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Double point of self-transverse immersions of $M^{2n} \looparrowright \boldsymbol{R}^{4n-5}$

By Mohammad A. Asadi-Golmankhaneh

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Abstract. A self-transverse immersion of a smooth manifold M^{2n} in $\boldsymbol{R}^{4n-5}$ for $n > 5$ has a double point self-intersection set which is the image of an immersion of a smooth 5-dimensional manifold, cobordant to Dold manifold V^5 or a boundary. We will show that the double point manifold of any such immersion is a boundary. The method of proof is to evaluate the Stiefel-Whitney numbers of the double point self-intersection manifold. By a certain method these numbers can be read off from spherical elements of $H_{4n-5}QMO(2n-5)$, corresponding to the immersions under the Pontrjagin-Thom construction.

1. Introduction.

The classification of manifolds and maps is a difficult problem. Recently, some mathematicians try to classify the immersions up to multiple point manifolds. This problem is not only equivalent to Hopf invariant one problem in codimension one (see [**9**]) but also closely related to Kervaire invariant problem.

The r-fold point manifold problem comes back to Banchoff's work about immersion $P^2 \looparrowright \boldsymbol{R}^3$ given by Boy's surface (see [**4**]). More generally P. J. Eccles in [**8**] describes the problem when the r-fold point manifold is 0-dimensional. For more information in this dimension see also [**9**] and [**10**]. Since the cobordism classes of 1-dimensional manifolds are boundaries, the problem when the r-fold point manifold of an immersion is 1-dimensional is clear up to cobordism. The problem has been investigated by A. Szucs in [**17**], when the r-fold point manifolds are surfaces; with correction by the author and P. J. Eccles in [**3**]. In this paper we will consider the problem when the double point manifolds are 5-dimensional. The cobordism class of 5-dimensional manifolds is generated by boundaries, and 5-dimensional Dold manifold, denoted by V^5. We will show that the Dold manifold V^5 cannot occur as a double point manifold for a given even dimensional manifold M^n and any immersion $f : M^n \looparrowright \boldsymbol{R}^{2n-5}$. We use the algebraic topology and in particular the correspondence between cobordism groups

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and homotopy groups of Thom complexes. In [**2**] we have described a general approach to these problems which gives a method for determining the bordism class of the self-intersection manifold of any immersion. The unoriented bordism class of a manifold can be detected by its Stiefel-Whitney numbers and the Stiefel-Whitney numbers of the self-intersection manifolds of an immersion can be read off from certain homological information about the immersion. Although the introduction for this problem can be found in [**3**], we will give a short introduction.

Let $f : M^n \looparrowright \boldsymbol{R}^{2n-5}$ be a self-transverse immersion of a compact closed smooth n-dimensional manifold M in $(2n-5)$-dimensional Euclidean space. A point of $\boldsymbol{R}^{2n-5}$ is an *r-fold self-intersection point* of the immersion if it is the image under f of r distinct points of the manifold. The self-transversality of f implies that the set of r-fold self-intersection points is itself the image of an immersion

$$\theta_r(f) : \Delta_r(f) \looparrowright \boldsymbol{R}^{2n-5}$$

of a compact manifold $\Delta_r(f)$, the r-fold self-intersection manifold, of dimension $n-(n-5)(r-1)$. Since we have supposed $n > 10$, the above number will be negative if $r > 2$. Therefore, we will investigate the problem when $r = 2$. In the cases $n \leq 10$, we have the multiple point manifolds and detecting of spherical elements needs different techniques. Note that this problem is valid for $n \geq 6$, the cases $n = 6, 8, 9, 10$ has been investigated. If $n = 6, 8, 10$ the double point manifolds are boundaries, but for $n = 9$ there is an immersion of a boundary with double point cobordant to Dold manifold V^5. The cases when the dimension of a manifold is odd and $n > 10$ is still open, but if $n \equiv 1 \mod 8$, I have some idea to solve the problem. If $n \equiv 3, 5, 7 \mod 8$, the problem should be difficult. However, here we will work with arbitrary dimension to find some general results for future references. Our main result is the following.

Theorem 1.1. *Let $f : M^{2n} \looparrowright \boldsymbol{R}^{4n-5}$ be a self-transverse immersion. Then, for $n > 5$ the double point manifolds of such immersions are boundaries.*

The paper is organized as follows: In Section 2 we will describe how we can use the general technique introduced in [**2**] to solve this problem. In Section 3 we will calculate the primitive and $\mathscr{A}_2$-annihilated elements of $H_{2n-5}QMO(n-5)$ and finally in Section 4 we will prove Theorem 1.1 and we will detect the spherical elements which involve height tow elements.

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2. The Stiefel-Whitney numbers.

Let $\mathrm{Imm}(n,k)$ denote the group of bordism classes of immersions $M^n \looparrowright \boldsymbol{R}^{n+k}$ of compact closed smooth manifolds in Euclidean $(n+k)$-space. By general position every immersion is regularly homotopic and so bordant to a self-transverse immersion and so each element of $\mathrm{Imm}(n,k)$ can be represented by a self-transverse immersion. In the same way bordism between self-transverse immersion can be taken to be self-transverse; it is clear that such a bordism will induce a bordism of the immersions of the double point self-intersection map

$$\theta_2 : \mathrm{Imm}(n,k) \to \mathrm{Imm}(n-k,2k).$$

Let $MO(k)$ denote the Thom complex of a universal $O(k)$-bundle $\gamma^k : EO(k) \to BO(k)$. Using the Pontrjagin-Thom construction, Wells in [**19**] describes an isomorphism

$$\phi : \mathrm{Imm}(n,k) \cong \pi^S_{n+k} MO(k).$$

But the stable homotopy group $\pi^S_{n+k}MO(k)$ is isomorphic to the homotopy group $\pi_{n+k}QMO(k)$, where QX denotes the direct limit $\Omega^\infty\Sigma^\infty X = \lim \Omega^n\Sigma^n X$, where Σ denotes the reduced suspension functor and Ω denotes the loop space functor. By considering the $\boldsymbol{Z}/2$-homology Hurewicz homomorphism

