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Chain-connected component decomposition of curves on surfaces

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Abstract. We prove that an arbitrary reducible curve on a smooth surface has an essentially unique decomposition into chain-connected curves. Using this decomposition, we give an upper bound of the geometric genus of a numerically Gorenstein surface singularity in terms of certain topological data determined by the canonical cycle. We show also that the fixed part of the canonical linear system of a 1-connected curve is always rational, that is, the first cohomology of its structure sheaf vanishes.

Introduction.

In the study of algebraic surfaces, we often encounter with reducible non-reduced curves. Typical examples are various cycles supported by the exceptional set of a normal surface singularity and singular fibres in a fibred surface. Needless to say, any reducible curve decomposes into a sum of irreducible curves uniquely up to the order. As one may see from the success of 1-connected curves ([**11**], [**3**]), however, it is sometimes more convenient and even natural to treat a connected reducible curve as if it were a single irreducible curve. In other words, some coarser decompositions could be better suited to certain problems than the decomposition into irreducible components.

The purpose of the paper is to revive and recast another canonical way to decompose reducible curves on a smooth surface used by Miyaoka in [**10**]. Our atomic curves are chain-connected curves [**12**] (called s-connected divisors in [**10**]) which themselves are reducible in general. The decomposition theorem (Corollary 1.7) states that every effective divisor on a smooth surface decomposes into a sum of chain-connected curves enjoying nice numerical relations. Furthermore, such an ordered decomposition is essentially unique. We call it a chain-connected component decomposition (a CCC decomposition for short). We know that 1-connectivity is a very important notion in the surface theory. However, the class

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of 1-connected curves is not big enough to cover some important classes like fundamental cycles of singularities. The chain-connectivity, a notion which dates back to Kodaira [**5**], is defined by a weaker condition and covers a considerably wider range.

The present paper is organized as follows. In Section 1, after stating several properties of chain-connected curves, we show that every curve has a CCC decomposition. Though its essential part is roughly stated in [**10**], the relation between chain-connected curves derived from Proposition 1.5 (1) seems overlooked or slighted there. In Section 2, we study the space of global sections of a nef line bundle on a chain-connected curve and show that the dimension is bounded from above by the degree plus one. Unlike irreducible curves, however, curves attaining the bound are not necessarily rational, usually with a large fixed part of the canonical linear system. In Section 3, we consider the minimal model problem for chain-connected curves. Here, a minimal model is defined to be a subcurve with nef dualizing sheaf and of the same arithmetic genus as the original curve. We show that the minimal model uniquely exists for any chain-connected curve with positive arithmetic genus. The procedure obtaining the model is quite similar to that for a global surface, that is, the subtraction of "(-1)-curves" one by one. The rest of the paper is devoted to exhibiting applications of CCC decompositions in some concrete situations. In Section 4, we study the canonical cycles of numerically Gorenstein surface singularities. Recall that, the canonical cycle of a weakly elliptic, numerically Gorenstein singularity has a natural decomposition, called the elliptic sequence, introduced by S. S. T. Yau [**16**]. Among other things, he succeeded in bounding the geometric genus by the length of the sequence. It is shown that our decomposition by chain-connected curves coincides with the elliptic sequence in this case. For the other singularities, we give in Theorem 4.1 an upper bound of the geometric genus with the quantity which can be determined by the weighted dual graph of the canonical cycle. It generalizes Yau's result as well as a bound given by Tomaru [**14**]. In Section 5, we study subcurves of a 1-connected curve, especially the fixed part of the canonical linear system. We reprove a theorem in [**7**] which asserts that the canonical fixed part of a 1-connected curve is rational in the sense that the first cohomology group of the structure sheaf vanishes. Finally in Section 6, we consider subcurves of fibres in a fibred algebraic surface.

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1. Chain-connected curves.

By a *curve*, we mean an effective non-zero divisor on a non-singular surface. A line bundle (or an invertible sheaf) on a curve is called *nef* if it is of non-negative degree on any irreducible components. For a curve D, the arithmetic genus is defined by $p_a(D) := 1 - \chi(D, \mathscr{O}_D)$. If D is on a non-singular surface X, then the dualizing sheaf ω_D is defined to be $\mathscr{O}_D(K_X + D)$ and we have $2p_a(D) - 2 = \deg \omega_D = D(K_X + D)$. If $D = A + B$ with two curves A and B, then $p_a(D) = p_a(A) + p_a(B) - 1 + AB$.

