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Estimates for eigenvalues of a clamped plate problem on Riemannian manifolds

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Abstract. In this paper we study eigenvalues of a clamped plate problem on a bounded domain in an n-dimensional complete Riemannian manifold. By making use of Nash's theorem and introducing k free constants, we derive a universal bound for eigenvalues, which solves a problem proposed by Wang and Xia [**16**].

1. Introduction.

Let Ω be a bounded domain in an n-dimensional complete Riemannian manifold M. The following is called a Dirichlet eigenvalue problem of Laplacian:

$$\begin{cases} \Delta u = -\lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Δ is the Laplacian on M. This eigenvalue problem has a real and discrete spectrum:

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \longrightarrow \infty,$$

where each eigenvalue is repeated according to its multiplicity.

When M is a Euclidean space $\boldsymbol{R}^n$, namely, when Ω is a bounded domain in $\boldsymbol{R}^n$, Payne, Pólya and Weinberger [**15**] proved

$$\lambda_{k+1} - \lambda_k \leq \frac{4}{kn} \sum_{i=1}^{k} \lambda_i. \tag{1.2}$$

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Hile and Protter [**11**] generalized the above result to

$$\sum_{i=1}^{k} \frac{\lambda_i}{\lambda_{k+1} - \lambda_i} \geq \frac{kn}{4}. \tag{1.3}$$

In 1991, a much sharper inequality was obtained by Yang [**17**] (cf. [**7**]):

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)\lambda_i, \tag{1.4}$$

which is called Yang's first inequality (see [**1**] and [**2**]). According to the inequality, one can infer

$$\lambda_{k+1} \leq \frac{1}{k}\left(1 + \frac{4}{n}\right) \sum_{i=1}^{k} \lambda_i, \tag{1.5}$$

which is called Yang's second inequality.

For the Dirichlet eigenvalue problem on a complete Riemannian manifold M, Chen and Cheng [**3**] and El Soufi, Harrell and Ilias [**9**] have proved, independently,

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)\left(\lambda_i + \frac{n^2}{4} H_0^2\right), \tag{1.6}$$

where H_0^2 is an nonnegative constant which only depends on M and Ω. When M is the unit sphere, the above inequality is best possible, which has been obtained in [**5**]. In particular, when M is an n-dimensional hypersurface in $\boldsymbol{R}^{n+1}$, Harrell [**10**] has also proved the above inequality.

On the other hand, we consider an eigenvalue problem of the biharmonic operator Δ^2 on a bounded domain in an n-dimensional complete Riemannian manifold M, which is also called a clamped plate problem:

$$\begin{cases} \Delta^2 u = \Gamma u & \text{in } \Omega \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where Δ^2 denotes the biharmonic operator on M, and ν is the outward unit normal of $\partial\Omega$.

When Ω is a bounded domain in $\boldsymbol{R}^n$, Payne, Pólya and Weinberger [**15**] proved

$$\Gamma_{k+1} - \Gamma_k \leq \frac{8(n+2)}{n^2 k} \sum_{i=1}^{k} \Gamma_i. \tag{1.8}$$

Chen and Qian [**4**] and Hook [**12**], independently, extended the above inequality to

$$\frac{n^2 k^2}{8(n+2)} \leq \sum_{i=1}^{k} \frac{\Gamma_i^{\frac{1}{2}}}{\Gamma_{k+1} - \Gamma_i} \sum_{i=1}^{k} \Gamma_i^{\frac{1}{2}} \quad (\text{ cf. } [\mathbf{13}]). \tag{1.9}$$

Recently, answering a question of Ashbaugh [**1**], Cheng and Yang [**6**] have proved the following remarkable estimate:

$$\sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_i) \leq \left(\frac{8(n+2)}{n^2} \right)^{\frac{1}{2}} \sum_{i=1}^{k} (\Gamma_i(\Gamma_{k+1} - \Gamma_i))^{\frac{1}{2}}, \tag{1.10}$$

which is analogous to Yang's first inequality.