$$h : \pi^S_{n+k}MO(k) \cong \pi_{n+k}QMO(k) \longrightarrow H_{n+k}QMO(k)$$

we describe in [**3**] how for a self-transverse immersion $f : M^n \looparrowright \boldsymbol{R}^{n+k}$ corresponding to $\alpha \in \pi^S_{n+k}MO(k)$, the Hurewicz image $h(\alpha) \in H_{n+k}QMO(k)$ determines the normal Stiefel-Whitney numbers of the self-intersection manifold $\Delta_r(f)$. In the case of double point self-intersection manifold may be outlined as:
The quadratic construction on a pointed space X is defined to be

$$D_2X = X \wedge X \rtimes_{\boldsymbol{Z}/2} S^\infty = X \wedge X \times_{\boldsymbol{Z}/2} S^\infty / * \times_{\boldsymbol{Z}/2} S^\infty,$$

where the non-trivial element of the group $\boldsymbol{Z}/2$ acts on $X \wedge X$ by permuting the coordinates and on the infinite sphere S^∞ by the antipodal action. There is a natural map $h^2 : QX \to QD_2X$ known as the stable James-Hopf map which induces stable Hopf invariant $h^2_* : \pi^S_n X \to \pi^S_n D_2X$ (see [**5**] and [**6**]). If the self-transverse immersion $f : M^n \looparrowright \boldsymbol{R}^{n+k}$ corresponds to an element $\alpha \in \pi^S_{n+k}MO(k)$, then the immersion of the double point self-intersection manifold $\theta_2(f) : \Delta_2(f) \looparrowright \boldsymbol{R}^{n+k}$ corresponds to the element $h^2_*(\alpha) \in \pi^S_{n+k}D_2MO(k)$ given by the stable

Hopf invariant (see [**12**], [**15**], [**16**], [**18**]). The immersion $\theta_2(f)$ corresponds to an element in the stable homotopy of $D_2MO(k)$ because the immersion of the double point self-intersection manifold automatically acquires additional structure on its normal bundle, namely at each point $f(x_1) = f(x_2)$ the normal $2k$-dimensional space is decomposed as the direct sum of the two (unordered) k-dimensional normal spaces of f at the points x_1 and x_2. The universal bundle for this structure is

$$\gamma^k \times \gamma^k \times_{\boldsymbol{Z}/2} 1 : EO(k) \times EO(k) \times_{\boldsymbol{Z}/2} S^\infty \to BO(k) \times BO(k) \times_{\boldsymbol{Z}/2} S^\infty$$

which has the Thom complex $D_2MO(k)$. We consider the map

$$\xi_* : \pi^s_{n+k} D_2MO(k) \to \pi^s_{n+k} MO(2k)$$

induced by the map of Thom complexes $\xi : D_2MO(k) \to MO(2k)$ which comes from the map $BO(k) \times BO(k) \times_{\boldsymbol{Z}/2} S^\infty \to BO(2k)$ classifying the bundle $\gamma^k \times \gamma^k \times_{\boldsymbol{Z}/2} 1$. In homology, observe that by adjointness, the stable James-Hopf map $h^2 : QX \to QD_2X$ gives a stable map $\Sigma^\infty QX \to \Sigma^\infty D_2X$ inducing a map in homology $H_{n+k}QX \to H_{n+k}D_2X$. These give the following commutative diagram.

$$\begin{tikzcd}
\mathrm{Imm}(n,k) \arrow[rr, "\theta_2"] \arrow[d, "\phi", "\cong"'] & & \mathrm{Imm}(n-k,2k) \arrow[d, "\phi", "\cong"'] \\
\pi^S_{n+k} MO(k) \arrow[r, "h^2_*"] \arrow[d, "h"] & \pi^S_{n+k} D_2MO(k) \arrow[r, "\xi_*"] \arrow[d, "h^S"] & \pi^S_{n+k} MO(2k) \arrow[d, "h^S"] \\
H_{n+k} QMO(k) \arrow[r, "h^2_*"] & H_{n+k} D_2MO(k) \arrow[r, "\xi_*"] & H_{n+k} MO(2k)
\end{tikzcd}$$

Diagram (1).

In this diagram the vertical maps ϕ are the Wells isomorphisms, second and third vertical maps in the bottom squares are the stable Hurewicz homomorphism defined by using the fact that the Hurewicz homomorphism commutes with suspension. The first square on the bottom commutes by the definition of the stable Hurewicz map and by naturality, and the second square commutes by naturality. Notice that the normal Stiefel-Whitney numbers (and so bordism class) of the double point self-intersection manifold $\Delta_2(f)$ of an immersion $f : M^{n-k} \looparrowright \boldsymbol{R}^n$ corresponding to $\alpha \in \pi^S_{n+k}MO(k)$ are determined by (and determine) the Hurewicz image $h^S(\beta)$ of the element $\beta = \xi_* h^2_*(\alpha) \in \pi^S_{n+k}MO(2k)$ corresponding to the immersion $\theta_2(f)$. To recognize this we recall the structure of $H_*MO(k)$.

Homology of $\boldsymbol{MO(k)}$ and $\boldsymbol{QMO(k)}$.

Let $e_i \in H_iBO(1) \cong \boldsymbol{Z}_2$ be the non-zero element (for $i \geq 0$). For each sequence $I = (i_1, i_2, \dots, i_k)$ of non-negative integers we define

$$e_I = e_{i_1}e_{i_2}\dots e_{i_k} = (\mu_k)_*\big(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}\big) \in H_*BO(k)$$

where $\mu_k : BO(1)^k \to BO(k)$ is the map which classifies the product of the universal line bundles. The dimension of e_I is $|I| = i_1 + i_2 + \cdots + i_k$.

