J. Math. Soc. Japan
Vol. 63, No. 2 (2011) pp. 419–442
doi: 10.2969/jmsj/06320419

Bishop-Gromov type inequality on Ricci limit spaces

By Shouhei Honda

(Received Aug. 14, 2009)
(Revised Jan. 10, 2010)

Abstract. In this paper, we study limit spaces of a sequence of n-dimensional complete Riemannian manifolds whose Ricci curvatures have definite lower bound. We will give several measure theoretical properties of such limit spaces.

1. Introduction.

In this paper, we study a pointed metric space (Y, y) that is pointed Gromov-Hausdorff limit of a sequence of complete, pointed, connected n-dimensional Riemannian manifolds, $\{(M_i, m_i)\}_i$, with $\mathrm{Ric}_{M_i} \geq -(n-1)$. We call such a metric space (Y, y) *Ricci limit space* in this paper. See Section 4.1 in [**16**]. In the papers [**4**], [**5**] and [**6**], Cheeger-Colding studied such limit spaces, showed many important results. There exists a Borel measure υ on a Ricci limit space (Y, y), which is called *the limit measure*. See Definition 2.3. They developed the structure theory by using the limit measure υ and results in [**4**], [**5**] and [**6**]. Most of this paper, we will study measure theoretical properties on Ricci limit spaces for the limit measure. In the other papers [**13**] and [**14**], we will discuss several geometric applications of the results in this paper to Ricci limit spaces.

First, we study cut loci on Ricci limit spaces in Section 3. We prove that the measure of cut locus is equal to zero. See Theorem 3.2. We will study cut locus geometrically in Section 8 in [**13**]. We also give a relationship between "the limit space of cut loci" and "cut locus of the limit space". See Theorem 3.5.

Cheeger-Colding defined the measure of codimension one of υ in [**5**], we denote it by υ_{-1}. See Definition 4.1 for the definition of υ_{-1}. If Y is isometric to a k-dimensional smooth Riemannian manifold, then the measure υ_{-1} and the $(k-1)$-dimensional Hausdorff measure $\mathscr{H}^{k-1}$ are mutually absolutely continuous. We will give several properties of υ_{-1}. For example, we will show the following Bishop-Gromov type inequality for υ_{-1}:

2000 *Mathematics Subject Classification.* Primary 53C20; Secondary 53C43.

Key Words and Phrases. Ricci curvature, Gromov-Hausdorff convergence, geometric measure theory.

This research was partly supported by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists.

Theorem 1.1. *Let (Y, y) be a Ricci limit space. Then, there exists a positive constant $C(n) > 0$ depending only on n, such that for every positive numbers $0 < s < t < \infty$, every point $x \in Y$ and for every Borel set $A \subset \partial B_t(x)$,*

$$\frac{\upsilon_{-1}(A)}{\operatorname{vol} \partial B_t(\underline{p})} \leq C(n) \frac{\upsilon_{-1}(\partial B_s(x) \cap C_x(A))}{\operatorname{vol} \partial B_s(\underline{p})}$$

holds.

Here, $\underline{p}$ is a point in the standard n-dimensional hyperbolic space $\boldsymbol{H}^n(-1)$ and $C_x(A) = \{z \in Y \mid \text{There exists } w \in A \text{ such that } \overline{x,z} + \overline{z,w} = \overline{x,w} \text{ holds.}\}$, $\overline{x,z}$ is the distance between x and z on Y. This is like Laplacian comparison theorem on Riemannian manifolds. If Y is isometric to a smooth Riemannian manifold, then Theorem 1.1 corresponds to area comparison theorem (see [**1**, Theorem 0.7] and (2.13) in [**1**]). See [**13**, Theorem 1.2] for a geometric application of Theorem 1.1 to low dimensional Ricci limit spaces.

Also we will show a finiteness result (Theorem 4.2) and a positivity result of the measure υ_{-1} (Corollary 4.7). It means that the measure υ_{-1} is a good measure on the set $\partial B_r(x) \setminus C_x$. Here, C_x is the cut locus of $x \in Y$. These properties are similar to those on Riemannian manifolds.

We will give a relationship between the limit measure υ and the measure υ_{-1} in Section 5. Theorem 5.2 is like co-area formula for Lipschitz maps on Euclidean spaces (see [**8**, 3.2.12. Theorem]). We will discuss an application of Theorem 5.2 to a rectifiability of Ricci limit spaces in [**14**].

Finally, we also consider the subset of Ricci limit space (Y, y), $A_Y(\alpha)$ consists of points $x \in Y$ satisfying $\upsilon(B_r(x)) \sim r^\alpha$ as $r \to 0$. See Definition 6.1. The limit measure υ on $A_Y(\alpha)$ and α-dimensional Hausdorff measure $\mathscr{H}^\alpha$ are mutually absolutely continuous. We will give an upper bound of Hausdorff dimension of the set. As a corollary, we will give an easy new proof of Corollary 6.4.

Acknowledgements. The author is grateful to Professor Kenji Fukaya for his numerous suggestions and advice. He is also grateful to the referee for many valuable advice on the first version.

2. Notation.

In this section, we recall some fundamental notion on metric spaces and define the notion of Ricci limit spaces.

Definition 2.1. We say that a metric space X is *proper* if every bounded closed set is compact. A metric space X is said to be a *geodesic space* if for every points $x_1, x_2 \in X$, there exists an isometric embedding $\gamma : [0, \overline{x_1, x_2}] \to X$ such that $\gamma(0) = x_1$, $\gamma(\overline{x_1, x_2}) = x_2$. Here $\overline{x_1, x_2}$ is the distance between x_1 and x_2 on X. We say that γ is a *minimal geodesic from x_1 to x_2*.

For a proper geodesic space X, a point $x \in X$, a set $A \subset X$, and for positive numbers $0 < r < R$, we use the following notations; $B_r(x) = \{z \in X \mid \overline{x, z} < r\}$, $\overline{B}_r(x) = \{z \in X \mid \overline{x, z} \le r\}$, $A_{r,R}(x) = \overline{B}_R(x) \setminus B_r(x)$, $\partial B_r(x) = \{z \in X \mid \overline{x, z} = r\}$, $C_x(A) = \{z \in X \mid$ There exists $w \in A$ such that $\overline{x, z} + \overline{z, w} = \overline{x, w}$ holds.$\}$. Throughout the paper, we fix a positive integer $n > 0$.

Definition 2.2. Let (Y, y) be a pointed proper geodesic space ($y \in Y$), K a real number. We say that (Y, y) is a *(n, K)-Ricci limit space* if there exist a sequence of real numbers $\{K_i\}$ and a sequence of complete, pointed, connected n-dimensional Riemannian manifolds $\{(M_i, m_i)\}_i$ with $\mathrm{Ric}_{M_i} \ge K_i(n-1)$, such that K_i converges to K and that (M_i, m_i) converges to (Y, y) as $i \to \infty$ in the sense of pointed Gromov-Hausdorff topology.

Here, for a sequence of pointed proper geodesic spaces $\{(X_i, x_i)\}_i$, we say that (X_i, x_i) *converges to a pointed proper geodesic space* (X_∞, x_∞) *in the sense of pointed Gromov-Hausdorff topology* if there exist sequences of positive numbers ϵ_i, $R_i > 0$ and exists a sequence of maps $\phi_i : (B_{R_i}(x_i), x_i) \to (B_{R_i}(x_\infty), x_\infty)$ such that ϵ_i converges to 0, R_i converges to ∞, $|\overline{z_i, w_i} - \overline{\phi_i(z_i), \phi_i(w_i)}| < \epsilon_i$ holds for every points $z_i, w_i \in B_{R_i}(x_i)$, and that $B_{\epsilon_i}(\mathrm{Image}(\phi_i)) \supset B_{R_i}(x_\infty)$ holds. (See Definitions 6.1 and 6.2 in [**9**].) We say that ϕ_i *is a ϵ_i-Gromov-Hausdorff approximation.* Then for a sequence of points $z_i \in X_i$ such that the set $\{\overline{x_i, z_i} \mid i \in \boldsymbol{N}\}$ is bounded set in $\boldsymbol{R}$, we say that z_i *converges to a point $z_\infty \in X_\infty$ in the sense of pointed Gromov-Hausdorff topology* if $\overline{\phi_i(z_i), z_\infty} < \epsilon_i$. We denote it by either $z_i \to z_\infty$ or $\overline{z_i, z_\infty} < \epsilon_i$.

We remark that for every $K \neq 0$ and every (n, K)-Ricci limit space (Y, y), there exists a sequence of complete, connected n-dimensional Riemannian manifolds $\{(M_i, m_i)\}_i$ with $\mathrm{Ric}_{M_i} \ge K(n-1)$, such that (M_i, m_i) converges to (Y, y) by rescaling. *Throughout the paper, (Y, y) is always a $(n, -1)$-Ricci limit space and is not a single point. More simply, we say that (Y, y) is Ricci limit space.*

We shall give the definition of *limit measure.* The measure is a useful tool for studying Ricci limit spaces.

Definition 2.3. Let υ be a Borel measure on Y. We say that υ is a *limit measure* if there exists a sequence of complete, pointed, connected n-dimensional Riemannian manifolds $\{(M_i, m_i)\}_i$ with $\mathrm{Ric}_{M_i} \ge -(n-1)$, such that (M_i, m_i) converges to (Y, y) and that for every positive number $r > 0$ and every points $x \in Y$, $\hat{m}_j \in M_j$ satisfying $\hat{m}_j \to x$ in the sense of pointed Gromov-Hausdorff topology,

$$\frac{\operatorname{vol} B_r(\hat{m}_j)}{\operatorname{vol} B_1(m_j)} \to \upsilon(B_r(x))$$

holds. Then we say that $(M_j, m_j, \operatorname{vol}/\operatorname{vol} B_1(m_j))$ *converges to* (Y, y, υ) *in the sense of measured Gromov-Hausdorff topology.*

There exists a limit measure on Y (see [**4**, Theorem 1.6], [**4**, Theorem 1.10] and [**9**]). In general, it is *not* unique (see [**4**, Example 1.24]). *Throughout the paper, υ is always a fixed limit measure on Y.*

3. Cut locus.

In this section, we study a cut locus on Ricci limit spaces.

3.1. Measure of cut locus.

First, we give the definition of cut locus.

Definition 3.1. For a proper geodesic space X and every $w \in X$, we put $C_w = \{x \in X \mid \text{For every point } z \in Y \setminus \{x\}, \overline{w,x} + \overline{x,z} - \overline{w,z} > 0 \text{ holds.}\}$. If X is a single point, then $C_x = \emptyset$. We say that C_w is the *cut locus of w*.

The following theorem is the main result in this subsection.

