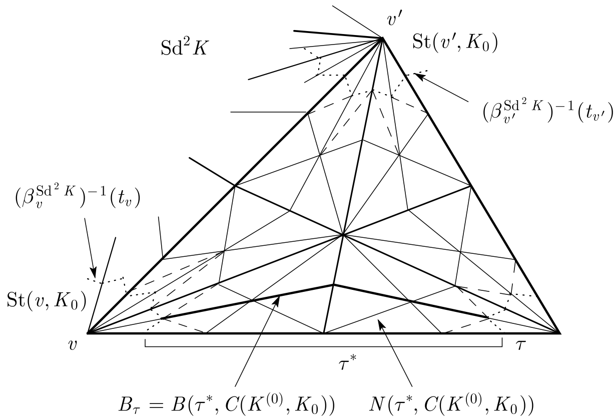

Figure 1. The subdivision K_0 of $\mathrm{Sd}^2 K$.

Assume that K_{n-1} has been obtained. For each n-simplex $\tau \in K$, we define

$$\tau^* = \tau \cap \left|C(K^{(n-1)}, K_{n-1})\right|.$$

Note that $K_{n-1}|\tau^*$ is a triangulation of τ^*. We can choose $n(\tau) \in \boldsymbol{N}$ so that $\mathrm{Sd}^{n(\tau)}(K_{n-1}|\tau^*) \prec \mathscr{U}$.[3] Let

$$B_\tau = B(\tau^*, C(K^{(n-1)}, K_{n-1})) \quad \text{and}$$
$$N_\tau = \mathrm{Sd}_{B_\tau}^{n(\tau)} N\big(\tau^*, C(K^{(n-1)}, K_{n-1})\big).$$

Then, N_τ is an admissible subdivision of $N(\tau^*, C(K^{(n-1)}, K_{n-1}))$, hence $|N_\tau|_{\mathrm{m}}$ is a subspace of $|K_{n-1}|_{\mathrm{m}} = |K|_{\mathrm{m}}$. Moreover,

$$N_\tau \mid \tau^* = \mathrm{Sd}^{n(\tau)}(K_{n-1} \mid \tau^*) \prec \mathscr{U},$$

hence each $\sigma \in N_\tau|\tau^*$ is contained in some $U_\sigma \in \mathscr{U}$. By Lemma 5, $(\beta_\sigma^{N_\tau})^{-1}([t,1]) \subset U_\sigma$ for some $1/2 < t < 1$. Since $N_\tau|\tau^*$ is finite, we can find $1/2 < t_\tau < 1$ such that

$$\left\{(\beta_\sigma^{N_\tau})^{-1}([t_\tau, 1]) \mid \sigma \in N_\tau \mid \tau^*\right\} \prec \mathscr{U}.$$

[3]In general, $n(\tau)$ cannot be chosen so that $\mathrm{Sd}^{n(\tau)}(N(\tau, K_{n-1}) \cap C(\partial\tau, K_{n-1})) \prec \mathscr{U}$ (Proposition 3).

For each $\sigma \in N_\tau[\tau^*] \setminus N_\tau|\tau^*$, we have $\sigma \cap \tau^* \in N_\tau|\tau^*$ and $\beta^{N_\tau}_{\sigma\cap\tau^*}|\sigma = \beta^{N_\tau}_{\tau^*}|\sigma$. Dividing each $\sigma \in N_\tau[\tau^*] \setminus N_\tau|\tau^*$ into two cells by $(\beta^{N_\tau}_{\tau^*})^{-1}(t_\tau)$, we have a cell complex L_τ subdividing N_τ, that is,

$$L_\tau = N_\tau|\tau^* \cup C(\tau^*, N_\tau) \cup \big\{\sigma \cap (\beta^{N_\tau}_{\tau^*})^{-1}(t_\tau),\ \sigma \cap (\beta^{N_\tau}_{\tau^*})^{-1}([0, t_\tau]),\\ \sigma \cap (\beta^{N_\tau}_{\tau^*})^{-1}([t_\tau, 1]) \mid \sigma \in N_\tau[\tau^*] \setminus N_\tau \mid \tau^*\big\}.$$

Then, $L^{(0)}_\tau$ is discrete in $|N_\tau|_{\mathrm{m}}$, so in $|K|_{\mathrm{m}}$. Indeed, $L^{(0)}_\tau$ consists of $N^{(0)}_\tau$ and the points

$$(1 - t_\tau)w + t_\tau v, \quad v \in N^{(0)}_\tau|\tau^*, \quad w \in \mathrm{Lk}(v, N_\tau)^{(0)} \setminus \tau^*,$$

where $N^{(0)}_\tau$ is discrete in $|N_\tau|_{\mathrm{m}}$. As is easily observed, we have

$$\mathrm{dist}_{\rho_{N_\tau}}\big(N^{(0)}_\tau, (\beta^{N_\tau}_{\tau^*})^{-1}(t_\tau)\big) \geq \min\{2t_\tau,\ 2(1 - t_\tau)\}.$$