From the definition of μ_k, $e_{i_1}e_{i_2}\dots e_{i_k} = e_{\sigma(1)}e_{\sigma(2)}\dots e_{\sigma(k)}$ for each $\sigma \in \Sigma_k$, where Σ_k denotes the permutation group on k elements. Thus each such element can be written as $e_{i_1}e_{i_2}\dots e_{i_k}$ where $i_1 \leq i_2 \leq \cdots \leq i_k$ and it follows by counting argument that $\{e_{i_1}e_{i_2}\dots e_{i_k} \mid 0 \leq i_1 \leq i_2 \leq \cdots \leq i_k\}$ is a basis for $H_*BO(k)$. The sphere bundle of the universal $O(k)$-bundle γ^k is given up to homotopy by the inclusion $BO(k-1) \to BO(k)$ and so the Thom complex $MO(k)$ is homotopy equivalent to the quotient space $BO(k)/BO(k-1)$. It follows that $\{e_{i_1}e_{i_2}\dots e_{i_k} \mid 1 \leq i_1 \leq i_2 \leq \cdots \leq i_k\}$ is a basis for $\tilde{H}_*MO(k)$.

Dyer and Lashof (see [**7**] or [**13**]) make use of the Kudo-Araki operations $Q^i : H_mQX \to H_{m+i}QX$ to describe the homology of QX. These operations are trivial for $i < m$ and equal to the Pontrjagin square for $i = m$. If I denotes the sequence $(i_1, i_2, \dots, i_r)$ then we write $Q^Ix = Q^{i_1}Q^{i_2}\dots Q^{i_r}x$. The sequence I is admissible if $i_j \leq i_{j+1}$ for $1 \leq j < r$ and its excess is given by $e(I) = i_1 - i_2 - \cdots - i_r$. With this notation we can give the description of H_*QX as a polynomial algebra: if $\{x_\lambda \mid \lambda \in \Lambda\}$ is a homogeneous basis for $\tilde{H}_*X \subseteq H_*QX$ where X is path-connected space, then

$$H_*QX = \boldsymbol{Z}_2\left[Q^Ix_\lambda \mid \lambda \in \Lambda, I \quad \text{admissible of excess} \quad e(I) > \frac{\dim}{x_\lambda}\right].$$

We may define a height function ht on the monomial generators of H_*QX by $ht(x_\lambda) = 1$, $ht(Q^iu) = 2ht(u)$ and $ht(u \cdot v) = ht(u) + ht(v)$ (where $u \cdot v$ represents the Pontrjagin product).

Now by Diagrams (1), the double point self-intersection manifold of an immersion $M^n \looparrowright \boldsymbol{R}^{2n-5}$ may be identified up to bordism by using the stable Hurewicz homomorphism

$$h^S : \pi^S_{2n-5}MO(2n-10) \to H_{2n-5}MO(2n-10).$$

To determine these note that from the above, $H_{2n-5}MO(2n-10)$ has a basis

$$e_1^{2n-11}e_6, \quad e_1^{2n-12}e_2e_5, \quad e_1^{2n-12}e_3e_4, \quad e_1^{2n-13}e_2^2e_4,$$
$$e_1^{2n-13}e_2e_3^2, \quad e_1^{2n-14}e_2^3e_3, \quad e_1^{2n-15}e_2^5.$$

On the other hand, since $\tilde{H}^*MO(k) \cong w_k\boldsymbol{Z}_2[w_1, w_2, \ldots, w_k]$, where $w_i \in H^iBO(k)$ is the i-th universal Stiefel-Whitney classes (see [**14**, Theorem 7.1]). So the cohomology group $H^{2n-5}MO(2n-10)$ has a basis

$$w_{2n-10}w_1^5, \quad w_{2n-10}w_1^3w_2, \quad w_{2n-10}w_1^2w_3, \quad w_{2n-10}w_1w_4$$
$$w_{2n-10}w_1w_2^2, \quad w_{2n-10}w_2w_3, \quad w_{2n-10}w_5.$$

Now from the vector space duality $H_{2n-5}MO(2n-10)$ has the dual basis

$$\big(w_{2n-10}w_1^5\big)^*, \quad \big(w_{2n-10}w_1^3w_2\big)^*, \quad \big(w_{2n-10}w_1^2w_3\big)^*, \quad (w_{2n-10}w_1w_4)^*$$
$$\big(w_{2n-10}w_1w_2^2\big)^*, \quad (w_{2n-10}w_2w_3)^*, \quad (w_{2n-10}w_5)^*,$$

By using [Theorem 3.4 of [**2**]] we can show that

$$\begin{aligned}
\big(w_{2n-10}w_1^5\big)^* &= e_1^{2n-11}e_6,\\
\big(w_{2n-10}w_1^3w_2\big)^* &= e_1^{2n-11}e_6 + e_1^{2n-12}e_2e_5,\\
\big(w_{2n-10}w_1^2w_2^2\big)^* &= e_1^{2n-11}e_6 + e_1^{2n-12}e_2e_5 + e_1^{2n-12}e_3e_4,\\
\big(w_{2n-10}w_1^2w_3\big)^* &= e_1^{2n-11}e_6 + e_1^{2n-12}e_2e_5 + e_1^{2n-13}e_2^2e_4,\\
(w_{2n-10}w_2w_3)^* &= e_1^{2n-11}e_6 + e_1^{2n-12}e_2e_5 + e_1^{2n-12}e_3e_4 + e_1^{2n-13}e_2e_3^2,\\
(w_{2n-10}w_1w_4)^* &= e_1^{2n-11}e_6 + e_1^{2n-12}e_2e_5 + e_1^{2n-12}e_3e_4 + e_1^{2n-13}e_2^2e_4\\
&\quad + e_1^{2n-13}e_2e_3^2 + e_1^{2n-14}e_2^3e_3,\\
(w_{2n-10}w_5)^* &= e_1^{2n-11}e_6 + e_1^{2n-12}e_2e_5 + e_1^{2n-12}e_3e_4 + e_1^{2n-13}e_2^2e_4\\
&\quad + e_1^{2n-13}e_2e_3^2 + e_1^{2n-14}e_2^3e_3 + e_1^{2n-15}e_2^5.
\end{aligned}$$