In 2007, Wang and Xia ([**16**, p. 336]) have proposed that for what kind of M, there exists a universal bound on the $(k+1)^{\text{th}}$ eigenvalue in terms of the first k eigenvalues of (1.7). When M is either a complete minimal submanifold in a Euclidean space or the unit sphere, Wang and Xia [**16**] have solved this problem. Namely, they have proved the following: when M is an n-dimensional minimal submanifold in a Euclidean space,

$$\sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_i)^2 \leq \frac{8(n+2)}{n^2} \sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_i) \Gamma_i \tag{1.11}$$

and when M is an n-dimensional unit sphere,

$$\sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_i)^2 \leq \frac{1}{n^2} \sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_i) \big(n^2 + (2n+4)\Gamma_i^{\frac{1}{2}}\big) \big(n^2 + 4\Gamma_i^{\frac{1}{2}}\big). \tag{1.12}$$

have been proved.

When M is a hyperbolic space $H^n(-1)$, Cheng and Yang [**8**] have also solved this problem, that is, they have proved

$$\sum_{i,j=1}^{k} (\Gamma_{k+1} - \Gamma_i)^2 \leq 24 \sum_{i,j=1}^{k} (\Gamma_{k+1} - \Gamma_i) \left\{ \Gamma_i^{\frac{1}{2}} - \frac{(n-1)^2}{4} \right\} \left\{ \Gamma_j^{\frac{1}{2}} - \frac{(n-1)^2}{6} \right\}. \tag{1.13}$$

In this paper, our purpose is to solve the problem proposed by Wang and Xia, completely. We derive that, for any complete Riemannian manifold M, there exists a universal bound on the $(k+1)^{\text{th}}$ eigenvalue in terms of the first k eigenvalues of (1.7). In order to prove our result, we make use of Nash's theorem [**14**] to construct trial functions and introduce k free constants to deal with the undesired terms.

Theorem. *Let Ω be a bounded domain in an n-dimensional complete Riemannian manifold M. Assume that Γ_i is the i^{th} eigenvalue of the clamped plate problem* (1.7)*. Then, there exists a constant H_0, which only depends on M and Ω such that*

$$\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 \leq \frac{1}{n^2}\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\big(n^2H_0^2+(2n+4)\Gamma_i^{\frac{1}{2}}\big)\big(n^2H_0^2+4\Gamma_i^{\frac{1}{2}}\big). \quad (1.14)$$

holds.

Remark 1. For a complete minimal submanifold M in a Euclidean space, we can infer $H_0=0$. For an n-dimensional unit sphere $M=S^n(1)$, which can be considered as a hypersurface in $\boldsymbol{R}^{n+1}$ with the mean curvature $H=1$, we have $H_0=1$. Hence, the results of Wang and Xia [**16**] are simple consequences of our result.

When M is the unit sphere $S^n(1)$ and Ω tends to $S^n(1)$, we know that Γ_1 tends to zero and Γ_i, for $i=2,\dots,n+1$, tends to n^2. Therefore, for $k=1,2,\dots,n$, our inequality (1.13) becomes equality.

Since our inequality (1.13) is a quadratic inequality of Γ_{k+1}, it is not difficult to derive an upper bound on Γ_{k+1} according to the first k eigenvalues and H_0^2.

Corollary 1. *Under the assumptions of the theorem, we have*

$$\Gamma_{k+1} \leq A_k + \sqrt{A_k^2 - B_k}, \quad (1.15)$$

where

$$A_k = \frac{1}{k}\left\{\sum_{i=1}^{k}\Gamma_i + \frac{1}{2n^2}\sum_{i=1}^{k}\big(n^2H_0^2+(2n+4)\Gamma_i^{\frac{1}{2}}\big)\big(n^2H_0^2+4\Gamma_i^{\frac{1}{2}}\big)\right\}$$

and

$$B_k = \frac{1}{k}\left\{ \sum_{i=1}^{k} \Gamma_i^2 + \frac{1}{n^2} \sum_{i=1}^{k} \Gamma_i \big(n^2 H_0^2 + (2n+4)\Gamma_i^{\frac{1}{2}}\big)\big(n^2 H_0^2 + 4\Gamma_i^{\frac{1}{2}}\big) \right\}.$$