Definition 1.1.

(1) Let D_1 be a non-trivial subcurve of D, i.e., $0 \prec D_1 \prec D = D_1 + D_2$. The ordered pair (D_1, D_2) is called a *chain-disconnected partition* of D if $\mathscr{O}_{D_2}(-D_1)$ is nef or, in other words, if $D_1 C \leq 0$ for every irreducible component C of D_2.

(2) An increasing sequence of curves $D_0, D_1, \ldots, D_m$ is called a *connecting chain* from D_0 to D_m if (i) the difference $D_i - D_{i-1}$ is an irreducible curve C_i and (ii) $C_i D_{i-1} > 0$ for $i = 1, \ldots, m$.

Proposition 1.2. *The following three conditions on a curve D are equivalent.*

(1) *D has no chain-disconnected partition.*
(2) *For any non-trivial subcurve D_0 of D, there exists a connecting chain $D_0, \ldots, D_m$ from D_0 to $D = D_m$.*
(3) *There exists a connecting chain $D_0, \ldots, D_m$ such that $D = D_m$ and D_0 is an irreducible curve.*

Proof.

(1) $\Rightarrow$ (2): Pick up any non-trivial subcurve $D_0 \prec D$. We inductively construct a connecting chain from D_0 to D. If we have $0 \prec D_i \prec D$, then $(D_i, D - D_i)$ is not a chain-disconnected partition of D by (1). Hence, there exists an irreducible component C_{i+1} of $D - D_i$ with $D_i C_{i+1} > 0$. Define D_{i+1} to be $D_i + C_{i+1}$, eventually arriving at $D = D_m$ for some m.

(2) $\Rightarrow$ (1): Take an arbitrary non-trivial decomposition $D = A + B$. Let $D_0, \ldots, D_m = D$ be a connecting chain starting from $D_0 = A$. Then $C_1 = D_1 - D_0$ is a component of B satisfying $0 < C_1 D_0 = C_1 A$. Thus $-A$ cannot be nef on B. Similarly, $-B$ is not nef on A.

(3) $\Rightarrow$ (1): Let $D_0, \ldots, D_m = D$ be a connecting chain starting from an irreducible curve D_0, where $C_i = D_i - D_{i-1}$ is an irreducible curve. We do the proof by induction on m. When $m = 0$, the assertion is clear. Assume that D_{m-1} has no chain-disconnected partition. We derive a contradiction by constructing a chain-disconnected partition of D_{m-1} from that of D_m. Let (A, B) be a chain-

disconnected partition of D_m. We have neither $A = C_m$ nor $B = C_m$ by the assumption $C_m D_{m-1} > 0$. If C_m is a component of B, then $(A, B - C_m)$ is a chain-disconnected partition of D_{m-1}. If C_m is not a component of B, then $C_m B \geq 0$, $C_m \prec A$ and $\mathscr{O}_B(-(A - C_m))$ is nef, implying that $(A - C_m, B)$ is a chain-disconnected partition of D_{m-1}.

The implication (2) $\Rightarrow$ (3) is clear. $\square$

Definition 1.3. D is said to be *chain-connected* when D satisfies the equivalent conditions (1), (2) and (3) in Proposition 1.2.

Remark 1.4. The notion of chain-connected curves was introduced in [**10**] as *s-connected divisors*. Our terminology is taken from [**12**].

Here are typical examples of chain-connected curves.

i) Let $\mathscr{A} = \bigcup_{i=1}^{N} A_i$ be a connected bunch of irreducible curves A_i. The intersection form is negative semi-definite on $\mathscr{A}$ if and only if there exists a curve D with support $\subseteq \mathscr{A}$ such that $-D$ is nef on $\mathscr{A}$. The smallest curve enjoying such a property exists and called the *numerical cycle* [**13**]. If the intersection form is negative definite, it is usually called the *fundamental cycle* ([**1**], [**2**]). Numerical cycles are chain-connected, as is easily seen. In fact, it is the biggest chain-connected curve with support $\mathscr{A}$.

ii) For an integer k, a curve D is called (numerically) k-connected, if $(D - \Gamma)\Gamma \geq k$ for any subcurve $0 \prec \Gamma \prec D$. Any nef and big curve is 1-connected by Hodge's index theorem. Every 1-connected curve is chain-connected. But the converse does not hold true in general. See, [**4**, Appendix] for further properties of 1-connected curves.

