

Compact quotients with positive algebraic dimensions of large domains in a complex projective 3-space

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Abstract. A domain in a complex 3-dimensional projective space is said to be large, if the domain contains a line, i.e., a projective linear subspace of dimension one. We study compact complex 3-manifolds defined as non-singular quotients of large domains. Any holomorphic automorphism of a large domain becomes an element of the projective linear transformations. In the first half, we study the limit sets of properly discontinuous groups acting on large domains. In the second half, we determine all compact complex 3-manifolds with positive algebraic dimensions which are quotients of large domains.

1. Introduction.

The theory of discrete subgroups of $PGL(2, \mathbf{C})$ has a long history. Let Γ be a discrete subgroup of $PGL(2, \mathbf{C})$. We say that the action of Γ at a point $z \in \mathbf{P}^1$ is *discontinuous*, if there is a neighborhood W of z such that $\gamma(W) \cap W = \emptyset$ for all but finitely many $\gamma \in \Gamma$. Following B. Maskit [Ms], we call a subgroup $\Gamma \subset PGL(2, \mathbf{C})$ whose action is discontinuous at some point $z \in \mathbf{P}^1$ by *Kleinian group*.

Let $\Gamma \subset PGL(2, \mathbf{C})$ be a Kleinian group. The set $\Omega(\Gamma)$ of points $z \in \mathbf{P}^1$ at which Γ acts discontinuously is called the *set of discontinuity* of Γ . The set $\Omega(\Gamma)$ is a Γ -invariant open subset in \mathbf{P}^1 on which Γ acts properly discontinuously. The geometry of the quotient space $\Omega(\Gamma)/\Gamma$ is one of the main theme in the classical Kleinian group theory. The celebrated finiteness theorem of L. Ahlfors says that, for a finitely generated Kleinian group Γ , $\Omega(\Gamma)/\Gamma$ is a finite union of compact Riemann surfaces which are punctured at finitely many points.

If we seek for a higher dimensional version of the Kleinian group theory, we must first define the set of discontinuity for a given discrete subgroup. Fix $n \geq 2$. Take a discrete subgroup $\Gamma \subset PGL(n+1, \mathbf{C})$ acting of \mathbf{P}^n . Consider, as above,

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the set $\Omega(\Gamma)$ of points $z \in \mathbf{P}^n$ at which Γ acts discontinuously. Then it is true that Γ acts on $\Omega(\Gamma)$, but the action is not properly discontinuous in general. Therefore, we must find another definition of the set of discontinuity to get a good quotient space.

In this paper, we restrict ourselves to the case $n = 3$. Some part of the following arguments may apply for every odd n . We say that a discrete subgroup Γ of $PGL(4, \mathbf{C})$ is of type **L**, if Γ has the property

(**L**) *there is an open subdomain $W \subset \mathbf{P}^3$ biholomorphic to*

$$\{[z_0 : z_1 : z_2 : z_3] \in \mathbf{P}^3 : |z_0|^2 + |z_1|^2 < |z_2|^2 + |z_3|^2\}$$

satisfying $\gamma(W) \cap W = \emptyset$ for any $\gamma \in \Gamma \setminus \{1\}$.

In the first half of this paper, we shall study some properties of groups of type **L**. For such discrete subgroups, lines in \mathbf{P}^3 play the same role as points in Kleinian group theory. Let Γ be a discrete subgroup of type **L**. Using lines, in stead of points, we define the set $\Omega(\Gamma)$ of discontinuity of Γ (Definition 5). The action of Γ on $\Omega(\Gamma)$ is properly discontinuous (Theorem 2.5), and $\Omega(\Gamma)/\Gamma$ becomes a good space. There are many discrete subgroups of type **L**. Indeed, most of the flat twistor spaces over conformally flat real 4-dimensional manifolds are the quotient spaces of subdomains in \mathbf{P}^3 by the actions of type **L** groups. Further, given two groups Γ_1 and Γ_2 of type **L**, we can get another type **L** group $\Gamma \simeq \Gamma_1 * \Gamma_2$ by an analogous operations of Klein combinations ([**K1**]).

A domain Ω in \mathbf{P}^3 is said to be *large*, if it contains a projective line. Let Ω be a large domain. Then any holomorphic automorphism of Ω appears to be an element of $PGL(4, \mathbf{C})$ ([**K1**]). A properly discontinuous group Γ of holomorphic automorphisms of Ω is of type **L** (Proposition 1). Further, if the action of Γ on Ω is cocompact, then Ω is a connected component of $\Omega(\Gamma)$ (Theorem 3.1). This fact seems to justify our definition of $\Omega(\Gamma)$.

In the latter half of this paper, we shall study the compact quotient Ω/Γ with positive algebraic dimension. By [**K5**, Theorem A], we see that Ω is dense in \mathbf{P}^3 in this case. Thus by Theorem 3.1, we have $\Omega = \Omega(\Gamma)$. To give the statement of our result, we recall the definitions of Blanchard manifolds and L-Hopf manifolds.

DEFINITION 1. A 3-dimensional compact complex manifold is called a *Blanchard manifold*, if its universal covering space is biholomorphic to the complement of a single line in \mathbf{P}^3 .

DEFINITION 2. A 3-dimensional compact complex manifold is called an *L-Hopf manifold*, if its universal covering space is biholomorphic to the complement of two disjoint lines in \mathbf{P}^3 .

For the structure of such manifolds, see [K3], [K2], and [KK]. Now we have the following.

THEOREM 1.1. *Let $X = \Omega/\Gamma$ be a compact complex manifold which is a quotient of a large domain $\Omega \subset \mathbf{P}^3$ by a fixed point free and properly discontinuous group Γ of holomorphic automorphisms of Ω . If X admits a non-constant meromorphic function, then X is biholomorphic to either \mathbf{P}^3 , a Blanchard manifold, or an L-Hopf manifold.*

In [K5], a proof of this theorem was given under an additional condition on the complement $\mathbf{P}^3 \setminus \Omega$. We use freely results of Sections 1, 2 and 4 in [K5], but not those of Sections 3 and 5.

In view of Theorem 1.1, if $\Omega(\Gamma)/\Gamma$ contains a compact component of positive algebraic dimension, Γ may be classified into a class analogous to the elementary groups in the Kleinian group theory.

We conclude Introduction by a remark that we cannot hope for an analogue of Ahlfors' finiteness theorem for $\Omega(\Gamma)/\Gamma$. A counter example can be given by the twistor construction from the finitely generated subgroup of conformal transformations of S^4 defined by M. E. Kapovich and L. D. Potyagailo [KP].

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2. Discontinuous groups in the projective 3-space.

By a *line*, we shall mean a complex projective linear subspace of dimension 1 in \mathbf{P}^3 . The lines in \mathbf{P}^3 are parametrized by the Grassmannian manifold $\text{Gr}(4, 2)$. A line ℓ in \mathbf{P}^3 corresponds to a point in $\text{Gr}(4, 2)$ denoted by $\hat{\ell}$. By using Plücker coordinates, $\text{Gr}(4, 2)$ can be identified with a quadric in \mathbf{P}^5 , and each element of $PGL(4, \mathbf{C})$ induces an automorphism of \mathbf{P}^5 which leaves the quadric invariant. Therefore we have the group homomorphism

$$PGL(4, \mathbf{C}) \xrightarrow{\rho} PGL(6, \mathbf{C}),$$

with

$$\rho(PGL(4, \mathbf{C})) \subset \text{Aut}(\text{Gr}(4, 2)).$$

In the following, for a subset $S \subset \mathbf{P}^3$, we denote by $\hat{S} \subset \text{Gr}(4, 2)$ the set of points

corresponding to lines contained in S .

Let U be the domain in \mathbf{P}^3 defined by

$$U = \{[z_0 : z_1 : z_2 : z_3] \in \mathbf{P}^3 : |z_0|^2 + |z_1|^2 < |z_2|^2 + |z_3|^2\}.$$

The notation U is used to indicate this domain throughout this paper. Note that the complement $\mathbf{P}^3 \setminus [U]$ is also biholomorphic to U . The lines in U constitute a bounded Stein domain $\hat{U} \subset \text{Gr}(4, 2)$, which is biholomorphic to

$$\{X \in M(2, \mathbf{C}) : X^* X \ll I\}.$$

For any line ℓ in \mathbf{P}^3 , the family of open sets containing ℓ and biholomorphic to U forms a fundamental system of neighborhoods of ℓ in \mathbf{P}^3 .

We recall here several results of Myrberg [My]. Let Γ be a subgroup of $PGL(n+1, \mathbf{C})$ and $\{\sigma_m\}$ be an infinite sequence of elements of Γ . Let $\tilde{\sigma}_m \in GL(n+1, \mathbf{C})$ be a representative of σ_m such that $\|\tilde{\sigma}_m\| = 1$, where for a matrix $A = (a_{jk})$ of size $n+1$, we put $\|A\| = \max_{0 \leq j, k \leq n} |a_{jk}|$. We say that $\{\sigma_m\}$ is a *normal sequence* if the following conditions are satisfied.

1. The sequence $\{\sigma_m\}$ consists of distinct elements of Γ .
2. The sequence of matrices $\{\tilde{\sigma}_m\}$ can be chosen to be convergent to a matrix $\tilde{\sigma}$.

The projective linear subspace defined by the image of the linear map $\tilde{\sigma} : \mathbf{C}^{n+1} \rightarrow \mathbf{C}^{n+1}$ is called the *limit image* of the normal sequence $\{\sigma_m\}$ and denoted by $I(\{\sigma_m\})$. Similarly the projective linear subspace defined by the kernel of $\tilde{\sigma}$ is called the *limit kernel* of $\{\sigma_m\}$ and denoted by $K(\{\sigma_m\})$. Here $r = \text{rank } \tilde{\sigma}$ is called the rank of the normal sequence. Note that $I(\{\sigma_m\})$, $K(\{\sigma_m\})$, and r are determined independently of the choice of representatives $\tilde{\sigma}_m$. Obviously, we have $\dim I(\{\sigma_m\}) = r - 1$ and $\dim K(\{\sigma_m\}) = n - r$.

THEOREM 2.1 ([My, Satz 8']). *Let $V = \{F(z) = 0\}$ be a non-singular hypersurface in \mathbf{P}^n . We assume that V is not cylindrical, i.e., the set of vectors $\{\text{grad}_x F : x \in V\}$ spans \mathbf{C}^{n+1} . Let $\{\sigma_m\}$ be a normal series in $PGL(n+1, \mathbf{C})$ such that every σ_m leaves V invariant. Choose a representative $\tilde{\sigma}_m \in GL(n+1, \mathbf{C})$ of σ_m with $\|\tilde{\sigma}_m\| = 1$. Suppose that the series of linear transformations $\{\tilde{\sigma}_m\}$ converges to $\tilde{\sigma} \in M(n+1, \mathbf{C})$ with $\text{rank } \tilde{\sigma} = 1$. Then the limit kernel $K(\{\sigma_m\})$ is tangential to V at some point.*

PROOF. Put $\tilde{\sigma} = (\tilde{\sigma}_k^\ell)$. Since $\text{rank } \tilde{\sigma} = 1$, there are non-zero constant vectors $c = (c^\ell)$ and $\tilde{\sigma} = (\tilde{\sigma}_k)$ in \mathbf{C}^{n+1} such that $\tilde{\sigma}_k^\ell = c^\ell \tilde{\sigma}_k$. Since V is not cylindrical, there is a point α of V , such that

$$\sum_{\ell=0}^n \frac{\partial F}{\partial z^\ell}(\alpha) c^\ell \neq 0 \tag{1}$$

holds. The tangent hyperplane $T_{\sigma_m^{-1}(\alpha)} = \sigma_m^{-1}(T_\alpha)$ is given by

$$\sum_{k=0}^n \left(\sum_{\ell=0}^n \frac{\partial F}{\partial z^\ell}(\alpha) \right) \tilde{\sigma}_{mk}^\ell z^k = 0.$$

The coefficients

$$\sum_{\ell=0}^n \frac{\partial F}{\partial z^\ell}(\alpha) \tilde{\sigma}_{mk}^\ell, \quad k = 0, \dots, n$$

of the tangent hyperplane tends to

$$\left(\sum_{\ell=0}^n \frac{\partial F}{\partial z^\ell}(\alpha) c^\ell \right) \tilde{\sigma}_k, \quad k = 0, \dots, n$$

as $m \rightarrow +\infty$, where all the limits are not zero by (1). Note that $K(\{\sigma_m\})$ is given by $\sum_{k=0}^n \tilde{\sigma}_k z^k = 0$. Therefore, the limit of the sequence of hyperplanes $\{T_{\sigma_m^{-1}(\alpha)}\}$ coincides with $K(\{\sigma_m\})$. At the same time, we see that $K(\{\sigma_m\})$ is tangential to V at some accumulation point of the sequence $\{\sigma_m^{-1}(\alpha)\}$. \square

THEOREM 2.2 ([My, Satz 8]). *Let $V = \{F(z) = 0\}$ be a non-singular hypersurface in \mathbf{P}^n . We assume that V is not cylindrical. Let $\{\sigma_m\}$ be a normal series in $PGL(n + 1, \mathbf{C})$ such that every σ_m leaves V invariant. Then the limit image $I(\{\sigma_m\})$ is contained in V .*

PROOF. Put $I = I(\{\sigma_m\})$ and $K = K(\{\sigma_m\})$ for short. Suppose that I is not contained in V and we shall derive a contradiction. Let σ the projection defined by the limit $\tilde{\sigma}$. Put $i = \dim I$, then $n - i - 1 = \dim K$. Take any point $w \in I$. Then the fibre of the projection σ through $w \in I$ is an $(n - i)$ -dimensional projective linear subspace L_w containing K . Suppose that $\sigma(V \setminus K)$ is contained in a proper subvariety, say V_0 , of I . Then we have $n - 1 = \dim V \leq \dim L_w + \dim V_0 \leq (n - i) + (i - 1) = n - 1$. Hence we see that $\dim V_0 = i - 1$ and V is contained in $\sigma^{-1}(V_0)$. This contradicts the assumption that V is not cylindrical. Hence L_w intersects V outside K . Since every σ_m leaves V invariant, we see that w is a limit point of some point in $V \setminus K$ and we have $I \subset V$. \square

From now on through out this paper, we assume that Γ is of type **L**, if not stated otherwise explicitly. A normal sequence $\{\sigma_m\}$ of Γ defines a normal sequence $\{\hat{\sigma}_m\}$, $\hat{\sigma}_m = \rho(\sigma_m)$, of $PGL(6, \mathbf{C})$. Thus $\{\sigma_m\}$ defines $I(\{\hat{\sigma}_m\})$ and $K(\{\hat{\sigma}_m\})$ in \mathbf{P}^5 .

DEFINITION 3. A line ℓ in \mathbf{P}^3 is called a *limit line* of Γ , if there is a normal sequence $\{\sigma_m\}$ of Γ with $\hat{\ell} \in I(\{\hat{\sigma}_m\})$.

Later, we shall see that $I(\{\hat{\sigma}_m\})$ consists of a single point for type **L** groups (see, Corollary 2). Let $\mathcal{L} = \mathcal{L}(\Gamma)$ denote the set of limit lines of Γ .

DEFINITION 4. The union

$$\Lambda = \Lambda(\Gamma) = \bigcup_{\ell \in \mathcal{L}(\Gamma)} |\ell|$$

of the support of limit lines of Γ is called the *limit set* of Γ .

Here we often indicate by $|\ell|$ the support of a line ℓ in \mathbf{P}^3 in order to express explicitly the set of points on the line.

DEFINITION 5. The set

$$\Omega(\Gamma) = \mathbf{P}^3 \setminus \Lambda(\Gamma)$$

is called the set of discontinuity of the group Γ .

By the property **L**, $\Omega(\Gamma)$ is non-empty and contains many lines.

THEOREM 2.3. Let $\{\sigma_m\}$ be a sequence of distinct elements of Γ . Then there are limit lines ℓ_I, ℓ_K , and a subsequence $\{\tau_m\}$ of $\{\sigma_m\}$, such that $\{\tau_m\}$ is uniformly convergent to ℓ_I on $\mathbf{P}^3 \setminus \ell_K$ in the following sense that, for any compact subset $M \subset \mathbf{P}^3 \setminus \ell_K$, and for any neighborhood V of ℓ_I , there is an integer m_0 such that $\tau_m(M) \subset V$ for any $m > m_0$.

PROOF. Put $s_m = \hat{\sigma}_m$ for simplicity. By replacing $\{s_m\}$ with its subsequence, we can assume that the sequence of representatives $\{\tilde{s}_m\} \subset GL(6, \mathbf{C})$ converges to a linear map $\tilde{s} : \mathbf{C}^6 \rightarrow \mathbf{C}^6$ with $\text{rank } \tilde{s} = k$. Let L_I, L_K be the projective linear subspaces in \mathbf{P}^5 defined by the limit image and the limit kernel of $\{s_m\}$, respectively. Here we have $\dim L_I + \dim L_K = 4$. Since $\text{Gr}(4, 2)$ is not cylindrical, we see that L_I is contained in $\text{Gr}(4, 2)$ by Theorem 2.2. If $\dim L_I \geq 1$, then $B = \bigcup_{\ell \in L_I} |\ell|$ is an algebraic surface contained in $\Lambda(\Gamma)$. This is absurd, since $\Omega(\Gamma)$ contains a line which does not intersect B . Consequently, we have that L_I

is a single point. Let ℓ_I be the line in \mathbf{P}^3 corresponding to the point L_I . By Theorem 2.1, L_K is a hyperplane tangent to $\text{Gr}(4, 2)$ at some point ∞ . Hence $\text{Gr}(4, 2) \cap L_K$ parametrizes lines in \mathbf{P}^3 intersecting the line ℓ_K which corresponds to $\infty \in \text{Gr}(4, 2)$. Let M be any compact subset of $\mathbf{P}^3 \setminus \ell_K$. Take a neighborhood W of ℓ_K which is biholomorphic to U and has no common points with M . Put $M_1 = (\mathbf{P}^3 \setminus W) \cap M$. Note that M_1 is filled with lines and the set $\hat{M}_1 \subset \text{Gr}(4, 2)$ which parametrizes lines in M_1 is a compact subset in $\text{Gr}(4, 2) \setminus L_K$. By another theorem of Myrberg [My, Satz 1], we see that $\{\sigma_m\}$ is uniformly convergent on any compact subset in $\mathbf{P}^5 \setminus L_K$.

It is clear that ℓ_I is a limit line. We shall show that ℓ_K also is a limit line. Let $[z_0 : \dots : z_5]$ be a system of homogeneous coordinates on \mathbf{P}^5 such that $\text{Gr}(4, 2)$ is given by $\langle z, z \rangle = 0$, where

$$\langle z, z \rangle = \sum_{i=0}^5 z_i^2.$$

Then, for each $\tilde{s}_m \in GL(6, \mathbf{C})$, there is $\lambda_m \in \mathbf{C}^*$ such that

$${}^t\tilde{s}_m \cdot \tilde{s}_m = \lambda_m I.$$

Hence we see that s_m^{-1} is represented by the transposition ${}^t\tilde{s}_m$ of \tilde{s}_m and therefore the sequence $\{s_m^{-1}\}$ is also a normal sequence. Obviously we see that $\{{}^t\tilde{s}_m\}$ converges to the linear map ${}^t\tilde{s} : \mathbf{C}^6 \rightarrow \mathbf{C}^6$, and that \tilde{s} and ${}^t\tilde{s}$ are matrices with rank 1 which satisfy ${}^t\tilde{s} \cdot \tilde{s} = \tilde{s} \cdot {}^t\tilde{s} = 0$. Let L'_I, L'_K be the projective linear subspaces in \mathbf{P}^5 defined by the limit image and the limit kernel of $\{{}^t\tilde{s}_m\}$, respectively. Namely, L'_I is the projective linear subspace corresponding to $\text{Im } {}^t\tilde{s}$, and L'_K is the one corresponding to $\text{Ker } {}^t\tilde{s}$. Since $\text{Ker } {}^t\tilde{s} = (\text{Im } \tilde{s})^\perp$, L'_K coincides with the tangent hyperplane to $\text{Gr}(4, 2)$ at the point $L_I = \hat{\ell}_I$. Since $\text{Im } {}^t\tilde{s} = (\text{Ker } \tilde{s})^\perp$, L'_I coincides with the point $\hat{\ell}_K$. Thus, in particular, ℓ_K is a limit line of the sequence $\{\sigma_m^{-1}\}$, and the theorem is proved. \square

In the course of the proof above, we have shown the following corollaries.

COROLLARY 1. *Let ℓ_0 be a limit line of Γ . Then there are a limit line ℓ_∞ , and a normal sequence $\{\sigma_m\}$ of distinct elements of Γ such that $\{\sigma_m\}$ is uniformly convergent to ℓ_0 on any compact set in $\mathbf{P}^3 \setminus \ell_\infty$ and that $\{\sigma_m^{-1}\}$ is uniformly convergent to ℓ_∞ on any compact set in $\mathbf{P}^3 \setminus \ell_0$.*

COROLLARY 2. *Let Γ be a properly discontinuous group acting on a large domain. Then, for any normal sequence $\{\sigma_m\}$ in Γ , the limit image $I(\{\hat{\sigma}_m\})$ in $\text{Gr}(4, 2)$ consists of a single point.*

THEOREM 2.4. $\Lambda(\Gamma)$ is a closed, nowhere dense Γ -invariant subset in \mathbf{P}^3 .