For each $v, v' \in N^{(0)}_\tau|\tau^*$, $w \in \mathrm{Lk}(v, N_\tau)^{(0)} \setminus \tau^*$ and $w' \in \mathrm{Lk}(v', N_\tau)^{(0)} \setminus \tau^*$, if $v \neq v'$ or $w \neq w'$ then

$$\rho_{N_\tau}\big((1 - t_\tau)w + t_\tau v, (1 - t_\tau)w' + t_\tau v'\big) \geq \min\{2t_\tau,\ 2(1 - t_\tau)\}.$$

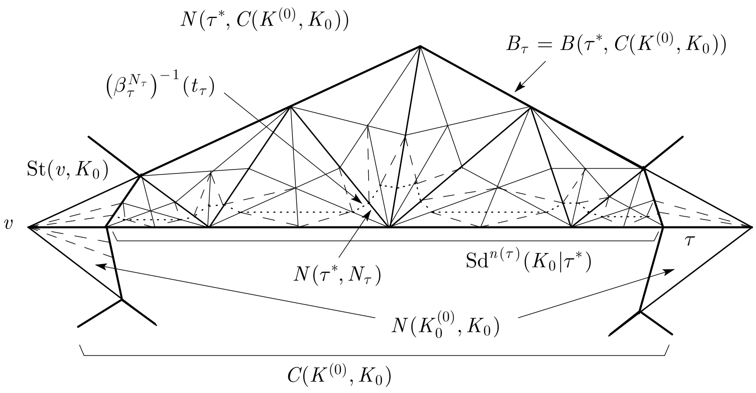

Figure 2. The subdivision N_τ of $N(\tau, K_0)$.

Now, for each $\tau \in K(n)$, let K_τ be a simplicial subdivision of L_τ with $K^{(0)}_\tau = L^{(0)}_\tau$. Observe

$$B_\tau = K_\tau \cap C(K^{(n)}, K_{n-1}) \text{ and } |B_\tau| = |K_\tau| \cap \big|C(K^{(n)}, K_{n-1})\big|.$$

Then, the following is a simplicial complex subdividing $C(K^{(n-1)}, K_{n-1})$:

$$C' = C(K^{(n)}, K_{n-1}) \cup \bigcup_{\tau \in K(n)} K_\tau.$$

By Lemma 7, we have a simplicial subsdivision N' of $N(K^{(n-1)}, K_{n-1})$ such that

$$N' | |B(K^{(n-1)}, K_{n-1})| = C' | |B(K^{(n-1)}, K_{n-1})| \quad \text{and}$$
$$N'^{(0)} = N\big(K^{(n-1)}, K_{n-1}\big)^{(0)} \cup B'^{(0)}.$$

Then, $K_n = C' \cup B'$ is a simplicial subdivision of K_{n-1} such that

$$\big|N(K^{(n-1)}, K_{n-1})\big| = \big|N(K^{(n-1)}, K_n)\big|,$$

that is, K_n satisfies the conditions (1) and (4). Note that

$$\begin{aligned} K_n^{(0)} &= N\big(K^{(n-1)}, K_{n-1}\big)^{(0)} \cup C\big(K^{(n)}, K_{n-1}\big)^{(0)} \cup \bigcup_{\tau \in K(n)} K_\tau^{(0)} \\ &= K_{n-1}^{(0)} \cup \bigcup_{\tau \in K(n)} N_\tau^{(0)}, \end{aligned}$$

which is discrete in $|K|_{\mathrm{m}}$. This means that K_n is an admissible subdivision of K by Theorem 6. By our construction, we have $K_n | |K^{(n-1)}| = K_{n-1} | |K^{(n-1)}|$, that is, K_n satisfies (2). Moreover, $K_n[K^{(n)}] \prec \mathscr{U}$ because

$$K_n[K^{(n-1)}] \prec K_{n-1}[K^{(n-1)}] \prec \mathscr{U} \quad \text{and}$$
$$K_n[K^{(n)}] \setminus K_n[K^{(n-1)}] \subset \bigcup_{\tau \in K(n)} N_\tau \prec \mathscr{U}.$$

Thus, K_n satisfies (3). The proof is completed. □

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