Theorem 3.2. *We have $\upsilon(C_w) = 0$ for every point $w \in Y$.*

Proof. We shall give only a proof of the case $w = y$. There exists a sequence of complete pointed, connected n-dimensional Riemannian manifolds, $\{(M_j, m_j)\}_j$ such that $\operatorname{Ric}_{M_j} \geq -(n-1)$ and that $(M_j, m_j, \operatorname{vol}/\operatorname{vol} B_1(m_j))$ converges to (Y, y, υ) in the sense of measured Gromov-Hausdorff topology. For every positive number $r > 0$ and every positive integer $N \in \boldsymbol{N}$, we put $C_y(r) = \{x \in Y \mid \text{For every } z \in Y \setminus B_r(x), \overline{y,x} + \overline{x,z} - \overline{y,z} > 0 \text{ holds.}\}$ and $C_y(r,N) = \{x \in Y \mid \text{For every } z \in Y \setminus B_r(x), \overline{y,x} + \overline{x,z} - \overline{y,z} \geq N^{-1} \text{ holds.}\}$. By the definition, $C_y(r,N)$ is closed.

Claim 3.3. *We have $C_y(r) = \bigcup_{N \in \boldsymbol{N}} C_y(r,N)$.*

It suffices to see that $C_y(r) \subset \bigcup_{N \in \boldsymbol{N}} C_y(r,N)$. This proof is done by a contradiction. We assume that there exists a point $x \in C_y(r) \setminus \bigcup_{N \in \boldsymbol{N}} C_y(r,N)$. Then, for every positive integer N, there exists a point $y_N \in Y \setminus B_r(x)$ such that $\overline{y,x} + \overline{x,y_N} - \overline{y,y_N} < N^{-1}$ holds. Clearly, for every positive integer N, there exists a point $z_N \in \partial B_r(x)$ such that $\overline{x,z_N} + \overline{z_N,y_N} = \overline{x,y_N}$ holds. Then, by triangle inequality, we have $\overline{y,x} + \overline{x,z_N} - \overline{y,z_N} < N^{-1}$. Since $\partial B_r(x)$ is compact, there exists a subsequence $\{z_{k(N)}\}_N$ and a point $z_\infty \in \partial B_r(x)$ such that $z_{k(N)}$

converges to z_∞ in Y. Therefore, we have $\overline{y,x} + \overline{x,z_\infty} = \overline{y,z_\infty}$. This contradicts the assumption. Thus we have Claim 3.3.

By the definition, we have $C_y = \bigcap_{r>0} C_y(r)$. We fix a positive number $r > 0$ and a positive integer $N \in \boldsymbol{N}$. Let $l \in \boldsymbol{N}$ be a positive integer, $\delta > 0$ a sufficiently small positive number satisfying $0 < \delta \ll \min\{2^{-l}, r, N^{-1}\}$. Let $\{x_i\}_{i=1}^k$ be a maximal 100δ-separated subset of the set $C_y(r,N) \cap A_{2^{-l},2^l}(y)$. For every positive integers $i, j > 0$ $(1 \le i \le k)$, we take $x_i(j) \in M_j$ such that $x_i(j)$ converges to x_i as $j \to \infty$ in the sense of pointed Gromov-Hausdorff topology. In general, for a complete pointed, connected n-dimensional Riemannian manifold (M, m), we put $S_m M = \{u \in T_m M \mid |u| = 1\}$ and define $t(u) > 0$ as the supremum of $t \in (0,\infty)$ such that $\exp_m su|_{[0,t]}$ is a minimal geodesic segment from m to $\exp_m tu$ for $u \in S_m M$. For every positive numbers $0 < r_1 < r_2$ and $\eta > 0$, we put $X(m, r_1, r_2, \eta) = \{\exp_m tu \in M \mid u \in S_m M, t(u) - \eta \le t < t(u), \exp_m tu \in A_{r_1,r_2}(m)\}$.

Claim 3.4. *We have $\bigcup_{i=1}^k B_{10\delta}(x_i(j)) \setminus C_{m_j} \subset X(m_j, 2^{-l-1}, 2^{l+1}, 100r)$ for every sufficiently large j.*

We take $x \in B_{10\delta}(x_i(j)) \setminus C_{m_j}$. For every point $z \in M_j \setminus B_{40r}(x)$, we take $w \in Y$ such that $\overline{z,w} < \epsilon_j$ in the sense of pointed Gromov-Hausdorff topology $(\epsilon_j \to 0)$. Then, we have

$$\begin{aligned} \overline{m_j,x} + \overline{x,z} - \overline{m_j,z} &\ge \overline{m_j,x_i(j)} + \overline{x_i(j),z} - \overline{m_j,z} - 20\delta \\ &\ge \overline{y,x_i} + \overline{x_i,w} - \overline{y,w} - 20\delta - 7\epsilon_j \qquad (*) \end{aligned}$$

and $\overline{w,x_i} \ge \overline{z,x_i(j)} - 3\epsilon_j \ge \overline{z,x} - \overline{x,x_i(j)} - 3\epsilon_j \ge 40r - 10\delta - 3\epsilon_j > 30r$. By the definition of x_i, we have

$$(*) \ge N^{-1} - 20\delta - 7\epsilon_j \ge (2N)^{-1} > 0.$$

Thus there exist $u \in S_{m_j} M_j$ and positive number $t > 0$ such that $t(u) - 50r \le t < t(u)$ and $x = \exp_{m_j} tu$ hold. Because, if we assume that $t < t(u) - 50r$, then there exists $\alpha \in M_j$ such that $\overline{x,\alpha} = 45r$ and that $\overline{m_j,x} + \overline{x,\alpha} - \overline{m_j,\alpha} = 0$. On the other hand, since $\alpha \in M_j \setminus B_{40r}(x)$, by the argument above, we have $\overline{m_j,x} + \overline{x,\alpha} - \overline{m_j,\alpha} > 0$. This is a contradiction. Thus we have $t(u) - 50r \le t$.

Since $\overline{x,x_i(j)} < 10\delta$ holds, we have $x \in A_{2^{-l-1},2^{l+1}}(m_j)$. Therefore, we have $x \in X(m_j, 2^{-l-1}, 2^{l+1}, 100r)$. Hence, we have Claim 3.4.

Since $\{B_{10\delta}(x_i(j))\}_i$ are pairwise disjoint for every sufficiently large j, Claim 3.4 yields

$$\sum_{i=1}^{k} \underline{\mathrm{vol}}\, B_{10\delta}(x_i(j)) \le \underline{\mathrm{vol}}\, X(m_j, 2^{-l-1}, 2^{l+1}, 100r).$$

Here, $\underline{\mathrm{vol}} = \mathrm{vol}/\mathrm{vol}\, B_1(m_j)$. By the proof of [**4**, Lemma 2.16], there exists a positive constant $C = C(l,n) > 0$ depending only on l, n, such that $\underline{\mathrm{vol}}\, X$ $(m_j, 2^{-l-1}, 2^{l+1}, 100r) \le C(l,n)r$ holds. Thus, we have

$$\begin{aligned} \upsilon\big(C_y(r,N) \cap A_{2^{-l},2^l}(y)\big) &\le \sum_{i=1}^{k} \upsilon(B_{100\delta}(x_i)) \\ &\le C \sum_{i=1}^{k} \upsilon(B_{10\delta}(x_i)) \\ &\le Cr. \end{aligned}$$

Therefore, by letting $\delta \to 0$, $N \to \infty$, $r \to 0$, and $l \to \infty$, we have $\upsilon(C_y) = 0$. $\square$

We remark that $\mathscr{W}\mathscr{E}_0(w) \subset C_w$ holds for every $w \in Y$. (See [**4**, Definition 2.10] for the definition of $\mathscr{W}\mathscr{E}_0(w)$.) Therefore, Theorem 3.2 differs from [**4**, Proposition 2.13].

3.2. Convergence of cut loci.

In this subsection, we give a relationship between "the limit space of cut loci" and "the cut locus of the limit space". Roughly speaking, we will show that "the limit space of cut loci" contains "the cut locus of the limit space". Let $\{(M_i, m_i)\}_i$ be a sequence of complete pointed, connected n-dimensional Riemannian manifolds with $\mathrm{Ric}_{M_i} \ge -(n-1)$. For every positive number $R > 0$, the sequence of pointed compact metric spaces $(\overline{B}_{2R}(m_i) \cap (C_{m_i} \cup \{m_i\}), m_i)_{i \in \boldsymbol{N}}$ is precompact in the sense of pointed Gromov-Hausdorff topology. We assume that there exist a pointed proper geodesic space (Y, y) and a pointed compact metric space (X_R, x_R) such that $(\overline{B}_{2R}(m_i) \cap (C_{m_i} \cup \{m_i\}), m_i)$ conveges to (X_R, x_R) and that (M_i, m_i) converges to (Y, y).

Theorem 3.5. *Under the notation above, there exists an isometric embedding* $\Phi : (\overline{B}_R(y) \cap (C_y \cup \{y\}), y) \to (X_R, x_R)$.

Proof. First, we shall prove that for every finite points $x_1, x_2, \dots, x_N \in C_y \cap \overline{B}_R(y)$, there exists an isometric embedding $\phi_N : (\{x_1, x_2, \dots, x_N, y\}, y) \to (X_R, x_R)$. We fix finite points $x_1, x_2, \dots, x_N \in C_y \cap \overline{B}_R(y)$. For every sufficiently large $k \in \boldsymbol{N}$, there exists a positive number $\tau > 0$ such that $\overline{y, x_j} + \overline{x_j, x} - \overline{y, x} \ge \tau$ holds for every $1 \le j \le N$ and every point $x \in \overline{B}_{10R}(y) \setminus B_{k^{-1}}(x_j)$. We take