PROOF. Let x be any point in $\Lambda(\Gamma)$. Since x is on a limit line, say ℓ_0 , there is a normal sequence $\{\sigma_m\}$ of Γ with $I(\{\hat{\sigma}_m\}) = \hat{\ell}_0$ by Corollary 1. Then $\{\sigma \circ \sigma_m\}$ is a normal sequence with $I(\{\hat{\sigma} \circ \hat{\sigma}_m\}) = \hat{\sigma}(\hat{\ell}_0)$. Since the limit line $\sigma(\ell_0)$ passes through the point $\sigma(x)$, $\Lambda(\Gamma)$ is Γ -invariant.

To show that $\Lambda(\Gamma)$ is closed, let $\{x_m\}$ be a sequence of points of $\Lambda(\Gamma)$ such that $\lim_m x_m = x$ for some point $x \in \mathbf{P}^3$. Let ℓ_m be a limit line through x_m . By Corollary 1, for each m , we can find a limit kernel line $\ell_{m,\infty}$ and a normal sequence $\{\sigma_{m,k}\}_k$ such that $I(\{\hat{\sigma}_{m,k}\}_k) = \hat{\ell}_m$ and such that the sequence $\{\sigma_{m,k}\}_k$ is uniformly convergent to ℓ_m on compact sets in $\mathbf{P}^3 \setminus \ell_{m,\infty}$. Taking a subsequence of $\{\sigma_m\}$, we can assume that the ℓ_m are all distinct and that $\{\hat{\ell}_m\}$ and $\{\ell_{m,\infty}\}_m$ are convergent in $\text{Gr}(4, 2)$. Since $\{\hat{\ell}_{m,\infty}\}_m$ is convergent, we can choose a line ℓ such that $|\ell| \cap \bigcup_m |\ell_{m,\infty}| = \emptyset$ and that $\lim_k \hat{\sigma}_{m,k}(\hat{\ell}) = \hat{\ell}_m$ by Corollary 1. Fix any metric on $\text{Gr}(4, 2)$ and consider distance of points on $\text{Gr}(4, 2)$. Let δ_m be the minimal distance from $\hat{\ell}_m$ to any other $\hat{\ell}_j$ in $\text{Gr}(4, 2)$. Choose $k(m)$ such that $\text{dist}(\hat{\sigma}_{m,k(m)}(\hat{\ell}), \hat{\ell}_m) < \delta_m/2$ and that the $\sigma_{m,k(m)}$ are all distinct. Then $\{\sigma_{m,k(m)}\}_m$ is a sequence of distinct elements of Γ , and $\{\sigma_{m,k(m)}(\ell)\}_m$ is convergent to a limit line passing through x . Thus $\Lambda(\Gamma)$ is closed.

Lastly, we shall show that $\Lambda(\Gamma)$ is nowhere dense. Let x be any point in $\Lambda(\Gamma)$. By Corollary 1, there are lines ℓ_0, ℓ_∞ in \mathbf{P}^3 and a normal sequence $\{\sigma_m\}$ such that $x \in \ell_0$ and that $\lim_m \hat{\sigma}_m(\hat{K}) = \hat{\ell}_0$ for any compact set $K \subset \mathbf{P}^3 \setminus \ell_\infty$. By the property **L**, we can set K as a single line ℓ contained in $\Omega(\Gamma)$. Then, for every neighborhood W of x , there is an integer m_0 such that $W \cap \sigma_m(\ell) \neq \emptyset$ for $m \geq m_0$. Hence W contains a point in $\Omega(\Gamma)$. Thus $\Lambda(\Gamma)$ is nowhere dense. \square

THEOREM 2.5. The action of Γ on $\Omega(\Gamma)$ is properly discontinuous.

PROOF. Take any compact set M in $\Omega(\Gamma)$. Suppose that there is an infinite sequence $\{\sigma_m\}_m$ of distinct elements of Γ such that $M \cap \sigma_m(M) \neq \emptyset$ for any m . By Corollary 1, replacing $\{\sigma_m\}$ with its normal subsequence, we can assume that there are limit lines ℓ_K and ℓ_I such that $\{\sigma_m\}$ converges uniformly on $\mathbf{P}^3 \setminus \ell_K$ to ℓ_I . Since $\Omega(\Gamma)$ has no intersection with limit lines, we see that $M \cap (\ell_I \cup \ell_K) = \emptyset$. Therefore $\{\sigma_m(M)\}$ converges to a subset on ℓ_I . This contradicts the assumption that $M \cap \sigma_m(M) \neq \emptyset$ for any m . \square

By the argument above, given a group Γ of type **L**, we can define canonically the quotient space $\Omega(\Gamma)/\Gamma$, which we denote by $X(\Gamma)$,

$$X(\Gamma) = \Omega(\Gamma)/\Gamma.$$

There are examples of Γ of type **L** for which $X(\Gamma)$ is not connected. Such an example can be constructed easily by using the group given in [K4].

3. Discontinuous group actions on large domains.

In this section, we shall show that a large domain which covers a compact manifold is a connected component of $\Omega(\Gamma)$.

PROPOSITION 1. *Let Γ be a group of holomorphic automorphisms of a large domain Ω in \mathbf{P}^3 . Suppose that the action of Γ on Ω is properly discontinuous. Then Γ is of type **L**.*

PROOF. Fix a system of homogeneous coordinates on \mathbf{P}^5 such that $\text{Gr}(4, 2)$ is given by $\langle z, z \rangle = \sum_{i=0}^5 z_i^2 = 0$. By the assumption that Ω is large, there is an relatively compact subdomain $W \subset \Omega$ which is biholomorphic to U . The lines in W are parametrized by $\hat{W} \subset \text{Gr}(4, 2)$. Since the action of Γ on Ω is properly discontinuous, the set

$$S = \{ \sigma \in \Gamma \setminus \{1\} : \hat{\sigma}(\hat{W}) \cap \hat{W} \neq \emptyset \}$$

is finite. Let ℓ be a line in W . For $\sigma \in S$, if ℓ intersects $\sigma(\ell)$, we have $\langle \hat{\ell}, \hat{\sigma}(\hat{\ell}) \rangle = 0$. Suppose that the set

$$Y_\sigma = \{ \zeta \in \text{Gr}(4, 2) : \langle \zeta, \hat{\sigma}(\zeta) \rangle = 0 \}$$

is a proper analytic subset of $\text{Gr}(4, 2)$. Then the set

$$V = \hat{W} \setminus \bigcup_{\sigma \in S} Y_\sigma$$

is not empty. Take a point $\hat{\ell}' \in V$. Then, we can choose a neighborhood W' of ℓ' which is biholomorphic to U and satisfies $\sigma(W') \cap W' = \emptyset$ for all σ in S , and hence in Γ . Thus it is enough to show the following.

LEMMA 1. *Under the assumption of Proposition 1, the set*

$$Y = \{ \zeta \in \text{Gr}(4, 2) : \langle \zeta, \hat{\sigma}(\zeta) \rangle = 0 \}$$

is a proper subvariety for any element $\sigma \in \Gamma$ of infinite order.

PROOF. To prove the lemma by contradiction, we assume that

$$\langle \zeta, \hat{\sigma}(\zeta) \rangle = 0 \tag{2}$$

holds for any $\zeta \in \text{Gr}(4, 2)$. Suppose that $\hat{\sigma}$ is represented by $S \in \text{SL}(6, \mathbf{C})$. Since $\hat{\sigma}$ leaves $\text{Gr}(4, 2)$ invariant, we have

$$\langle Sz, Sz \rangle = \langle z, z \rangle \quad \text{for all } z \in \mathbf{C}^6. \tag{3}$$

By the assumption (2), there is a constant $c \in \mathbf{C}^*$ such that

$$\langle z, Sz \rangle = c\langle z, z \rangle \quad \text{for all } z \in \mathbf{C}^6. \tag{4}$$

LEMMA 2. *Under the conditions (3) and (4), the matrix S is conjugate in $GL(6, \mathbf{C})$ to the diagonal matrix of the form*

$$\begin{pmatrix} \alpha I_3 & 0 \\ 0 & \alpha^{-1} I_3 \end{pmatrix},$$

where I_3 is the identity matrix of size 3, α satisfies $\alpha^2 - 2c\alpha + 1 = 0$, and $|\alpha| \neq 1$.

Before giving the proof of Lemma 2, we shall complete the proof of Lemma 1. By Lemma 2, we see that both $\{\hat{\sigma}^m\}_m$ and $\{\hat{\sigma}^{-m}\}_m$ are normal sequences in $PGL(6, \mathbf{C})$, both of which have 2-planes as limit images. Let V_I^+ (resp. V_I^-) be the limit image of $\{\hat{\sigma}^m\}_m$ (resp. $\{\hat{\sigma}^{-m}\}_m$). Note that $V_I^+ \cap V_I^- = \emptyset$ and that $\hat{\sigma}$ fixes every point of V_I^\pm . By Theorem 2.2, the 2-planes V_I^\pm are both contained in $\text{Gr}(4, 2)$. By Griffiths-Harris [GH, pp.756–759], we know that a 2-plane in $\text{Gr}(4, 2)$ is one of the Schubert cycles $\sigma_2(p)$ or $\sigma_{1,1}(h)$, i.e., the set of all lines through a fixed point $p \in \mathbf{P}^3$ or the set of all lines lying on a fixed plane $h \subset \mathbf{P}^3$. Suppose that one of V_I^\pm is $\sigma_{1,1}(h)$. Since $\hat{\sigma}$ fixes every point of V_I^\pm , σ fixes every lines on h . This implies that σ fixes every point of h . Hence we have $h \subset \Lambda(\Gamma)$. This contradicts the assumption that Ω is large. Hence both V_I^\pm are of type $\sigma_2(p)$. Suppose that $V_I^+ = \sigma_2(p)$ and $V_I^- = \sigma_2(q)$. Since the line ℓ_{pq} passing through both p and q is a member of $V_I^+ \cap V_I^-$, this contradicts $V_I^+ \cap V_I^- = \emptyset$. Thus we obtain Lemma 1. □

PROOF OF LEMMA 2. By (3) and (4) we have

$${}^tSS = I, \quad {}^tS + S = 2cI. \tag{5}$$

Choose $P \in GL(6, \mathbf{C})$ such that

$$J = P^{-1}SP \tag{6}$$

is the Jordan canonical form of S . By (5) and (6), we have

$$2cI - J = P^{-1}{}^tSP. \tag{7}$$

Put $K = 2cI - J$. Then we have

$$KJ = I, \quad K + J = 2cI, \tag{8}$$

and also

$$J^2 - 2cJ + I = 0, \quad K^2 - 2cK + I = 0. \tag{9}$$

Recall that $S = \rho(\tilde{\sigma})$, where $\tilde{\sigma}$ is a representative of $\sigma \in \Gamma$. Let J_σ be the Jordan canonical form of $\tilde{\sigma}$. In view of (9), we can check that the sizes of Jordan blocks of J_σ are at most 2 by using Plücker coordinates. Suppose that J_σ is not diagonal. Then J_σ is of one of the following forms;

$$u'_0 = \alpha_0 u_0 + u_1, \quad u'_1 = \alpha_0 u_1, \quad u'_2 = \alpha_2 u_2 + u_3, \quad u'_3 = \alpha_2 u_3 \tag{10}$$

$$u'_0 = \alpha_0 u_0 + u_1, \quad u'_1 = \alpha_0 u_1, \quad u'_2 = \alpha_2 u_2, \quad u'_3 = \alpha_3 u_3. \tag{11}$$

In the case (10), the line $\ell : u_0 = u_1, u_2 = u_3$ satisfies $\sigma(\ell) \cap \ell = \emptyset$. Suppose that we are in the case (11). If $\alpha_0 = \alpha_2 = \alpha_3$ holds, then J_σ fixes every point on the plane $u_1 = 0$. This contradicts the fact that Γ is properly discontinuous on Ω , since Ω is large and hence any plane intersects Ω . If $\alpha_2 \neq \alpha_3$ holds, then $\ell : u_0 = u_1, u_2 = u_3$ satisfies $\sigma(\ell) \cap \ell = \emptyset$. If $\alpha_0 \neq \alpha_2 = \alpha_3$ holds, then $\ell : u_0 = u_2, u_1 = u_3$ satisfies $\sigma(\ell) \cap \ell = \emptyset$. Thus we infer that J_σ is diagonal. Hence J and K are also diagonal, and we have

$$J = \begin{pmatrix} \alpha I_p & 0 \\ 0 & \alpha^{-1} I_q \end{pmatrix}, \quad K = \begin{pmatrix} \alpha^{-1} I_p & 0 \\ 0 & \alpha I_q \end{pmatrix} \tag{12}$$

for some $p \geq 0$ and $q \geq 0$ with $p + q = 6$, where I_p, I_q are identity matrices. By the relation $PJP^{-1} = {}^t(PKP^{-1})$, we have easily $p = q = 3$. If $|\alpha| = 1$, then for any point $x \in \mathbf{P}^3$ and any neighborhood V of x , there would be infinite number of integers m with $\sigma^m(V) \cap V \neq \emptyset$. Since Γ is properly discontinuous and σ is of the infinite order, this is absurd. Hence we have $|\alpha| \neq 1$. Thus we obtain the lemma. □

For a properly discontinuous group Γ of holomorphic automorphisms of a large domain Ω in \mathbf{P}^3 , we can define $\Omega(\Gamma)$ and $\Lambda(\Gamma)$ by using Proposition 1.

THEOREM 3.1. *Let Γ be a group of holomorphic automorphisms of a large domain Ω in \mathbf{P}^3 . Suppose that the action of Γ on Ω is properly discontinuous. If the quotient space Ω/Γ is compact, then Ω coincides with a connected component of $\Omega(\Gamma)$.*

PROOF. We claim that $\Omega \subset \Omega(\Gamma)$. To verify this, suppose contrarily that there is a point $x \in \Omega \cap \Lambda(\Gamma)$. Then there are limit lines ℓ_I, ℓ_K such that $x \in \ell_I$, and a sequence $\{\sigma_m\}$ of distinct elements of Γ such that $\{\sigma_m\}$ converges uniformly on $\mathbf{P}^3 \setminus \ell_K$ to ℓ_I . Let ℓ be a line contained in Ω . Displacing ℓ a little if necessary, we can assume that $\ell \cap \ell_K = \emptyset$. Let K_x be a compact neighborhood of x contained in Ω . Put $K = K_x \cup \ell$, which is a compact set contained in Ω . Since $\{\sigma_m(\ell)\}$ converges to ℓ_I , we see that $\sigma_m(K) \cap K \neq \emptyset$ for infinitely many m . This contradicts the assumption that Γ is properly discontinuous on Ω . Thus the claim is verified.

Thus Ω is contained in a connected component, say Ω_0 , of $\Omega(\Gamma)$. Since Ω is Γ -invariant, so is Ω_0 . Therefore, by Theorem 2.5, Ω_0/Γ is a connected complex spaces which contains Ω/Γ . Since Ω/Γ is compact, we infer that $\Omega/\Gamma = \Omega_0/\Gamma$. Hence $\Omega = \Omega_0$. □

4. Free abelian group actions on P^2 and P^3 .

This is a preliminary section for the later arguments. First we shall study free abelian actions on domains in \mathbf{P}^2 .

LEMMA 3. *Let G be a free abelian subgroup in $PGL(3, \mathbf{C})$. Then for a suitable conjugate group G_0 of G in $PGL(3, \mathbf{C})$, one of the following occurs.*

(A)
$$G_0 \subset \left\{ \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix} : \lambda, \mu, \nu \in \mathbf{C}^* \right\}.$$

(B)
$$G_0 \subset \left\{ \begin{pmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu \end{pmatrix} : \lambda \in \mathbf{C}, \mu \in \mathbf{C}^* \right\} \quad \text{with} \quad J_a = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix} \in G_0,$$

 where $a \neq 1$.

(C)
$$G_0 \subset \left\{ \begin{pmatrix} 1 & \lambda & \mu \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \lambda, \mu \in \mathbf{C} \right\} \quad \text{with} \quad J(2, 1) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in G_0.$$

$$(D) \quad G_0 \subset \left\{ \begin{pmatrix} 1 & 0 & \lambda \\ 0 & 1 & \mu \\ 0 & 0 & 1 \end{pmatrix} : \lambda, \mu \in \mathbf{C} \right\} \quad \text{with} \quad J(1, 2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \in G_0.$$

$$(E) \quad G_0 \subset \left\{ \begin{pmatrix} 1 & \lambda & \mu \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{pmatrix} : \lambda, \mu \in \mathbf{C} \right\} \quad \text{with} \quad J(3) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \in G_0.$$

PROOF. In the following, we discuss as if G is a subgroup of $GL(3, \mathbf{C})$. Since G is abelian, we can assume that G_0 is contained in the set of upper triangular matrices. Suppose that G_0 contains $J(3)$. Since any element which commutes with $J(3)$ is of the form $\begin{pmatrix} 1 & \lambda & \mu \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{pmatrix}$, G_0 is in Case (E).

Suppose that G_0 contains $J(2, 1)$. Then G contains no elements conjugate to $J(3)$. Take any $S, T \in G_0$. Since S, T commute with $J(2, 1)$, we can set $S = \begin{pmatrix} s_1 & s_2 & s_3 \\ 0 & s_1 & 0 \\ 0 & 0 & s_4 \end{pmatrix}$ and $T = \begin{pmatrix} t_1 & t_2 & t_3 \\ 0 & t_1 & 0 \\ 0 & 0 & t_4 \end{pmatrix}$. If $s_1 \neq s_4$ for some $S \in G_0$, then replacing G_0 with its suitable conjugate group in the upper triangular group, we can assume that $s_3 = 0$. Then, by $ST = TS$, it follows that $t_3 = 0$ for any $T \in G_0$. Hence G_0 is in Case (B). If $s_1 = s_4$ for any $S \in G_0$, G_0 is in Case (C).

Suppose that G_0 contains $J(1, 2)$. Take any $S, T \in G_0$. Since S, T commute with $J(1, 2)$, we can set $S = \begin{pmatrix} s_1 & 0 & s_3 \\ 0 & s_2 & s_4 \\ 0 & 0 & s_2 \end{pmatrix}$ and $T = \begin{pmatrix} t_1 & 0 & t_3 \\ 0 & t_2 & t_4 \\ 0 & 0 & t_2 \end{pmatrix}$. If $s_1 \neq s_2$ for some $S \in G_0$, then replacing G_0 with its suitable conjugate group in the upper triangular group, we can assume that $s_3 = 0$. Then, by $ST = TS$, it follows that $t_3 = 0$ for any $T \in G_0$. Hence G_0 is in Case (B). If $s_1 = s_2$ for any $S \in G_0$, G_0 is in Case (D).

Suppose that $J_a \in G_0$ for some $a \in \mathbf{C}^* - \{1\}$. Take any $S \in G_0$. By $SJ_a = J_aS$, S can be written as $S = \begin{pmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu \end{pmatrix}$. Hence G_0 is in Case (B). If $J_1 \in G_0$ but $J_a \notin G_0$ for any $a \neq 1$, we are in Case (C). If G_0 contains $\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ instead of J_a , we can replace G_0 with another upper triangular conjugate group of G in $PGL(3, \mathbf{C})$ such that $J_a \in G_0$. Hence this case can be settled as above.

The remaining case is that every element of G_0 is conjugate to a diagonal matrix. In this case, all the elements of G_0 can be diagonalized at the same time. Hence G_0 is in Case (A). \square

Let X, Y be locally connected topological spaces. A surjective continuous map $f : X \rightarrow Y$ is called an *even covering* if every point on Y admits a connected neighborhood V such that every connected component \tilde{V} of $f^{-1}(V)$ is homeomorphic to V by $f|_{\tilde{V}}$.

LEMMA 4. Let W_j , $j = 1, 2$, be (connected) topological manifolds, and $\varphi : W_1 \rightarrow W_2$ a continuous map which is locally homeomorphic everywhere. Let G_j be a group of homeomorphisms of W_j whose action on W_j is free and properly discontinuous. Assume that, for every $\sigma_1 \in G_1$, there is a unique element $\sigma_2 \in G_2$ such that $\varphi \circ \sigma_1 = \sigma_2 \circ \varphi$. If the quotient space W_1/G_1 is compact, and if the correspondence $\varphi_* : G_1 \rightarrow G_2$ which sends σ_1 to σ_2 is a group isomorphism, then φ is surjective and an even covering. Further, if φ is injective, then φ is a homeomorphism of W_1 onto W_2 .

Proof is easy. This lemma will be used throughout the proof of the next lemma.