Note that the cohomology group $H^5(V^5;\boldsymbol{Z}_2) = \langle cd^2\rangle$, where $c \in H^1(V^5;\boldsymbol{Z}_2)$ and $d \in H^2(V^5;\boldsymbol{Z}_2)$ so by vector space duality, $H_5(V^5;\boldsymbol{Z}_2) \cong \langle (cd^2)^*\rangle$. It is well known that the total normal Stiefel-Whitney class of V^5 is

$$\overline{w} = 1 + d + cd.$$

So $\overline{w}_1 = 0$, $\overline{w}_2 = d$, $\overline{w}_3 = cd$ and $\overline{w}_i = 0$, for all $i \geq 4$. Since the only nonzero class in dimension 5 is $\overline{w}_2\overline{w}_3$ we have $\overline{w}_2\overline{w}_3[V^5] = 1$ and the other normal Stiefel-Whitney numbers are zero, by Whitney duality theorem. Note that $\overline{w}_1 = 0$ means that V^5 is oriented up to cobordism. We collect the above in the following theorem.

THEOREM 2.1. *Let* $f : M^n \longrightarrow \boldsymbol{R}^{2n-5}$ *be a self-transverse immersion. Then the double point manifold of* f *is the Dold manifold* V^5, *(i.e. is not null-cobordant) if and only if*

$$\xi_* h_*^2(h(\alpha)) = e_1^{2n-11}e_6 + e_1^{2n-12}e_2e_5 + e_1^{2n-12}e_3e_4 + e_1^{2n-13}e_2e_3^2,$$

And it is a boundary, (i.e. is null-cobordant) if and only if $\xi_* h_*^2(h(\alpha)) = 0$. *Where* α *denotes the representation of* f *in* $\pi_{2n-5}^S MO(n-5) \cong \pi_{2n-5}QMO(n-5)$ *under the Wells isomorphism.*

PROOF. Since by Diagram (1), $\xi_* h_*^2(h(\alpha))$ represents the double point manifold of immersion $f : M^n \longrightarrow \boldsymbol{R}^{2n-5}$ in $H_{2n-5}MO(2n-10)$ and from the above calculations the only non zero normal Stiefel-Whitney number corresponding to Dold manifold V^5 is the element w_2w_3, dual to $e_1^{2n-11}e_6 + e_1^{2n-12}e_2e_5 + e_1^{2n-12}e_3e_4 + e_1^{2n-13}e_2e_3^2$, the theorem follows. □

For the evaluation of $\xi_* h_*^2(h(\alpha))$ from Diagram (1) and then the cobordism class of the manifold $\Delta_2(f)$ from Theorem 2.1, we need the following lemma which is the special case of Lemma 2.3 in [**3**].

LEMMA 2.2. *The homomorphism* $h_*^2 : \tilde{H}_*QX \to \tilde{H}_*D_2X$ *is given by the projection onto the monomial generators of height* 2. *The kernel is spanned by the set of height other than* 2.

COROLLARY 2.3. *A basis for* $H_{2n-5}D_2MO(n-5)$ *is given by the following elements.*

$$\begin{array}{cccc}
e_1^{n-5} \cdot e_1^{n-6}e_6, & e_1^{n-5} \cdot e_1^{n-7}e_2e_5, & e_1^{n-5} \cdot e_1^{n-7}e_3e_4, & e_1^{n-5} \cdot e_1^{n-8}e_2^2e_4, \\
e_1^{n-5} \cdot e_1^{n-8}e_2e_3^2, & e_1^{n-5} \cdot e_1^{n-9}e_2^3e_3, & e_1^{n-5} \cdot e_1^{n-10}e_2^5, & e_1^{n-6}e_2 \cdot e_1^{n-6}e_5, \\
e_1^{n-6}e_2 \cdot e_1^{n-7}e_2e_4, & e_1^{n-6}e_2 \cdot e_1^{n-7}e_3^2, & e_1^{n-6}e_2 \cdot e_1^{n-8}e_2^2e_3, & e_1^{n-6}e_2 \cdot e_1^{n-9}e_2^4, \\
e_1^{n-6}e_3 \cdot e_1^{n-8}e_2^3, & e_1^{n-6}e_3 \cdot e_1^{n-7}e_2e_3, & e_1^{n-6}e_3 \cdot e_1^{n-6}e_4, & e_1^{n-7}e_2^2 \cdot e_1^{n-8}e_2^3, \\
e_1^{n-7}e_2^2 \cdot e_1^{n-7}e_2e_3, & e_1^{n-7}e_2^2 \cdot e_1^{n-6}e_4, & Q^n e_1^{n-5}, & Q^{n-1}e_1^{n-6}e_2, \\
 & Q^{n-2}e_1^{n-6}e_3, & Q^{n-2}e_1^{n-7}e_2^2. &
\end{array}$$

Finally, we should determine the spherical classes in $H_{2n-5}QMO(n-5)$, i.e.

the classes in the image of

$$h : \pi_{2n-5}QMO(n-5) \to H_{2n-5}QMO(n-5).$$

Then, the images of these classes under the map $\xi_* \circ h_*^2$ determine the double point manifolds, (see Diagram (1)). Although the description of these spherical classes is so difficult, it is not necessary. For it is sufficient to observe the following well-known properties of spherical classes which are immediate from H_*S^n by naturality. Note that this is not sufficient to find the spherical elements but it is helpful to reduce the number of elements to be checked.

Lemma 2.4.