ϵ_i-Gromov-Hausdorff approximations ($\epsilon_i \to 0$), $\phi_i : (\overline{B}_{R_i}(m_i), m_i) \to (\overline{B}_{R_i}(y), y)$, $\hat{\phi}_i : (\overline{B}_{R_i}(y), y) \to (\overline{B}_{R_i}(m_i), m_i)$, $\psi_i : (\overline{B}_{2R}(m_i) \cap (C_{m_i} \cup \{m_i\}), m_i) \to (X_R, x_R)$ and $\hat{\psi}_i : (X_R, x_R) \to (\overline{B}_{2R}(m_i) \cap (C_{m_i} \cup \{m_i\}), m_i)$ such that $\overline{\phi_i \circ \hat{\phi}_i, \mathrm{id}} < \epsilon_i$, $\overline{\hat{\phi}_i \circ \phi_i, \mathrm{id}} < \epsilon_i$, $\overline{\psi_i \circ \hat{\psi}_i, \mathrm{id}} < \epsilon_i$ hold and that $\overline{\hat{\psi}_i \circ \psi_i, \mathrm{id}} < \epsilon_i$ holds. Here, the inequality $\overline{\phi_i \circ \hat{\phi}_i, \mathrm{id}} < \epsilon_i$ means that $\overline{\phi_i \circ \hat{\phi}_i(x), x} < \epsilon_i$ holds for every $x \in \overline{B}_{2R}(y)$. Since $\phi_i(\overline{B}_{2R}(m_i) \setminus B_{100k^{-1}}(\hat{\phi}_i(x_j))) \subset B_{2R+\epsilon_i}(y) \setminus B_{100k^{-1}-2\epsilon_i}(x_j)$, we have $\overline{m_i, \hat{\phi}_i(x_j)} + \overline{\hat{\phi}_i(x_j), z_i} - \overline{m_i, z_i} > \tau/100$ for every sufficiently large i, every $1 \le j \le N$ and every point $z_i \in \overline{B}_{2R}(m_i) \setminus B_{100k^{-1}}(\hat{\phi}_i(x_j))$. Thus, there exists a point $x_j(i,k) \in C_{m_i} \cap \overline{B}_{2R}(m_i)$ such that $\overline{\hat{\phi}_i(x_j), x_j(i,k)} < 100k^{-1}$ holds. By taking a subsequence, we can assume that the sequence $\{\psi_i(x_j(i,k))\}_i$ is a Cauchy sequence in X_R for every $1 \le j \le N$. We put $x(j,k) = \lim_{i\to\infty} \psi_i(x_j(i,k))$. Similarly, without loss of generality, we can assume that the sequence $\{x(j,k)\}_k$ is a Cauchy sequence for every j. We put $x(j,\infty) = \lim_{k\to\infty} x(j,k)$ and put $\phi_N(x_j) = x(j,\infty)$. Then we have an isometric embedding $\phi_N : (\{x_1, x_2, \dots, x_N, y\}, y) \to (X_R, x_R)$. By using ϕ_N and diagonal argument, it is not difficult to construct the map Φ. $\square$

Clearly, in general, the cut locus of the limit space is *not* isometric to the limit space of cut loci. For example, consider the situation that the flat tori $\boldsymbol{S}^1(r) \times \boldsymbol{S}^1$ converges to $\boldsymbol{S}^1$ as $r \to 0$. Here, $\boldsymbol{S}^1(r) = \{x \in \boldsymbol{R}^2 \mid |x| = r\}$.

4. The measure of codimension one.

In this section, we recall the definition of the measure υ_{-1} on Ricci limit spaces, and give several properties of υ_{-1}.

4.1. Definition and finiteness.

First, we recall the definition of υ_{-1}. (See (2.1) and (2.2) in [**5**].)

Definition 4.1. For positive numbers $\beta, \delta > 0$ and a subset $A \subset Y$, we put

$$(\upsilon_{-\beta})_\delta(A) = \inf\left\{ \sum_{i\in I} r_i^{-\beta} \upsilon(B(x_i)) \,\middle|\, \sharp I \le \aleph_0,\ A \subset \bigcup_{i\in I} B_{r_i}(x_i),\ r_i < \delta \right\},$$

$$\upsilon_{-\beta}(A) = \lim_{\delta\to 0} (\upsilon_{-\beta})_\delta(A).$$

By Carathéodory criterion, $\upsilon_{-\beta}$ is a Borel measure on Y. We remark that $\upsilon_{-\beta}(\{x\}) > 0$ holds if and only if $\liminf_{r\to 0} \upsilon(B_r(x))/r^\beta > 0$ holds. The following theorem is the main result in this subsection. However, we will prove a result where sharpens the conclusion in the following theorem later. See Corollary 5.7. The following theorem is used in the proof of Theorem 1.1.

Theorem 4.2. *There exists a positive constant $C(n) > 0$ depending only on n such that for every positive number $t > 0$ and every point $x \in Y$,*

$$\frac{\upsilon_{-1}(\partial B_t(x))}{\upsilon(B_t(x))} \le C(n) \frac{\operatorname{vol} \partial B_t(\underline{p})}{\operatorname{vol} B_t(\underline{p})}$$

holds. Here, $\underline{p}$ is a point in the standard n-dimensional hyperbolic space $\boldsymbol{H}^n(-1)$. Especially, we have $\upsilon_{-1}(\partial B_t(x)) < \infty$.

Proof. We can assume that $\partial B_t(x) \neq \emptyset$. There exists a sequence of complete, pointed, connected n-dimensional Riemannian manifolds $\{(M_j, m_j)\}_j$ with $\mathrm{Ric}_{M_j} \ge -(n-1)$ such that $(M_j, m_j, \operatorname{vol}/\operatorname{vol} B_1(m_j))$ converges to (Y, y, υ) in the sense of measured Gromov-Hausdorff topology. We fix a sufficiently small positive number $\delta > 0$ satisfying $\sinh^{n-1}(t+2\delta)/\sinh^{n-1}(t-2\delta) < 2$. Let $\{x_i\}_{i=1}^N$ be a maximal 100δ-separated subset of $\partial B_t(x)$. For every positive integers $i, j > 0$ $(1 \le i \le N)$, we take $x(j), x_i(j) \in M_j$ such that $x_i(j)$ converges to x_i as $j \to \infty$ and that $x(j)$ converges to x as $j \to \infty$. We put $S_j^i = \{u \in S_{x(j)} M_j \mid$ There exists $0 < s < t(u)$ such that $\exp_{x(j)} su \in B_\delta(x_i(j))$ holds.$\}$. Also we put $I_j^i(u) = \{s \in (0, t(u)) \mid \exp_{x(j)} su \in B_\delta(x_i(j))\}$ for $u \in S_j^i$, $\underline{k}(s) = \sinh(s)$ and put $\theta(s,u) = s^{n-1}\sqrt{\det(g_{ij}|_{\exp_{x(j)} su})}$. Here, $g_{ij} = g(\partial/\partial x_i, \partial/\partial x_j)$ for a normal coordinate around $x(j)$, $(x_1, x_2, \dots, x_n)$. Then, we have

$$\begin{aligned}
\operatorname{vol} B_\delta(x_i(j)) &= \int_{S_j^i} \int_{I_j^i(u)} \theta(s,u) ds du \\
&\le \int_{S_j^i} \int_{I_j^i(u)} \theta(t-2\delta, u) \frac{\underline{k}^{n-1}(s)}{\underline{k}^{n-1}(t-2\delta)} ds du \\
&\le 2 \int_{S_j^i} \int_{I_j^i(u)} \theta(t-2\delta, u) ds du \\
&\le 8\delta \int_{S_j^i} \theta(t-2\delta, u) du \\
&\le 8\delta \operatorname{vol}\big(\partial B_{t-2\delta}(x(j)) \cap C_{x(j)}(B_\delta(x_i(j)))\big).
\end{aligned}$$

Since the set $\{\partial B_{t-2\delta}(x(j)) \cap C_x(B_\delta(x_i(j)))\}_i$ are pairwise disjoint for every sufficiently large j, we have

$$\sum_{i=1}^N \delta^{-1} \underline{\operatorname{vol}} B_\delta(x_i(j)) \le 4 \underline{\operatorname{vol}} \big(\partial B_{t-2\delta}(x(j)) \setminus C_{x(j)}\big).$$

By Bishop-Gromov volume comparison theorem, we have

$$\begin{aligned}\underline{\mathrm{vol}}(\partial B_{t-2\delta}(x(j)) \setminus C_{x(j)}) &= \frac{\mathrm{vol}\, B_{t-2\delta}(x(j))}{\mathrm{vol}\, B_1(m_j)} \frac{\mathrm{vol}(\partial B_{t-2\delta}(m_j) \setminus C_{x(j)})}{\mathrm{vol}\, B_{t-2\delta}(x(j))} \\ &\le \frac{\mathrm{vol}\, B_{t-2\delta}(x(j))}{\mathrm{vol}\, B_1(m_j)} \frac{\mathrm{vol}\, \partial B_{t-2\delta}(\underline{p})}{\mathrm{vol}\, B_{t-2\delta}(\underline{p})}.\end{aligned}$$

Thus, we have

$$\sum_{i=1}^{N} \delta^{-1}\, \underline{\mathrm{vol}}\, B_\delta(x_i(j)) \le 8 \frac{\mathrm{vol}\, B_{t-2\delta}(x(j))}{\mathrm{vol}\, B_1(m_j)} \frac{\mathrm{vol}\, \partial B_{t-2\delta}(\underline{p})}{\mathrm{vol}\, B_{t-2\delta}(\underline{p})}.$$

By letting $j \to \infty$, we have

$$\begin{aligned}(\upsilon_{-1})_{1000\delta}(\partial B_t(x)) &\le \sum_{i=1}^{N} (1000\delta)^{-1} \upsilon(B_{1000\delta}(x_i)) \\ &\le \frac{C(n)}{8} \sum_{i=1}^{N} \delta^{-1} \upsilon(B_\delta(x_i)) \\ &\le C(n) \upsilon(B_t(x)) \frac{\mathrm{vol}\, \partial B_{t-2\delta}(\underline{p})}{\mathrm{vol}\, B_{t-2\delta}(\underline{p})}\end{aligned}$$

for some $C(n) > 0$. Therefore, by letting $\delta \to 0$, we have

$$\upsilon_{-1}(\partial B_t(x)) \le C(n) \upsilon(B_t(x)) \frac{\mathrm{vol}\, \partial B_t(\underline{p})}{\mathrm{vol}\, B_t(\underline{p})}. \qquad \square$$

We shall state the following proposition stated in (4.3) of [**6**]. Since there is no proof of the following proposition in [**6**], we give a proof of it.

PROPOSITION 4.3. *We assume that $\partial B_1(y) \neq \emptyset$. Then for every positive number $R > 0$ and every point $x \in B_R(y)$, we have*

$$\upsilon(B_s(x)) \le C(R, n) s$$

for every positive number $0 < s < 1$. Here, $C(R, n) > 0$ is a positive constant depending only on R and n.

PROOF. By an argument simular to the proof of [**13**, Proposition 5.2]. $\square$

As a corollary of Theorem 4.2 and Proposition 4.3, we have a universal upper bound for $\upsilon_{-1}(\partial B_t(x))$:

COROLLARY 4.4. *We assume that $\partial B_1(y) \neq \emptyset$. Then for every positive number $R > 0$ and every point $x \in B_R(y)$, we have*

$$\upsilon_{-1}(\partial B_s(x)) \le C(R, n).$$

for every positive number $0 < s < 1$.

4.2. Bishop-Gromov type inequality.

In this subsection, we shall give a proof of Theorem 1.1.