LEMMA 5. Let W be a subdomain in \mathbf{P}^2 . Suppose that an finitely generated free abelian subgroup G of $\text{PGL}(3, \mathbf{C})$ acts on W . If the action of G is free and properly discontinuous, and if the quotient manifold W/G is compact, then, one of the following occurs, where the homogeneous coordinates of \mathbf{P}^2 is that of Lemma 3.

- I. (a) $G \simeq \mathbf{Z}$ and G is in Case (A), or Case (B) with $|a| \neq 1$,
 (b) $W = \mathbf{P}^2 \setminus \{[z_0 : z_1 : z_2] : z_2 = 0 \text{ or } [0 : 0 : 1]\}$,
 (c) W/G is biholomorphic to a Hopf surface,
- II. (a) $G \simeq \mathbf{Z}^2$ and G is in Case (A),
 (b) $W = \mathbf{P}^2 \setminus \{[z_0 : z_1 : z_2] : z_0 z_1 z_2 = 0\}$,
 (c) W/G is biholomorphic to a complex torus,
- III. (a) $G \simeq \mathbf{Z}^3$ and G is in Case (B),
 (b) $W = \mathbf{P}^2 \setminus \{[z_0 : z_1 : z_2] : z_1 z_2 = 0\}$,
 (c) W/G is biholomorphic to a complex torus,
- IV. (a) $G \simeq \mathbf{Z}^4$ and G is in Case (D) or Case (E),
 (b) $W = \mathbf{P}^2 \setminus \{[z_0 : z_1 : z_2] : z_2 = 0\}$,
 (c) W/G is biholomorphic to a complex torus.

PROOF. Denote by ℓ_j the line defined by $z_j = 0$ on \mathbf{P}^2 . We check the cases of Lemma 3. If $\text{rank } G = 1$, it is known that W/G is a Hopf surface and Case I occurs (see, [K2, Proposition 4.2]). Therefore we assume that $n = \text{rank } G \geq 2$ in the following.

Step 1: Suppose that G is in Case (C). Since $J(2, 1)$ fixes every point on ℓ_1 , we see that $\ell_1 \cap \Omega = \emptyset$. Put $x = z_0/z_1$ and $y = z_2/z_1$. Then G contains only elements of the forms

$$\begin{cases} x' = x + 1 \\ y' = y \end{cases} \quad \text{or} \quad \begin{cases} x' = x + \lambda + \mu y \\ y' = y \end{cases} .$$

This implies that the holomorphic function y is well-defined on the compact manifold W/G . Hence y is constant on W/G and hence on W . This is absurd. Therefore Case (C) does not occur.

Step 2: Suppose that G is in Case (E). First we consider the case where $\ell_2 \cap W = \emptyset$. On $\mathbf{P}^2 \setminus \ell_2$, we introduce new coordinates (u, v) by

$$\begin{cases} u = \frac{z_0}{z_2} - \frac{1}{2} \left(\frac{z_1}{z_2} \right)^2 \\ v = \frac{z_1}{z_2} \end{cases} .$$

Let $\{\sigma_j\}_{j=1, \dots, n}$ be a basis of G , where $\sigma_1 = J(3)$. Then

$$\sigma_j = \begin{pmatrix} 1 & \lambda_j & \mu_j \\ 0 & 1 & \lambda_j \\ 0 & 0 & 1 \end{pmatrix}$$

can be written as a translation

$$\tau_j : (u, v) \mapsto \left(u + \mu_j - \frac{1}{2} \lambda_j^2, v + \lambda_j \right).$$

Since the group of translations $\{\tau_j\}_{j=1, \dots, n}$ acts properly discontinuously on W , we see that the 2-vectors

$$\left(\mu_j - \frac{1}{2} \lambda_j^2, \lambda_j \right), \quad j = 1, \dots, n$$

are linearly independent and span \mathbf{C}^2 over \mathbf{R} . Hence $\text{rank } G = n = 4$ and $W = \mathbf{P}^2 \setminus \ell_2$. Thus G is in Case IV.

Next we consider the case where $\ell_2 \cap W \neq \emptyset$. We shall show that this case does not occur. Since the action of G on $\ell_2 \cap W$ is properly discontinuous, we have $n = 2$ and $\lambda_2 \notin \mathbf{R}$. Since the quotient manifold $E = (\ell_2 \cap W)/G$ is a finite union of compact curves which admit holomorphic affine structures, we see that E consists of a single elliptic curve and that $\ell_2 \cap W = \ell_2 \setminus \{[1 : 0 : 0]\}$ by Lemma 4. Take an open covering $\{U_j\}_{j=0,1,2}$ of \mathbf{P}^2 such that $U_j = \{z_j \neq 0\}$. Let $(x_0, y_0) = (z_1/z_0, z_2/z_0)$ be coordinates on U_0 . Similarly, we put $(x_1, y_1) = (z_2/z_1, z_0/z_1)$ on U_1 , and $(x_2, y_2) = (z_0/z_2, z_1/z_2)$ on U_2 . Then the meromorphic 2-form on \mathbf{P}^2 defined by

$$\omega = dx_2 \wedge dy_2 = x_1^{-3} dx_1 \wedge dy_1 = y_0^{-3} dx_0 \wedge dy_0$$

is G -invariant. Therefore ω defines a meromorphic 2-form on the compact surface $S = W/G$. Hence the canonical divisor K_S is given by $K_S = -3E$. By Nishiguchi [N, Proposition 1.2], a compact non-Kähler surface with a non-zero effective anti-canonical divisor is of class VII. Hence, if S is non-Kähler, then S is of class VII. Then, we have, in particular, that the first Betti number of S equals to 1. This contradicts the assumption $n \geq 2$. Suppose that S is Kähler. By the construction, S admits a holomorphic projective structure. It is known by Kobayashi-Ochiai [KoOc2] that a compact Kähler surface with a holomorphic projective structure is either \mathbf{P}^2 , complex torus, or a compact quotient of a unit ball in \mathbf{C}^2 . The surface S is, however, non-simply connected, has non-trivial canonical bundle, and contains the elliptic curve E . This is a contradiction. Thus Kähler case doesn't occur either.

Step 3: Suppose that G is in Case (B). Let $\{\sigma_j\}_{j=1, \dots, n}$ be a basis of G , where $\sigma_1 = J_a$. Put

$$\sigma_j = \begin{pmatrix} 1 & \lambda_j & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu_j \end{pmatrix}.$$

First consider the case where $\ell_1 \cap W \neq \emptyset$. Since $[1 : 0 : 0]$ and $[0 : 0 : 1]$ are fixed by G , these two points are excluded from $\ell_1 \cap W$. Put $z = z_2/z_0$. Then each σ_j acts on $\ell_1 \cap W$ as $z \mapsto \mu_j z$. Since the action of G on $\ell_1 \cap W \subset \mathbf{C}^*$ is properly discontinuous, we see that $n = 1$, $|a| \neq 1$, $\ell_1 \cap W = \mathbf{C}^*$ by Lemma 4, and that $G = \langle \sigma_1 \rangle \simeq \mathbf{Z}$. This case is settled at the beginning of the proof.

It remains to consider the case where $W \subset \mathbf{P}^2 \setminus \ell_1 \simeq \mathbf{C}^2$. Put $(x, y) = (z_0/z_1, z_2/z_1)$. Then the σ_j are of the form

$$\begin{cases} x' = x + \lambda_j \\ y' = \mu_j y \end{cases}.$$

Suppose that $\ell_2 \cap W \neq \emptyset$. Then the quotient manifold $E = (\ell_2 \cap W)/G$ is a finite union of compact curves which admit holomorphic affine structures, we see that E consists of a single elliptic curve, $\ell_2 \cap W = \mathbf{C}$ by Lemma 4, and that $G = \langle J_a, \sigma_2 \rangle \simeq \mathbf{Z}^2$. The quotient space $V = \mathbf{C}^2/G$ is a line bundle over the elliptic curve defined by $E = \mathbf{C}/\langle 1, \lambda_1 \rangle$. The inclusion map $j : W \rightarrow \mathbf{P}^2 \setminus \ell_1$ induces an open embedding $j : W/G \rightarrow V$. Since W/G is compact, this is absurd. Therefore we have $W \subset \mathbf{P}^2 \setminus \{z_1 z_2 = 0\}$. Define $q : \mathbf{C}^2 \rightarrow \mathbf{P}^2 \setminus \{z_1 z_2 = 0\}$ by

$$q(u, v) = [u : 1 : e^{2\pi iv}].$$

Suppose that $\log y$ has a single-valued branch on W , which we denote also by $\log y$. The holomorphic map $\varphi : W \rightarrow \mathbf{C}^2$, $\varphi(x, y) = (x, \log y)$, is an open embedding. Since G is properly discontinuous on W , each element of G induces a transformation of \mathbf{C}^2 by the 2-vectors

$$(\lambda_j, \log \mu_j), \quad j = 1, \dots, n. \tag{13}$$

Since the action of G on $\varphi(W)$ is properly discontinuous, these 2-vectors are linearly independent over \mathbf{R} . Hence by Lemma 4, we have $\text{rank } G = 4$ and $\varphi(W) = \mathbf{C}^2$. Therefore $W = q \circ \varphi(W) = \mathbf{P}^2 \setminus \{z_1 z_2 = 0\}$ follows. This contradicts the assumption that $\log y$ is single-valued on W . Thus $\log y$ is multi-valued on W .

Let $\tilde{W} \subset \mathbf{C}^2$ be a connected component of $q^{-1}(W)$, where $\log y$ is single-valued. Let \tilde{G} be the group generated by the translations of \mathbf{C}^2 defined by

$$\tau_0(u, v) = (u, v + 2\pi i), \quad \tau_j(u, v) = (u + \lambda_j, v + \log \mu_j), \quad j = 1, \dots, n$$

and put

$$\tilde{G}_W = \{\tau \in \tilde{G} : \tau(\tilde{W}) \subset \tilde{W}\}.$$

Since \tilde{W}/\tilde{G}_W is compact, we have $\text{rank } \tilde{G}_W = 4$. Since $\log y$ is multi-valued on W , $\tau_0^m \in \tilde{G}_W$ for some non-zero integer m . Hence we have $\text{rank } G = \text{rank } q(\tilde{G}_W) = 3$. Thus we are in Case III.

Step 4: Suppose that G is in Case (D). Let $\{\sigma_j\}_{j=1, \dots, n}$ be a basis of G , where $\sigma_1 = J(1, 2)$. Put

$$\sigma_j = \begin{pmatrix} 1 & 0 & \lambda_j \\ 0 & 1 & \mu_j \\ 0 & 0 & 1 \end{pmatrix}.$$

Every element of G fixes every point on ℓ_2 . Hence $\ell_2 \cap W = \emptyset$. Put $(x, y) = (z_0/z_2, z_1/z_2)$. Then the σ_j are of the form

$$\begin{cases} x' = x + \lambda_j \\ y' = y + \mu_j \end{cases}.$$

Since the action of G is properly discontinuous, we see that the 2-vectors

$$\{(\lambda_j, \mu_j), \quad j = 1, \dots, n\} \tag{14}$$

are linearly independent and span \mathbf{C}^2 over \mathbf{R} . Hence $\text{rank } G = 4$ and we have $W/G \simeq \mathbf{C}^2/L$, where L is the lattice in \mathbf{C}^2 spanned by (14). Hence $W = \mathbf{P}^2 \setminus \{z_2 = 0\}$ and we are in Case IV.

Step 5: Suppose that G is in Case (A). First consider the case where $\ell_0 \cap W \neq \emptyset$. Then we see that $\ell_0 \cap W = \ell_0 \setminus \{[0 : 1 : 0], [0 : 0 : 1]\}$ and $G \simeq \mathbf{Z}$ as above cases. Hence this case is settled at the beginning of the proof. The other cases where $\ell_j \cap W \neq \emptyset$, $j = 1, 2$, can be settled similarly.

Next we consider the case where $W \subset \mathbf{P}^2 \setminus \{z_0 z_1 z_2 = 0\} \simeq \mathbf{C}^* \times \mathbf{C}^*$. Put $(x, y) = (z_0/z_2, z_1/z_2)$. Then the σ_j are of the form

$$\begin{cases} x' = a_j x \\ y' = b_j y \end{cases} .$$

We have to show that $W = \mathbf{P}^2 \setminus \{z_0 z_1 z_2 = 0\}$ and $\text{rank } G = 2$. By a similar argument in Step 3, we see that both $\log x$ and $\log y$ are multi-valued on W . As in Step 3, we define $q : \mathbf{C}^2 \rightarrow \mathbf{C}^* \times \mathbf{C}^*$ by $q(u, v) = (e^{2\pi i u}, e^{2\pi i v})$ and \tilde{W} be a connected component of $q^{-1}(W)$. Let \tilde{G} be the group generated by the transformations of \mathbf{C}^2 defined by

$$\begin{aligned} \sigma_0(u, v) &= (u + 2\pi i, v) \\ \sigma_1(u, v) &= (u, v + 2\pi i) \\ \tau_j(u, v) &= (u + \log a_j, v + \log b_j), \quad j = 1, \dots, n \end{aligned}$$

and put

$$\tilde{G}_W = \{\tau \in \tilde{G} : \tau(\tilde{W}) \subset \tilde{W}\}.$$

Since the action of \tilde{G}_W on \tilde{W} is properly discontinuous and its quotient by \tilde{G}_W is compact, \tilde{G}_W forms a lattice on \mathbf{C}^2 and $\text{rank } \tilde{G}_W = 4$. Since $\log x$ and $\log y$ are multi-valued, certain non-zero powers of σ_0 and σ_1 are contained in \tilde{G}_W . Therefore $\text{rank } G = \text{rank } q(\tilde{G}_W) = 2$ and the action of G on $\mathbf{C}^* \times \mathbf{C}^*$ is properly discontinuous. Therefore, by Lemma 4, W coincides with $\mathbf{P}^2 \setminus \{z_0 z_1 z_2 = 0\}$. Thus G is in Case II. □

Now we go to the three dimensional case. Let Ω and Γ be as in Section 1. Let $\pi : \Omega \rightarrow \Omega/\Gamma$ be the canonical projection. By [K2, Theorem 1.3], we have

PROPOSITION 2. *If Ω is large, $\Gamma \simeq \mathbf{Z}$, and if Ω/Γ is compact, then Ω/Γ is an L-Hopf manifold.*

The following fact is useful.

PROPOSITION 3. *If Ω is large and if Ω/Γ is compact, then Γ is not isomorphic to \mathbf{Z}^2 .*

PROOF. Assuming $\Gamma \simeq \mathbf{Z}^2$, we derive a contradiction. Since Γ is abelian, there is a system of homogeneous coordinates $[z_0 : z_1 : z_2 : z_3]$ on \mathbf{P}^3 such that H_3 is Γ -invariant, where by H_j we indicate the plane defined by $z_j = 0$.

SUBLEMMA 1. Γ consists of diagonal matrices.

PROOF. Since Ω contains a line, we have $\Omega \cap H_j \neq \emptyset$ for any j . Consider the restriction of Γ to H_3 , which we denote by Γ_{H_3} . Obviously, Γ_{H_3} is in Case II of Lemma 5. Let $\{\sigma, \tau\}$ be a basis of Γ . Then σ and τ are of the following form.

$$\sigma = \begin{pmatrix} a_0 & 0 & 0 & s_0 \\ 0 & a_1 & 0 & s_1 \\ 0 & 0 & a_2 & s_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} b_0 & 0 & 0 & t_0 \\ 0 & b_1 & 0 & t_1 \\ 0 & 0 & b_2 & t_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Introducing new coordinates

$$w_3 = z_3, \quad w_j = z_j + \lambda_j z_3 \quad j = 0, 1, 2$$

for suitable constants $\lambda_j \in \mathbf{C}$, we can assume that

$$(a_0 - 1)s_0 = (a_1 - 1)s_1 = (a_2 - 1)s_2 = 0.$$

Since the action of σ on H_3 is not trivial, some a_k , say a_0 , is not equal to 1, and hence $s_0 = 0$. By $\sigma\tau = \tau\sigma$, we have also $t_0 = 0$. Hence H_0 is Γ -invariant. Since $\Gamma_{H_0} \simeq \mathbf{Z}^2$ is conjugate to the group of Case (A) of Lemma 3 by Lemma 5, we have $s_1 = s_2 = t_1 = t_2 = 0$. □

By ℓ_{jk} , we indicate lines defined by $z_j = z_k = 0$. Put $e_j = [\delta_j^0 : \delta_j^1 : \delta_j^2 : \delta_j^3]$, $0 \leq j \leq 3$, where δ_j^i is the Kronecker's delta.

SUBLEMMA 2. $\Omega = \mathbf{P}^3 \setminus \bigcup_{j < k} \ell_{jk}$.

PROOF. Put $W = \mathbf{P}^3 \setminus \bigcup_{j < k} \ell_{jk}$. Since Γ_{H_j} is in Case II of Lemma 5, we

have easily that $\Omega \subset W$. Thus it is enough to show that any boundary point x of Ω is contained in some ℓ_{jk} . Suppose that $x \in W \cap \partial\Omega$. Take a sequence $\{N_j\}_{j=1}^\infty$ of neighborhood of x such that $W \supset N_j \supset N_{j+1}$ and that $\bigcap_j^\infty N_j = \{x\}$. Let F be a closed fundamental region on Ω with respect to Γ . For each N_j there is an element $\sigma_j \in \Gamma$ such that $\sigma_j(F) \cap N_j \neq \emptyset$. For each j , there is a point $y_j \in F$ such that $\sigma_j(y_j) \in N_j$. By the definition of N_j , we have

$$\lim_{j \rightarrow \infty} \sigma_j(y_j) = x, \quad y_j \in F \subset \Omega.$$

Since F is compact, replacing $\{y_j\}_j$ with its subsequence, we can assume that there is a point $y \in F$ such that

$$\lim_{j \rightarrow \infty} y_j = y. \tag{15}$$

For an element $A = (a_{ij}) \in GL(4, \mathbf{C})$, define the norm of A by $|A| = \max\{|a_{ij}|\}$. We indicate by $\tilde{\sigma}$ a representative in $GL(4, \mathbf{C})$ of $\sigma \in PGL(4, \mathbf{C})$. Replacing $\{\sigma_j\}_{j=1}^\infty$ with its subsequence, we consider the convergent matrix sequence $\{\tilde{\sigma}_j\}_{j=1}^\infty$, $|\tilde{\sigma}_j| = 1$. Put $\tilde{\sigma} = \lim_{j \rightarrow \infty} \tilde{\sigma}_j$. Let $\sigma : \mathbf{P}^3 \rightarrow \mathbf{P}^3$ be the rational map defined by the linear map $\tilde{\sigma} : \mathbf{C}^4 \rightarrow \mathbf{C}^4$. Note that the limit $\tilde{\sigma}$ is also diagonal matrix, since Γ consists of diagonal matrices by Sublemma 1. Let $K(\sigma)$ be the projective linear subspace in \mathbf{P}^3 corresponding to the kernel of $\tilde{\sigma}$, and $I(\sigma)$ the one corresponding to the image of $\tilde{\sigma}$. Then we have $\dim I(\sigma) + \dim K(\sigma) = 2$. It is not difficult to check that the sequence $\{\sigma_j\}_{j=1}^\infty$ is uniformly convergent to $I(\sigma)$ on any compact subset of $\mathbf{P}^3 \setminus K(\sigma)$ ([My]). We have $\text{rank } \tilde{\sigma} \leq 3$, since Γ is properly discontinuous on a non-empty open subset of \mathbf{P}^3 .

If $\text{rank } \tilde{\sigma} = 3$, then $K(\sigma)$ is a single point and $I(\sigma)$ is a plane. Since $\tilde{\sigma}$ is a diagonal matrix, $K(\sigma) = e_m$ for some m , $0 \leq m \leq 3$, and $I(\sigma) = H_m$. Since y in (15) is a point in $F \subset \Omega$, and since $e_m \notin \Omega$, we obtain $e_m \neq y$. Hence $y \in \mathbf{P}^3 \setminus K(\sigma)$. Since $\{\sigma_j\}_j$ is uniformly convergent on $\mathbf{P}^3 \setminus K(\sigma)$, we have

$$x = \lim_{j \rightarrow \infty} \sigma_j(y_j) = \lim_{j \rightarrow \infty} \sigma_j(y) \in I(\sigma) = H_j.$$

This shows that $H_j \cap \partial\Omega$ is not empty. But this contradicts Lemma 5 Case II, since in this case, we have $H_j \setminus (\bigcup_k \ell_{jk}) \subset \Omega$. Thus $\text{rank } \tilde{\sigma} \neq 3$.

If $\text{rank } \tilde{\sigma} = 2$, then $K(\sigma)$ and $I(\sigma)$ are lines. Since $\tilde{\sigma}$ is a diagonal matrix, $K(\sigma) = \ell_{ij}$ for some i, j , $0 \leq i < j \leq 3$, and $I(\sigma) = \ell_{km}$, $k < m$, where $\{i, j, k, m\} = \{0, 1, 2, 3\}$. Since y in (15) is a point in $F \subset \Omega$, $y \notin \ell_{km} = K(\sigma)$. Therefore, we have $x \in I(\sigma) = \ell_{ij}$. This contradicts the assumption that $x \in W$. Thus $\text{rank } \tilde{\sigma} \neq 2$.