Proposition 1.5. *The following hold.*

(1) *Let D be a chain-connected curve and Δ a curve. If $\mathscr{O}_D(-\Delta)$ is nef, then either* $\mathrm{Supp}(D) \cap \mathrm{Supp}(\Delta) = \emptyset$ *or $D \preceq \Delta$.*
(2) *Let D be a chain-connected curve and C an irreducible curve with $DC > 0$. Then $D' = D + C$ is again chain-connected.*
(3) *Let D be a curve with connected support. Then there exists the greatest chain-connected subcurve D_1 of D. Furthermore,* $\mathrm{Supp}(D_1) = \mathrm{Supp}(D)$, *and $-D_1$ is nef on $D - D_1$.*

Proof. (1) Assume that $\mathrm{Supp}(D) \cap \mathrm{Supp}(\Delta) \neq \emptyset$. Then, since $D\Delta \leq 0$, we can write $D = A + B$, $\Delta = A + \Gamma$, where $A \succ 0$, $B \succeq 0$, $\Gamma \succeq 0$ and the two cycles B, Γ contain no common component. We show that $B = 0$. By assumption $\mathscr{O}_D(-\Delta)$ is nef and so is $\mathscr{O}_B(-\Delta)$. On the other hand, since B has no component in common with Γ, $\mathscr{O}_B(\Gamma)$ is nef. Hence $\mathscr{O}_B(-A) = \mathscr{O}_B(-\Delta + \Gamma)$ is nef. If B were non-zero, the pair (A, B) would be a chain-disconnected partition.

(2) If $D_0, \dots, D_m = D$ is a connecting chain starting from an irreducible curve, then so is $D_0, \dots, D_{m+1} = D + C$.

(3) Let D_1, D_2 be maximal chain-connected subcurves of D. The assertion (2) above (plus the connectivity of D) shows that $-D_i$ is nef on $D - D_i$ and that $\mathrm{Supp}(D_i) = \mathrm{Supp}(D)$. Let us prove that $D_1 = D_2$. We can write $D_i = A + B_i$, where $A \succ 0$ and B_1, B_2 have no common irreducible component. In particular, $A + B_1 + B_2 \preceq D$, that is, $B_2 \preceq D - D_1$. Hence $-D_1$ is nef on B_2, so that $-A = -D_1 + B_1$ is nef on B_2. Then, in view of the chain-connectivity of $D_2 = A + B_2$, we conclude that $B_2 = 0$, i.e., $D_1 \succeq D_2$. By the maximality of D_2, this shows the equality $D_1 = D_2$. □

Definition 1.6. Let D be a curve with connected support. The greatest chain-connected subcurve of D is called *the chain-connected component* of D. If D is a curve with several connected components, *a chain-connected component* of D will mean the chain-connected component of some connected component of D.

From our definition it follows that a chain-connected component of a subcurve $D' \preceq D$ is a subcurve of a chain-connected component of D.

Corollary 1.7. *Let D be a curve. Then there are a sequence $D_1, D_2, \dots, D_r$ of chain-connected subcurves of D and a sequence $m_1, \dots, m_r$ of positive integers which satisfy*

(1) $D = m_1 D_1 + \cdots + m_r D_r$.
(2) *For $i < j$, the divisor $-D_i$ is nef on D_j.*
(3) *If $m_i \geq 2$, then $-D_i$ is nef on D_i.*
(4) *For $i < j$, either* $\mathrm{Supp}(D_i) \cap \mathrm{Supp}(D_j) = \emptyset$ *or* $D_i \succ D_j$.

Sequences as above are unique up to suitable permutations of the indices $1, \dots, r$ *and the number* $n(D) := \sum_{i=1}^{r} m_i$ *is uniquely determined.*

Definition 1.8. The ordered decomposition $D = m_1 D_1 + \cdots + m_r D_r$ as in Corollary 1.7 is called a *chain-connected component decomposition* or a *CCC decomposition* of D.