PROOF OF THEOREM 1.1. First, we assume that A is compact. There exists a sequence of complete, pointed, connected n-dimensional Riemannian manifolds $\{(M_j, m_j)\}_j$ with $\mathrm{Ric}_{M_j} \ge -(n-1)$ such that $(M_j, m_j, \mathrm{vol}/\mathrm{vol}\, B_1(m_j))$ converges to (Y, y, υ) in the sense of measured Gromov-Hausdorff topology. We fix a sufficiently small positive number $\delta > 0$ and put $C_x(A, s, \delta) = \{z \in \partial B_s(x) \mid$ There exists $p \in \overline{B}_\delta(A)$ such that $\overline{x,z} + \overline{z,p} - \overline{x,p} \le \delta$ holds.$\}$. Clearly, $C_x(A, s, \delta)$ is compact and $\bigcap_{\delta > 0} C_x(A, s, \delta) = \partial B_s(x) \cap C_x(A)$ holds. Let $\epsilon > 0$ be a positive number satisfying $\epsilon \ll \min\{s, t-s, \delta\}$. There exists a sequence of sets $\{B_{r_i}(x_i)\}_{i=1}^N$ such that $|\upsilon_{-1}(C_x(A, s, 7\delta)) - \sum_{i=1}^N r_i^{-1} \upsilon(B_{r_i}(x_i))| < \epsilon$, $C_x(A, s, 7\delta) \subset \bigcup_{i=1}^N B_{r_i}(x_i)$ and $0 < r_i < \min\{s, t-s, \delta\}/1000 (1 \le i \le N)$ hold, and that $C_x(A, s, 7\delta) \cap B_{r_i}(x_i) \neq \emptyset (1 \le i \le N)$ holds. We put $\tau = \min_{1 \le i \le N}\{r_i/1000\}$. Let $x(j)$, $x_i(j)$ be points in M_j satisfying $\overline{x_i(j), x_i} < \epsilon_j$, $\overline{x(j), x} < \epsilon_j$ in the sense of pointed Gromov-Hausdorff topology ($\epsilon_j \to 0$). For every positive integers i, j ($1 \le i \le N$), we put $S_i^j = \{u \in S_{x(j)} M_j \mid$ There exists $0 < r < t(u)$ such that $\exp_{x(j)} ru \in B_{4r_i}(x_i(j))$ holds.$\}$ and $\hat{S}_i^j = \{u \in S_i^j \mid t(u) > t - 100\delta\}$. Also we put $I_i^j(u) = \{r \in (0, t(u)) \mid \exp_{x(j)} ru \in B_{4r_i}(x_i(j))\}$ and $\hat{I}_i^j(u) = B_{r_i}(I_i^j(u))$ $(\subset (0, t(u)))$ for $u \in \hat{S}_i^j$. Then, we have

$$\begin{aligned}
\mathrm{vol}\, B_{10r_i}(x_i(j)) &\ge \int_{\hat{S}_i^j} \int_{\hat{I}_i^j} \theta(r, u) dr du \\
&\ge \int_{\hat{S}_i^j} \int_{\hat{I}_i^j} \underline{k}^{n-1}(r) \frac{\theta(t - 10^4\delta, u)}{\underline{k}^{n-1}(t - 10^4\delta)} dr du \\
&\ge \frac{\underline{k}^{n-1}(s - 10^4\delta)}{\underline{k}^{n-1}(t - 10^4\delta)} r_i \int_{\hat{S}_i^j} \theta(t - 10^4\delta, u) du \\
&= \frac{\underline{k}^{n-1}(s - 10^4\delta)}{\underline{k}^{n-1}(t - 10^4\delta)} r_i \,\mathrm{vol}\left(\partial B_{t - 10^4\delta}(x(j)) \cap S_{x(j)}(B_{4r_i}(x_i(j)))\right).
\end{aligned}$$

Here, $S_{x(j)}(\hat{A}) = \{\alpha \in M_j \setminus C_{x(j)} \mid$ There exists $\beta \in \hat{A}$ such that $\overline{x(j), \beta} + \overline{\beta, \alpha} = \overline{x(j), \alpha}$ holds or $\overline{x(j), \alpha} + \overline{\alpha, \beta} = \overline{x(j), \beta}$ holds.$\}$ for every subset $\hat{A} \subset Y$. Therefore, we have

$$\begin{aligned}
&\sum_{i=1}^{N} r_i^{-1} \underline{\mathrm{vol}}\, B_{10r_i}(x_i(j)) \\
&\qquad \geq \frac{\underline{k}^{n-1}(s - 10^4\delta)}{\underline{k}^{n-1}(t - 10^4\delta)} \sum_{i=1}^{N} \underline{\mathrm{vol}} \left(\partial B_{t-10^4\delta}(x_j) \cap S_{x(j)}(B_{4r_i}(x_i(j)))\right).
\end{aligned}$$

Let $\{\hat{x}_i\}_{i=1}^{\hat{N}}$ be a maximal $10^{10}\delta$-separated subset of A and $\hat{x}_i(j) \in M_j$ a point satisfying $\overline{\hat{x}_i(j), \hat{x}_i} < \epsilon_j$.

Claim 4.5. *For every sufficiently large $j > 0$, every point $z_j \in B_\delta(\hat{x}_i(j))$ and every minimal geodesic from $x(j)$ to z_j, $\gamma : [0, \overline{x(j), z_j}] \to M_j$, we have*

$$\mathrm{Image}(\gamma) \cap \left(\bigcup_{i=1}^{N} B_{4r_i}(x_i(j)) \right) \neq \emptyset.$$

We take $\alpha_j \in Y$ such that $\overline{\gamma(s+\tau), \alpha_j} < \epsilon_j$. Since $s < \overline{x, \alpha_j} < s + 2\tau$ for every sufficiently large j, we can take $w \in \partial B_s(x)$ such that $\overline{x, w} + \overline{w, \alpha_j} = \overline{x, \alpha_j}$. Then, we have $\overline{x, w} + \overline{w, \hat{x}_i} - \overline{x, \hat{x}_i} \leq \overline{x, \alpha_j} + \overline{\alpha_j, \hat{x}_i} - \overline{x, \hat{x}_i} + 4\tau \leq \overline{x(j), \gamma(s+\tau)} + \overline{\gamma(s+\tau), \hat{x}_i(j)} - \overline{x(j), \hat{x}_i(j)} + 5\tau \leq 2\delta + 5\tau \leq 7\delta$. Therefore, we have $w \in C_x(A, s, 7\delta)$. Thus, there exists a ball $B_{r_l}(x_l)$ such that $w \in B_{r_l}(x_l)$. Therefore, we have $\alpha_j \in B_{2r_l}(x_l)$. Thus, we have $\gamma(s+\tau) \in B_{4r_l}(x_l(j))$. We have Claim 4.5.

For every ball $B_\delta(\hat{x}_i(j))$, we put $\acute{S}_i^j = \{u \in S_{x(j)}M_j \mid$ There exists $0 < r < t(u)$ such that $\exp_{x(j)} ru \in B_\delta(\hat{x}_i(j))$ holds.$\}$ and $\acute{I}_i^j(u) = \{r \in (0, t(u)) \mid \exp_{x(j)} ru \in B_\delta(\hat{x}_i(j))\}$ for $u \in \acute{S}_i^j$. Then, we have

$$\begin{aligned}
\mathrm{vol}\, B_\delta(\hat{x}_i(j)) &= \int_{\acute{S}_i^j} \int_{\acute{I}_i^j} \theta(r, u)\, dr du \\
&\leq \int_{\acute{S}_i^j} \int_{\acute{I}_i^j} \underline{k}^{n-1}(r) \frac{\theta(t - 10^4\delta, u)}{\underline{k}^{n-1}(t - 10^4\delta)}\, dr du \\
&\leq 2 \frac{\underline{k}^{n-1}(t + 10^4\delta)}{\underline{k}^{n-1}(t - 10^4\delta)} \delta \int_{\acute{S}_i^j} \theta(t - 10^4\delta, u)\, du
\end{aligned}$$

$$= 2\frac{\underline{k}^{n-1}(t+10^4\delta)}{\underline{k}^{n-1}(t-10^4\delta)}\delta \operatorname{vol}\big(\partial B_{t-10^4\delta}(x(j)) \cap C_{x(j)}(B_\delta(\hat{x}_i(j)))\big).$$

Therefore, we have

$$\begin{aligned}
&\sum_{i=1}^{\hat{N}} \delta^{-1}\, \underline{\operatorname{vol}}\, B_\delta(\hat{x}_i(j)) \\
&\quad \le 2\sum_{i=1}^{\hat{N}} \frac{\underline{k}^{n-1}(t+10^4\delta)}{\underline{k}^{n-1}(t-10^4\delta)}\, \underline{\operatorname{vol}}\big(\partial B_{t-10^4\delta}(x(j)) \cap C_{x(j)}(B_\delta(\hat{x}_i(j)))\big) \\
&\quad = 2\frac{\underline{k}^{n-1}(t+10^4\delta)}{\underline{k}^{n-1}(t-10^4\delta)}\, \underline{\operatorname{vol}}\left(\partial B_{t-10^4\delta}(x(j)) \cap C_{x(j)}\left(\bigsqcup_{i=1}^{\hat{N}} B_\delta(\hat{x}_i(j))\right)\right).
\end{aligned}$$

By Claim 4.5, we have

$$\partial B_{t-10^4\delta}(m_j) \cap C_{x(j)}\left(\bigsqcup_{i=1}^{\hat{N}} B_\delta(\hat{x}_i(j))\right) \subset \partial B_{t-10^4\delta}(x(j)) \cap S_{x(j)}\left(\bigcup_{i=1}^{N} B_{4r_i}(x_i(j))\right).$$

Thus, we have

$$\sum_{i=1}^{\hat{N}} \delta^{-1}\, \underline{\operatorname{vol}}\, B_\delta(\hat{x}_i(j)) \le 3\frac{\underline{k}^{n-1}(t-10^4\delta)}{\underline{k}^{n-1}(s-10^4\delta)} \sum_{i=1}^{N} r_i^{-1}\, \underline{\operatorname{vol}}\, B_{10r_i}(x_i(j)).$$

Therefore, by letting $j \to \infty$, we have

$$\begin{aligned}
(\upsilon_{-1})_{10^{11}\delta}(A) &\le \sum_{i=1}^{\hat{N}} (10^{11}\delta)^{-1} \upsilon(B_{10^{11}\delta}(\hat{x}_i)) \\
&\le C(n) \sum_{i=1}^{\hat{N}} \delta^{-1} \upsilon(B_\delta(\hat{x}_i)) \\
&\le C(n) \frac{\underline{k}^{n-1}(t-10^4\delta)}{\underline{k}^{n-1}(s-10^4\delta)} \sum_{i=1}^{N} r_i^{-1} \upsilon(B_{10r_i}(x_i)) \\
&\le C(n) \frac{\underline{k}^{n-1}(t-10^4\delta)}{\underline{k}^{n-1}(s-10^4\delta)} \big(\upsilon_{-1}(C_x(A,s,7\delta)) + \epsilon\big).
\end{aligned}$$

By letting $\epsilon \to 0$ and $\delta \to 0$, we have Theorem 1.1 for every compact set A. By standard covering argument, it is easy to prove Theorem 1.1 for every Borel set A. $\square$

See [**13**, Theorem 1.2] for an application of Theorem 1.1 to low dimensional Ricci limit spaces.