If $\text{rank } \tilde{\sigma} = 1$, then consider $\{\sigma^{-1}\}_{j=1}^{\infty}$ and consider the limit matrix, which we denote by $\tilde{\sigma}'$. Then $\text{rank } \tilde{\sigma}' = 3$ and apply the same argument of rank = 3 case above. Then we can conclude that this case does not occur either.

Thus there is no boundary point of Ω in W . □

SUBLEMMA 3. *The action of $\Gamma \simeq \mathbf{Z}^2$ on $\mathbf{P}^3 \setminus \bigcup_{j < k} \ell_{jk}$ is not properly discontinuous.*

PROOF. By Sublemma 1, every element of Γ is represented by a diagonal matrix in $SL(4, \mathbf{C})$. For $\sigma \in \Gamma$, let $a_j = a_j(\sigma)$, $j = 0, 1, 2, 3$, be the diagonal components of a representative of σ . Namely we have

$$\sigma([z_0 : z_1 : z_2 : z_3]) = [a_0 z_0 : a_1 z_1 : a_2 z_2 : a_3 z_3].$$

Consider the set of positive real numbers $\{|a_j(\sigma)| : 0 \leq j \leq 3\}$, which we indicate by $\Delta(\sigma)$. By $\delta(\sigma)$, we indicates the numbers of distinct elements of $\Delta(\sigma)$.

Step 1: Consider the case where $\Gamma \setminus \{1\}$ contains an element σ with $\delta(\sigma) = 1$. The restriction Γ_{H_3} of Γ to $W \cap H_3$ is given by

$$\sigma_{H_3} : [z_0 : z_1 : z_2 : 0] \rightarrow [a_0 z_0 : a_1 z_1 : a_2 z_2 : 0], \quad |a_0| = |a_1| = |a_2|,$$

which is not properly discontinuous on $W \cap H_3$.

Step 2: Next consider the case where $\Gamma \setminus \{1\}$ contains an element σ with $\delta(\sigma) = 4$. Suppose that

$$|a_0(\sigma)| < |a_1(\sigma)| < |a_2(\sigma)| < |a_3(\sigma)|. \tag{16}$$

Put

$$U = \{[z_0 : z_1 : z_2 : z_3] \in \mathbf{P}^3 : |z_1|^2 + |z_2|^2 < |z_0|^2 + |z_3|^2\}$$

$$\Sigma = \{[z_0 : z_1 : z_2 : z_3] \in \mathbf{P}^3 : |z_1|^2 + |z_2|^2 = |z_0|^2 + |z_3|^2\}$$

and $[U] = U \cup \Sigma$. Then U is a tubular neighborhood of the line ℓ_{12} . We cover the domain $\mathbf{P}^3 \setminus \ell_{03}$ by two open sets U_1, U_2 , where $U_j \simeq \mathbf{C}^3$. Define systems of local coordinates

$$(x, y, s) = \left(\frac{z_1}{z_3}, \frac{z_2}{z_3}, \frac{z_0}{z_3} \right)$$

on U_1 , and

$$(u, v, t) = \left(\frac{z_1}{z_0}, \frac{z_2}{z_0}, \frac{z_3}{z_0} \right)$$

on U_2 . Then we have

$$u = s^{-1}x, \quad v = s^{-1}y, \quad t = s^{-1}$$

on $U_1 \cap U_2$. We consider tubular neighborhoods T_x, T_y, T_u, T_v of x -axis, y -axis, u -axis, and v -axis, respectively, in $\mathbf{P}^3 \setminus (\ell_{12} \cup \ell_{03})$ as follows. Take positive numbers p_0, p_1, p_2 so that the equality

$$\left| \frac{a_1}{a_3} \right|^{p_0} = \left| \frac{a_2}{a_3} \right|^{p_1} = \left| \frac{a_0}{a_3} \right|^{p_2}$$

holds. Similarly, take positive numbers q_0, q_1, q_2 so that the equality

$$\left| \frac{a_1}{a_0} \right|^{q_0} = \left| \frac{a_2}{a_0} \right|^{q_1} = \left| \frac{a_3}{a_0} \right|^{q_2}$$

holds. We define

$$T_x = \{(x, y, s) \in U_1 : |y|^{p_1} + |s|^{p_2} < r|x|^{p_0}\} \quad (17)$$

$$T_y = \{(x, y, s) \in U_1 : |x|^{p_0} + |s|^{p_2} < r|y|^{p_1}\} \quad (18)$$

$$T_u = \{(u, v, t) \in U_2 : |v|^{q_1} + |t|^{q_2} < r|u|^{q_0}\} \quad (19)$$

$$T_v = \{(u, v, t) \in U_2 : |u|^{q_0} + |t|^{q_2} < r|v|^{q_1}\}, \quad (20)$$

where r is a small positive number such that the closures of these four tubular neighborhoods do not intersect each other. Now consider the set

$$K = \Sigma \setminus \{T_x \cup T_y \cup T_u \cup T_v\}.$$

Then K is a connected compact set contained in Ω . Topologically, K is the real 5-manifold $S^2 \times S^3$ with four 5-dimensional open disks deleted which do not intersect each other. We claim that $\sigma^n(K) \cap K \neq \emptyset$ for any $n \neq 0$. Indeed, on U_1 , we have

$$\sigma(x, y, s) = \left(\frac{a_1}{a_3}x, \frac{a_2}{a_3}y, \frac{a_0}{a_3}s \right).$$

On U_2 , we have

$$\sigma(u, v, t) = \left(\frac{a_1}{a_0}u, \frac{a_2}{a_0}v, \frac{a_3}{a_0}t \right).$$

Therefore the closures of the four tubular neighborhoods T_x, \dots, T_v are σ -invariant. Further, by the inequalities (16), we see that, for $n > 0$, σ^n is (weakly) shrinking on U_1 to the center $[0 : 0 : 0 : 1]$, and (weakly) expanding on U_2 from the center $[1 : 0 : 0 : 0]$. In particular, we have

$$\begin{aligned} \sigma^n(\Sigma \cap T_x) &\subset [U] \\ \sigma^n(\Sigma \cap T_y) &\subset [U] \\ \sigma^n(\Sigma \cap T_u) &\subset \mathbf{P}^3 \setminus U \\ \sigma^n(\Sigma \cap T_v) &\subset \mathbf{P}^3 \setminus U. \end{aligned}$$

Hence we infer that $\sigma^n(K) \cap K \neq \emptyset$, since K is connected. By the similar argument, we can verify the claim for all $n < 0$.

Step 3: By the argument above, we see that $\delta(\sigma) = 2$ or 3 holds for any $\sigma \in \Gamma \setminus \{1\}$. Suppose that there is an element $\sigma \in \Gamma \setminus \{1\}$ such that $\delta(\sigma) = 2$. If $|a_i(\sigma)| = |a_j(\sigma)| = |a_k(\sigma)| \neq |a_l(\sigma)|$ for some $\{i, j, k, l\} = \{0, 1, 2, 3\}$, we have a contradiction by the same argument as in Step 1. Therefore, by a permutation of z_0, \dots, z_3 , we can assume that

$$|a_0(\sigma)| = |a_1(\sigma)| \neq |a_2(\sigma)| = |a_3(\sigma)|$$

holds. We claim that there is an element $\tau \in \Gamma$ such that $|a_0(\tau)| \neq |a_1(\tau)|$. Indeed, if contrary, the set

$$K = \{[z_0 : z_1 : z_2 : z_3] \in W : |z_0| = |z_1|\}$$

would be a Γ -invariant set. Since $K \simeq S^1 \times (\mathbf{C}^2 \setminus \{(0, 0)\}) \simeq S^1 \times S^3 \times \mathbf{R}$ has two ends, the quotient of K by a free properly discontinuous action of rank = 2 free abelian group cannot be a compact manifold by a theorem of Hopf [H]. Thus Γ contains τ such that $|a_0(\tau)| \neq |a_1(\tau)|$. Then $\delta(\sigma^n \tau) = 3$ holds for sufficiently large $n > 0$. Therefore Γ always contains an element σ such that $\delta(\sigma) = 3$.

Step 4: For $\sigma \in \Gamma$ with $\delta(\sigma) = 3$, there are two possibilities,

$$|a_0(\sigma)| < |a_1(\sigma)| = |a_2(\sigma)| < |a_3(\sigma)| \tag{21}$$

and

$$|a_0(\sigma)| = |a_1(\sigma)| < |a_2(\sigma)| < |a_3(\sigma)|. \tag{22}$$

If σ satisfies (21), then by the same argument as in Step 3, we have an element $\tau \in \Gamma$ such that $|a_1(\tau)| \neq |a_2(\tau)|$. Similarly, if σ satisfies (22), then we have an element $\tau \in \Gamma$ such that $|a_0(\tau)| \neq |a_1(\tau)|$. Then, in both cases, $\delta(\sigma^n\tau) = 4$ for sufficiently large $n > 0$. This contradicts the conclusion of Step 2. Thus we have disproved all possibilities of $\delta(\sigma)$, and the sublemma is proved. \square

Proposition 3 follows immediately from Sublemmas 2 and 3. \square

LEMMA 6. *Let S be a compact reduced irreducible 2-dimensional complex space and C an irreducible curve on S . Let Ω_S be a domain in S such that the Hausdorff dimension of $S \setminus \Omega_S$ is less than 1, and hence $C \cap \Omega_S \neq \emptyset$. Suppose that a properly discontinuous infinite group Γ of holomorphic automorphisms of S is acting freely on Ω_S and C and that the quotient spaces Ω_S/Γ and $(C \cap \Omega_S)/\Gamma$ are compact. Then there is a subgroup Γ_0 of Γ which have the following properties.*

- (1) Γ_0 has index less than three in Γ .
- (2) Γ_0 is isomorphic to either \mathbf{Z} or \mathbf{Z}^2 .
- (3) $C \cap \Omega_S$ is biholomorphic to either \mathbf{C} or \mathbf{C}^* .
- (4) $(C \cap \Omega_S)/\Gamma_0$ is a non-singular elliptic curve.

PROOF. Let $\nu : \tilde{S} \rightarrow S$ be the normalization of S . Put $\tilde{\Omega}_S = \nu^{-1}(\Omega_S)$. Take an irreducible component \tilde{C} of $\nu^{-1}(C)$ such that $C = \nu(\tilde{C})$. Let $a \in \tilde{C}$ be a point on the boundary of $\tilde{\Omega}_S$. Assume that both \tilde{S} and \tilde{C} are non-singular at a . Let $W = \{(z, w) : |z| < 1, |w| < 1\}$ be a polydisk on \tilde{S} centered at $a = (0, 0)$ such that $\tilde{C} \cap W = \{w = 0\}$. Take a point $(z_0, 0) \in \tilde{C} \cap \tilde{\Omega}_S \cap W$. Then there are small constants $0 < \varepsilon < 1 - |z_0|$ and $0 < \delta < 1$ such that the set

$$T_\delta = \{(z, w) \in W : |z - z_0| < \varepsilon, |w| < \delta\}$$

is contained in $\tilde{\Omega}_S$. Since the Hausdorff dimension of $S \setminus \Omega_S$ is less than 1 by assumption, so is $\tilde{S} \setminus \tilde{\Omega}_S$. Therefore, we can choose $0 < \delta_1 < \delta$ such that the image by ν of the set

$$\Sigma = \{(z, w) \in W : |z| < 1, |w| = \delta_1\}$$

does not intersects $S \setminus \Omega_S$. Then, there are positive constants δ_2, δ_3 with $\delta_2 < \delta_1 < \delta_3 < \delta$ such that

$$R = \{(z, w) \in W : |z| < 1, \delta_2 < |w| < \delta_3\} \supset \Sigma$$

is contained in $\tilde{\Omega}_S$.

The union $H = R \cup T_{\delta_3}$ is a Hartogs domain contained in $\tilde{\Omega}_S$ whose envelope of holomorphy is $\Delta = \{(z, w) \in W : |z| < 1, |w| < \delta_3\}$. Let $\pi : \Omega_S \rightarrow \Omega_S/\Gamma$ denote the natural projection. By a theorem of Ivashkovich [Iv, Corollary 1], the holomorphic map $(\pi \circ \nu)|_H : H \rightarrow \Omega_S/\Gamma$ extends to a holomorphic map from $\Delta \setminus A$, where A is a closed set of isolated points in Δ .

We claim that $\pi \circ \nu : \tilde{\Omega}_S \rightarrow \Omega_S/\Gamma$ cannot be extended outside of $\tilde{\Omega}_S$ (cf. [K5, Lemma 2.2]). If the claim is verified, then we see that

$$\Delta \cap (\tilde{C} \setminus \tilde{\Omega}_S) \subset A. \tag{23}$$

To verify the claim, suppose that there is a point $c \in \partial\tilde{\Omega}_S \subset \tilde{S}$, and a neighborhood \tilde{N} of c in \tilde{S} such that $\pi \circ \nu$ extends holomorphically to a map $\varphi : \tilde{N} \rightarrow \Omega_S/\Gamma$. The group Γ lifts to a group $\tilde{\Gamma}$ of automorphisms of \tilde{S} . The domain $\tilde{\Omega}_S$ is $\tilde{\Gamma}$ -invariant and the quotient space $\tilde{\Omega}_S/\tilde{\Gamma}$ gives the normalization $\tilde{\nu} : \tilde{\Omega}_S/\tilde{\Gamma} \rightarrow \Omega_S/\Gamma$. Let $\tilde{\pi} : \tilde{\Omega}_S \rightarrow \tilde{\Omega}_S/\tilde{\Gamma}$ be the canonical projection. Obviously, we have $\pi \circ \nu = \tilde{\nu} \circ \tilde{\pi}$. Since \tilde{N} is normal, the extended map φ lifts to $\tilde{\varphi} : \tilde{N} \rightarrow \tilde{\Omega}_S/\tilde{\Gamma}$. Put $b = \tilde{\varphi}(c)$.

We can take a relatively compact small connected neighborhood B centered at b such that each connected component of $\tilde{\pi}^{-1}(B)$ is biholomorphic to B via $\tilde{\pi}$. Since $\tilde{\varphi}$ is continuous, there is a connected neighborhood $W \subset \tilde{N}$ of c in \tilde{S} such that $\tilde{\varphi}(W) \subset B$. Since the Hausdorff dimension of $S \setminus \Omega_S$ is less than 1 by assumption, so is $\tilde{S} \setminus \tilde{\Omega}_S$. Therefore we can assume that $W \cap \tilde{\Omega}_S$ is connected. Since $W \cap \tilde{\Omega}_S$ is connected, it is contained in a connected component, say \tilde{B} , of $\tilde{\pi}^{-1}(B)$. Thus we have a lifting $\psi : W \rightarrow \tilde{B}$ of $\tilde{\varphi}|_W$. Obviously ψ is the holomorphic extension of the inclusion $j : W \cap \tilde{\Omega}_S \rightarrow \tilde{\Omega}_S$. Since \tilde{B} is relatively compact in $\tilde{\Omega}_S$, $j(W \cap \tilde{\Omega}_S) = \psi(W \cap \tilde{\Omega}_S)$ is relatively compact in $\tilde{\Omega}_S$. This is absurd. Thus our claim is verified.

Since a is on the boundary of $\tilde{C} \cap \tilde{\Omega}_S$ and is a non-singular point of \tilde{S} and \tilde{C} , the inclusion relation (23) implies that $\tilde{C} \setminus \tilde{\Omega}_S$ has no accumulation points other than the singular points of \tilde{S} and \tilde{C} . Hence, $C \setminus \Omega_S$ is a non-empty closed countable subset of C .

Take the normalization $\mu : \hat{C} \rightarrow C$ of C . Then $\mu^{-1}(C \setminus \Omega_S)$ is a non-empty closed countable set. The group Γ induces an infinite group $\hat{\Gamma}$ of holomorphic automorphisms of \hat{C} and the quotient $\mu^{-1}(C \cap \Omega_S)/\hat{\Gamma}$ is compact. By a theorem on the cardinality of ends due to Hopf [H], the complement $\mu^{-1}(C \setminus \Omega_S)$ should be a set of cardinality less than three.

Since \hat{C} admits an infinite group of holomorphic automorphisms, \hat{C} is either \mathbf{P}^1 or an elliptic curve. Since $\mu^{-1}(C \setminus \Omega_S)$ is non-empty and $\hat{\Gamma}$ -invariant, \hat{C} is not an elliptic curve. Hence $\hat{C} \simeq \mathbf{P}^1$ and the set $\mu^{-1}(C \setminus \Omega_S)$ consists of at most 2 points and so does $C \setminus \Omega_S$. If $C \setminus \Omega_S$ consists of a single point, then $\Gamma_0 = \Gamma \simeq \mathbf{Z}^2$

or \mathbf{Z} . If $C \setminus \Omega_S$ consists of two points, then there is a subgroup Γ_0 of Γ with index at most 2 such that $\Gamma_0 \simeq \mathbf{Z}$. In both cases, $(C \cap \Omega_S)/\Gamma_0$ is an elliptic curve. Thus we have the lemma. \square

COROLLARY 3. *Let Γ be a group of holomorphic automorphisms acting on a large domain Ω in \mathbf{P}^3 . Suppose that there are a curve C and a surface S in \mathbf{P}^3 which are Γ -invariant and satisfying $C \subset S$. We assume the following.*

- (1) *The action of Γ on Ω is fixed point free, properly discontinuous and cocompact.*
- (2) *The Hausdorff dimension of $S \setminus \Omega$ is less than 1.*

Then $\Gamma \simeq \mathbf{Z}$ and Ω/Γ is an L-Hopf manifold.

PROOF. Since Ω is large, Γ is a subgroup of $PGL(4, \mathbf{C})$. The corollary follows from Propositions 2, 3 and Lemma 6. \square

COROLLARY 4. *Let Γ be a group of holomorphic automorphisms acting on a large domain Ω in \mathbf{P}^3 . Let C be a curve in \mathbf{P}^3 . We assume the following.*

- (1) *The action of Γ on Ω is fixed point free, properly discontinuous and cocompact.*
- (2) *C is Γ -invariant and $(C \cap \Omega)/\Gamma$ is an elliptic curve.*

Then $\Gamma \simeq \mathbf{Z}$ and Ω/Γ is an L-Hopf manifold.

PROOF. By the assumption (2), Γ is an abelian with rank ≤ 2 . Then the lemma follows from Propositions 2 and 3. \square

5. Compact quotients with positive algebraic dimensions.

In this section, we shall prove our main Theorem 1.1. Let Ω and Γ be as in Section 1 and put $\Lambda = \mathbf{P}^3 \setminus \Omega$. Let $\pi : \Omega \rightarrow \Omega/\Gamma$ be the canonical projection. We assume that the complex manifold $X = \Omega/\Gamma$ admits a non-constant meromorphic function. By a *variety*, we shall mean an irreducible reduced complex space. By a *curve* (resp. *surface*), we shall mean a variety of dimension 1 (resp. 2), unless stated otherwise.

In [K5], we have shown the following fact.

THEOREM 5.1 ([K5, Theorem A]). *Suppose that $X = \Omega/\Gamma$ admits a non-constant meromorphic function. Then the complement $\mathbf{P}^3 \setminus \Omega$ is contained in $S \cup A$, where S is a finite union of complex hypersurfaces in \mathbf{P}^3 , and A is a closed subset of $\mathbf{P}^3 \setminus S$ with the Hausdorff dimension of A not more than 2. In particular, Ω is dense in \mathbf{P}^3 .*

By Theorems 5.1 and 3.1, we have $\Omega = \Omega(\Gamma)$ in this case. Hence $\Lambda = \Lambda(\Gamma)$. Thus we have the following.

PROPOSITION 4. Λ is a union of lines.

Note that Λ may well contain uncountably many lines.