(a) *If a homology class $u \in H_nX$ is spherical then it is primitive with respect to the cup coproduct, that is $\psi(u) = u \otimes 1 + 1 \otimes u$, where $\psi : H_nX \to H_n(X \times X) \cong \Sigma_i H_iX \otimes H_{n-i}X$ is the map induced by the diagonal map.*

(b) *If a homology class $u \in H_nX$ is spherical (or stably spherical, i.e. in the image of $h^S : \pi_n^S X \to H_nX$) then it is $\mathscr{A}_2$-annihilated by the reduced Steenrod algebra, i.e. $Sq_*^i(u) = 0$, for all $i > 0$, where $Sq_*^i : H_nX \to H_{n-i}X$ is the vector space dual of the usual Steenrod square cohomology operation $Sq^i : H^{n-i}X \to H^nX$.*

Remark. Since the r-fold intersection manifold of any immersion $M^n \looparrowright \boldsymbol{R}^{2n-5}$ for $n > 10$ is empty for all $r \geq 3$ we have,

$$H_{2n-5}QMO(n-5) = H_{2n-5}MO(n-5) \oplus H_{2n-5}D_2MO(n-5).$$

The homology group $H_{2n-5}MO(n-5)$ is generated by $e_{i_1}e_{i_2}\cdots e_{i_{n-5}}$ where, $1 \leq i_1 \leq i_2 \leq \cdots \leq i_{n-5}$ and $i_1 + i_2 + \cdots + i_{n-5} = 2n-5$ and the homology group $H_{2n-5}D_2MO(n-5)$ is described in Corollary 2.3. So for $n > 10$ the homology group $H_{2n-5}QMO(n-5)$ corresponding to immersions $M^n \looparrowright \boldsymbol{R}^{2n-5}$ is completely determined.

3. Primitive $\mathscr{A}_2$-annihilated elements.

In order to find the spherical elements of $H_{2n-5}QMO(n-5)$ using Lemma 2.4, first we try to find its primitive submodule. Let ψ denote the cup coproduct and note that $a \in H_*X$ is primitive if and only if

$$\psi(x) = (x \otimes 1 + 1 \otimes x).$$

We need the following lemma from [**1**].

LEMMA 3.1. *A height one element is primitive if and only if the element contains e_1. Moreover if a is primitive, then $Q^k a$ is also primitive.*

But it is possible the linear combination of non-primitive elements to be primitive element. To see which linear combination of non-primitive elements is primitive, note that the height one non-primitive elements by Lemma 3.1 are the following elements. Note that $\psi(e_n) = \sum e_i \otimes e_j$ and external and internal Cartan formula hold (see [**13**, Theorem 1.1]).

$$e_2^{n-6}e_7, \quad e_2^{n-7}e_3e_6, \quad e_2^{n-7}e_4e_5, \quad e_2^{n-8}e_3^2e_5, \quad e_2^{n-8}e_3e_4^2, \quad e_2^{n-9}e_3^3e_4, \quad e_2^{n-10}e_3^5.$$

And the height two non-primitive elements are the elements of Corollary 2.3 except the following primitive elements.

$$Q^n e_1^{n-5}, \quad Q^{n-1}e_1^{n-6}e_2, \quad Q^{n-2}e_1^{n-6}e_3, \quad Q^{n-2}e_1^{n-7}e_2^2,$$

The action of ψ on these elements shows that the following combinations are primitive. Note also that the calculations are so long and therefore we omit them.

$$\begin{aligned}
A &= e_1^{n-5}\cdot e_1^{n-6}e_6 + e_1^{n-6}e_2\cdot e_1^{n-6}e_5 + e_1^{n-6}e_3\cdot e_1^{n-6}e_4 + e_2^{n-6}e_7,\\
B &= e_1^{n-5}\cdot e_1^{n-7}e_2e_5 + e_1^{n-6}e_2\cdot e_1^{n-7}e_2e_4 + e_1^{n-6}e_3\cdot e_1^{n-7}e_2e_3\\
&\quad + e_1^{n-6}e_4\cdot e_1^{n-7}e_2^2 + e_1^{n-6}e_5\cdot e_1^{n-6}e_2 + e_2^{n-7}e_3e_6,\\
C &= e_1^{n-5}\cdot e_1^{n-7}e_3e_4 + e_1^{n-6}e_2\cdot e_1^{n-7}e_3^2 + e_1^{n-6}e_3\cdot e_1^{n-7}e_2e_3 + e_1^{n-6}e_4\cdot e_1^{n-6}e_3\\
&\quad + e_1^{n-6}e_2\cdot e_1^{n-7}e_2e_4 + e_1^{n-7}e_2^2\cdot e_1^{n-7}e_2e_3 + e_2^{n-7}e_4e_5,\\
D &= e_1^{n-5}\cdot e_1^{n-8}e_2^2e_4 + e_1^{n-6}e_2\cdot e_1^{n-8}e_2^2e_3 + e_1^{n-6}e_3\cdot e_1^{n-8}e_2^3 + e_1^{n-6}e_4\cdot e_1^{n-7}e_2^2\\
&\quad + e_2^{n-8}e_3^2e_5,\\
E &= e_1^{n-5}\cdot e_1^{n-8}e_2e_3^2 + e_1^{n-7}e_2^2\cdot e_1^{n-8}e_2^3 + e_1^{n-7}e_3^2\cdot e_1^{n-6}e_2 + e_2^{n-8}e_3e_4^2,\\
F &= e_1^{n-5}\cdot e_1^{n-9}e_2^3e_3 + e_1^{n-6}e_2\cdot e_1^{n-9}e_2^4 + e_1^{n-6}e_3\cdot e_1^{n-8}e_2^3 + e_2^{n-9}e_3^3e_4\\
&\quad + e_1^{n-6}e_2\cdot e_1^{n-8}e_2^2e_3 + e_1^{n-7}e_2^2\cdot e_1^{n-8}e_2^3 + e_1^{n-7}e_2^2\cdot e_1^{n-7}e_2e_3,\\
G &= e_1^{n-5}\cdot e_1^{n-10}e_2^5 + e_1^{n-6}e_2\cdot e_1^{n-9}e_2^4 + e_2^{n-10}e_3^5.
\end{aligned}$$

Therefore we have the following Corollary.