4.3. Positivity result of the measure υ_{-1}.

First, we shall prove the following theorem:

THEOREM 4.6. *There exists a positive constant $C(n) > 0$ depending only on n such that for every positive numbers $0 < s \le r < t$, every point $x \in Y$ and every Borel set $A \subset A_{r,t}(x)$,*

$$\frac{\upsilon(A)}{\operatorname{vol} B_t(\underline{p}) - \operatorname{vol} B_r(\underline{p})} \le C(n) \frac{\upsilon_{-1}(\partial B_s(x) \cap C_x(A))}{\operatorname{vol} \partial B_s(\underline{p})}$$

holds. Especially, if $\upsilon(A) > 0$ holds, then $\upsilon_{-1}(\partial B_s(x) \cap C_x(A)) > 0$ holds.

PROOF. Without loss of generality, we can assume that $s < r$ holds and A is compact. We fix a sufficiently small positive number δ. Let $\epsilon > 0$ be a positive number. There exists $\{B_{r_i}(x_i)\}_{i=1}^N$ such that $|\upsilon_{-1}(C_x(A, s, 7\delta)) - \sum_{i=1}^N r_i^{-1} \upsilon(B_{r_i}(x_i))| < \epsilon$, $C_x(A, s, 7\delta) \subset \bigcup_{i=1}^N B_{r_i}(x_i)$, $0 < r_i < \min\{r, t-s, \delta\}/1000$ hold for every i, and that $C_x(A, s, 7\delta) \cap B_{r_i}(x_i) \neq \emptyset$ holds. By taking a maximal 100δ-separated subset of A and by an argument simular to the proof of Theorem 1.1, we have

$$\upsilon(A) \le C(n) \frac{\operatorname{vol} B_{t+100\delta}(\underline{p}) - \operatorname{vol} B_{r-100\delta}(\underline{p})}{\operatorname{vol} \partial B_{s-100\delta}(\underline{p})} (\upsilon_{-1}(C_x(A, s, 7\delta)) + \epsilon).$$

Therefore, by letting $\epsilon \to 0$, $\delta \to 0$, we have Theorem 4.6. $\square$

Next corollary is a positivity result similar to the properties on Riemannian manifolds.

COROLLARY 4.7. *Let x be a point in Y and $R > 0$ a positive number. Assume that $\partial B_R(x) \setminus C_x \neq \emptyset$. Then for every $z \in \partial B_R(x) \setminus C_x$ and every positive number $\epsilon > 0$, $\upsilon_{-1}(B_\epsilon(z) \cap \partial B_R(x) \setminus C_x) > 0$ holds.*

PROOF. There exist a sufficiently small positive number $0 < \tau < \epsilon/1000$ and a point $w \in Y$ such that $\overline{x,z} + \overline{z,w} = \overline{x,w}$ and $\overline{z,w} = \tau$ hold. Then, since $\partial B_R(x) \cap C_x(B_{\tau/1000}(w)) \subset B_\epsilon(z) \cap \partial B_R(x) \setminus C_x$, Corollary 4.7 follows from Theorem 4.6. $\square$

Finally, we shall give the following theorem.

Corollary 4.8. *For every points $x, z \in Y$ such that $x \neq z$, the following conditions are equivalent*:

(1) $\upsilon(C_x(\{z\})) > 0$ *holds.*
(2) $\upsilon_{-1}(\partial B_t(x) \cap C_x(\{z\})) > 0$ *holds for every* $0 < t < \overline{x, z}$.
(3) $\upsilon_{-1}(\partial B_t(x) \cap C_x(\{z\})) > 0$ *holds for some* $0 < t < \overline{x, z}$.

Proof. First, we assume that $\upsilon(C_x(\{z\})) > 0$ holds. We put $r = \overline{x, z} > 0$. There exists a positive integer $N \in \boldsymbol{N}$ such that $\upsilon(C_x(\{z\}) \cap A_{(N+1)^{-1}r,\ N^{-1}r}(x)) > 0$ holds. Thus, by Theorem 4.6, we have $\upsilon_{-1}(\partial B_t(x) \cap C_x(\{z\})) > 0$ for every $0 < t < (N+1)^{-1}r$. Since $\partial B_s(x) \cap C_x(\{z\}) = \partial B_{r-s}(z) \cap C_z(\{x\})$ holds for every $0 < s < r$, by Theorem 1.1, we have $\upsilon_{-1}(\partial B_t(x) \cap C_x(\{z\})) > 0$ for every $0 < t < r$.

Next, we assume that $\upsilon(C_x(\{z\})) = 0$. Then, by Corollary 5.7 below, there exists $t \in (0, \overline{x, z})$ such that $\upsilon_{-1}(\partial B_t(x) \cap C_x(\{z\})) = 0$ holds. □

See [**13**, Theorem 6.5] for an application of Corollary 4.8.

5. Co-area formula for distance function.

In this section, we give a relationship between the limit measure υ and the measure υ_{-1}. Let x be a point in Y and $A \subset Y$ a subset. We define $\Phi_A : \boldsymbol{R}_{\geq 0} \to \boldsymbol{R}_{\geq 0}$ by

$$\Phi_A(t) = \upsilon_{-1}(\partial B_t(x) \cap A).$$

Proposition 5.1. *For every Borel set $A \subset Y$, the map Φ_A is a Lebesgue measurable function.*

We will give a proof of Proposition 5.1 in Appendix. The following theorem is the main result in this subsection.

Theorem 5.2. *Let x be a point in Y. There exists a non-negative valued function $f \in L_\infty(Y)$ and a constant $C(n) \geq 1$ depending only on n, such that $C(n)^{-1} \leq f(w) \leq C(n)$ holds for every $w \in Y$ and*

$$\int_0^\infty \int_{\partial B_t(x) \setminus C_x} g d\upsilon_{-1} dt = \int_Y g f d\upsilon$$

holds for every $g \in L_1(Y)$.

Proof. Recall that there exists a sequence of complete, pointed, connected n-dimensional Riemannian manifolds $\{(M_j, m_j)\}_j$ with $\mathrm{Ric}_{M_j} \geq -(n-1)$ such that $(M_j, m_j, \mathrm{vol}/\mathrm{vol}\, B_1(m_j))$ converges to (Y, y, υ) in the sense of measured Gromov-Hausdorff topology. For every positive number $\tau > 0$, we put $\mathscr{D}_\tau = Y \setminus C_x(\tau) = \{w \in Y \mid \text{There exists } z \in Y \setminus B_\tau(w) \text{ such that } \overline{x,w} + \overline{w,z} = \overline{x,z} \text{ holds.}\}$. Clearly, $\bigcup_{\tau>0} \mathscr{D}_\tau = Y \setminus C_x$. We fix $\tau > 0$. Let $s, t, r, R, \delta > 0$ be positive numbers satisfying $0 < \delta \ll \tau, s$ and $\delta \ll r < t < R$. We assume that $A_{r,R}(x) \neq \emptyset$. We take a point $w \in A_{r,R}(x)$. Let $\{x_i\}_{i=1}^N$ be a maximal 100δ-separated subset of $\partial B_t(x) \cap \overline{B}_s(w)$. We take a positive number $\hat{t} > 0$ such that $|t - \hat{t}| \leq \delta$ and $\hat{t} \in [r, R]$ hold.

Claim 5.3. *We have* $\partial B_{\hat{t}}(x) \cap \mathscr{D}_\tau \cap \overline{B}_{s-100\delta}(w) \subset \bigcup_{i=1}^N B_{300\delta}(x_i)$.

Let z be a point in $\partial B_{\hat{t}}(x) \cap \mathscr{D}_\tau \cap \overline{B}_{s-100\delta}(w)$. First, we assume $\hat{t} \geq t$. Then there exists $\alpha \in \partial B_t(x) \cap \overline{B}_s(w)$ such that $\overline{x,\alpha} + \overline{\alpha,z} = \overline{x,z}$ and $\overline{\alpha,z} \leq \delta$ hold. Thus, there exists a positive integer $1 \leq i \leq N$ such that $\alpha \in B_{250\delta}(x_i)$ holds. Therefore, we have $z \in B_{300\delta}(x_i)$.

Next, we assume that $\hat{t} < t$. Since $\delta \ll \tau$, there exists $\alpha \in \partial B_t(x) \cap \overline{B}_s(w)$ such that $\overline{x,z} + \overline{z,\alpha} = \overline{x,\alpha}$ and $\overline{\alpha,z} \leq \delta$ hold. Thus there exists a positive integer $1 \leq i \leq N$ such that $\alpha \in B_{200\delta}(x_i)$ holds. Hence, we have $z \in B_{300\delta}(x_i)$. Therefore, we have Claim 5.3.

For every positive integers $i, j > 0$ $(1 \leq i \leq N)$, let $x_i(j), x(j) \in M_j$ be points satisfying $\overline{x_i(j), x_i} < \epsilon_j$ and $\overline{x(j), x} < \epsilon_j$ $(\epsilon_j \to 0)$. We put $S^i_j = \{u \in S_{x(j)}M_j \mid \text{There exists } 0 < r < t(u) \text{ such that } \exp_{x(j)} ru \in B_\delta(x_i(j)) \text{ holds.}\}$ and $I_i(u) = \{r \in (0, t(u)) \mid \exp_{x(j)} ru \in B_\delta(x_i(j))\}$ for $u \in S^i_j$. Then, we have

$$\begin{aligned}
\mathrm{vol}\, B_\delta(x_i(j)) &= \int_{S^i_j} \int_{I_i(u)} \theta(\hat{s}, u) d\hat{s} du \\
&\leq \int_{S^i_j} \int_{I_i(u)} \underline{k}^{n-1}(\hat{s}) \frac{\theta(\hat{t} - 10\delta, u)}{\underline{k}^{n-1}(\hat{t} - 10\delta)} d\hat{s} du \\
&\leq 2 \int_{S^i_j} \int_{I_i(u)} \theta(\hat{t} - 10\delta, u) d\hat{s} du \\
&\leq 5\delta \int_{S^i_j} \theta(\hat{t} - 10\delta, u) du \\
&\leq 5\delta\, \mathrm{vol}\left(\partial B_{\hat{t}-10\delta}(x(j)) \cap C_{x(j)}(B_\delta(x_i(j))) \cap B_{20\delta}(x_i(j)) \setminus C_{x(j)}\right).
\end{aligned}$$

Claim 5.4. *For every* $i_1, i_2 \in \{1, 2, \cdots, N\}$ *such that* $i_1 \neq i_2$, *for every sufficiently large integer* j, *we have* $C_{x(j)}(B_{2\delta}(x_{i_1}(j))) \cap C_{x(j)}(B_{2\delta}(x_{i_2}(j))) \cap$

$B_{20\delta}(x_{i_2}(j)) = \emptyset$.