Since k is an any integer, we know that (1.13) also holds if we replace $k+1$ with k, that is, we have

$$\sum_{i=1}^{k-1} (\Gamma_k - \Gamma_i)^2 \le \frac{1}{n^2} \sum_{i=1}^{k-1} (\Gamma_k - \Gamma_i)\big(n^2 H_0^2 + (2n+4)\Gamma_i^{\frac{1}{2}}\big)\big(n^2 H_0^2 + 4\Gamma_i^{\frac{1}{2}}\big).$$

Therefore, we infer

$$\sum_{i=1}^{k} (\Gamma_k - \Gamma_i)^2 \le \frac{1}{n^2} \sum_{i=1}^{k} (\Gamma_k - \Gamma_i)\big(n^2 H_0^2 + (2n+4)\Gamma_i^{\frac{1}{2}}\big)\big(n^2 H_0^2 + 4\Gamma_i^{\frac{1}{2}}\big).$$

Namely, Γ_k also satisfies the same quadratic inequality. We derive

$$\Gamma_k \ge A_k - \sqrt{A_k^2 - B_k}.$$

Thus, we can obtain an estimate on $\Gamma_{k+1} - \Gamma_k$ as following:

Corollary 2. *Under the assumptions of the theorem, we have*

$$\Gamma_{k+1} - \Gamma_k \le 2\sqrt{A_k^2 - B_k}, \tag{1.16}$$

where A_k and B_k are given in the Corollary 1.

2. Proof of Theorem.

In order to prove our theorem, the following Nash's theorem plays an important role.

Nash's theorem ([**14**]). *Each complete Riemannian manifold M can be isometrically immersed into a Euclidean space $\boldsymbol{R}^N$.*

Let M be an n-dimensional isometrically immersed submanifold in $\boldsymbol{R}^N$. For an arbitrary point $p \in M$, let $(x^1, \dots, x^n)$ be an arbitrary coordinate system in a neighborhood U of $p \in M$. Let y be the position vector of $p \in M$ which is defined by

$$y = \big(y^1(x^1, \dots, x^n), \dots, y^N(x^1, \dots, x^n)\big).$$

Since M is isometrically immersed in $\boldsymbol{R}^N$,

$$g_{ij} = g\bigg(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\bigg) = \bigg\langle \sum_{\alpha=1}^{N} \frac{\partial y^\alpha}{\partial x^i}\frac{\partial}{\partial y^\alpha}, \sum_{\beta=1}^{N} \frac{\partial y^\beta}{\partial x^j}\frac{\partial}{\partial y^\beta} \bigg\rangle = \sum_{\alpha=1}^{N} \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\alpha}{\partial x^j}, \tag{2.1}$$

where g denotes the induced metric of M from $\boldsymbol{R}^N$ and $\langle\ ,\ \rangle$ is the standard inner product in $\boldsymbol{R}^N$. The following lemma of [**3**] is necessary for proving our theorem. For reader's convenience, we will give its proof.

Lemma (Chen and Cheng [**3**]). *For any function $u \in C^\infty(M)$,*

$$\sum_{\alpha=1}^{N} \big(g(\nabla y^\alpha, \nabla u)\big)^2 = |\nabla u|^2, \tag{2.2}$$

$$\sum_{\alpha=1}^{N} g(\nabla y^\alpha, \nabla y^\alpha) = \sum_{\alpha=1}^{N} |\nabla y^\alpha|^2 = n, \tag{2.3}$$

$$\sum_{\alpha=1}^{N} (\Delta y^\alpha)^2 = n^2|H|^2, \tag{2.4}$$

$$\sum_{\alpha=1}^{N} \Delta y^\alpha \nabla y^\alpha = 0, \tag{2.5}$$

where ∇ denotes the gradient operator on M, and $|H|$ is the mean curvature of M.