Proof of Corollary 1.7. We inductively construct a decomposition as above. Define D_1 to be a chain-connected component of D and let m_1 be the maximum of the integers k such that $kD_1 \preceq D$. (For $k \leq m_1 - 1$, the curve D_1 is a chain-connected component of $D - kD_1$.) Then define D_2 to be a chain-connected component of $D - m_1 D_1$ and m_2 be the largest integer such that $D - m_1 D_1 - m_2 D_2 \succeq 0$. Similar steps give rise to a decomposition which satisfies (1), (2) and (3).

The property (4) immediately follows from (2) and Proposition 1.5 (1).

Let us show the unicity of the decomposition (up to suitable permutations). Let $D = m_1D_1 + \cdots + m_rD_r$ be a decomposition into chain-connected curves with the properties (1) through (4). Consider the natural partial order $\preceq$ among the D_i. Then by (4), D_1 is a maximal member and by (1) and (2), D_1 is necessarily a chain-connected component of D. In particular, the choice of D_1 is exactly the same as the choice of a connected component of D. By definition, $D - m_1D_1 = m_2D_2 + \cdots + m_rD_r$ is a chain-connected component decomposition of $D - m_1D_1$. Then obvious induction (on the total number of components) shows the weak unicity. The ambiguity of the order does not arise if the curves $D - m_1D_1 - \cdots - m_sD_s$ have connected supports for $s = 0, \ldots, r$. $\square$

In practice, it is sometimes convenient to express a CCC decomposition as $D = \Gamma_1 + \cdots + \Gamma_n$ by putting $\Gamma_i := D_j$ for $\sum_{k<j} m_k < i \leq \sum_{k\leq j} m_k$, $n = n(D) = \sum_{k=1}^{r} m_k$. Then, for $i < j$, $\mathcal{O}_{\Gamma_j}(-\Gamma_i)$ is nef and, either $\Gamma_j \preceq \Gamma_i$ or $\mathrm{Supp}(\Gamma_i) \cap \mathrm{Supp}(\Gamma_j) = \emptyset$.

2. Nef line bundles on chain-connected curves.

Let D be a chain-connected curve. It is shown in [**10**] that $\dim H^0(D, \mathcal{O}_D) = 1$, so that $p_a(D) = \dim H^1(D, \mathcal{O}_D)$. Furthermore, for a nef line bundle L on D, $\dim H^0(D, -L) \neq 0$ if and only if $L \simeq \mathcal{O}_D$ (see, [**12**] and [**6**, Lemma 2.2]).

Theorem 2.1. *Let D be a chain-connected curve. Let L be a nef line bundle on D and put $d = \deg L \geq 0$. Then $\dim H^0(D, L) \leq d + 1$. If $\dim H^0(D, L)$ attains the maximum $d + 1$, then L is generated by global sections. When $d \geq 1$ and $\dim H^0(D, L) = d + 1$, there exists a decomposition $D = A + B$ which satisfies the following conditions*:

(1) *$A \succ 0$, $B \succeq 0$ and the two curves have no common components.*
(2) *$L|_B \simeq \mathcal{O}_B$ and $\dim H^1(B, \mathcal{O}_B) = \dim H^1(D, \mathcal{O}_D)$.*
(3) *L is ample on A, $H^1(A, \mathcal{O}_A) = 0$ and each irreducible component of A is isomorphic to $\boldsymbol{P}^1$.*

Proof. Let $D = \sum_i \mu_i A_i$ be the irreducible decomposition. For each irreducible component A_i, we pick up $d_i := \deg L|_{A_i}$ general points $p_{i,1}, \ldots, p_{i,d_i}$ on A_i and put $\delta = \sum_i \mu_i(p_{i,1} + \cdots + p_{i,d_i})$. Then δ is an effective Cartier divisor such that L and $\mathcal{O}_D(\delta)$ are numerically equivalent. By the chain-connectivity of D, we have $\dim H^0(D, \delta - L) \leq 1$ with equality holding only if $L \simeq \mathcal{O}_D(\delta)$. Then $\dim H^0(D, K_D - L) \leq \dim H^0(D, K_D - L + \delta) \leq p_a(D)$. It follows from the Riemann-Roch theorem that $\dim H^0(D, L) = \dim H^0(D, K_D - L) + \deg L + 1 - p_a(D) \leq \deg L + 1$.