Assume that the assertion is false. We take $z_j \in C_{x(j)}(B_{2\delta}(x_{i_1}(j))) \cap C_{x(j)}(B_{2\delta}(x_{i_2}(j))) \cap B_{20\delta}(x_{i_2}(j))$. There exist $y_{i_1}(j) \in B_{2\delta}(x_{i_1}(j))$, $y_{i_2}(j) \in B_{2\delta}(x_{i_2}(j))$ such that $\overline{x(j), z_j} + \overline{z_j, y_{i_1}(j)} = \overline{x(j), y_{i_1}(j)}$ and $\overline{x(j), z_j} + \overline{z_j, y_{i_2}(j)} = \overline{x(j), y_{i_2}(j)}$ hold. Then, by triangle inequality, we have

$$\begin{aligned}
\overline{x_{i_1}(j), x_{i_2}(j)} &\le \overline{x_{i_1}(j), y_{i_1}(j)} + \overline{y_{i_1}(j), z_j} + \overline{z_j, y_{i_2}(j)} + \overline{y_{i_2}(j), x_{i_2}(j)} \\
&\le 2\delta + \overline{y_{i_1}(j), z_j} + \overline{z_j, y_{i_2}(j)} + 2\delta \\
&\le 4\delta + t + 5\delta - \overline{x(j), z_j} + \overline{z_j, y_{i_2}(j)} \\
&\le 9\delta + t - (\overline{x(j), y_{i_2}(j)} - \overline{z_j, y_{i_2}(j)}) + \overline{z_j, y_{i_2}(j)} \\
&\le 9\delta + t - \overline{x(j), y_{i_2}(j)} + 50\delta \\
&\le 9\delta + 5\delta + 50\delta = 64\delta.
\end{aligned}$$

Thus, we have $\overline{x_{i_1}, x_{i_2}} < 70\delta$. This is a contradiction. Therefore, we have Claim 5.4.

Let $w(j) \in M_j$ be a point satisfying $\overline{w(j), w} < \epsilon_j$. By Claim 5.4 and $B_{20\delta}(x_i(j)) \subset B_{s+100\delta}(w(j))$, we have

$$\sum_{i=1}^{N} \underline{\mathrm{vol}}\, B_\delta(x_i(j)) \le 10\delta\, \underline{\mathrm{vol}}\left(\partial B_{\hat{t}-10\delta}(x(j)) \cap B_{s+100\delta}(w(j)) \setminus C_{x(j)}\right).$$

On the other hand, for every sufficiently large j, we have

$$\left| \sum_{i=1}^{N} \upsilon(B_\delta(x_i)) - \sum_{i=1}^{N} \underline{\mathrm{vol}}\, B_\delta(x_i(j)) \right| < \delta^2.$$

Therefore, for every sufficiently large j, we have

$$\begin{aligned}
&(\upsilon_{-1})_{1000\delta}\left(\partial B_{\hat{t}}(x) \cap \overline{B}_{s-100\delta}(w) \cap \mathscr{D}_\tau\right) \\
&\quad \le \sum_{i=1}^{N} (1000\delta)^{-1} \upsilon(B_{1000\delta}(x_i)) \\
&\quad \le C(n) \sum_{i=1}^{N} \delta^{-1} \upsilon(B_\delta(x_i))
\end{aligned}$$

$$
\begin{aligned}
&\leq C(n)\biggl(\delta + \sum_{i=1}^{N} \delta^{-1} \underline{\mathrm{vol}}\, B_\delta(x_i(j))\biggr) \\
&\leq C(n)\delta + C(n)\, \underline{\mathrm{vol}}\, \bigl(\partial B_{\hat{t}-10\delta}(x(j)) \cap B_{s+100\delta}(w(j)) \setminus C_{x(j)}\bigr). \qquad (*)
\end{aligned}
$$

Let $\{t_i\}_{i=1}^k \subset [r, R]$ be a subset satisfying $[r, R] \subset \bigcup_{i=1}^k B_{\delta/2}(t_i)$. For every $i = 1, 2, \ldots, k$, we have that inequality $(*)$ holds for every sufficiently large integer j and every $\hat{t} \in [r, R]$ satisfying $|\hat{t} - t_i| < \delta$. Hence, inequality $(*)$ holds for every sufficiently large j and every $\hat{t} \in [r, R]$. Therefore, for such j, we have

$$
\begin{aligned}
&\int_r^R (\upsilon_{-1})_{1000\delta}\bigl(\partial B_{\hat{t}}(x) \cap \overline{B}_{s-\tau}(w) \cap \mathscr{D}_\tau\bigr) d\hat{t} \\
&\quad \leq \int_r^R (\upsilon_{-1})_{1000\delta}\bigl(\partial B_{\hat{t}}(x) \cap \overline{B}_{s-100\delta}(w) \cap \mathscr{D}_\tau\bigr) d\hat{t} \\
&\quad \leq C(n)(R-r)\delta + C(n) \int_r^R \underline{\mathrm{vol}}\, \bigl(\partial B_{\hat{t}-10\delta}(x(j)) \cap B_{s+100\delta}(w(j)) \setminus C_{x(j)}\bigr) d\hat{t} \\
&\quad \leq C(n)(R-r)\delta + C(n) \int_{\tau-10\delta}^{R-10\delta} \underline{\mathrm{vol}}\, \bigl(\partial B_\alpha(x(j)) \cap B_{s+100\delta}(w(j)) \setminus C_{x(j)}\bigr) d\alpha \\
&\quad \leq C(n)(R-r)\delta + C(n)\, \underline{\mathrm{vol}}\, B_{s+100\delta}(w(j)).
\end{aligned}
$$

By letting $j \to \infty$, $\delta \to 0$, $R \to \infty$, $r \to 0$ and letting $\tau \to 0$, we have

$$
\int_0^\infty \upsilon_{-1}\bigl(\partial B_{\hat{t}}(x) \cap \overline{B}_s(w) \setminus C_x\bigr) d\hat{t} \leq C(n)\upsilon(\overline{B}_s(w)).
$$

Here, we remark that the map $\hat{\Psi} : \mathscr{B}(Y) \to \boldsymbol{R}_{\geq 0} \cup \{\infty\}$ defined by

$$
\hat{\Psi}(A) = \int_0^\infty \upsilon_{-1}(\partial B_t(x) \cap A \setminus C_x) dt
$$

is an additive set function on $\mathscr{B}(Y) = \{A \in 2^Y \mid A \text{ is a Borel subset of } Y\}$. By standard covering argument, for every $A \in \mathscr{B}(Y)$, we have

$$
\int_0^\infty \upsilon_{-1}(\partial B_t(x) \cap A \setminus C_x) dt \leq C(n)\upsilon(A).
$$

By Radon-Nikodym theorem, there exists $f \in L_\infty(Y)$ such that $0 \leq f \leq C(n)$ holds and that

$$\int_0^\infty \int_{\partial B_t(x) \setminus C_x} g d\upsilon_{-1} dt = \int_Y g f d\upsilon$$

holds for every $g \in L_1(Y)$.

Claim 5.5. *Let w be a point in Y and $0 < \tau < 1 < R$ positive numbers. Assume $w \in B_R(x) \setminus (B_\tau(x) \cup C_x(\tau))$. Then, for every $0 < \epsilon < \tau/1000$, we have*

$$\upsilon(B_\epsilon(w)) \le C(n, R) \int_0^\infty \upsilon_{-1}(\partial B_t(x) \cap B_\epsilon(w) \setminus C_x) dt.$$

Here, $C(n, R)$ is a positive constant depending only on n, R.

Because, for every $0 < \epsilon < \tau/100$, we take a minimal geodesic $\gamma : [0, \overline{x,w} + 10\epsilon] \to Y$ such that $\gamma(0) = x$ and that $\gamma(\overline{x,w}) = w$. Then, by Theorem 4.6, for every $t \in [\overline{x,w} - \epsilon/100, \overline{x,w}]$, we have

$$\begin{aligned}\upsilon(B_{10\epsilon}(w)) &\le C(n)\upsilon\big(B_\epsilon(\gamma(\overline{x,w} + 5\epsilon))\big) \\ &\le C(n) \operatorname{vol} A_{\overline{x,w}, \overline{x,w}+20\epsilon}(\underline{p}) \frac{\upsilon_{-1}\big(\partial B_t(x) \cap C_x(B_\epsilon(\gamma(\overline{x,w} + 5\epsilon))\big)}{\operatorname{vol} \partial B_t(\underline{p})} \\ &\le C(n, R)\epsilon\upsilon_{-1}(\partial B_t(x) \cap B_{10\epsilon}(w) \setminus C_x).\end{aligned}$$

By integrating this inequality on $[\overline{x,w} - \epsilon/100, \overline{x,w}]$, we have Claim 5.5.

Claim 5.6. *For every Borel subset A of Y, we have*

$$\upsilon(A) \le C(n) \int_0^\infty \upsilon_{-1}(\partial B_t(x) \cap A \setminus C_x) dt.$$

For every $\tau > 0$ and every $0 < R_1 < R_2 < \infty$, we put $A(\tau, R_1, R_2, x) = A \cap A_{R_1,R_2}(x) \setminus C_x(\tau)$. We fix $\tau > 0$, $0 < R_1 < 1 < R_2$ such that $y \in B_{R_2}(x)$. By standard covering argument, for every $\epsilon > 0$, there exists a sequence $\{\overline{B}_{r_i}(x_i)\}_{i \in \boldsymbol{N}}$ of balls such that $x_i \in \mathrm{Leb}(A(\tau, R_1, R_2, x))$ and that $0 < r_i < \min\{\tau, \epsilon, R_1, R_2 - R_1\}/100$ and that $\{\overline{B}_{r_i}(x_i)\}_{i \in \boldsymbol{N}}$ is pairwise disjoint and that $\upsilon(B_{r_i}(x_i) \cap A(\tau, R_1, R_2, x))/\upsilon(B_{r_i}(x_i)) \ge 1 - \epsilon$ holds and that $\mathrm{Leb}(A(\tau, R_1, R_2, x)) \setminus \bigcup_{i=1}^N \overline{B}_{r_i}(x_i) \subset \bigcup_{i=N+1}^\infty \overline{B}_{5r_i}(x_i)$ holds for every N. Here, $\mathrm{Leb}(A) = \{x \in Y \mid \lim_{r \to 0} \upsilon(A \cap B_r(x))/\upsilon(B_r(x)) = 1\}$ for each Borel subset A of Y. We take N such that $\sum_{i=N+1}^\infty \upsilon(B_{r_i}(x_i)) < \epsilon$ holds. Then, we have