Proof. For any point p, we define $\overline{y} = (\overline{y}^1, \dots, \overline{y}^N)$ by $y - y(p) = \overline{y}A$ such that $\big(\frac{\partial}{\partial \overline{y}^1}\big)_p, \dots, \big(\frac{\partial}{\partial \overline{y}^n}\big)_p$ span T_pM and $g\big(\frac{\partial}{\partial \overline{y}^i}, \frac{\partial}{\partial \overline{y}^j}\big) = \delta_{ij}$, where $A = (a^\alpha_\beta) \in O(N)$ is an orthogonal matrix. For any function $u \in C^\infty(M)$, at p,

$$\begin{aligned}
\sum_{\alpha=1}^{N} \big(g(\nabla y^\alpha, \nabla u)\big)^2 &= \sum_{\alpha=1}^{N} \bigg[g\bigg(\nabla\bigg(y^\alpha(p) + \sum_{\beta=1}^{N} a^\alpha_\beta \overline{y}^\beta\bigg), \nabla u\bigg)\bigg]^2 \\
&= \sum_{\alpha=1}^{N} \bigg[g\bigg(\nabla \sum_{\beta=1}^{N} a^\alpha_\beta \overline{y}^\beta, \nabla u\bigg)\bigg]^2
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\alpha=1}^{N} \left(\sum_{\beta=1}^{N} \sum_{i=1}^{n} a_{\beta}^{\alpha} \frac{\partial \overline{y}^{\beta}}{\partial \overline{y}^{i}} \frac{\partial u}{\partial \overline{y}^{i}} \right)^{2} \\
&= \sum_{i=1}^{n} \sum_{\alpha=1}^{N} \frac{\partial \overline{y}^{\alpha}}{\partial \overline{y}^{i}} \frac{\partial \overline{y}^{\alpha}}{\partial \overline{y}^{i}} \frac{\partial u}{\partial \overline{y}^{i}} \frac{\partial u}{\partial \overline{y}^{i}} \\
&= |\nabla u|^{2}. \qquad (2.6)
\end{aligned}
$$

This completes the proof of (2.2).

By definition,

$$
\sum_{\alpha=1}^{N} g(\nabla y^{\alpha}, \nabla y^{\alpha}) = \sum_{\alpha=1}^{N} \sum_{i,j}^{n} \frac{\partial y^{\alpha}}{\partial x^{i}} \frac{\partial y^{\alpha}}{\partial x^{j}} g^{ij} = \sum_{i,j}^{n} g_{ij} g^{ij} = n. \qquad (2.7)
$$

Since y is the position vector of M, we have

$$
\Delta y = nH. \qquad (2.8)
$$

Thus we can derive

$$
\sum_{\alpha=1}^{N} (\Delta y^{\alpha})^{2} = n^{2} |H|^{2}. \qquad (2.9)
$$

Since ∇y is tangent to M, we have

$$
\sum_{\alpha=1}^{N} \Delta y^{\alpha} \nabla_{i} y^{\alpha} = 0. \qquad (2.10)
$$

Therefore,

$$
\sum_{\alpha=1}^{N} \Delta y^{\alpha} \nabla y^{\alpha} = 0. \qquad (2.11)
$$

□

PROOF OF THEOREM. Since M is a complete Riemannian manifold, Nash's theorem implies that there exists an isometric immersion from M into a Euclidean space $\boldsymbol{R}^{N}$. Thus, M can be considered as an n-dimensional complete isometrically immersed submanifold in $\boldsymbol{R}^{N}$. We denote, by $y = (y^{\alpha})$, the position vector of M

in $\boldsymbol{R}^N$. Let u_i be an eigenfunction corresponding to the eigenvalue Γ_i such that

$$\begin{cases} \Delta^2 u_i = \Gamma_i u_i & \text{in } \Omega \\ u_i = \frac{\partial u_i}{\partial \nu} = 0 & \text{on } \partial\Omega \\ \int_\Omega u_i u_j = \delta_{ij} \ (i, j = 1, 2, \ldots). \end{cases} \tag{2.12}$$