Suppose that $\dim H^0(D, L) = \deg L + 1$. As the above argument shows, we have $\mathscr{O}_D(L) \simeq \mathscr{O}_D(\delta)$. Since the points defining δ can be chosen arbitrarily (as far as δ satisfies the requirement), we see that $|L|$ is free from base points. Suppose further that $d \geq 1$. Let B be the biggest subcurve of D such that $\deg L|_B = 0$, and put $A = D - B$. Then $\mathscr{O}_B(L) \simeq \mathscr{O}_B$ and $L|_A$ is ample. Since $H^0(D, K_D - \delta) \simeq H^0(D, K_D)$ and the support of δ can move on A, the restriction map $H^0(D, K_D) \to H^0(A, K_D)$ should be the zero map. Hence it follows from the exact sequence

$$0 \to \mathscr{O}_B(K_B) \to \mathscr{O}_D(K_D) \to \mathscr{O}_A(K_D) \to 0$$

that $\dim H^1(B, \mathscr{O}_B) = \dim H^0(B, K_B) = \dim H^0(D, K_D) = p_a(D)$. The restriction map $H^0(D, K_D) \to H^0(B, K_D)$ should be injective, because A, B have no common components and A is in the fixed part of $|K_D|$. Hence $H^0(A, K_A)$, which is isomorphic to the kernel, vanishes. In particular, every irreducible component of A is isomorphic to $\boldsymbol{P}^1$. $\square$

COROLLARY 2.2. *Let D be a curve and $D = m_1 D_1 + \cdots + m_r D_r$ a CCC decomposition. For a nef line bundle L on D, we have the following estimate of the dimension of global sections of L;*

$$\begin{aligned}\dim H^0(D, L) &\leq \deg L + \sum_{i=1}^{r} m_i - \frac{1}{2}\left(D^2 - \sum_{i=1}^{r} m_i D_i^2\right) \\ &= \deg L + \sum_{i=1}^{r} m_i - \sum_{i<j} m_i m_j D_i D_j - \sum_{i=1}^{r} \frac{m_i(m_i - 1)}{2} D_i^2.\end{aligned}$$

If the equality is attained in the upper bound, then L is generated by global sections. If $\deg L = 0$, $D^2 = \sum_{i=1}^{r} m_i D_i^2$ and $\dim H^0(D, L) = \sum_{i=1}^{r} m_i$, then $L \simeq \mathscr{O}_D$ and D_i is linearly equivalent to 0 on $(m_i - 1)D_i + \sum_{j>i} m_j D_j$.

PROOF. Consider the decreasing sequence of ideals

$$\mathscr{O}_X,\ \mathscr{O}_X(-D_1), \ldots, \mathscr{O}_X(-m_1 D_1), \ldots, \mathscr{O}_X(-m_1 D_1 - m_2 D_2), \ldots, \mathscr{O}_X(-D).$$

By dividing out by $\mathscr{O}_X(-D)$ and tensoring with L, this sequence defines a filtration of L, of which the associated graded module is of the form

$$L(-m_1 D_1 - \cdots - m_{k-1} D_{k-1} - j D_k)|_{D_k} \quad (1 \leq k \leq r,\ 0 \leq j \leq m_k - 1),$$

which is a nef line bundle on D_k. Applying Theorem 2.1 to each of these modules and almost everything is obvious. The final statement follows from:

1) If a divisor is linearly equivalent to 0 on a curve, so it is on any subcurve.

2) For $0 \leq n_i < m_i$, the divisor $n_i D_i + \sum_{j<i} m_j D_j$ is linearly equivalent to 0 on the curve $(m_i - n_i)D_i + \sum_{j>i} m_j D_j$. □

Remark 2.3. The inequality in Theorem 2.1 and Corollary 2.2 were already obtained in [**10**, Corollaries (3.8) and (3.10)].

3. Minimal models.

Definition 3.1. Let D be a curve on a smooth surface X.

(1) A *minimal model* of D is a subcurve $D_{\min}$ which satisfies the following two conditions:
   (a) $\chi(D_{\min}, \mathcal{O}_{D_{\min}}) = \chi(D, \mathcal{O}_D)$.
   (b) $K_{D_{\min}} = (K_X + D_{\min})|_{D_{\min}}$ is nef.