COROLLARY 3.2. *The primitive submodule of $H_{2n-5}(QMO(n-5))$ is gen-*

erated by the following elements.

$$Q^n e_1^{n-5}, \quad Q^{n-1} e_1^{n-6} e_2, \quad Q^{n-2} e_1^{n-7} e_2^2, \quad Q^{n-2} e_1^{n-6} e_3,$$
$$A, \quad B, \quad C, \quad D, \quad E, \quad F, \quad G, \quad \delta.$$

Here δ runs over a basis of primitive height one elements.

If a height one element is spherical, then the double point manifold is a boundary. So we are going to show which of the elements involving height two element of the above Lemma are $\mathscr{A}_2$-annihilated. The action of Steenrod Algebra is given by the Nishida relation and external and internal Cartan formula hold (see [**13**, Theorem 1.1]). Since the mod 2 Steenrod Algebra is generated by Sq^{2^i} and because of dimensional reason the action of $Sq_*^{2^i}$ on the above elements are zero for $i \geq 3$ we will look them, when $i = 0, 1, 2$. Note that this action preserves the height and the following formulas are useful in calculations.

$$Sq_*^i Q^j(a) = \sum_{2k \leq i} \binom{j-i}{i-2k} Q^{j-i+k}(Sq_*^k(a)); \quad Sq_*^i e_j = \binom{j-i}{i} e_{j-i}$$

where Sq_*^i denotes the dual of Sq^i.

From now on we suppose that $n \equiv 0 \mod 2$.

LEMMA 3.3. *Let $n \equiv 0 \mod 2$. The action of Sq_*^1 on the elements of Corollary 3.2 is given by*

$$\begin{aligned}
Sq_*^1 Q^n e_1^{n-5} &= Q^{n-1} e_1^{n-5}, \\
Sq_*^1 Q^{n-1} e_1^{n-6} e_2 &= 0, \\
Sq_*^1 Q^{n-2} e_1^{n-7} e_2^2 &= e_1^{n-7} e_2^2 \cdot e_1^{n-7} e_2^2, \\
Sq_*^1 Q^{n-2} e_1^{n-6} e_3 &= e_1^{n-6} e_3 \cdot e_1^{n-6} e_3, \\
Sq_*^1(A) &= e_1^{n-6} e_3 \cdot e_1^{n-6} e_3 + \delta, \\
Sq_*^1(B) &= e_1^{n-5} \cdot e_1^{n-7} e_2 e_4 + e_1^{n-6} e_2 \cdot e_1^{n-6} e_4 + e_1^{n-6} e_2 \cdot e_1^{n-7} e_2 e_3 \\
&\quad + e_1^{n-6} e_3 \cdot e_1^{n-7} e_2^2 + e_1^{n-6} e_3 \cdot e_1^{n-6} e_3 + \delta, \\
Sq_*^1(C) &= e_1^{n-5} \cdot e_1^{n-7} e_2 e_4 + e_1^{n-6} e_2 \cdot e_1^{n-6} e_4 + e_1^{n-6} e_2 \cdot e_1^{n-7} e_2 e_3 \\
&\quad + e_1^{n-6} e_3 \cdot e_1^{n-7} e_2^2 + \delta,
\end{aligned}$$

$$Sq^1_*(D) = \delta,$$

$$Sq^1_*(E) = e_1^{n-7}e_2^2 \cdot e_1^{n-7}e_2^2 + \delta,$$

$$Sq^1_*(F) = e_1^{n-5} \cdot e_1^{n-9}e_2^4 + e_1^{n-7}e_2^2 \cdot e_1^{n-7}e_2^2 + \delta,$$

$$Sq^1_*(G) = \delta.$$

Here δ denotes a linear combination of primitive height one elements.

PROOF. Just use the formulas mentioned before the lemma. □

From Lemma 3.3 one can show the following elements to be $\mathscr{A}_2$-annihilated.

$$Q^{n-1}e_1^{n-6}e_2, \quad E + Q^{n-2}e_1^{n-7}e_2^2, \quad A + Q^{n-2}e_1^{n-6}e_3, \quad A + B + C, \quad D, \quad G.$$

LEMMA 3.4. *The action of Sq^2_* on the remaining elements are as follows*:

$$Sq^2_*\big(Q^{n-1}e_1^{n-6}e_2\big) = \begin{cases} Q^{n-2}e_1^{n-5} & \text{if } n \equiv 0 \mod 4 \\ Q^{n-3}e_1^{n-6}e_2 + Q^{n-2}e_1^{n-5} & \text{if } n \equiv 2 \mod 4 \end{cases}$$

$$Sq^2_*\big(Q^{n-2}e_1^{n-7}e_2^2\big) = 0,$$

$$Sq^2_*\big(Q^{n-2}e_1^{n-6}e_3\big) = 0,$$

$$Sq^2_*(A) = \delta,$$

$$Sq^2_*(B) = e_1^{n-6}e_3 \cdot e_1^{n-6}e_2 + \delta,$$

$$Sq^2_*(C) = e_1^{n-5} \cdot e_1^{n-6}e_4 + e_1^{n-6}e_2 \cdot e_1^{n-7}e_2^2 + e_1^{n-5} \cdot e_1^{n-7}e_2e_3 + \delta,$$

$$Sq^2_*(D) = e_1^{n-7}e_2^2 \cdot e_1^{n-6}e_2 + e_1^{n-5} \cdot e_1^{n-8}e_2^3 + \delta,$$

$$Sq^2_*(E) = e_1^{n-7}e_2^2 \cdot e_1^{n-6}e_2 + e_1^{n-5} \cdot e_1^{n-8}e_2^3 + \delta,$$

$$Sq^2_*(G) = \delta.$$

Here δ denotes a linear combination of primitive height one elements.