For $i = 1, \ldots, k$, and $\alpha = 1, \ldots, N$, we define

$$\phi_i^\alpha := y^\alpha u_i - \sum_{j=1}^k r_{ij}^\alpha u_j, \tag{2.13}$$

where $r_{ij}^\alpha = \int_\Omega y^\alpha u_i u_j$. By a simple calculation, we obtain

$$\int_\Omega u_j \phi_i^\alpha = 0, \quad i, j = 1, \ldots, k. \tag{2.14}$$

From the Rayleigh-Ritz inequality, we have

$$\Gamma_{k+1} \le \frac{\int_\Omega \phi_i^\alpha \Delta^2 \phi_i^\alpha}{\int_\Omega (\phi_i^\alpha)^2}, \quad 1 \le i \le k. \tag{2.15}$$

Since

$$\begin{aligned} \int_\Omega \phi_i^\alpha \Delta^2 \phi_i^\alpha &= \int_\Omega \phi_i^\alpha \Delta^2 \Big(y^\alpha u_i - \sum_{j=1}^k r_{ij}^\alpha u_j \Big) \\ &= \int_\Omega \phi_i^\alpha \big\{ \Delta^2 y^\alpha \cdot u_i + 2\nabla(\Delta y^\alpha) \cdot \nabla u_i + 2\Delta y^\alpha \Delta u_i \\ &\qquad + 2\Delta(\nabla y^\alpha \cdot \nabla u_i) + 2\nabla y^\alpha \cdot \nabla(\Delta u_i) + \Gamma_i y^\alpha u_i \big\}, \end{aligned} \tag{2.16}$$

we infer from (2.14), (2.15) and (2.16)

$$(\Gamma_{k+1} - \Gamma_i) \|\phi_i^\alpha\|^2 \le \int_\Omega p_i^\alpha \phi_i^\alpha = \omega_i^\alpha - \sum_{j=1}^k r_{ij}^\alpha s_{ij}^\alpha, \tag{2.17}$$

where

$$p_i^\alpha = \Delta^2 y^\alpha \cdot u_i + 2\nabla(\Delta y^\alpha)\cdot\nabla u_i + 2\Delta y^\alpha \Delta u_i + 2\Delta(\nabla y^\alpha \cdot \nabla u_i) + 2\nabla y^\alpha \cdot \nabla(\Delta u_i),$$

$$s_{ij}^\alpha = \int_\Omega p_i^\alpha u_j, \qquad \omega_i^\alpha = \int_\Omega p_i^\alpha y^\alpha u_i.$$

From Stokes' theorem, we infer

$$2\int_\Omega y^\alpha u_i \nabla(\Delta y^\alpha)\cdot\nabla u_i = \int_\Omega \big\{2u_i\Delta y^\alpha \nabla u_i \cdot \nabla y^\alpha + u_i^2(\Delta y^\alpha)^2 - y^\alpha u_i^2 \Delta^2 y^\alpha\big\},$$

$$\begin{aligned} 2&\int_\Omega y^\alpha u_i \Delta(\nabla y^\alpha \cdot \nabla u_i) \\ &= \int_\Omega \big\{2u_i \Delta y^\alpha \nabla y^\alpha \cdot \nabla u_i + 4(\nabla y^\alpha \cdot \nabla u_i)^2 + 2y^\alpha \Delta u_i \nabla y^\alpha \cdot \nabla u_i\big\}, \\ 2&\int_\Omega y^\alpha u_i \nabla y^\alpha \cdot \nabla(\Delta u_i) \\ &= -2\int_\Omega \big(|\nabla y^\alpha|^2 u_i \Delta u_i + y^\alpha \Delta u_i \nabla y^\alpha \cdot \nabla u_i + y^\alpha \Delta y^\alpha u_i \Delta u_i\big). \end{aligned}$$