$$
\begin{aligned}
&\Big|\sum_{i=1}^{N}\int_0^\infty \upsilon_{-1}(\partial B_t(x)\cap \overline{B}_{r_i}(x_i)\setminus C_x)dt \\
&\quad -\sum_{i=1}^{N}\int_0^\infty \upsilon_{-1}(\partial B_t(x)\cap \overline{B}_{r_i}(x_i)\cap A(\tau,R_1,R_2,x))dt\Big| \\
&\quad = \int_0^\infty \upsilon_{-1}\left(\partial B_t(x)\cap\left(\bigcup_{i=1}^{N}\overline{B}_{r_i}(x_i)\setminus (A(\tau,R_1,R_2,x)\cup C_x)\right)\right)dt \\
&\quad \le C(n)\upsilon\left(\bigcup_{i=1}^{N}\overline{B}_{r_i}(x_i)\setminus A(\tau,R_1,R_2,x)\right) \\
&\quad \le C(n)\epsilon\sum_{i=1}^{N}\upsilon(B_{r_i}(x_i)) \le C(n)\epsilon\upsilon(B_{R_2}(x)).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\int_0^\infty \upsilon_{-1}(\partial B_t(x)\cap A(\tau,R_1,R_2,x))dt \\
&\quad \ge \int_0^\infty \upsilon_{-1}\left(\partial B_t(x)\cap\left(\bigcup_{i=1}^{N}\overline{B}_{r_i}(x_i)\cap A(\tau,R_1,R_2,x)\right)\right)dt \\
&\quad = \sum_{i=1}^{N}\int_0^\infty \upsilon_{-1}(\partial B_t(x)\cap \overline{B}_{r_i}(x_i)\cap A(\tau,R_1,R_2,x))dt \\
&\quad \ge \sum_{i=1}^{N}\int_0^\infty \upsilon_{-1}(\partial B_t(x)\cap \overline{B}_{r_i}(x_i)\setminus C_x)dt - \epsilon C(n,R_2) \\
&\quad \ge C(n)\sum_{i=1}^{N}\upsilon(B_{r_i}(x_i)) - \epsilon C(n,R_2) \\
&\quad \ge C(n)\upsilon\left(\bigcup_{i=1}^{N}\overline{B}_{r_i}(x_i)\cup\bigcup_{i=N+1}^{\infty}\overline{B}_{5r_i}(x_i)\right) - \epsilon C(n,R_2) \\
&\quad \ge C(n)\upsilon(\mathrm{Leb}(A(\tau,R_1,R_2,x))) - \epsilon C(n,R_2) \\
&\quad = C(n)\upsilon(A(\tau,R_1,R_2,x)) - \epsilon C(n,R_2).
\end{aligned}
$$

By letting $\epsilon\to 0$, $R_1\to 0$, $R_2\to\infty$, $\tau\to 0$, we obtain Claim 5.6. The statement $f\ge C(n)>0$ follows from Claim 5.6. $\square$

See [**14**] for several applications of Theorem 5.2 to a rectifiability and Laplacian comparison theorems on Ricci limit spaces. For example, in [**14**], we will give the following: Let H be a real number, (Y, y) a (n, H)-Ricci limit space ($n \geq 2$), υ a limit measure on Y, x a point in Y, R a positive number and f a non-negative valued Lipschitz function on $B_R(x)$. Then we have

$$\int_{B_R(x)} \langle df, dr_x \rangle d\upsilon \geq -(n-1) \int_{B_R(x)} \frac{\underline{k}'_H(\overline{x,w})}{\underline{k}_H(\overline{x,w})} f(w) d\upsilon.$$

Here, the function $\underline{k}_H$ on $\boldsymbol{R}_{\geq 0}$ defined by $\underline{k}''_H(r) + H\underline{k}_H(r) = 0$, $\underline{k}_H(0) = 0$ and $\underline{k}'_H(0) = 1$. If Y is an n-dimensional C^∞-Riemannian manifold, then this implies Laplacian comparison theorem on Riemannian manifolds: $\Delta r_x(w) \geq -(n-1)\underline{k}'_H(\overline{x,w})/\underline{k}_H(\overline{x,w})$ on $Y \setminus (C_x \cup \{x\})$. Thus we have $\mathrm{Ric}_Y \geq H(n-1)$. The formulation above was given in [**15**, Theorem 3.1]. Roughly speaking, this statement implies that *Laplacian comparison theorems are closed in the sense of measured Gromov-Hausdorff topology, or lower bounds of Ricci curvature are stable in the sense of measured Gromov-Hausdorff topology.* See [**15**], [**16**], [**17**], [**18**] and [**19**] for related results. We will use Theorem 5.2 in the proof of the statement above.

We give the next inequality which sharpens the conclusion in Theorem 4.2.

Corollary 5.7. *For every positive numbers $0 < r_1 < r_2 \leq R$, every point $x \in Y$ and every Borel set $A \subset \partial B_R(x)$,*

$$\frac{\upsilon_{-1}(A)}{\mathrm{vol}\, \partial B_R(\underline{p})} \leq C(n) \frac{\upsilon(A_{r_1,r_2}(x) \cap C_x(A))}{\mathrm{vol}\, B_{r_2}(\underline{p}) - \mathrm{vol}\, B_{r_1}(\underline{p})}$$

holds.

Proof. It follows from Theorems 1.1 and 5.2, immediately. □

6. Ahlfors α-regular set and the Hausdorff dimension.

We consider a set that the limit measure υ on the set and a Hausdorff measure are mutually absolutely continuous.

Definition 6.1. For non-negative numbers $\alpha \geq 0$, $C > 1$, we put

$$A_Y(\alpha, C) = \left\{x \in Y \mid C^{-1}s^\alpha \leq \upsilon(B_s(x)) \leq Cs^\alpha \text{ for every } 0 < s < 1\right\},$$

$$A_Y(\alpha) = \bigcup_{C>1} A_Y(\alpha, C).$$

We call the set $A_Y(\alpha)$ *Ahlfors α-regular set.*

Note that $A_Y(\alpha, C)$ is closed. The limit measure υ and α-dimensional Hausdorff measure are mutually absolutely continuous on $A_Y(\alpha)$. Next, we shall define the notion of tangent cone.

DEFINITION 6.2. Let (W, w), (Z, z) be pointed proper geodesic spaces. We say that (W, w) is a *tangent cone at $\alpha \in Z$* if there exists a sequence of positive numbers $r_i > 0$ such that r_i converges to 0 and that rescaled pointed proper geodesic spaces $(Z, r_i^{-1} d_Z, \alpha)$ converges to (W, w) in the sense of pointed Gromov-Hausdorff topology. Here, d_Z is the distance on Z.

We shall give an upper bound of Hausdorff dimension of Ahlfors α-regular set.

THEOREM 6.3. *We have* $\dim_{\mathscr{H}} A_Y(\alpha) \leq [\alpha]$ *for every positive number* $\alpha > 0$. *Here* $[\alpha] = \sup\{k \in \boldsymbol{Z} \mid k \leq \alpha\}$.

PROOF. This proof is done by a contradiction. We assume that $\dim_{\mathscr{H}} A_Y(\alpha) > [\alpha]$ holds. Then, there exist a sufficiently small positive number $0 < \beta < 1$ and a positive number $C > 1$ such that $\mathscr{H}^{\alpha+\beta}(A_Y(\alpha, C)) > 0$ holds. By standard covering argument, there exist $x \in Y$, a tangent cone $(T_x Y, 0_x)$ at x, and a sequence of positive numbers $r_i > 0$ such that r_i converges to 0, $\lim_{i \to 0} \mathscr{H}_\infty^{\alpha+\beta}(\overline{B}_{r_i}(x))/{r_i}^{\alpha+\beta} > 0$ holds and that $(Y, r_i^{-1} d_Y, x)$ converges to $(T_x Y, 0_x)$ (for example, see (1.39) and (10.7) in [**5**] for the definition of the $(\alpha + \beta)$-dimensional spherical Hausdorff content, $\mathscr{H}_\infty^{\alpha+\beta}$). Without loss of generality, we can assume that there exist a compact metric space Z, a limit measure υ_∞ on $(T_x Y, 0_x)$, a positive number $\hat{C} > 1$ and an isometric embedding $\phi : Z \to A_{T_x Y}(\alpha, \hat{C}) \cap \overline{B}_1(0_x)$ for υ_∞ such that $H^{\alpha+\beta}(Z) > 0$ holds and that $(\overline{B}_{r_i}(x) \cap A_Y(\alpha, C), r_i^{-1} d_Y)$ converges to Z in the sense of Gromov-Hausdorff topology. Especially, $\mathscr{H}^{\alpha+\beta}(\overline{B}_1(0_x) \cap A_{T_x Y}(\alpha, \hat{C})) > 0$ holds. By [**2**, Proposition 2.5], we have $\mathscr{H}^{\alpha+\beta}(\overline{B}_1(0_x) \cap A_{T_x Y}(\alpha, \hat{C}) \setminus \mathscr{W}\mathscr{D}_0(0_x)) > 0$ (see [**4**, Definition 2.10] for the definition of $\mathscr{W}\mathscr{D}_0(x)$). We put $(Y_1, y_1) = (T_x Y, 0_x)$. Then, there exist a point $z \in A_{Y_1}(\alpha, \hat{C}) \setminus \mathscr{W}\mathscr{D}_0(y_1)$, a sequence of positive numbers s_i and a pointed proper geodesic space (W, w) such that s_i converges to 0, $\lim_{i \to 0} \mathscr{H}_\infty^{\alpha+\beta}(\overline{B}_{s_i}(z))/s_i^{\alpha+\beta} > 0$ and $(Y_1, s_i^{-1} d_{Y_1}, z)$ converges to $(\boldsymbol{R} \times W, (0, w))$.

By iterating this argument, there exist an iterated tangent cone (T, t) of Y, a limit measure $\tilde{\upsilon}_\infty$ on (T, t), a positive constant $\tilde{C} > 1$ and a proper geodesic space X such that $\mathscr{H}^{\alpha+\beta}(\overline{B}_1(t) \cap A_T(\alpha, \tilde{C})) > 0$ holds for $\tilde{\upsilon}_\infty$ and T is isometric to $\boldsymbol{R}^{[\alpha]+1} \times X$. Therefore, there exists a point $w \in T$ such that $\liminf_{r \to 0} \tilde{\upsilon}_\infty(B_r(w))/r^\alpha > 0$ holds. This contradicts [**4**, Proposition 1.35]. $\square$

Next corollary follows from [**6**, Theorem 5.5], immediately. We shall give a new proof.

Corollary 6.4. *We assume that $\upsilon(A_Y(\alpha)) > 0$ holds. Then α is an integer.*

Proof. By the assumption, we have $\mathscr{H}^\alpha(A(\alpha)) > 0$. Hence, $\dim_{\mathscr{H}} A(\alpha) \geq \alpha$. Therefore, by Theorem 6.3, we have $\alpha = [\alpha]$. $\square$

7. Appendix: A proof of Proposition 5.1.

In this section, we shall give a proof of Proposition 5.1. We fix $0 < r < R$. For every $t \in \boldsymbol{Q}_{>0}$, let $\{x_i^t\}_{i \in \boldsymbol{N}}$ be a countable dense subset of $\partial B_t(x)$. For every positive integer $N \in \boldsymbol{N}$ and every positive number $\delta > 0$, we put $\mathscr{B} = \{\overline{B}_s(x_i^t) \mid i \in \boldsymbol{N},\ s, t \in \boldsymbol{Q}_{>0}\}$, $\mathscr{B}_\delta^N = \{(\overline{B}_{r_i}(x_i))_{i=1,2,\cdots,N} \in \mathscr{B}^N \mid r_i < \delta\}$ and put $\mathscr{B}_\delta = \bigcup_{N \in \boldsymbol{N}} \mathscr{B}_\delta^N$. Clearly, these are countable sets.