According to the above lemma one can show the following elements to be $\mathscr{A}_2$-annihilated.

$$Q^{n-2}e_1^{n-6}e_3 + A + \delta, \quad Q^{n-2}e_1^{n-7}e_2^2 + D + E + \delta, \quad G + \delta.$$

LEMMA 3.5. *The action of Sq^4_* on the remaining elements are as follows*:

$$Sq^4_*\big(Q^{n-2}e_1^{n-7}e_2^2\big) = Q^{n-4}e_1^{n-5},$$
$$Sq^4_*\big(Q^{n-2}e_1^{n-6}e_3\big) = 0,$$
$$Sq^4_*(A) = \delta,$$
$$Sq^4_*(D) = \delta,$$
$$Sq^4_*(E) = e_1^{n-5} \cdot e_1^{n-6}e_2 + \delta,$$
$$Sq^4_*(G) = \delta.$$

Here δ denotes a linear combination of height one elements.

We sum up all of the above in the following theorem.

THEOREM 3.6. *Let $n \equiv 0 \mod 2$, then the primitive $\mathscr{A}_2$-annihilated submodule of $H_{2n-5}QMO(n-5)$ is generated by the following elements.*

$$Q^{n-2}e_1^{n-6}e_3 + A + \delta, \quad G, \quad \delta,$$

Here δ denotes a primitive combination of height one elements.

4. Detecting spherical elements.

COROLLARY 4.1. *Let $n \equiv 0 \mod 2$. If the element $Q^{n-2}e_1^{n-6}e_3 + A + \delta$ is spherical, then the double point manifolds are boundaries.*

PROOF. By Theorem 3.6 it is primitive and $\mathscr{A}_2$-annihilated. Now if it is spherical then there is an element $\alpha \in \pi_{2n-5}QMO(n-5)$ such that $h(\alpha) = Q^{n-2}e_1^{n-6}e_3 + A + \delta$. But by Theorem 3.1 of [**2**] we have

$$\xi_*Q^{n-2}e_1^{n-6}e_3 = e_1^{2n-12}e_3e_4 + e_1^{2n-12}e_2e_5 + e_1^{2n-11}e_6,$$
$$\xi_*A = e_1^{2n-12}e_3e_4 + e_1^{2n-12}e_2e_5 + e_1^{2n-11}e_6.$$

Now from Diagram (1)

$$\xi_*\big(h^2_*(h(\alpha))\big) = \xi_*\big(Q^{n-2}e_1^{n-6}e_3 + A\big) = 0.$$

Therefore, by Theorem 2.1 the double point manifolds are boundaries. □

COROLLARY 4.2. *If the element G is spherical then the double point manifolds are boundaries.*

Proof. It is primitive and $\mathscr{A}_2$-annihilated. If it is spherical, then there is an element $\alpha \in \pi_{2n-5}QMO(n-5)$ such that

$$h(\alpha) = e_1^{n-5} \cdot e_1^{n-10}e_2^5 + e_1^{n-6}e_2 \cdot e_1^{n-9}e_2^4 + e_2^{n-10}e_3^5 + \delta,$$

So by Theorem 3.1 of [**2**] we have $\xi_* h_*^2 h(\alpha) = 0$. Then the corollary follows from Theorem 2.1. □

Proof of Theorem 1.1.
It follows from Corollaries 4.1 and 4.2.

Some Comments. It is good idea to know these elements are spherical or not. We claim that the element $G + \delta$ is spherical for some height one element δ. The element $Q^{n-2}e_1^{n-6}e_3 + A + \delta$ is not spherical for any height one element δ, when the corresponding manifold is embedded.

Lemma 4.3. *The element $G + \delta$ is spherical for some height one element δ.*

Proof. We prove this lemma in two cases. If n is odd, write $n = 5 + 2^{r_1} + 2^{r_2} + 2^{r_3}$, then consider the manifold $M^n = V^5 \times P^{2^{r_1}} \times P^{2^{r_2}} \times P^{2^{r_3}}$. This manifold is immersed in $\boldsymbol{R}^{2n-5}$ since $\overline{w}(V^5) = 1 + d + cd$ and $\overline{w}(P^{2^{r_i}}) = (1 + a_i)^{2^{r_i}-1}$ for $i = 1, 2, 3$, where $c \in H^1(V^5; \boldsymbol{Z})$, $d \in H^2(V^5; \boldsymbol{Z})$ and $a_i \in H^1(P^{2^{r_i}})$. Therefore $\overline{w}_{n-5}(M) = cd \otimes a_1^{2^{r_1}-1} \otimes a_2^{2^{r_2}-1} \otimes a_3^{2^{r_3}-1}$ and $\overline{w}_5(M) = d \otimes a_1 \otimes a_2 \otimes a_3$. From which we deduce that $\overline{w}_{n-5}\overline{w}_5(M) = cd^2 \otimes a_1^{2^{r_1}} \otimes a_2^{2^{r_2}} \otimes a_3^{2^{r_3}} \neq 0$. This shows that the Hurewicz image of this immersion involves the element $e_2^{n-10}e_3^5$. Let α represent this immersion in $\pi_{2n-5}QMO(n-5)$. So necessarily we have

$$h(\alpha) = G + \delta.$$

This proves lemma in this case. If n is even we write $n = 2^{r_1} + 2^{r_2} + 2^{r_3} + 2^{r_4} + 2^{r_5}$. Then by a similar argument as above we can show that $G + \delta$ is also spherical. □

To prove the element $Q^{n-2}e_1^{n-6}e_3 + A + \delta$ is not spherical we need the following theorem.