Thus, we obtain

$$\omega_i^\alpha = \int_\Omega \big\{(\Delta y^\alpha)^2 u_i^2 + 4(\nabla y^\alpha \cdot \nabla u_i)^2 - 2|\nabla y^\alpha|^2 u_i \Delta u_i + 4u_i \Delta y^\alpha \nabla y^\alpha \cdot \nabla u_i\big\}. \tag{2.18}$$

Since

$$\begin{aligned} 2&\int_\Omega \Delta u_j \nabla y^\alpha \cdot \nabla u_i - \Delta u_i \nabla y^\alpha \cdot \nabla u_j \\ &= (\Gamma_j - \Gamma_i) r_{ij}^\alpha - \int_\Omega u_i \Delta u_j \Delta y^\alpha + \int_\Omega u_j \Delta u_i \Delta y^\alpha, \end{aligned}$$

we can infer

$$s_{ij}^\alpha = (\Gamma_j - \Gamma_i) r_{ij}^\alpha. \tag{2.19}$$

Then (2.17) can be written as

$$(\Gamma_{k+1} - \Gamma_i)\|\phi_i^\alpha\|^2 \le \omega_i^\alpha + \sum_{j=1}^k (\Gamma_i - \Gamma_j)\big(r_{ij}^\alpha\big)^2. \tag{2.20}$$

On the other hand, defining

$$v_i^\alpha = -2\int_\Omega y^\alpha u_i\left(\nabla y^\alpha \cdot \nabla u_i + \frac{u_i \Delta y^\alpha}{2}\right)$$

and

$$t_{ij}^\alpha = \int_\Omega u_j\left(\nabla y^\alpha \cdot \nabla u_i + \frac{u_i \Delta y^\alpha}{2}\right) = -t_{ji}^\alpha,$$

then it follows that

$$\int_\Omega -2\phi_i^\alpha\left(\nabla y^\alpha \cdot \nabla u_i + \frac{u_i \Delta y^\alpha}{2}\right) = v_i^\alpha + 2\sum_{j=1}^k r_{ij}^\alpha t_{ij}^\alpha. \tag{2.21}$$

Multiplying (2.21) by $(\Gamma_{k+1}-\Gamma_i)^2$, we obtain, from (2.20),

$$\begin{aligned}
&(\Gamma_{k+1}-\Gamma_i)^2\left(v_i^\alpha + 2\sum_{j=1}^k r_{ij}^\alpha t_{ij}^\alpha\right)\\
&= (\Gamma_{k+1}-\Gamma_i)^2\int_\Omega -2\phi_i^\alpha\left\{\left(\nabla y^\alpha \cdot \nabla u_i + \frac{u_i \Delta y^\alpha}{2}\right) - \sum_{j=1}^k t_{ij}^\alpha u_j\right\}\\
&\le \delta_i(\Gamma_{k+1}-\Gamma_i)^3\|\phi_i^\alpha\|^2 + \frac{\Gamma_{k+1}-\Gamma_i}{\delta_i}\left\|\nabla y^\alpha \cdot \nabla u_i + \frac{u_i \Delta y^\alpha}{2} - \sum_{j=1}^k t_{ij}^\alpha u_j\right\|^2\\
&= \delta_i(\Gamma_{k+1}-\Gamma_i)^3\|\phi_i^\alpha\|^2 + \frac{\Gamma_{k+1}-\Gamma_i}{\delta_i}\left\{\left\|\nabla y^\alpha \cdot \nabla u_i + \frac{u_i \Delta y^\alpha}{2}\right\|^2 - \sum_{j=1}^k (t_{ij}^\alpha)^2\right\}\\
&\le \delta_i(\Gamma_{k+1}-\Gamma_i)^2\left\{\omega_i^\alpha + \sum_{j=1}^k(\Gamma_i-\Gamma_j)\big(r_{ij}^\alpha\big)^2\right\}\\
&\quad + \frac{\Gamma_{k+1}-\Gamma_i}{\delta_i}\left\{\left\|\nabla y^\alpha \cdot \nabla u_i + \frac{u_i \Delta y^\alpha}{2}\right\|^2 - \sum_{j=1}^k \big(t_{ij}^\alpha\big)^2\right\},
\end{aligned} \tag{2.22}$$