Lemma 7.1. *Let $A \subset Y$ be a compact set. Then the function $t \mapsto (\upsilon_{-1})_\delta(\partial B_t(x) \cap A)$ is a Borel function for every positive number $\delta > 0$. Especially, the map $\Phi_A\,|_{[r,R]}$ is a Borel function.*

Proof. For every $F = (\overline{B}_{r_i}(x_i))_{i=1,2,\cdots,N} \in \mathscr{B}_\delta$, we define a map Ψ_F from $[r, R]$ to $\boldsymbol{R}_{\geq 0} \cup \{\infty\}$ by $\Psi_F(t) = \sum_{i=1}^N r_i^{-1} \upsilon(B_{r_i}(x_i))$ if $\partial B_t(x) \cap A \subset \bigcup_{i=1}^N \overline{B}_{r_i}(x_i)$ holds, $\Psi_F(t) = \infty$ otherwise. Since $\partial B_t(x) \cap A$ is a compact set, Ψ_F is a Borel function. Therefore, $\Psi = \inf_{F \in \mathscr{B}_\delta} \Psi_F$ is a Borel function. By the definition of $(\upsilon_{-1})_\delta$, we have $\Psi(t) = (\upsilon_{-1})_\delta(\partial B_t(x) \cap A)$. $\square$

Therefore, we have the following corollary:

Corollary 7.2. *Let $O \subset Y$ be an open set. Then the map $\Phi_O|_{[r,R]}$ is a Borel function.*

Here we put $\hat{\mathscr{B}} = \{A \in \mathscr{B}(Y) \mid$ For every positive number $\epsilon > 0$, there exist a sequence of compact sets $K_i \subset A$, a sequence of open sets $A \subset O_i$ and exists a Lebesgue measurable set $E_\epsilon \subset [r,\ R]$ such that $\mathscr{H}^1([r, R] \setminus E_\epsilon) < \epsilon$ holds and that $\sup_{t \in E_\epsilon} \upsilon_{-1}(\partial B_t(x) \cap A \setminus K_i)$ and $\sup_{t \in E_\epsilon} \upsilon_{-1}(\partial B_t(x) \cap O_i \setminus A)$ converge to 0 as $i \to \infty\}$. Note that for every $A_i \in \hat{\mathscr{B}}$, $\Phi_A|_{[r,R]}$ is a Lebesgue measurable function for every set $A = \bigcup_{i \in \boldsymbol{N}} A_i$.

Lemma 7.3. *$\hat{\mathscr{B}}$ is a σ-algebra.*

Proof. It suffices to show $\bigcup_{i \in \boldsymbol{N}} A_i \in \hat{\mathscr{B}}$ for every sets $A_i \in \hat{\mathscr{B}}$. We take a sequence $A_i \in \hat{\mathscr{B}}$. Let $\epsilon > 0$ be a positive number. For every $i \in \boldsymbol{N}$, there exist a sequence of compact sets $K_i(j) \subset A_i$, a sequence of open sets $A_i \subset O_i(j)$, and exists a Lebesgue measurable set $E_\epsilon(i) \subset [r, R]$ such that $\mathscr{H}^1([r, R] \setminus E_\epsilon(i)) < 2^{-i}\epsilon$ holds and that $\sup_{t \in E_\epsilon(i)} \upsilon_{-1}(\partial B_t(x) \cap O_i(j) \setminus A_i)$ and $\sup_{t \in E_i(l)} \upsilon_{-1}(\partial B_t(x) \cap A_i \setminus K_i(j))$ converge to 0 as $j \to \infty$. Thus, for every $l \in \boldsymbol{N}$, there exists a sufficiently large integer $N(l) \in \boldsymbol{N}$ such that for every $1 \le i \le l$, $\sup_{t \in E_\epsilon(i)} \upsilon_{-1}(\partial B_t(x) \cap A_i \setminus K_i(N(l))) \le l^{-1}2^{-i}$ holds. Since $\upsilon_{-1}(\partial B_t(x) \cap (\bigcup_{i=1}^l A_i))$ converges to $\upsilon_{-1}(\partial B_t(x) \cap (\bigcup_{i \in \boldsymbol{N}} A_i))$ as $l \to \infty$ for every $t \in [r, R]$, by Egoroff's theorem, there exists a Lebesgue measurable set $E_\epsilon \subset [r, R]$ such that $\mathscr{H}^1([r, R] \setminus E_\epsilon) < \epsilon$ holds and that $\sup_{t \in E_\epsilon} \upsilon_{-1}(\partial B_t(x) \cap (\bigcup_{i \in \boldsymbol{N}} A_i \setminus \bigcup_{i=1}^l A_i))$ converges to 0 as $l \to \infty$. We put $\hat{E}_\epsilon = \bigcap_{i \in \boldsymbol{N}} E_\epsilon(i) \cap E_\epsilon$. Then, we have, $\mathscr{H}^1([r, R] \setminus \hat{E}_\epsilon) \le \sum_{i \in \boldsymbol{N}} \mathscr{H}^1([r, R] \setminus E_\epsilon(i)) + \mathscr{H}^1([r, R] \setminus E_\epsilon) < 2\epsilon$. We also put a compact set $\hat{K}_l = \bigcup_{i=1}^l K_i(N(l))$. Then, $\sup_{t \in \hat{E}_\epsilon} \upsilon_{-1}(\partial B_t(x) \cap (\bigcup_{i \in \boldsymbol{N}} A_i \setminus \hat{K}_l))$ converges to 0 as $l \to \infty$. For each $l, i \in \boldsymbol{N}$, there exists a sufficiently large $j(l, i) \in \boldsymbol{N}$ such that $\sup_{t \in E_\epsilon(i)} \upsilon_{-1}(\partial B_t(x) \cap (O_i(j(l, i)) \setminus A_i)) < l^{-1}2^{-i}$ holds. We put an open set $O_l = \bigcup_{i \in \boldsymbol{N}} O_i(j(l, i))$. Then $\sup_{t \in \hat{E}_\epsilon} \upsilon_{-1}(\partial B_t(x) \cap (O_l \setminus \bigcup_{i \in \boldsymbol{N}} A_i))$ converges to 0 as $l \to \infty$. Therefore $\bigcup_{i \in \boldsymbol{N}} A_i \in \hat{\mathscr{B}}$ holds. $\square$

LEMMA 7.4. *$\hat{\mathscr{B}} = \mathscr{B}(Y)$ holds.*

PROOF. For every open set $O \subset Y$, there exists a sequence of compact sets $K_i \subset O$ such that $\bigcup_{i \in \boldsymbol{N}} K_i = O$. By Egoroff's theorem, for every positive number $\epsilon > 0$, there exists a Lebesgue measurable set $E_\epsilon \subset [r, R]$ such that $\sup_{t \in E_\epsilon} \upsilon_{-1}(\partial B_t(x) \cap O \setminus K_i))$ converges to 0 as $i \to \infty$. Thus, $O \in \hat{\mathscr{B}}$. Therefore we have Lemma 7.4. $\square$

Proposition 5.1 follows from these lemmas above, immediately.

References

[1] J. Cheeger, Degeneration of Rimannian metrics under Ricci curvature bounds, Lezioni Fermiane, Scuola Normale Superiore, Pisa, 2001.

[2] J. Cheeger, Differentiability of Lipschitz functions on metric measure spaces, Geom. Funct. Anal., **9** (1999), 428–517.

[3] J. Cheeger and T. H. Colding, Lower bounds on Ricci curvature and the almost rigidity of warped products, Ann. of Math., **144** (1996), 189–237.

[4] J. Cheeger and T. H. Colding, On the structure of spaces with Ricci curvature bounded below, I, J. Differential Geom., **45** (1997), 406–480.

[5] J. Cheeger and T. H. Colding, On the structure of spaces with Ricci curvature bounded below, II, J. Differential Geom., **54** (2000), 13–35.

[6] J. Cheeger and T. H. Colding, On the structure of spaces with Ricci curvature bounded below, III, J. Differential Geom., **54** (2000), 37–74.

[7] T. H. Colding, Ricci curvature and volume convergence, Ann. of Math. (2), **145** (1997), 477–501.

[8] H. Federer, Geometric measure theory, Springer, New York, 1969.

[9] K. Fukaya, Hausdorff convergence of Riemannian manifolds and its applications, Recent topics in differential and analytic geometry, Adv. Stud. Pure Math., **18-I**, Academic Press, Boston, MA, 1990, pp. 143–238.

[10] K. Fukaya, Metric Riemannian geometry, Handbook of differential geometry, **II**, Elsevier/North-Holland, Amsterdam, 2006, 189–313.

[11] K. Fukaya, Y. Otsu, T. Yamaguchi, T. Shioya, T. Sakai and A. Kasue, Riemannian manifolds and its limit, (in Japanese), MSJ Mem., **3**, Math. Soc. Japan, Tokyo, 2004.

[12] M. Gromov, Metric Structures for Riemannian and Non-Riemannian Spaces, Birkhäuser Boston Inc, Boston, MA, 1999, Based on the 1981 French original [MR 85e:53051], With appendices by M. Katz, P. Pansu, and S. Semmes, Translated from the French by Sean Michael Bates.

[13] S. Honda, On Low Dimensional Ricci limit spaces, preprint. http://www.math.kyoto-u.ac.jp/preprint/preprint2008.html

[14] S. Honda, Ricci curvature and convergence of Lipschitz functions, to appear in Comm. Anal. Geom.

[15] K. Kuwae and T. Shioya, A topological splitting theorem for weighted Alexandrov spaces, preprint. http://arxiv.org/abs/0903.5150

[16] J. Lott, Optimal transport and Ricci curvature for metric-measure spaces, in Surveys in Differential Geometry, **XI**, Metric and Comparison Geometry, (eds. J. Cheeger and K. Grove), International Press, Somerville, MA, 2007, p. 229–257.

[17] S. Ohta, On the measure contraction property of metric measure spaces, Comment. Math. Helv., **82** (2007), 805–828.

[18] C. Villani, Topics in optimal transportation, American Mathematical Society, Providence, RI, 2003.

[19] C. Villani, Optimal transport, old and new, Springer-Verlag, 2008.

Shouhei Honda
Research Institute for Mathematical Sciences
Kyoto University
Kyoto 606-8502, Japan
E-mail: honda@kurims.kyoto-u.ac.jp