Theorem 4.4. *Given $\alpha \in \pi_{2m}(QX) \cong \pi_{2m}^S(X)$, if $h(\alpha) = (u_m^*)^2$ and $u_m^* \in H_m(X)$, then $Sq^{m+1}u_m \neq 0 \in H^{2m+1}C_\alpha$. Here the stable space C_α is the mapping cone of the stable map $\alpha : S^n \to X$.*

Proof. See [**11**] notes of Proposition 4.4 and for more details when $X = P^\infty$ see [**10**]. Also the interested readers can see the proof of Proposition 5.8 of [**3**]. □

Let $Q^{n-2}e_1^{n-6}e_3 + A + \delta$ be spherical. Then there is an element $\alpha \in \pi_{2n-5}QMO(n-5)$ such that

$$h(\alpha) = Q^{n-2}e_1^{n-6}e_3 + A + \delta.$$

If we put this immersion in $\boldsymbol{R}^{2n-4}$, then we will have another immersion with Hurewicz image

$$e_1^{n-5}e_3 \cdot e_1^{n-5}e_3 + e_1e_2^{n-6}e_7 + e_1\delta,$$

Now if the corresponding manifold M is embedded up to cobordism in $\boldsymbol{R}^{2n-4}$, there is an embedding $M^n \hookrightarrow \boldsymbol{R}^{2n-4}$ with Hurewicz image equal to $e_1e_2^{n-6}e_7 + e_1\delta$. This shows that when n is even the element $e_1^{n-5}e_3 \cdot e_1^{n-5}e_3$ is spherical in $H_{2n-4}QMO(n-4)$. But the following lemma shows that if the manifold is embedded, the element $Q^{n-2}e_1^{n-6}e_3 + A + \delta$ can not to be spherical.

LEMMA 4.5. *Let $n = 2m$. Then the element $e_1^{n-5}e_3 \cdot e_1^{n-5}e_3$ is primitive $\mathscr{A}_2$-annihilated but it is not spherical.*

PROOF. It is clearly primitive $\mathscr{A}_2$-annihilated. Suppose it is spherical. Since $e_1^{n-5}e_3 \cdot e_1^{n-5}e_3 = (w_{n-4}w_1^2)^* \cdot (w_{n-4}w_1^2)^*$, then by Theorem 4.4 $e_1^{n-5}e_3 \cdot e_1^{n-5}e_3$ is spherical if and only if $Sq^{n-1}w_{n-4}w_1^2 \neq 0$ in $H^{2n-3}C_\alpha$, where C_α is the mapping cone of stable map of α. But since $n = 2m$ we have

$$\begin{aligned}
Sq^{n-1}w_{n-4}w_1^2 &= Sq^{2m-1}w_{2m-4}w_1^2 \\
&= Sq^1Sq^{2m-2}w_{2m-4}w_1^2 \quad \text{(Adem)} \\
&= Sq^1w_{2m-4}^2w_1^4 \quad \text{(dimension)} \\
&= Sq^1Sq^1w_{2m-4}^2w_1^3 \\
&= 0 \quad \text{(Adem)}.
\end{aligned}$$

This is a contradiction. So it is not spherical. □

References

[1] M. A. Asadi-Golmankhaneh, Self-intersection manifolds of immersions, Ph.D. Thesis, University of Manchester, 1998.

[2] M. A. Asadi-Golmankhaneh and P. J. Eccles, Determining the characteristic numbers of self-intesection manifolds, J. London Math. Soc. (2), **62** (2000), 278–290.

[3] M. A. Asadi-Golmankhaneh and P. J. Eccles, Double point surfaces of immersions, Geom.

Topol., **4** (2000), 149–170.

[4] T. F. Banchoff, Triple points and surgery of immersed surfaces, Proc. Amer. Math. Soc., **46** (1974), 407–413.

[5] M. G. Barratt and P. J. Eccles, Γ^{+}-structures I: A free group functor for stable homotopy theory, Topology, **13** (1974), 25–45.

[6] M. G. Barratt and P. J. Eccles, Γ^{+}-structures III: The stable structure of $\Omega^{\infty}\Sigma^{\infty}A$, Topology, **13** (1974), 407–413.

[7] E. Dyer and R. K. Lashof, Homology of iterated loop spaces, Amer. J. Math., **84** (1982), 35–88.

[8] P. J. Eccles, Multiple points of codimension one immersions, Lecture Notes in Math., **788**, Springer, 1980, pp. 23–38.

[9] P. J. Eccles, Multiple points of codimension one immersions of oriented manifolds, Math. Proc. Cambridge Philos. Soc., **87** (1980), 213–220.

[10] P. J. Eccles, Codimension one immersions and the Kervaire invariant one problem, Math. Proc. Cambridge Philos. Soc., **90** (1981), 483–493.

[11] P. J. Eccles, Characteristic numbers of immersions and self-intersection manifolds, Topology with Applications, Szekszard, Hungary, Bolyai Soc. Math. Stud., **4** (1993), 197–216.

[12] U. Koschorke and B. Sanderson, Self intersections and higher Hopf invariants, Topology, **17** (1978), 283–290.

[13] J. P. May, The homology of E_{∞} spaces, Lecture Notes in Math., **533**, Springer, 1976, pp. 1–68.

[14] J. P. May, A general algebraic approach to Steenrod operations, Lecture Notes in Math., **168**, Springer, 1970.

[15] A. Szucs, Gruppy kobordismov l-pogruzenii I, Acta Math. Acad. Sci. Hungar., **27** (1976), 343–358.

[16] A. Szucs, Gruppy kobordismov l-pogruzenii II, Acta Math. Acad. Sci. Hungar., **28** (1976), 93–102.

[17] A. Szucs, Double point surfaces of smooth immersions $M^{n} \to \boldsymbol{R}^{2n-2}$, Math. Proc. Cambridge Philos. Soc., **113** (1993), 601–613.

[18] P. Vogel, Cobordisme d'immersions, Ann. Sci. École Norm. Sup. (4), **7** (1974), 317–358.

[19] R. Wells, Cobordism groups of immersions, Topology, **5** (1966), 281–294.

Mohammad A. Asadi-Golmankhaneh
Department of Mathematics
University of Urmia
P.O. Box 165, Urmia, Iran
E-mail: m.asadi@mail.urmia.ac.ir