5.1. The algebraic reduction.

A non-constant meromorphic function f defines a meromorphic map $X \dashrightarrow \mathbf{P}^1$. Since Ω is large, π^*f extends to a Γ -invariant rational function F on \mathbf{P}^3 . Thus we have a commutative diagram of meromorphic maps

$$\begin{array}{ccc} \Omega & \xrightarrow{i} & \mathbf{P}^3 \\ \pi \downarrow & & \downarrow F \\ X & \dashrightarrow f & \mathbf{P}^1, \end{array}$$

where i is the natural inclusion, and π is the canonical projection. We eliminate the base locus of F by successive blowing-ups of \mathbf{P}^3 to obtain a non-singular 3-manifold M and a bimeromorphic holomorphic map $u : M \rightarrow \mathbf{P}^3$. Then $u^*F : M \rightarrow \mathbf{P}^1$ is holomorphic. Consider the Stein factorization of u^*F , then we obtain a ramified covering $v : C \rightarrow \mathbf{P}^1$ with the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\tilde{F}} & C \\ u \downarrow & & \downarrow v \\ \mathbf{P}^3 & \dashrightarrow F & \mathbf{P}^1, \end{array} \tag{24}$$

where $C \simeq \mathbf{P}^1$ and \tilde{F} is a surjective holomorphic map with connected fibres. Each element of Γ induces a bimeromorphic map of M and a biholomorphic map of C . Since the group of automorphisms γ of C induced by Γ which satisfy $v \circ \gamma = v$ is finite, we can choose a normal subgroup Γ_1 of Γ with a finite index such that each element $g \in \Gamma_1$ induces an identity map on C . Thus replacing $X = \Omega/\Gamma$ with $X_1 = \Omega/\Gamma_1$, we can assume that each member of the pencil

$$S_t = \{z \in \mathbf{P}^3 : F(z) = t\}, \quad t \in \mathbf{P}^1,$$

is Γ -invariant. Further, replacing Γ_1 with its subgroup of finite index if necessary, we assume that all irreducible components of the members of the pencil are Γ -invariant. In the course of the proof of Theorem 5.1, we see that the analytic set S appeared in that theorem is a finite union of members of the pencil which correspond to singular fibres of the algebraic reduction.

Let B denote the base locus of the pencil. Except for a finite number of points, say $t \in \mathbf{P}^1$, $t = a_1, \dots, a_s$, $s \geq 0$, $F(z) = t$ defines a reduced irreducible algebraic set and S_t is non-singular outside B . For $t \in \mathbf{P}^1$, we put

$$\Omega_t = \Omega \cap S_t, \quad \Lambda_t = \Lambda \cap S_t.$$

LEMMA 7. *The set*

$$\mathcal{E} = \{t \in \mathbf{P}^1 : \Lambda_t \setminus B \text{ has positive Hausdorff dimension}\}.$$

has Lebesgue measure zero in \mathbf{P}^1 .

PROOF. See [K5, Proposition 2.1].

PROPOSITION 5. *If X is not an L-Hopf manifold, then $B \subset \Lambda$.*

PROOF. Suppose that there is an irreducible component C of B such that $C \not\subset \Lambda$. Since B is Γ -invariant, and since it has only finite number of irreducible components, there is a subgroup Γ_1 of Γ with finite index such that C is Γ_1 -invariant. By Lemma 7, there is a member S_t with $t \in \mathbf{P}^1 \setminus (\mathcal{E} \cup \{a_1, \dots, a_s\})$. Since $C \subset S_t$, we have the proposition by Corollary 3. \square

REMARK 1. For L-Hopf manifolds, B is not necessarily contained in Λ .

LEMMA 8. *If X is not an L-Hopf manifold, then the meromorphic function field of X is isomorphic to the pure transcendental extension of \mathbf{C} of degree 1.*

PROOF. Suppose that there are two meromorphic functions f^1, f^2 on X , which are algebraically independent one another. Then, there are 2 pencils

$$S_t^\nu = \{z \in \mathbf{P}^3 : F^\nu(z) = t\}, \quad t \in \mathbf{P}^1,$$

where F^ν is the rational function on \mathbf{P}^3 obtained by extending $\pi^* f^\nu$, $\nu = 1, 2$. Choosing $t_1, t_2 \in \mathbf{P}^1$ suitably, we have a Γ_1 -invariant curve C in the intersection $S_{t_1}^1 \cap S_{t_2}^2$, where Γ_1 is a subgroup of Γ with a finite index. Then, by the same argument as in the proof of Proposition 5, X is an L-Hopf manifold. Let $\varphi : X \dashrightarrow C$ be the algebraic reduction of X over a curve C . Since X contains many lines, $\varphi|_\ell$ is non-trivial for some line ℓ . Hence C is rational. \square

REMARK 2. For L-Hopf manifolds, the algebraic dimensions can be 0, 1, and 2.

5.2. Singular fibres of the algebraic reduction.

In the following in Section 5, we impose on X the following

ASSUMPTION. X is neither \mathbf{P}^3 , an L-Hopf manifold, nor Blanchard manifold. (25)

Our aim is to disprove the existence of X under the assumption above. By Proposition 5, we have

$$B \subset \Lambda. \tag{26}$$

By (26), we can choose a meromorphic function f on X such that $f : X \rightarrow \mathbf{P}^1$ gives the *holomorphic* algebraic reduction of X . In the following, we indicate by $f^{-1}(t)$ the reduced complex analytic subset in X defined by $f(x) = t$. We indicate $f^{-1}(t)$ by $f^*(t)$ when we consider it as a complex space with the structure sheaf \mathcal{O}_X/f^*m_t , or the effective divisor on X defined by f^*m_t , where m_t is the maximal ideal in $\mathcal{O}_{\mathbf{P}^1,t}$. Put

$$\mathcal{A} = \{t \in \mathbf{P}^1 : f^*(t) \text{ is singular}\}. \tag{27}$$

For any $t \in \mathbf{P}^1$, we have $S_t \cap \Omega \subset S_t \setminus B$ by (26), and

$$f^{-1}(t) = (S_t \cap \Omega)/\Gamma.$$

PROPOSITION 6. *Any fibre of f contains no positive dimensional images of a simply connected manifold.*

PROOF. This follows from Proposition 5, and consequently, the fact that $S_t \cap \Omega$ is a subdomain of the affine variety $S_t \setminus B$. □

We recall the following key fact.

LEMMA 9 ([K1, Lemma 5.9]).*) *If the algebraic dimension of X is positive, then there is a subgroup Γ_1 of finite index in Γ such that Γ_1 leaves invariant a plane H in \mathbf{P}^3 .*

Replacing Γ with Γ_1 , we can assume that Γ leaves the plane H invariant. Put

$$M = (H \cap \Omega)/\Gamma.$$

*) Another proof will appear in [K6].

LEMMA 10. *M is contained in a single fibre of the algebraic reduction.*

PROOF. Suppose that a connected component M_0 of M is not contained in any fibres of f . Then, for any $t \in \mathbf{P}^1$, $H \cap S_t$ is a closed algebraic curve in \mathbf{P}^3 . For $t \in \mathbf{P}^1 \setminus (\mathcal{E} \cup \mathcal{A})$, the Hausdorff dimension of $(S_t \setminus B) \cap \Lambda$ is equal to zero. Hence the curve $C_t = H \cap S_t$ intersects Λ in a set of Hausdorff dimension zero outside B . Therefore C_t contains an irreducible component C such that $C \cap (B \cup \Lambda)$ is of Hausdorff dimension zero. Since there is a finite index subgroup Γ_1 of Γ such that Γ_1 leaves C invariants, we see that X would be an L-Hopf manifold by Corollary 3. Thus, by Assumption (25), each connected component of M is contained in a fibre of f .

Now suppose that there are connected components M_1 and M_2 of M such that $f(M_1) \neq f(M_2)$. Let $a_j = f(M_j)$, $j = 1, 2$. Then the meromorphic function $F|_H$ has distinct constant values a_1 and a_2 on non-empty open sets of H . This is absurd. Therefore all the connected components of M are contained in a single fibre of the algebraic reduction. \square

By Lemma 10, we can assume

$$M \subset f^{-1}(0) \tag{28}$$

without loss of generality.

We insert here an easy lemma. Suppose that an infinite group $G \subset PSL(3, \mathbf{C})$ acting on \mathbf{P}^2 leaves invariant a curve $C \subset \mathbf{P}^2$. By a theorem of Burnside on infinite subgroups of matrices, G contains an element $\gamma \in G$ of the infinite order. Replacing γ with its suitable power, and choosing a suitable system of homogeneous coordinates $[z_0 : z_1 : z_2]$ on \mathbf{P}^2 , we can write γ as one of the following matrices.

$$\begin{array}{lll}
 \text{(a)} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} & \text{(b)} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{(c)} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \alpha \end{pmatrix} \\
 \text{(d)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{pmatrix} & \text{(e)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha^p & 0 \\ 0 & 0 & \alpha^q \end{pmatrix} & \text{(f)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}
 \end{array} \tag{29}$$

Here $p, q \in \mathbf{Z}$ with $1 \leq p < q$, $\text{gcd}(p, q) = 1$, and α, β are constant complex numbers which satisfy no relations such that $\alpha^r = 1$, $\beta^s = 1$, $\alpha^r \beta^s = 1$ with $r, s \in \mathbf{Z}$, $rs \neq 0$.

LEMMA 11. Let $\varphi(z_0, z_1, z_2)$ be the defining homogeneous polynomial of C . Put $m = \deg \varphi$. Then we have the following.

1. If Γ contains an element γ of the form (a), then φ is of the form

$$\varphi(z_0, z_1, z_2) = \sum_{i+2j=m} a_{ij} z_0^i w^j, \quad w = z_0 z_1 + 2z_0 z_2 - z_1^2$$

where $a_{ij} \in \mathbf{C}$.

2. If Γ contains an element γ of the form (b), then φ is of the form

$$\varphi(z_0, z_1, z_2) = G(z_0, z_2),$$

where $G(x, y) \in \mathbf{C}[x, y]$ is a homogeneous polynomial of $\deg G = m$.

3. If Γ contains an element γ of the form (c), then φ is a monomial of the form

$$\varphi(z_0, z_1, z_2) = a z_0^i z_2^{m-i}$$

where $a \in \mathbf{C}^*$.

4. If Γ contains an element γ of the form (d), then φ is of the form

$$\varphi(z_0, z_1, z_2) = z_2^k G(z_0, z_1)$$

where $G(x, y) \in \mathbf{C}[x, y]$ is a homogeneous polynomial with $\deg G = m - k$.

5. If Γ contains an element γ of the form (e), then φ is of the form

$$\varphi(z_0, z_1, z_2) = \sum_{i=0}^m a_i z_0^{m-i} z_1^{j_i} z_2^{k_i}$$

where $a_i \in \mathbf{C}$, $j_i + k_i = i$ and $p j_i + q k_i = n$. Here $n \in \mathbf{N}$ is a constant independent of i .

6. If Γ contains an element γ of the form (f), then φ is a monomial

$$\varphi(z_0, z_1, z_2) = a z_0^i z_1^j z_2^k$$

where $a \in \mathbf{C}^*$ and $i + j + k = m$.

PROOF. Easy by calculation. □

Now we go back to studying the fibre $f^{-1}(0)$ which contains M . Suppose that

$f^{-1}(0)$ contains an irreducible component D which is not contained in M and $D \cap M \neq \emptyset$. Then, the hypersurface $S_0 \subset \mathbf{P}^3$ contains H and \tilde{D} as its irreducible components, where $\pi(\Omega \cap \tilde{D}) = D$. The set $C = H \cap \tilde{D}$ is non-empty. Put $B_H = B \cap H$. Replacing Γ with its subgroup of a finite index, we can assume that Γ leaves invariant every irreducible components of C and B_H . Note that $\emptyset \neq B_H \subset H$, $B_H \cap \Omega = \emptyset$, $\emptyset \neq C \subset H$, and $C \cap \Omega \neq \emptyset$. Note also that every compact curve on H intersects B_H , since $H \setminus B_H$ is an affine open set. Let Γ_H denote the restrictions of Γ to H . Note that the restriction of Γ to Γ_H is an isomorphism.

LEMMA 12. *If Γ_H contains an element of an infinite order which is conjugate to one of (b), (c), (d), (f) in the list of (29), then Γ is an Abelian group with rank ≤ 2 .*

PROOF. Suppose that $\gamma \in \Gamma_H$ is the element of the infinite order which is conjugate in $PSL(3, \mathbf{C})$ to one of (b), (c), (d), (f) in the list (29). Then plane curves left invariant by Γ are among the curves defined by φ of the forms (2), (3), (4) of (6) in Lemma 11. Therefore we see that $C \cup B_H$ is a finite union of projective lines. This implies that Γ leaves invariant at least a projective line ℓ_0 in B_H and a non-empty union of projective lines C . Note that $H \cap \Omega \subset H \setminus B_H \subset H \setminus \ell_0 = \mathbf{C}^2$. Therefore Γ acts on $H \cap \Omega$ and on $C \cap \Omega$ as an affine transformation group. Hence $(C \cap \Omega)/\Gamma$ is a union of compact curves with holomorphic affine structures, namely, a union of elliptic curves. Therefore Γ is an Abelian with rank ≤ 2 . \square

LEMMA 13. *If Γ_H contains an element which is conjugate in $PGL(2, \mathbf{C})$ to (a) or (e) in the list of (29), then Γ contains an Abelian subgroup of a finite index with rank ≤ 2 .*

PROOF. The Γ_H -invariant set $C \cup B_H$ is contained in the set $\{\varphi = 0\}$, where φ is a homogeneous polynomial of Lemma 11.

First consider the case (e). In this case, φ is given by Lemma 11(5). Note that j_i, k_i are positive integers given by

$$j_i = \frac{q}{q-p}i - \frac{n}{q-p}, \quad k_i = \frac{-p}{q-p}i - \frac{n}{q-p}.$$

Therefore φ is rewritten as

$$\varphi(z_0, z_1, z_2) = \left(z_1^{\frac{-n}{q-p}} z_2^{\frac{n}{q-p}} \right) \sum_{i=0}^m a_i z_0^{m-i} \left(z_1^{\frac{q}{q-p}} z_2^{\frac{-p}{q-p}} \right)^i.$$

Therefore outside $\{z_1 z_2 = 0\}$, curves defined by $\varphi = 0$ are given locally by $b'z_0 =$

$c'z_1^{\frac{q}{q-p}}z_2^{\frac{-p}{q-p}}$ for some $b', c' \in \mathbf{C}$. Thus the irreducible curves in $\{\varphi = 0\}$ are contained in curves of the form

$$C_{b,c} : bz_0^{q-p}z_2^p = cz_1^q,$$

where b, c are constants in \mathbf{C} . Since $C \cap \Omega \neq \emptyset$, there is a curve $C_{b,c} \subset \{\varphi = 0\}$ such that $C_{b,c} \cap \Omega \neq \emptyset$.

Suppose that $bc = 0$. Then $C_{b,c}$ consists of projective lines. There is a line ℓ in $C_{b,c}$ such that $\ell \cap \Omega \neq \emptyset$. The action of Γ on $\ell \cap \Omega$ is free and properly discontinuous. Since Γ fixes points $\ell \cap B_H \neq \emptyset$, we see that $(\ell \cap \Omega)/\Gamma$ is a compact curve with a holomorphic affine structure, namely, an elliptic curve. Therefore Γ is an Abelian with rank ≤ 2 .

Suppose that $bc \neq 0$. Then $C_{b,c}$ is an irreducible rational curve. Since $C_{b,c} \cap \Omega \neq \emptyset$, the action of Γ on $C_{b,c} \cap \Omega$ is free and properly discontinuous. Hence Γ is isomorphic to a subgroup of $\text{Aut}(C_{b,c}) \simeq PGL(2, \mathbf{C})$. Since Γ leaves invariant the set $B_H \cap C_{b,c}$, which is non-empty. Therefore Γ is an elementary Kleinian group. Therefore Γ contains an Abelian subgroup with rank ≤ 2 of finite index.

Next consider the case (a). In this case φ is of the form Lemma 11(1). Thus $C \cup B_H$ is contained in a finite union of non-singular curves C_a of the form

$$az_0^2 = w, \quad a \in \mathbf{C}$$

and the line $\ell_0 = \{z_0 = 0\}$. Since $C \cap \Omega$ is non-empty, either $\ell_0 \subset C$ and $\ell_0 \cap \Omega \neq \emptyset$, or $C_a \subset C$ and $C_a \cap \Omega \neq \emptyset$ for some a . The rest of the argument is exactly the same as in the case (e). □

Combining Propositions 2, 3, and Lemmas 12, 13, we have the following.

PROPOSITION 7. *The fibre which contains M is irreducible, i.e., $M = f^{-1}(0)$.*

PROPOSITION 8. *$X \setminus f^{-1}(0)$ admits a holomorphic affine structure.*

PROOF. By Proposition 7 and Lemma 10, we have

$$X \setminus f^{-1}(0) = X \setminus M = (\Omega \setminus H)/\Gamma.$$

Since Γ acts on $\Omega \setminus H \subset \mathbf{P}^3 \setminus H = \mathbf{C}^3$ as an affine transformation group, we have the lemma. □

REMARK 3. If $0 \notin \mathcal{A}$, then X is an L-Hopf manifold or a Blanchard manifold. This fact would be proved later in a more general setting.

5.3. Singularities of singular fibres.

We use the following easy fact.

LEMMA 14. *Let Z be a compact variety with $\dim Z \geq 1$, and Ω a non-empty open (connected non-singular) submanifold of Z . Let Y be a closed analytic subset in Z with $\dim Y \leq \dim Z - 1$. Assume that there is a group Γ of holomorphic automorphisms of Z which leaves Ω and Y invariant. If the action of Γ on Ω is fixed point free and properly discontinuous, then $Y \cap \Omega$ has no isolated points.*

PROOF. Suppose that $y \in Y \cap \Omega$ be an isolated point of Z . Then any point of the Γ -orbit of y is isolated $Z \cap \Omega$, since Γ is properly discontinuous on Ω . This implies that Y has an infinite number of connected components in Ω , since the action of Γ on Ω is fixed point free on Ω . Since Y is an analytic set in a compact space Z , this is absurd. \square

As an immediate consequence of Lemma 14, we have

LEMMA 15. *Let Z be a compact 3-dimensional variety and Ω a non-empty (non-singular, connected) open submanifold of Z . Let Y_j , $j = 1, \dots, m$, be compact surfaces in Z such that $Y_j \cap \Omega \neq \emptyset$. Assume that there is a group Γ of holomorphic automorphisms of Z which leaves Ω and each Y_j invariant. If the action of Γ on Ω is fixed point free and properly discontinuous, then the singular locus of $Y \cap \Omega$ is a disjoint union of non-singular curves, where $Y = \bigcup_{j=1}^m Y_j$.*

The following lemma is easily derived from Lemma 15 and the universal property of monoidal transformations (see, [Fs, p.162] for the definition of monoidal transformations).

LEMMA 16. *Let \bar{Z}_1 be a compact complex 3-dimensional variety and Ω_1 a non-empty open (non-singular, connected) submanifold of \bar{Z}_1 . Let \bar{V}_{1j} , $j = 1, \dots, m$, be compact surfaces in \bar{Z}_1 such that $\bar{V}_{1j} \cap \Omega_1 \neq \emptyset$. Let $\{\bar{C}_{1,k}\}_{k=1}^n$ be the set of compact curves which are contained in the singular locus of $\bar{V}_1 = \bigcup_{j=1}^m \bar{V}_{1,j}$, and intersect Ω_1 .*

I. *Assume that there is a group Γ_1 of holomorphic automorphisms of \bar{Z}_1 which has the following properties.*

- (a) *The group Γ_1 leaves invariant Ω_1 , each \bar{V}_{1j} , and each \bar{C}_{1k} .*
- (b) *The action of Γ_1 on Ω_1 is properly discontinuous and fixed point free.*

Then we have the following.

- (i) *The analytic set \bar{V}_1 has no isolated singular points in Ω_1 .*
- (ii) *Each $\bar{C}_{1,j} \cap \Omega_1$ has no singular points.*
- (iii) *For any $j \neq k$, $\bar{C}_{1,j} \cap \bar{C}_{1,k} \cap \Omega_1$ is empty.*

- II. Let $\mu : \bar{Z}_2 \rightarrow \bar{Z}_1$ be the monoidal transformation of \bar{Z}_1 with the center $\bar{C}_{1,1}$. Put $\Omega_2 = \mu^{-1}(\Omega_1)$. Let $\bar{V}_{2,j}$, $j = 1, \dots, m$, be the proper transformation of $\bar{V}_{1,j}$, $j = 1, \dots, m$, and set $\bar{V}_{2,m+1} = \mu^{-1}(\bar{C}_{1,1})$. Let $\{\bar{C}_{2,k}\}_{k=1}^r$ be the set of all compact curves which are contained in the singular locus of $\bar{V}_2 = \bigcup_{j=1}^{m+1} \bar{V}_{2,j}$, and intersect Ω_2 . Then, under the conditions (a) and (b) above, we have the following.
- (iv) There is a group Γ_2 of holomorphic automorphisms of \bar{Z}_2 and a surjective homomorphism $\mu_* : \Gamma_2 \rightarrow \Gamma_1$ which sends $\sigma \in \Gamma_2$ to $\tau \in \Gamma_1$ such that $\tau \circ \mu = \mu \circ \sigma$.
 - (v) Ω_2 is a (non-singular connected) open submanifold in \bar{Z}_2 .
 - (vi) The group Γ_2 leaves invariant Ω_2 , each $\bar{V}_{2,j}$, and each $\bar{C}_{2,k}$.
 - (vii) The action of Γ_2 on Ω_2 is properly discontinuous and fixed point free.
 - (viii) The analytic set \bar{V}_2 has no isolated singular points in Ω_2 .
 - (ix) Each $\bar{C}_{2,j} \cap \Omega_2$ has no singular points.
 - (x) For any $j \neq k$, $\bar{C}_{2,j} \cap \bar{C}_{2,k} \cap \Omega_2$ is empty.
 - (xi) $\{\bar{C}_{2,j}\}_{j=1}^r = \{\mu^{-1}(\bar{C}_{1,j})\}_{j=2}^m \cup \{\bar{C}'_1, \dots, \bar{C}'_s\}$, where $\bar{C}'_1, \dots, \bar{C}'_s$ are the curves which appear in the intersections of $\bar{V}_{2,m+1} \cap \bar{V}_{2,j}$, $j = 1, \dots, m$, and intersect Ω_2 .
 - (xii) For $j = 1, \dots, s$, μ maps \bar{C}'_j onto $\bar{C}_{1,1}$.
 - (xiii) The induced map $\Omega_2/\Gamma_2 \rightarrow \Omega_1/\Gamma_1$ defines the blowing-up with the center $\pi_1(\bar{C}_{1,1} \cap \Omega_1)$, where π_1 is the canonical projection $\Omega_1 \rightarrow \Omega_1/\Gamma_1$.