where δ_i is a positive constant. By the Stokes' theorem and the Schwarz's inequality, we have

$$\int_\Omega |\nabla u_i|^2 \leq \Gamma_i^{\frac{1}{2}}.$$

From (2.18) and the lemma, we have

$$\begin{aligned}\sum_{\alpha=1}^{N} \omega_i^\alpha &= n^2 \int_\Omega |H|^2 u_i^2 + (2n+4) \int_\Omega |\nabla u_i|^2 \\ &\leq n^2 \sup_\Omega |H|^2 + (2n+4)\Gamma_i^{\frac{1}{2}} \end{aligned} \tag{2.23}$$

and

$$\sum_{\alpha=1}^{N} \left\| \nabla y^\alpha \cdot \nabla u_i + \frac{u_i \Delta y^\alpha}{2} \right\|^2 \leq \frac{1}{4} n^2 \sup_\Omega |H|^2 + \Gamma_i^{\frac{1}{2}}. \tag{2.24}$$

By a simple calculation, we derive

$$\sum_{\alpha=1}^{N} v_i^\alpha = \sum_{\alpha=1}^{N} \int_\Omega |\nabla y^\alpha|^2 u_i^2 = n. \tag{2.25}$$

Summing on i from 1 to k for (2.22), we have

$$\begin{aligned}&\sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_i)^2 v_i^\alpha - 2 \sum_{i,j}^{k} (\Gamma_{k+1} - \Gamma_i)(\Gamma_i - \Gamma_j) r_{ij}^\alpha t_{ij}^\alpha \\ &\quad \leq \sum_{i=1}^{k} \delta_i (\Gamma_{k+1} - \Gamma_i)^2 \omega_i^\alpha + \sum_{i=1}^{k} \frac{1}{\delta_i} (\Gamma_{k+1} - \Gamma_i) \left\| \nabla y^\alpha \cdot \nabla u_i + \frac{u_i \Delta y^\alpha}{2} \right\|^2 \\ &\qquad + \sum_{i,j}^{k} \delta_i (\Gamma_{k+1} - \Gamma_i)^2 (\Gamma_i - \Gamma_j) \big(r_{ij}^\alpha\big)^2 - \sum_{i,j}^{k} \frac{1}{\delta_i} (\Gamma_{k+1} - \Gamma_i)(t_{ij}^\alpha)^2. \end{aligned} \tag{2.26}$$

Putting

$$\delta_i = \frac{\delta}{n^2 \sup_\Omega |H|^2 + (2n+4)\Gamma_i^{\frac{1}{2}}}, \ \delta \text{ is a positive constant,}$$

then, we have

$$
\begin{aligned}
&-\sum_{i,j}^{k}\delta_i(\Gamma_{k+1}-\Gamma_i)^2(\Gamma_i-\Gamma_j)\big(r_{ij}^{\alpha}\big)^2-\sum_{i,j}^{k}\delta_i(\Gamma_{k+1}-\Gamma_i)(\Gamma_i-\Gamma_j)^2\big(r_{ij}^{\alpha}\big)^2\\
&=-\sum_{i,j}^{k}\delta_i(\Gamma_{k+1}-\Gamma_i)(\Gamma_{k+1}-\Gamma_j)(\Gamma_i-\Gamma_j)\big(r_{ij}^{\alpha}\big)^2\\
&=-\frac{1}{2}\sum_{i,j}^{k}(\Gamma_{k+1}-\Gamma_i)(\Gamma_{k+1}-\Gamma_j)(\Gamma_i-\Gamma_j)(\delta_i-\delta_j)\big(r_{ij}^{\alpha}\big)^2\\
&\geq 0. \qquad (2.27)
\end{aligned}
$$