LEMMA 17. For any $t \in \mathbf{P}^1$, $S_t \cap \Omega$ is connected.

PROOF. Fix any $t \in \mathbf{P}^1$. We can find an open neighborhood N of $f^{-1}(t)$ in X such that $f^{-1}(t)$ is a deformation retract of N . Put $\tilde{N} = \pi^{-1}(N)$. Then $S_t \cap \Omega$ is a deformation retract of \tilde{N} . Since $\mathbf{P}^1 \setminus \mathcal{E}$ is dense in \mathbf{P}^1 , there is a point $s \in \mathbf{P}^1 \setminus \mathcal{E}$ such that $f^{-1}(s) \subset N$. By the choice of s , the Hausdorff dimension of $(S_s \setminus B) \cap \Omega$ is of Hausdorff dimension zero. Therefore $S_s \cap \Omega = (S_s \setminus B) \cap \Omega$ is connected. Note that $S_s \cap \Omega \subset \tilde{N}$ and that $S_s \cap \Omega = \pi^{-1}(f^{-1}(s))$ has common points with every connected component of \tilde{N} . Therefore the connectedness of $S_s \cap \Omega$ implies that of \tilde{N} . Hence $S_t \cap \Omega$ is connected. \square

Suppose that \mathcal{A} contains points other than 0. Let $f^*(a)$, $a \neq 0$, be a singular fibre and express it as

$$f^*(a) = \sum_{j=1}^{m_a} s_j V_{a,j}, \quad s_j \in \mathbf{N},$$

where m_a is a positive integer, and the $V_{a,j}$ are the irreducible components of $f^{-1}(a)$.

For any V_{aj} , let \bar{V}_{aj} be an irreducible component of the hypersurface $S_a \subset \mathbf{P}^3$ such that $V_{aj} = \pi(\Omega \cap \bar{V}_{aj})$. Since Γ fixes every component of $F^{-1}(a)$, \bar{V}_{aj} is uniquely determined by V_{aj} . Put $V_a = \bigcup_{j=1}^{m_a} V_{aj}$, $\bar{V}_a = \bigcup_{j=1}^{m_a} \bar{V}_{aj}$, $C_{ajk} = V_{aj} \cap V_{ak}$. $\bar{C}_{ajk} = \bar{V}_{aj} \cap \bar{V}_{ak}$. Then $S_a = \bar{V}_a$, $f^{-1}(a) = V_a = \pi(\bar{V}_a \cap \Omega)$, and $C_{ajk} = \pi(\bar{C}_{ajk} \cap \Omega)$ hold.

LEMMA 18. *If the reduced analytic set S_a has singular points, then the singular locus of V_a is a finite disjoint union of non-singular curves with genera ≥ 2 , and all the irreducible components of V_a are algebraic surfaces.*

PROOF. For simplicity, we omit the subscript a in the proof. If $m \geq 2$, then, by Lemma 17 and by the fact that every \bar{V}_j intersects Ω^\dagger , for any integer j , $1 \leq j \leq m$, there is an integer $k \neq j$, $1 \leq k \leq m$, such that $\bar{V}_j \cap \bar{V}_k \cap \Omega \neq \emptyset$. Since V_j contains no rational curves, the curves $C_{jk} = \pi(\bar{V}_j \cap \bar{V}_k \cap \Omega)$ are non-singular with genera ≥ 1 . If there are elliptic curves among C_{jk} 's, then X is an L-Hopf manifold by Corollary 4. Hence all the C_{jk} 's are with genera ≥ 2 . Since the normalization V_j^* of V_j is non-singular, free from rational curves, and since it contains non-singular curves with genus ≥ 2 , we infer that V_j^* is projective algebraic. If $m = 1$ and if V_1 has singular points, then, since the singular locus is a finite set of disjoint non-singular curves of genera ≥ 2 , we see that V_1^* is projective algebraic by the same reason. \square

We construct a sequence of blowing-ups

$$\mu_k : Z_k \rightarrow Z_{k-1}, \quad k = 1, \dots, r, \quad Z_0 = X$$

so that the singular fibres of $f \circ \mu : Z_r \rightarrow \mathbf{P}^1$, $\mu = \mu_1 \circ \dots \circ \mu_r$, define *simple normal crossing divisors without self-intersections*. The sequence of blowing-ups are defined inductively as follows.

Consider the set of surfaces which appear in the singular fibres of f ,

$$\mathcal{V}_0 = \{V \subset Z_0 : V = V_{aj} \text{ for some } a \in \mathcal{A}, \text{ and some } 1 \leq j \leq m_a\}.$$

Note that singular loci of surfaces in \mathcal{V}_0 are union of non-singular curves. Consider also the set of curves which appear as singular loci of the singular fibres of f ,

$$\mathcal{C}_0 = \{C \subset Z_0 : C \text{ is a singular locus of some } f^{-1}(a), a \in \mathcal{A}\}.$$

If \mathcal{C}_0 is empty, then every singular fibre is a multiple of a non-singular surface.

[†]because a line contained in Ω intersects \bar{V}_j

Hence there is nothing to prove. Therefore we assume that $\mathcal{C}_0 \neq \emptyset$.

By Lemma 18, \mathcal{V}_0 consists of algebraic surfaces. We know also that \mathcal{C}_0 consists of non-singular curves of genera ≥ 2 , and that any distinct two curves in \mathcal{C}_0 are disjoint each other. We set $\bar{Z}_0 = \mathbf{P}^3$, $\Omega_0 = \Omega$, $\Gamma_0 = \Gamma$, and $f_0 = f$. Recall that $Z_0 = X$.

As the initial induction step, we have

$$\clubsuit_0 : \quad Z_0, f_0 : Z_0 \rightarrow \mathbf{P}^1, \Omega_0 \subset \bar{Z}_0, \Gamma_0, \mathcal{V}_0, \mathcal{C}_0.$$

We construct $\mu_1 : Z_1 \rightarrow Z_0$ as follows. Let \mathcal{C}'_0 be the set of curves C in \mathcal{C}_0 which satisfy the following three conditions.

1. C is not a singular locus of any single surface in \mathcal{V}_0 ,
2. C is contained in exactly two surfaces in \mathcal{V}_0 , say V' and V'' ,
3. V' and V'' are crossing normally and $C = V' \cap V''$.

If $\mathcal{C}_0 = \mathcal{C}'_0$, then we are done. Otherwise choose any $C_0 \in \mathcal{C}_0 \setminus \mathcal{C}'_0$ and consider the blowing-up $\mu_1 : Z_1 \rightarrow Z_0$ with the center C_0 .

Let $\pi_0 : \Omega_0 \rightarrow Z_0$ be the canonical projection. We can find a compact curve \bar{C}_0 in $\bar{Z}_0 = \mathbf{P}^3$ such that $C_0 = \pi_0(\bar{C}_0 \cap \Omega_0)$. Consider the monoidal transformation $\bar{\mu}_1 : \bar{Z}_1 \rightarrow \bar{Z}_0$ with the center \bar{C}_0 . Put $\Omega_1 = \bar{\mu}_1^{-1}(\Omega_0)$. Then, by Lemma 16, II, Γ_0 induces a group Γ_1 of holomorphic automorphisms of \bar{Z}_1 which acts on Ω_1 , and the action is properly discontinuous and fixed point free. Further, the induced map $\Omega_1/\Gamma_1 \rightarrow \Omega_0/\Gamma_0$ from $\bar{\mu}_1$ coincides with the blowing-up $\mu_1 : Z_1 \rightarrow Z_0$ with the center C_0 .

Define

$$\mathcal{V}_1 = \{V_1 \subset Z_1 : V_1 \text{ is a proper transform of some } V \in \mathcal{V}_0\} \cup \{\mu_1^{-1}(C_0)\}$$

and

$$\begin{aligned} \mathcal{C}_1 = & \{C_1 \subset Z_1 : C_1 \text{ is a proper transform of some } C \in \mathcal{C} \setminus \{C_0\}\} \\ & \cup \{C_1 \subset Z_1 : C_1 \text{ is a component of } \mu_1^{-1}(C_0) \cap V \text{ for some } V \in \mathcal{V}_0\}. \end{aligned}$$

Note that $\mu_1^{-1}(C_0)$ is a non-singular surface, since C_0 is non-singular. Here and from now on, we write by the same symbol the variety and its proper transform of a blowing-up. By Lemma 16, II, \mathcal{V}_1 consists of algebraic surfaces whose singular loci are disjoint union of non-singular curves. We have also that \mathcal{C}_1 consists of non-singular curves of genera ≥ 2 , and that any distinct two curves in \mathcal{C}_1 are disjoint each other. Put $f_1 = f \circ \mu_1$. Thus we get to the next stage,

$$\clubsuit_1 : \quad \mu_1 : Z_1 \rightarrow Z_0, f_1 : Z_1 \rightarrow \mathbf{P}^1, \Omega_1 \subset \bar{Z}_1, \Gamma_1, \mathcal{V}_1, \mathcal{C}_1.$$

Continuing this process, we obtain a sequence of blowing-ups $\{\mu_k\}_{k=1}^r$,

$$\clubsuit_k : \quad \mu_k : Z_k \rightarrow Z_{k-1}, f_k : Z_k \rightarrow \mathbf{P}^1, \Omega_k \subset \bar{Z}_k, \Gamma_k, \mathcal{V}_k, \mathcal{C}_k.$$

By a theorem of Hironaka, $\mathcal{C}_r = \mathcal{C}'_r$ for some integer r . Namely, all the singular fibres of f_r define simple normal crossing divisors without self-intersections. At the same time, we see by the construction above that the singular loci of the singular fibres of f_r are disjoint union of non-singular curves with genera ≥ 2 .

For a divisor D , we indicate by $[D]$ the associated class in $H^2(X, \mathbf{Q})$. For simplicity, a divisor and its proper transform by blowing-up are indicated by the same symbol D .

LEMMA 19. *If the analytic set $f^{-1}(a)$, $a \neq 0$, has a singular point, then every irreducible component of $f^{-1}(a)$ is a rational surface.*

PROOF. Let V be a surface contained in $f^{-1}(a)$. We shall show that the canonical bundle of the proper transform of V in Z_r is numerically equivalent to the negative of a non-trivial effective divisor. Since V is algebraic, this implies that V is rational. In $f_r^{-1}(a)$, there are some surfaces V_{a_1}, \dots, V_{a_p} other than V , and some exceptional divisors E_{a_k} of the blowing-ups $\{\mu_k\}$ such that

$$s[V] \equiv -\sum_{j=1}^p s_j[V_{a_j}] - \sum_{k=1}^r m_k[E_{a_k}],$$

$$s, s_1, \dots, s_p, m_1, \dots, m_r : \text{positive integers.} \quad (30)$$

By Proposition 8, $X \setminus f^{-1}(0)$ has a holomorphic affine structure. Therefore the canonical bundle of K_{Z_0} is numerically trivial on $X \setminus f^{-1}(0)$. Hence on a tubular neighborhood of $f_r^{-1}(a)$, we have

$$K_{Z_r} \equiv \sum_{k=1}^r [E_{a_k}]. \quad (31)$$

Hence we have

$$K_{Z_r} + [V] \equiv \sum_{k=1}^r [E_{a_k}] - \frac{1}{s} \left(\sum_{j=1}^p s_j [V_{a_j}] + \sum_{k=1}^r m_k [E_{a_k}] \right)$$

$$\equiv -\frac{1}{s} \sum_{j=1}^p s_j [V_{a_j}] - \sum_{k=1}^r \left(\frac{m_k}{s} - 1 \right) [E_{ak}].$$

Note that, if $E_{ak} \cap V \neq \emptyset$, then $m_k > s$ holds, since the center of the blowing-up at each step is a singular locus of the fibre which is lying on V . Hence we have

$$K_V = (K_{Z_r} + [V]) \cdot V \equiv -\frac{1}{s} \sum_{j=1}^p s_j [V_{a_j}] \cdot [V] - \sum_{k=1}^r \left(\frac{m_k}{s} - 1 \right) [E_{ak}] \cdot [V].$$

Thus K_V is numerically equivalent to the negative of a non-trivial effective divisor. □

PROPOSITION 9. *Every singular fibre of the algebraic reduction $f : X \rightarrow \mathbf{P}^1$ is a multiple fibre of a non-singular surface.*

PROOF. The fibre $f^{-1}(0)$ is non-singular, since $f^{-1}(0) = M = (\Omega \cap H)/\Gamma$ by Proposition 7, and since $\Omega \cap H$ is non-singular. If the fibre $f^{-1}(a), a \neq 0$, has a singular point, then by Lemma 19, $f^{-1}(a)$ contains a rational surface. This contradicts Proposition 6. □

5.4. Multiple fibres.

By a general result on regular fibres of algebraic reductions of codimension two due to Kawai [Kw], Ueno [U, Remark 12.5], and Fujiki [Fj1], [Fj2], together with Proposition 6, we have the following.

PROPOSITION 10. *A regular fibre of the algebraic reduction $f : X \rightarrow \mathbf{P}^1$ is biholomorphic to either a complex torus, a hyperelliptic surface, a Kodaira surface (i.e., a non-Kähler surface with a trivial canonical bundle), or a surface of Class VII₀.*

On the other hand, we recall that M has a holomorphic projective structure. By Proposition 9, we obtain a complex family of small deformations of M , by a base change at a singular fibre of the algebraic reduction. Therefore, by [KoOc1], [KoOc2], and Proposition 10, we have the following.

PROPOSITION 11. *A regular fibre of the algebraic reduction $f : X \rightarrow \mathbf{P}^1$ is biholomorphic to one of the following.*

- (a) *A complex torus,*
- (b) *a hyperelliptic surface,*
- (c) *a Kodaira surface,*
- (d) *a Hopf surface, or*

(e) an Inoue surface with $b_2 = 0$.

In Proposition 11, all surfaces other than case (d) have Stein surfaces as their universal coverings. First we settle case (d).

PROPOSITION 12. *A regular fibre of the algebraic reduction $f : X \rightarrow \mathbf{P}^1$ is not a Hopf surface.*

PROOF. If a regular fibre is a Hopf surface, the fundamental group of X admits an infinite cyclic subgroup of a finite index. Hence X is an L-Hopf manifold by Proposition 2. This contradicts Assumption (25). \square

Now we consider the remaining cases (a), (b), (c) and (e), where the regular fibres admit Stein surfaces as their universal coverings. For a subset $W \subset \mathbf{P}^1$, we put

$$X_W = f^{-1}(W), \quad S_W = \bigcup_{t \in W} S_t, \quad \Omega_W = S_W \setminus \Lambda.$$

For a complex manifold Y étale over a Stein manifold, we denote by Y_{env} the envelope of holomorphy of Y . Recall that we are working in the cases where $B \subset \Lambda$ holds, see (26).

LEMMA 20. *Let $h : Y \rightarrow D$ be a complex family of deformations over a unit disk D such that $h^{-1}(0)$ is one of the cases (a), (b), (c), and (e). Then, there is a neighborhood Δ of 0 such that the universal covering of $h^{-1}(\Delta)$ is of Stein.*

PROOF. For the cases (a) and (b) are well-known. For the case (e), see Inoue [In]. The case (c) is proved in Appendix, see Proposition 16. \square

LEMMA 21. *Assume that a regular fibre of f is one of the cases (a), (b), (c), and (e). Let W be a complex manifold of dimension 3 étale over a Stein manifold. Then, any $t \in \mathcal{A}$ has a neighborhood $\Delta \subset \mathbf{P}^1$ such that every étale holomorphic map $W \rightarrow X_\Delta$ extends to $W_{\text{env}} \rightarrow X_\Delta$.*

PROOF. Let $\Delta \subset \mathbf{P}^1$ be a small disk such that $\Delta \cap \mathcal{A} = \{t\}$. Since $f^*(t)$ is a multiple of a non-singular surface by Proposition 9, we can get a family of small deformations of $f^{-1}(t)$ by a base change. Since regular fibres are one of (a), (b), (c), and (e), we see that so is the central fibre $f^{-1}(t)$ by topological conditions. Let $\varpi : X'_{\Delta'} \rightarrow \Delta'$ be the fibre space obtained by the base change. Then by Lemma 20, $X'_{\Delta'}$ is 3-probable and $\text{Hex}_3(X'_{\Delta'}) = +\infty$. Put $\Delta^* = \Delta \setminus \{t\}$. Since, any $s \in \Delta^*$ has a neighborhood $\Delta_s \subset \Delta^*$ such that $\text{Hex}_3(X_{\Delta_s}) = +\infty$, we

have $\text{Hex}_3(X_\Delta \setminus X_t) = +\infty$ by [KaOk, Theorem 3]. Therefore by Corollary 5 in Appendix, we have the lemma. \square

LEMMA 22. *Assume that a regular fibre of f is one of the cases (a), (b), (c), and (e). Then, any point $t \in \mathbf{P}^1$ has a neighborhood $\Delta \subset \mathbf{P}^1$ such that $(S_\Delta \setminus \Lambda)_{\text{env}} = S_\Delta \setminus B$.*

PROOF. By Theorem 5.1, we see that the Hausdorff dimension of $(S_{\mathbf{P}^1 \setminus \mathcal{A}} \setminus B) \cap \Lambda$ is not more than 2. Hence, we have

$$(S_\Delta \setminus \Lambda)_{\text{env}} = S_\Delta \setminus B$$

for any open subset $\Delta \subset \mathbf{P}^1 \setminus \mathcal{A}$. Thus the lemma holds if $t \notin \mathcal{A}$. Suppose that $t \in \mathcal{A}$. Let W be a small disk centered at t and $W \cap \mathcal{A} = \{t\}$. Note that $S_W \setminus B$ and $S_W \setminus f^{-1}(t)$ are of Stein. Therefore every holomorphic function on $((S_W \setminus B) \setminus f^{-1}(t)) \cup (\Omega \cap f^{-1}(t))$ extends holomorphically on $S_W \setminus B$ by a theorem of Thullen. Thus we have the lemma also for $t \in \mathcal{A}$. \square

LEMMA 23. *Assume that a regular fibre of f is one of the cases (a), (b), (c), and (e). Then $B = \Lambda$. In particular, Λ is a finite union of lines which are on a Γ -invariant plane in \mathbf{P}^3 .*

PROOF. Since $B \subset \Lambda$, it is enough to show that, any $t \in \mathbf{P}^1$ has a neighborhood Δ such that

$$S_\Delta \setminus B \subset S_\Delta \setminus \Lambda. \tag{32}$$

Take any $t \in \mathbf{P}^1$. If $t \notin \mathcal{A}$, let Δ be the one in Lemma 20. If $t \in \mathcal{A}$, let Δ be the one in Lemma 21. In any case, the canonical map

$$\pi : S_\Delta \setminus \Lambda \rightarrow X_\Delta,$$

which is étale, extends holomorphically to

$$\pi : (S_\Delta \setminus \Lambda)_{\text{env}} \rightarrow X_\Delta,$$

by Lemmas 20 and 21. We know that $(S_\Delta \setminus \Lambda)_{\text{env}} = S_\Delta \setminus B$ by Lemma 22. But, since π does not extend even continuously across any boundary point of $\Omega_\Delta = S_\Delta \setminus \Lambda$ in $S_\Delta \setminus B$, we see that $S_\Delta \setminus \Lambda \supset S_\Delta \setminus B$. This proves (32). Recall that Λ is a union of lines by Proposition 4. Hence, since $B \subset H$, Λ is a finite union of lines which are a Γ -invariant plane in \mathbf{P}^3 . \square

5.5. Proof of the main Theorem.

Recall that we are working under Assumption (25). To prove the main Theorem 1.1, it suffices to show by Propositions 11 and 12 that the regular fibre of the algebraic reduction $f : X \rightarrow \mathbf{P}^1$ is neither a torus (case (a)), a hyperelliptic surface (case (b)), a Kodaira surface (case (c)), nor an Inoue surfaces with $b_2 = 0$ (case (e)).