It is clear that

$$
\begin{aligned}
&-\sum_{i,j}^{k}\delta_i(\Gamma_{k+1}-\Gamma_i)(\Gamma_i-\Gamma_j)^2(r_{ij}^{\alpha})^2-\sum_{i,j}^{k}\frac{1}{\delta_i}(\Gamma_{k+1}-\Gamma_i)\big(t_{ij}^{\alpha}\big)^2\\
&\leq -2\sum_{i,j}^{k}(\Gamma_{k+1}-\Gamma_i)(\Gamma_i-\Gamma_j)r_{ij}^{\alpha}t_{ij}^{\alpha}. \qquad (2.28)
\end{aligned}
$$

Therefore, it follows that

$$
\begin{aligned}
&\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2v_i^{\alpha}\\
&\leq\sum_{i=1}^{k}\delta_i(\Gamma_{k+1}-\Gamma_i)^2\omega_i^{\alpha}+\sum_{i=1}^{k}\frac{1}{\delta_i}(\Gamma_{k+1}-\Gamma_i)\left\|\nabla y^{\alpha}\cdot\nabla u_i+\frac{u_i\Delta y^{\alpha}}{2}\right\|^2. \qquad (2.29)
\end{aligned}
$$

Summing on α from 1 to N for (2.29), we infer, from (2.23), (2.24) and (2.25),

$$
\begin{aligned}
&n\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\\
&\leq\sum_{i=1}^{k}\delta_i(\Gamma_{k+1}-\Gamma_i)^2\Big(n^2\sup_{\Omega}|H|^2+(2n+4)\Gamma_i^{\frac{1}{2}}\Big)\\
&\quad+\sum_{i=1}^{k}\frac{1}{\delta_i}(\Gamma_{k+1}-\Gamma_i)\bigg(\frac{1}{4}n^2\sup_{\Omega}|H|^2+\Gamma_i^{\frac{1}{2}}\bigg)
\end{aligned}
$$

$$= \delta \sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_i)^2$$
$$+ \frac{1}{\delta} \sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_i) \left(\frac{1}{4} n^2 \sup_{\Omega} |H|^2 + \Gamma_i^{\frac{1}{2}} \right) \left(n^2 \sup_{\Omega} |H|^2 + (2n+4) \Gamma_i^{\frac{1}{2}} \right). \tag{2.30}$$

Putting

$$\delta = \left[\frac{\sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_i) \left(\frac{1}{4} n^2 \sup_{\Omega} |H|^2 + \Gamma_i^{\frac{1}{2}} \right) \left(n^2 \sup_{\Omega} |H|^2 + (2n+4) \Gamma_i^{\frac{1}{2}} \right)}{\sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_i)^2} \right]^{\frac{1}{2}},$$

we obtain

$$\sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_i)^2$$
$$\leq \frac{1}{n^2} \sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_i) \left(n^2 \sup_{\Omega} |H|^2 + (2n+4) \Gamma_i^{\frac{1}{2}} \right) \left(n^2 \sup_{\Omega} |H|^2 + 4 \Gamma_i^{\frac{1}{2}} \right). \tag{2.31}$$

Since the spectrum of the clamped plate problem is an invariant of isometries, we know that the above inequality holds for any isometric immersion from M into a Euclidean space.

Now we define ψ as

$$\psi := \{\phi; \phi \text{ is an isometric immersion from } M \text{ into a Euclidean space}\}.$$

Putting

$$H_0^2 := \inf_{\phi \in \psi} \sup_{\Omega} |H|^2,$$

We infer (1.13). This completes the proof of the theorem. □

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