By Lemma 23, we know that, in the cases (a), (b), (c) and (e), Λ consists of finite number of lines on a Γ -invariant plane, say H , in \mathbf{P}^3 . Let Γ_H denote the restriction of Γ to H . Since $\Omega \cap H \neq \emptyset$ and Γ is fixed point free on Ω , the restriction $\Gamma \rightarrow \Gamma_H$ is an isomorphism.

Suppose that Λ contains at least three lines. If Λ contains three lines in a general position, then Γ_H consists of diagonal matrices. Hence Γ_H is abelian. Then $\text{rank } \Gamma = \text{rank } \Gamma_H \leq 4$ holds by Lemma 5. If $\text{rank } \Gamma_H = 4$, then we are in case IV of Lemma 5, and we see that B consists of a single line. This is a contradiction. If $\text{rank } \Gamma_H = 3$, then we are in case III of Lemma 5. In this case, however, two lines are outside $\Omega \cap H$. This is a contradiction. The cases $\text{rank } \Gamma_H = 2$ do not occur by Proposition 3. If $\text{rank } \Gamma_H = 1$, then X is an L-Hopf manifold, which is also excluded by Assumption (25).

If every triple of lines in Λ is not in a general position, there is a point $v \in H$ through which there are three lines. Then the affine transformations which leave these lines are of the form

$$\begin{cases} x' = x \\ y' = ax + by + c \end{cases},$$

where one of the three lines is the line at infinity. Therefore x defines a holomorphic function on the compact manifold $(\Omega \cap H)/\Gamma$, which reduces to be constant function. This is absurd.

Thus we infer that Λ contains less than three lines. If Λ consists of a single line, when X is a Blanchard manifold, which is excluded by Assumption. Suppose that Λ consists of two lines, say $\Lambda = \ell_1 \cup \ell_2$. Since $B \subset H$, $\Omega \cap H = H \setminus B$ is naturally identified with $\mathcal{C}^* \times \mathcal{C}$. Since the universal covering of $(\Omega \cap H)/\Gamma$ is \mathcal{C}^2 , it is not an Inoue surface with $b_2 = 0$. If $(\Omega \cap H)/\Gamma$ is covered by a torus, then Γ contains an abelian subgroup of finite index. Therefore this case is also excluded by the same argument as above. If $(\Omega \cap H)/\Gamma$ is not covered by a torus, it is a Kodaira surface. By Proposition 14 in Appendix, however, Kodaira surfaces are not covered by $\mathcal{C}^* \times \mathcal{C}$. Thus Theorem 1.1 is proved completely. \square

6. Appendix.

6.1. Holomorphic extension index.

Let $\pi : X \rightarrow \Delta$ be a proper surjective holomorphic map of an m -dimensional complex manifold X to a disk $\Delta = \{t \in \mathbf{C} : |t| < \varepsilon\}$ with $\pi^{-1}(t)$ connected for all t . We assume that π is of maximal rank at every point on $\pi^{-1}(\Delta^*)$, where $\Delta^* = \Delta \setminus \{0\}$. We assume in the following that $X_0 = \pi^{-1}(0)$ is non-singular and the divisor $\pi^*(0)$ is a multiple fibre with multiplicity $\mu \geq 2$.

Consider a disk $\Delta' = \{s \in \mathbf{C} : |s| < \varepsilon^{1/\mu}\}$. The set of points $X' = \{(x, s) | \pi(x) = s^\mu\} \subset X \times \Delta'$ forms a branched covering of X under the projection $\xi : X' \rightarrow X$ to the first factor. The projection $\pi' : X' \rightarrow \Delta'$ to the second factor is a family of $(m - 1)$ -manifold every fibre of which is regular. Then we have the following. See [KaOk], for the terms n -probable and Hex.

PROPOSITION 13. *Assume that X' is n -probable, $\text{Hex}_n(X') = \infty$, and that $\text{Hex}_n(X \setminus X_0) = \infty$. Let W be an étale domain over a Stein manifold of dimension n , and W_{env} the envelope of holomorphy of W . Then, for any holomorphic map $\sigma : W \rightarrow X$, there is an analytic subset A of codimension at least 2 in W_{env} such that σ extends holomorphically to $W_{\text{env}} \setminus A \rightarrow X$. In particular, $\text{Hex}_n(X) \geq 4$ holds.*

PROOF. Since X' is n -probable and since X' is a finite branched covering of X , we see that X is n -probable by [KaOk, Theorem 4]. Hence $X \setminus X_0$ is also n -probable by [K5, Lemma 1.2]. Let H be a Hartogs domain in \mathbf{C}^n , i.e., $H = G_1 \cup G_2$ and

$$G_1 = \{(z_1, \dots, z_n) : |z_j| < r_j, j = 1, \dots, n - 1, |z_n| < 1\},$$

$$G_2 = \{(z_1, \dots, z_n) : |z_j| < 1, j = 1, \dots, n - 1, r_n < |z_n| < 1\},$$

where $0 < r_j < 1, j = 1, \dots, n$. Consider any holomorphic map $\sigma : H \rightarrow X$. By [KaOk, Theorem 2], it is enough to show that σ extends to a holomorphic map $\hat{\sigma} : \hat{H} \rightarrow X$, where $\hat{H} = H_{\text{env}}$ is the unit polydisk associated with H . Consider the holomorphic function $\tau = (\pi \circ \sigma)^*t$ on H . Then τ extends to a holomorphic function $\hat{\tau}$ on \hat{H} . Let \hat{S} be a hypersurface in \hat{H} defined by $\hat{\tau} = 0$. Put $S = \hat{S} \cap H$. By a theorem of Dloussky [D], the envelope of holomorphy of $H \setminus S$ coincides with $\hat{H} \setminus \hat{S}$. Therefore, $\sigma|_{H \setminus S}$ extends to a holomorphic map $\hat{H} \setminus \hat{S} \rightarrow X \setminus X_0$, since $X \setminus X_0$ is n -probable with $\text{Hex}_n(X \setminus X_0) = \infty$ by the assumption. Therefore σ extends to a holomorphic map

$$\sigma_o : (\hat{H} \setminus \hat{S}) \cup H \rightarrow X.$$

Shrinking H a little bit, we can assume that \hat{S} is the restriction of a larger analytic subset \tilde{S} in a larger polydisk \tilde{H} to \hat{H} , $\hat{S} = \tilde{S} \cap \hat{H}$. Here we change coordinates (z_1, \dots, z_n) slightly so that $z_j, j = 1, \dots, z_{n-1}$, are not constant on any component of \hat{S} .

Let \tilde{S}_{sing} be the set of singular points of \tilde{S} . Let $\tilde{N}(\delta)$ be a tubular neighborhood of \tilde{S}_{sing} in \tilde{H} with small radius $\delta > 0$. Denote by $\partial\tilde{N}(\delta)$ the boundary of $\tilde{N}(\delta)$ in \tilde{H} . Put $\hat{N}(\delta) = \tilde{N}(\delta) \cap \hat{H}$ and $\partial\hat{N}(\delta) = \partial\tilde{N}(\delta) \cap \hat{H}$. Put

$$\begin{aligned} W^{n-1}(r) &= \{(z_1, \dots, z_{n-1}) \in \mathbf{C}^{n-1} : |z_1| < r, \dots, |z_{n-1}| < r\}, \\ D^{n-1} &= \{(z_1, \dots, z_{n-1}) \in \mathbf{C}^{n-1} : |z_1| < 1, \dots, |z_n| < 1\}, \\ D_n &= \{z_n \in \mathbf{C} : |z_n| < 1\}, \\ W(r, \delta) &= ((\hat{H} \setminus \hat{S}) \cup (W^{n-1}(r) \times D_n)) \setminus \hat{N}(\delta). \end{aligned}$$

Fix a small $\delta > 0$. Note that σ_o is holomorphic on $W(r, \delta)$ for $0 < r \leq \min\{r_1, \dots, r_{n-1}\}$. Let ρ be the supremum of r such that σ_o extends to a holomorphic map $\sigma_r : W(r, \delta) \rightarrow X$.

Suppose that $\rho < 1$. Then the boundary of $W(\rho, \delta)$ in \hat{H} contains points of $\hat{S} \setminus \partial\hat{N}(\delta)$. Take any point a on the boundary point of $W(\rho, \delta)$ on $\hat{S} \setminus \partial\hat{N}(\delta)$. Take a small open ball $B \subset \hat{H}$ centered at a such that $B \cap \tilde{S}_{\text{sing}} = \emptyset$. Using the local defining equation $s = 0$ of $B \cap \hat{S}$ on B , we define the cyclic branched covering $B' = \{(z, s) \mid z = s^\mu\} \subset B \times \mathbf{C}$ of B under the projection $\eta : B' \rightarrow B$ to the first factor. The map $\sigma_r|_{B \cap W(r, \delta)}$ lifts to a holomorphic map

$$\hat{\sigma}_{B'} : (B' \setminus \{s = 0\}) \cup \eta^{-1}(B \cap W(r, \delta)) \rightarrow X'$$

by $\hat{\sigma}_{B'}(z, s) = (\sigma_r(z), s) \in X' \subset X \times \Delta'$. Since $\text{Hex}_n(X') = +\infty$, $\hat{\sigma}_{B'}$ extends holomorphically to $B' \setminus \eta^{-1}(\hat{N}(\delta)) \rightarrow X'$. Since $\sigma_o \circ \eta = \pi' \circ (\hat{\sigma}_{B'})$ on $B' \setminus \eta^{-1}(\hat{N}(\delta))$, we see that σ_r extends holomorphically to $B \cup W(r, \delta)$. Since a is an arbitrary point on the boundary point of $W(\rho, \delta)$ on $\hat{S} \setminus \partial\hat{N}(\delta)$, we see that there is an $r > \rho$ such that σ_o extends to a holomorphic map $\sigma_r : W(r, \delta) \rightarrow X$. This contradicts the definition of ρ . Hence σ_o extends to a holomorphic map $\sigma_1 : W(1, \delta) = \hat{H} \setminus \hat{N}(\delta) \rightarrow X$.

Since $\delta > 0$ is arbitrary small, we see that σ_1 extend holomorphically to $\hat{H} \setminus \hat{S}_{\text{sing}} \rightarrow X$. At the beginning of the proof, we have shrunk H slightly. Since the shrink can be chosen arbitrary small, we have proved that, for the original H , $\sigma : H \rightarrow X$ can be extended to a holomorphic map $\hat{H} \setminus \hat{S}_{\text{sing}} \rightarrow X$. Thus we have $\text{Hex}_n(X) \geq 4$ and the proposition follows by [KaOk, Theorem 2]. \square

Using the notation above, we have the following corollary.

COROLLARY 5. *Assume that X' is n -probable, $\text{Hex}_n(X') = \infty$, and that $\text{Hex}_n(X \setminus X_0) = \infty$. Let W be an étale domain over a Stein manifold of dimension n , and W_{env} the envelope of holomorphy of W . Then, any étale holomorphic map $\varphi : W \rightarrow X$ extends to an étale holomorphic map $\tilde{\varphi} : W_{\text{env}} \rightarrow X$.*

PROOF. To prove this, we can assume that the map $\sigma : H \rightarrow X$ is étale. In this case the singular set \tilde{S}_{sing} is an empty set. Therefore σ extends holomorphically to $\hat{\sigma} : \hat{H} \rightarrow X$ and $\hat{\sigma}$ is étale. This implies the corollary. \square

6.2. Kodaira surfaces.

A compact non-Kähler surface with a trivial canonical bundle is called a *Kodaira surface*. By Kodaira ([**K**, pp. 785–788]), a Kodaira surface S is biholomorphic to the quotient manifold \mathbf{C}^2/G , where G be a properly discontinuous group of the affine transformations without fixed points of \mathbf{C}^2 generated by

$$g_j(w_1, w_2) = (w_1 + \alpha_j, w_2 + \bar{\alpha}_j w_1 + \beta_j), \quad j = 1, 2, 3, 4. \tag{33}$$

Here m is a fixed positive integer, and β_1, β_2 are linearly independent over \mathbf{R} . Further α_j and β_j satisfy

$$\alpha_1 = \alpha_2 = 0, \quad \bar{\alpha}_3 \alpha_4 - \bar{\alpha}_4 \alpha_3 = m\beta_2 \neq 0.$$

The surface S is an elliptic bundle over an elliptic curve and has numerical invariants $q(S) = \dim H^1(S, \mathcal{O}) = 2$, $h^{1,0}(S) = \dim H^0(S, \Omega^1) = 1$, $b_1(S) = \dim H^1(S, \mathbf{C}) = 3$. We put

$$Z = \mathbf{C}^* \times \mathbf{C} = \{(z_1, z_2) \in \mathbf{C}^2 : z_1 \neq 0\} \subset \mathbf{C}^2.$$

PROPOSITION 14. *A Kodaira surface S is not biholomorphic to a compact quotient manifold of the form Z/H , where H is a group of the affine transformations of \mathbf{C}^2 which acts on Z as a fixed point free properly discontinuous group.*

PROOF. We shall prove the proposition by contradiction. Put $M = Z/H$. Suppose that there is a biholomorphic map

$$\varphi : S \rightarrow M.$$

Then φ lifts to an unramified covering

$$\tilde{\varphi} : \mathbf{C}^2 \rightarrow Z.$$

Let G_0 be the covering transformation group of $\tilde{\varphi}$. Then G_0 is a normal subgroup of G isomorphic to $\pi_1(Z) \simeq \mathbf{Z}$. Let $g_0 \in G$ be a generator of G_0 .

LEMMA 24. *There are integers m_1, m_2 such that $g_0 = g_1^{m_1} g_2^{m_2}$.*

PROOF. Since G_0 is a normal subgroup of G , we have $g_j^{-1} g_0 g_j = g_0^{\pm 1}$ holds for each $j = 1, 2, 3, 4$. Then, by a direct calculation, we can check that g_0 is of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix},$$

where $\beta \in \mathbf{C}$ is a constant. Then the lemma follows easily. \square

Since H leaves the line $z_1 = 0$ invariant, any element $h \in H$ is written as

$$h(z_1, z_2) = (az_1, bz_2 + cz_1 + e), \quad (34)$$

where a, b, c, e are constants in \mathbf{C} depending on h , and $ab \neq 0$. For each $g_j \in G$, there is an element $h_j \in H$ such that

$$\tilde{\varphi} \circ g_j = h_j \circ \tilde{\varphi}. \quad (35)$$

Put $\tilde{\varphi}(w_1, w_2) = (u(w_1, w_2), v(w_1, w_2))$. Then u, v are holomorphic functions on \mathbf{C}^2 . Note that $u \neq 0$ everywhere on \mathbf{C}^2 . By (35), we have

$$u(g_j(w_1, w_2)) = a_j u(w_1, w_2) \quad (36)$$

$$v(g_j(w_1, w_2)) = b_j v(w_1, w_2) + c_j u(w_1, w_2) + e_j, \quad (37)$$

where we put

$$h_j = \begin{pmatrix} a_j & 0 & 0 \\ c_j & b_j & e_j \\ 0 & 0 & 1 \end{pmatrix}.$$

By (36), we have a d -closed holomorphic 1-form du/u on \mathbf{C}^2 , which satisfies

$$\frac{du(g(w))}{u(g(w))} = \frac{du(w)}{u(w)}, \quad w = (w_1, w_2),$$

for any $g \in G$. This relation implies that there is a d -closed holomorphic 1-form ϕ on S such that $\varpi^*\phi = du/u$, where $\varpi : \mathbf{C}^2 \rightarrow S$ is the canonical projection. On the other hand, we see easily that dw_1 is also a d -closed holomorphic 1-form on S . Hence, by $h^{1,0}(S) = 1$, we have

$$\phi = \lambda dw_1$$

for some constant $\lambda \in \mathbf{C}^*$. This shows that u is a holomorphic function on \mathbf{C}^2 of the form

$$u(w_1, w_2) = C_1 e^{\lambda w_1} \tag{38}$$

for some constant $C_1 \in \mathbf{C}^*$. Substituting (38) in (37), and applying $\partial/\partial w_2$ to (37), we have

$$\frac{\partial v}{\partial w_2}(g(w)) = b_j \frac{\partial v}{\partial w_2}(w).$$

Since $\tilde{\varphi}$ is locally bijective, $\partial v/\partial w_2$ never vanishes on \mathbf{C}^2 . Hence

$$\frac{1}{\frac{\partial v}{\partial w_2}(w)} d\left(\frac{\partial v}{\partial w_2}(w)\right)$$

is a G -invariant d -closed holomorphic 1-form on \mathbf{C}^2 . Hence, by $h^{1,0}(S) = 1$, we have

$$\frac{1}{\frac{\partial v}{\partial w_2}(w)} d\left(\frac{\partial v}{\partial w_2}(w)\right) = \mu dw_1$$

for some constant $\mu \in \mathbf{C}$. Integrating this equality, we have

$$v(w_1, w_2) = C_2 w_2 e^{\mu w_1} + \psi(w_1) \tag{39}$$

for a constant $C_2 \in \mathbf{C}^*$ and a holomorphic function ψ on \mathbf{C} . Since $\tilde{\varphi}(g_0(w)) = \tilde{\varphi}(w)$, we have by Lemma 24,

$$u(w_1, w_2 + \beta) = u(w_1, w_2), \quad v(w_1, w_2 + \beta) = v(w_1, w_2), \quad \beta = m_1 \beta_1 + m_2 \beta_2.$$

Hence, it follows from (39) that

$$C_2\beta e^{\mu w_1} = 0.$$

This implies $\beta = 0$ and hence $m_1 = m_2 = 0$. Consequently, we have $g_0 = 1$. This is absurd. □

Now we shall construct a complete family of small deformations of a Kodaira surface. Proposition 16 below is used in proving Lemma 20. The construction of the complete family may be well-known, but we describe it here for the readers convenience.

For $\varepsilon > 0$, we put $\Delta_\varepsilon = \{t \in \mathbf{C} : |t| < \varepsilon\}$. Let ε be a small positive number. Put $\mathscr{W} = \mathbf{C}^2 \times B$, where $B = \{(s, t) \in \mathbf{C}^2 : |s - \alpha_4| < \varepsilon, |t - \beta_4| < \varepsilon\}$. Let $\mathscr{G} = \langle \sigma_j \rangle_{j=1}^4$ be the group of holomorphic automorphisms of \mathscr{W} generated by the following four elements,

$$\sigma_1(w_1, w_2, s, t) = (w_1, w_2 + \beta_1, s, t), \tag{40}$$

$$\sigma_2(w_1, w_2, s, t) = (w_1, w_2 + \beta_2(s), s, t), \tag{41}$$

$$\sigma_3(w_1, w_2, s, t) = (w_1 + \alpha_3, w_2 + \bar{\alpha}_3 w_1 + \beta_3, s, t), \tag{42}$$

$$\sigma_4(w_1, w_2, s, t) = (w_1 + s, w_2 + \bar{\alpha}_4 w_1 + t, s, t), \tag{43}$$

which satisfy the condition

$$\bar{\alpha}_3 s - \bar{\alpha}_4 \alpha_3 = m\beta_2(s).$$

Since $\varepsilon > 0$ is sufficiently small, and since β_1 and $\beta_2(s)$ are linearly independent over \mathbf{R} for $s = \alpha_4$, so are β_1 and $\beta_2(s)$ for all $(s, t) \in B$. Note that $\sigma_j = g_j \times \text{id}_B$ for $j = 1, 2, 3$. We have the relations $\sigma_j \sigma_k = \sigma_k \sigma_j$ unless $\{j, k\} = \{3, 4\}$, and $\sigma_3 \sigma_4 = \sigma_2^m \sigma_4 \sigma_3$.

LEMMA 25. *The group \mathscr{G} is free and properly discontinuous on \mathscr{W} .*

PROOF. Any element $\sigma \in \mathscr{W}$ can be written as $\sigma = \sigma_1^{m_1} \sigma_2^{m_2} \sigma_3^{m_3} \sigma_4^{m_4}$ for some integers m_1, \dots, m_4 . Then σ is given by

$$\left(\begin{array}{ccc} 1 & 0 & m_3 \alpha_3 + m_4 s \\ m_3 \bar{\alpha}_3 + m_4 \bar{\alpha}_4 & 1 & \frac{m_3(m_3-1)}{2} |\alpha_3|^2 + \frac{m_4(m_4-1)}{2} \bar{\alpha}_4 s \\ & & + m_3 m_4 \bar{\alpha}_3 s + m_1 \beta_1 + m_2 \beta_2(s) + m_3 \beta_3 + m_4 t \\ 0 & 0 & 1 \end{array} \right). \tag{44}$$

Fix any ε_1 with $0 < \varepsilon_1 < \varepsilon$ and put $K_1 = \{(s, t) \in B : |s - \alpha_4| \leq \varepsilon_1, |t - \beta_4| \leq \varepsilon_1\}$. Take any positive number M and set $K_2 = \{(w_1, w_2) \in \mathcal{C}^2 : |w_1| \leq M, |w_2| \leq M\}$. Put $K = K_1 \times K_2$. To show that \mathcal{G} is properly discontinuous, it is enough to show that the set

$$\{\sigma \in \mathcal{G} : \sigma(K) \cap K \neq \emptyset\} \tag{45}$$

is finite. Fix any $(s, t) \in K_1$. Suppose that $\sigma(K \cap p_2^{-1}(s, t)) \cap K \neq \emptyset$, where $p_2 : \mathcal{W} \rightarrow B$ is the projection to the second component. Then there is a point $w = (w_1, w_2, s, t) \in K$ such that $\sigma(w) \in K$. Put $\sigma(w) = (w'_1, w'_2, s, t)$. Then we have

$$w'_1 = w_1 + m_3\alpha_3 + m_4s \tag{46}$$

$$w'_2 = (m_3\bar{\alpha}_3 + m_4\bar{\alpha}_4)w_1 + w_2 + \frac{m_3(m_3 - 1)}{2}|\alpha_3|^2 + \frac{m_4(m_4 - 1)}{2}\bar{\alpha}_4s + m_3m_4\bar{\alpha}_3s + m_1\beta_1 + m_2\beta_2(s) + m_3\beta_3 + m_4t. \tag{47}$$

By (46), we have

$$|m_3\alpha_3 + m_4s| \leq 2M. \tag{48}$$

Since α_3, s are linearly independent over \mathbf{R} , the set of the pairs (m_3, m_4) satisfying (49) is finite. Put

$$L = \max\{|m_3|, |m_4| : |m_3\alpha_3 + m_4s| \leq 2M\}.$$

Then, from (47), it follows that there is a constant C which depend only on $M, L, \varepsilon_1, \alpha_3, \alpha_4, \beta_3$ and β_4 such that

$$|m_1\beta_1 + m_2\beta_2(s)| \leq C \tag{49}$$

holds. Since β_1 and $\beta_2(s)$ are linearly independent over \mathbf{R} , the set of integers satisfying (49) is finite. Put

$$N(s, t) = \#\{\sigma \in \mathcal{G} : \sigma(K \cap p^{-1}(s, t)) \cap K \neq \emptyset\}.$$

Suppose that $\sup_{(s,t) \in K_1} N(s, t) = \infty$. Then there is a sequence $(s_n, t_n) \in K_1$ such that $\lim_n N(s_n, t_n) = \infty$. Taking a subsequence, we can assume that $\lim_n (s_n, t_n) = (s_0, t_0) \in K_1$. For each n , we take $\sigma_n \in \mathcal{G}$ such that

$\sigma_n(K \cap p^{-1}(s_n, t_n)) \cap K \neq \emptyset$. Taking a subsequence of (s_n, t_n) again if necessary, we can assume that the members of the sequence $\{\sigma_n\}$ are distinct each other by the fact that $\lim_n N(s_n, t_n) = \infty$. Put $\sigma_n = \sigma_1^{m(n)_1} \sigma_2^{m(n)_2} \sigma_3^{m(n)_3} \sigma_4^{m(n)_4}$. Since $\sigma_n(K \cap p^{-1}(s_n, t_n)) \cap K \neq \emptyset$, we have $|m(n)_3\alpha_3 + m(n)_4s_n| \leq 2M$. If $\lim_n |m(n)_3| = \infty$, then by

$$\left| \alpha_3 + \frac{m(n)_4}{m(n)_3} s_n \right| \leq \frac{2M}{|m(n)_3|},$$

we have

$$\alpha_3 + \left(\lim_n \frac{m(n)_4}{m(n)_3} \right) s_0 = 0.$$

This contradicts the fact that α_3 and s_0 is linearly independent over \mathbf{R} . The case $\lim_n |m(n)_4| = \infty$ is settled by the same manner. If both $\{|m(n)_3|\}$ and $\{|m(n)_4|\}$ are bounded, then either $\{|m(n)_1|\}$ or $\{|m(n)_2|\}$ is unbounded. Note that in this case we see by (47) that $\{|m(n)_1\beta_1 + m(n)_2\beta_2(s_n)|\}$ is bounded. This contradicts the fact that β_1 and $\beta_2(s_0)$ are linearly independent over \mathbf{R} by the same argument as above. Hence $N(s, t)$ is bounded on K_1 . Hence the set (45) is finite. Thus \mathcal{G} is properly discontinuous on \mathcal{W} . It is easy to check that \mathcal{G} is fixed point free on \mathcal{W} . □

Consider the following 4-dimensional complex manifolds

$$\mathcal{X} = \mathcal{W} / \mathcal{G}.$$

Then \mathcal{X} is a space of small deformations of S over B with the projection

$$p : \mathcal{X} \rightarrow B, \quad p(w_1, w_2, s, t) = (s, t), \tag{50}$$

where $S = p^{-1}(\alpha_4, \beta_4)$. Put $S_{(s,t)} = p^{-1}(s, t)$.

LEMMA 26. $\dim H^0(S_{(s,t)}, \Theta) = 1, \dim H^1(S_{(s,t)}, \Theta) = 2.$

PROOF. Since the canonical bundle of $S_{(s,t)}$ is trivial, $\dim H^2(S_{(s,t)}, \Theta) = \dim H^0(S_{(s,t)}, \Omega^1) = 1$ by Serre duality. To calculate $\dim H^0(S_{(s,t)}, \Theta)$, let θ be any vector field on $S_{(s,t)}$. Then θ defines a $\mathcal{G}_{(s,t)}$ -invariant vector field on \mathbf{C}^2 , which is indicated by the same symbol θ . Here $\mathcal{G}_{(s,t)}$ indicates the automorphism of \mathbf{C}^2 with fixed (s, t) . Put

$$\theta(w_1, w_2) = a(w_1, w_2) \frac{\partial}{\partial w_1} + b(w_1, w_2) \frac{\partial}{\partial w_2},$$

where $a(w_1, w_2)$, $b(w_1, w_2)$ are holomorphic functions on \mathbf{C}^2 . By $\mathcal{G}_{(s,t)}$ -invariance, we have

$$\begin{aligned} a(w_1, w_2) &= a(w_1, w_2 + \beta_1) \\ a(w_1, w_2) &= a(w_1, w_2 + \beta_2(s)) \\ a(w_1, w_2) &= a(w_1 + \alpha_3, \bar{\alpha}_3 w_1 + w_2 + \beta_3) \\ a(w_1, w_2) &= a(w_1 + s, \bar{\alpha}_4 w_1 + w_2 + t) \\ b(w_1, w_2) &= b(w_1, w_2 + \beta_1) \\ b(w_1, w_2) &= b(w_1, w_2 + \beta_2(s)) \\ b(w_1, w_2) &= b(w_1 + \alpha_3, \bar{\alpha}_3 w_1 + w_2 + \beta_3) - \bar{\alpha}_3 a(w_1 + \alpha_3, \bar{\alpha}_3 w_1 + w_2 + \beta_3) \\ b(w_1, w_2) &= b(w_1 + s, \bar{\alpha}_4 w_1 + w_2 + t) - \bar{\alpha}_4 a(w_1 + s, \bar{\alpha}_4 w_1 + w_2 + t). \end{aligned}$$

This implies that $a(w_1, w_2)$ reduces to constant, which we denote by a . Thus, for $b(w_1, w_2)$, we have

$$b(w_1, w_2) = b(w_1, w_2 + \beta_1) \tag{51}$$

$$b(w_1, w_2) = b(w_1, w_2 + \beta_2(s)) \tag{52}$$

$$b(w_1, w_2) = b(w_1 + \alpha_3, \bar{\alpha}_3 w_1 + w_2 + \beta_3) - \bar{\alpha}_3 a \tag{53}$$

$$b(w_1, w_2) = b(w_1 + s, \bar{\alpha}_4 w_1 + w_2 + t) - \bar{\alpha}_4 a. \tag{54}$$

By (51) and (52), $b(w_1, w_2)$ is constant with respect to w_2 . Put $b(w_1) = b(w_1, w_2)$. Then, by (53) and (54), we have

$$b(w_1) = b(w_1 + \alpha_3) - \bar{\alpha}_3 a \tag{55}$$

$$b(w_1) = b(w_1 + s) - \bar{\alpha}_4 a. \tag{56}$$

Hence the first order differential function $b'(w_1)$ is constant, and we have

$$b(w_1, w_2) = b(w_1) = aw_1 + b$$

for some constants $a, b \in \mathbf{C}$. Since $\bar{\alpha}_3 s - \bar{\alpha}_4 \alpha_3 \neq 0$, we have $a = 0$. Therefore we obtain

$$\theta(w_1, w_2) = b \frac{\partial}{\partial w_2}.$$

Hence $\dim H^0(S_{(s,t)}, \Theta) = 1$. By the Riemann-Roch theorem and by $c_1^2(S_{(s,t)}) = c_2(S_{(s,t)}) = 0$, we have the lemma. \square

PROPOSITION 15. *The family (50) is complete and effectively parametrized at every point of B .*

PROOF. The projection $\mathbf{C}^2 \rightarrow \mathbf{C}$, $(w_1, w_2) \mapsto w_1$, defines an elliptic bundle over an elliptic curve on each fibre of p . Thus we have a projection

$$q : \mathcal{X} \rightarrow C \times B, \quad C = \mathbf{C} / \langle \tau_{\alpha_3}, \tau_s \rangle,$$

where τ_c indicates the translation on \mathbf{C} defined by $z \mapsto z + c$. Let \mathcal{F} denote the subsheaf of germs of vector fields which are tangential to the fibres of q and \mathcal{Q} the quotient sheaf Θ / \mathcal{F} . Thus we have the following exact sequence of sheaves on \mathcal{X} ,

$$0 \rightarrow \mathcal{F} \rightarrow \Theta \rightarrow \mathcal{Q} \rightarrow 0.$$

By the form of transition functions of the tangent bundle of S , we see that \mathcal{F} is generated by the global vector field $\partial / \partial w_2$, and hence $\mathcal{F} \simeq \mathcal{O}$. At the same time, we see also that all the transition functions of \mathcal{Q} are 1, and the global section is given by the class $[\partial / \partial w_1]$. In particular, we have $\mathcal{Q} \simeq \mathcal{O}$. Thus we have the exact sequence

$$0 \rightarrow H^0(\mathcal{X}, \mathcal{Q}) \rightarrow H^1(\mathcal{X}, \mathcal{F}) \rightarrow H^1(\mathcal{X}, \Theta) \rightarrow H^1(\mathcal{X}, \mathcal{Q}) \rightarrow \dots \quad (57)$$

Let $\pi : \mathcal{W} \rightarrow \mathcal{X}$ be the canonical projection. We choose an open covering $\mathcal{U} = \{U_\lambda\}$ of \mathcal{X} such that each U_λ is evenly covered by π . Define a system of local coordinates $(w_{1\lambda}, w_{2\lambda})$ by $w_{j\lambda} = w_j \circ (\pi|_{\tilde{U}_\lambda})^{-1}$, where \tilde{U}_λ is a connected component of $\pi^{-1}(U_\lambda)$. The transition functions

$$w_\lambda = f_{\lambda\mu}(w_\mu), \quad w_\nu = (w_{1\nu}, w_{2\nu})$$

are given by

$$f_{\lambda\mu} = \sigma_1^{m_{1\lambda\mu}} \sigma_2^{m_{2\lambda\mu}} \sigma_3^{m_{3\lambda\mu}} \sigma_4^{m_{4\lambda\mu}}$$

for some integers $m_{j\lambda\mu}$. By (44), in terms of local coordinates we have

$$w_{1\lambda} = w_{1\mu} + m_{3\lambda\mu}\alpha_3 + m_{4\lambda\nu}s \tag{58}$$

$$w_{2\lambda} = w_{2\mu} + (m_{3\lambda\mu}\bar{\alpha}_3 + m_{4\lambda\mu}\alpha_4)w_{1\mu} + a_{\lambda\mu} + b_{\lambda\mu}s + m_{4\lambda\mu}t \tag{59}$$

for some constant numbers $a_{\lambda\mu}, b_{\lambda\mu} \in \mathbf{C}$.

Let $\rho : T_B \rightarrow H^1(\mathcal{X}, \Theta)$ be the Kodaira-Spencer map. Then, we have

$$\rho\left(\frac{\partial}{\partial s}\right) = \left\{ \frac{\partial w_{1\lambda\mu}}{\partial s} \frac{\partial}{\partial w_{1\lambda}} + \frac{\partial w_{2\lambda\mu}}{\partial s} \frac{\partial}{\partial w_{2\lambda}} \right\} = \left\{ m_{4\lambda\mu} \frac{\partial}{\partial w_{1\lambda}} + b_{\lambda\mu} \frac{\partial}{\partial w_{2\lambda}} \right\} \tag{60}$$

$$\rho\left(\frac{\partial}{\partial t}\right) = \left\{ \frac{\partial w_{2\lambda\mu}}{\partial t} \frac{\partial}{\partial w_{2\lambda}} \right\} = \left\{ m_{4\lambda\mu} \frac{\partial}{\partial w_{2\lambda}} \right\}. \tag{61}$$

The set $\{m_{4\lambda\mu}\}$ defines a 1-cocycle $m_4 \in H^1(\mathcal{X}, \mathbf{Z})$. By (58) and (59), we have that $\partial/\partial w_{2\lambda} = \partial/\partial w_{2\mu} = \dots$ defines the global vector field $\partial/\partial w_2 \in H^0(\mathcal{X}, \mathcal{F})$, and that $[\partial/\partial w_{1\lambda}] = [\partial/\partial w_{1\mu}] = \dots$ defines the global section $[\partial/\partial w_1] \in H^0(\mathcal{X}, \mathcal{D})$. Therefore, the set of integers $\{m_{4\lambda\mu}\}$ defines a 1-cocycle in $H^1(\mathcal{X}, \mathbf{Z})$. Thus to prove that $\rho(\partial/\partial s)$ and $\rho(\partial/\partial t)$ are cohomologically independent in $H^1(\mathcal{X}, \Theta)$, it is enough to show that $\{m_{4\lambda\mu}\}$ is not trivial in $H^1(\mathcal{X}, \mathcal{O})$. Suppose that there is a 0-cochain $\{a_\lambda\} \in C^0(\mathcal{U}, \mathcal{O})$ such that $m_{4\lambda\mu} = a_\mu - a_\lambda$. Then $e^{2\pi i a_\lambda} = e^{2\pi i a_\mu} = \dots$ defines a nowhere vanishing global holomorphic function on \mathcal{X} , hence a holomorphic function of (s, t) , which we denote by $A(s, t)$. Let $a = a(s, t)$ be a holomorphic function on B such that $A(s, t) = e^{2\pi i a(s, t)}$. Put $m_\lambda = a_\lambda - a$. Then we see that $\{m_\lambda\}$ is a set of integers satisfying $m_{4\lambda\mu} = m_\mu - m_\lambda$. We introduce a new system of local coordinates $(u_{1\lambda}, u_{2\lambda}, s, t)$ on U_λ by

$$(u_{1\lambda}, u_{2\lambda}, s, t) = \sigma_4^{m_\lambda}(w_{1\lambda}, w_{2\lambda}, s, t).$$

Then in terms of the new system of local coordinates, the transition functions are given by

$$g_{\lambda\mu} = \sigma_1^{m_{1\lambda\mu}} \sigma_2^{m_{2\lambda\mu}} \sigma_3^{m_{3\lambda\mu}}.$$

This implies that there is a holomorphic étale map $\phi : \mathcal{X} \rightarrow \mathcal{W}/\langle \sigma_1, \sigma_2, \sigma_3 \rangle$ with $p_2 \circ \phi = p$, where $p_2 : \mathcal{W}/\langle \sigma_1, \sigma_2, \sigma_3 \rangle \rightarrow B$ is induced map from the natural projection to the second component. This is absurd, since every compact fibre of p is mapped locally biholomorphically to a fibre of p_2 which is non-compact. Thus we have shown that $\rho(\partial/\partial s)$ and $\rho(\partial/\partial t)$ are cohomologically independent in $H^1(S_{(s,t)}, \Theta)$ for any $(s, t) \in B$. Hence they form a basis in $H^1(S_{(s,t)}, \Theta)$ for any $(s, t) \in B$, since $\dim H^1(S_{(s,t)}, \Theta) = 2$ by Lemma 26. \square

By the above construction of the complete family of small deformations of Kodaira surfaces, we see the following.

PROPOSITION 16. *Let $f : \mathcal{X} \rightarrow \Delta_1$ be a family of deformations of a Kodaira surface $S = f^{-1}(0)$ over a unit disk Δ_1 . Then there are a positive number $\varepsilon > 0$, a free and properly discontinuous group G of holomorphic automorphisms of $\mathbf{C}^2 \times \Delta_\varepsilon$, and a biholomorphic map $\psi : (\mathbf{C}^2 \times \Delta_\varepsilon)/G \rightarrow f^{-1}(\Delta_\varepsilon)$, which satisfy the following.*

1. G acts as a group of affine transformations on $\mathbf{C}^2 \times \{t\}$ for any $t \in \Delta_\varepsilon$,
2. G acts trivially on the second component Δ_ε ,
3. $f \circ \psi = p$ holds, where $p : (\mathbf{C}^2 \times \Delta_\varepsilon)/G \rightarrow \Delta_\varepsilon$ is the projection to the second component.

References

- [D] G. Dloussky, Envelopes d'holomorphie et prolongements d'hypersurfaces, *Seminaire Pierre Lelong, Année 1975–76, Lecture Notes in Math.*, **578**, Springer, 1977, pp. 217–235.
- [Fs] G. Fischer, *Complex Analytic Geometry*, Lecture Notes in Math., **538**, Springer-Verlag, 1976.
- [Fj1] A. Fujiki, Algebraic reduction of twistor spaces of Hopf surfaces, *Osaka J. Math.*, **37** (2000), 847–858.
- [Fj2] A. Fujiki, On the general fiber of an algebraic reduction of a compact complex manifolds of algebraic codimension two, *J. Math. Kyoto Univ.*, **42** (2002), 789–793.
- [GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Pure and Applied Mathematics, Wiley-Interscience [John Wiley & Sons], New York, 1978.
- [H] H. Hopf, Enden offener Räume und unendliche diskontinuierliche Gruppen, *Comment. Math. Helv.*, **16** (1944), 81–100.
- [In] M. Inoue, On surfaces of class VII₀, *Invent. Math.*, **24** (1974), 269–310.
- [Iv] S. M. Ivashkovich, Extension properties of meromorphic mappings with values in non-Kähler complex manifolds, *Ann. of Math.*, **160** (2004), 795–837.
- [K] K. Kodaira, On the structure of compact complex analytic surfaces, I, *Amer. J. Math.*, **86** (1964), 751–798.
- [K1] Ma. Kato, On compact complex 3-folds with lines, *Japanese J. Math.*, **11** (1985), 1–58.
- [K2] Ma. Kato, Compact complex 3-folds with projective structures, The infinite cyclic fundamental group case, *Saitama Math. J.*, **4** (1986), 35–49.
- [K3] Ma. Kato, Factorization of compact complex 3-folds which admit certain projective structures, *Tohoku Math. J.*, **41** (1989), 359–397.
- [K4] Ma. Kato, Examples on an extension problem of holomorphic maps and a holomorphic 1-dimensional foliation, *Tokyo J. Math.*, **13** (1990), 139–146.
- [K5] Ma. Kato, Compact quotients of large domains in a complex projective 3-space, *Tokyo J. Math.*, **29** (2006), 209–232.
- [K6] Ma. Kato, Existence of invariant planes in a complex projective 3-space under discrete projective transformation groups, to appear in *Tokyo J. Math.*
- [KK] Ma. Kato and K. Komada, On Blanchard manifolds, *Tokyo J. Math.*, **30** (2007), 397–401.

- [KaOk] Ma. Kato and N. Okada, On holomorphic maps into compact non-Kähler manifolds, *Ann. Inst. Fourier, Grenoble*, **54** (2004), 1827–1854.
- [KoOc1] S. Kobayashi and T. Ochiai, Holomorphic projective structures on compact complex surfaces, *Math. Ann.*, **249** (1980), 75–94.
- [KoOc2] S. Kobayashi and T. Ochiai, Holomorphic projective structures on compact complex surfaces, II, *Math. Ann.*, **255** (1981), 519–521.
- [KP] M. E. Kapovich and L. D. Potyagailo, On the absence of the Ahlfors and Sullivan theorems for Kleinian groups in higher dimensions, *Siberian math. J.*, **32** (1991), 227–237.
- [Kw] S. Kawai, On compact complex manifolds of complex dimension 3, II, *J. Math. Soc. Japan*, **17** (1965), 438–442.
- [Ms] B. Maskit, *Kleinian groups*, GMW 287, Springer-Verlag, 1988.
- [My] P. J. Myrberg, Untersuchungen über die Automorphen Funktionen Beliebiger Vieler Variablen, *Acta Math.*, **46** (1925), 215–336.
- [N] K. Nishiguchi, Degeneration of K3 surfaces, *J. Math. Kyoto Univ.*, **28** (1988), 267–300.
- [U] K. Ueno, *Classification theory of algebraic varieties and compact complex spaces*, *Lecture Notes Math.*, **439**, Springer-Verlag, 1975.

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