(2) Let D be a reducible curve. An irreducible component E of D is said to be a $(-m)_D$-*curve* if E is isomorphic to $\boldsymbol{P}^1$ and $E(D-E) = m$.

Lemma 3.2. *Let D be a reducible curve. Let $E \prec D$ be one of its irreducible components and assume that $ED' > 0$, where $D' = D - E$. Then* $\deg K_D|_E \geq -1$, $\chi(D', \mathcal{O}_{D'}) \geq \chi(D, \mathcal{O}_D)$. *Furthermore, the following four conditions on such E are equivalent*:

(1) $\deg K_D|_E = -1$.
(2) *E is a $(-1)_D$-curve, i.e., $ED' = 1$ and E is isomorphic to $\boldsymbol{P}^1$.*
(3) $\chi(D', \mathcal{O}_{D'}) = \chi(D, \mathcal{O}_D)$.
(4) *The restriction maps $H^0(D, \mathcal{O}_D) \to H^0(D', \mathcal{O}_{D'})$ and $H^1(D, \mathcal{O}_D) \to H^1(D', \mathcal{O}_{D'})$ are isomorphisms.*

Given a $(-1)_D$-curve E of D, we have:

(5) *If D contains another $(-1)_D$-curve $E' \neq E$ and $D \neq E + E'$, then E and E' are mutually disjoint and E' is again a $(-1)_{D'}$-curve of $D' = D - E$.*
(6) *If D is chain-connected, then the subcurve $D' = D - E$ is again chain-connected.*

Proof. The adjunction formula tells us $\deg(K_D|_E) = D'E + \deg K_E \geq D'E - 2 \geq -1$. This shows the equivalence of the conditions (1) and (2) as well. Furthermore, the exact sequence

$$0 \to \mathcal{O}_E(-D') \to \mathcal{O}_D \to \mathcal{O}_{D'} \to 0$$

induces the exact sequence

$$0 \to H^0(D, \mathscr{O}_D) \to H^0(D', \mathscr{O}_{D'}) \to H^1(E, \mathscr{O}_E(-D'))$$
$$\to H^1(D, \mathscr{O}_D) \to H^1(D', \mathscr{O}_{D'}) \to 0.$$

Then $\chi(D', \mathscr{O}_{D'}) = \chi(D, \mathscr{O}_D) + \dim H^1(E, \mathscr{O}_E(-D')) \geq \chi(D, \mathscr{O}_D)$. The conditions (3) and (4) are both equivalent to the vanishing of $H^1(E, \mathscr{O}_E(-D'))$, which amount to the condition (2).

If $C \neq E$ is irreducible, then $\deg(K_{D-E}|_C) \leq \deg(K_D|_C)$. Hence a $(-1)_D$-curve $E' \neq E$ is a $(-1)_{D-E}$-curve unless $D - E$ is irreducible. Thus E' is a $(-1)_{D-E}$-curve as well; in other words, $E'(D - E') = E'(D - E - E') = 1$, so that $EE' = 0$.

Suppose that D is chain-connected and that there is a chain-disconnected partition $D - E = A + B$, such that $-A$ is nef on B. Since D is chain-connected, $-A$ is not nef on $B + E$, which means that $EA > 0$ and that E cannot be a component of B (on which $-A$ is nef). In particular, $EB \geq 0$ and hence $1 \leq EA = E(D - E - B) = 1 - EB$. Thus E is disjoint from B, so that $-(A + E)$ is nef on B, contradicting the chain-connectivity of D. □

Corollary 3.3. *If D is a reducible, chain-connected curve with $p_a(D) = -\chi(D, \mathscr{O}_D) + 1 \geq 1$, then there exists one and only one minimal model $D_{\min}$ of D. The minimal model $D_{\min}$ of a chain-connected curve D has the following properties*:

(1) $D_{\min}$ *is chain-connected.*
(2) $D_{\min} \succeq \Delta$ *for any subcurve $\Delta \preceq D$ with K_Δ nef.*
(3) $D_{\min} \preceq \Delta$ *for any subcurve $\Delta \preceq D$ with $\chi(\Delta, \mathscr{O}_\Delta) = \chi(D, \mathscr{O}_D)$.*

Proof. The existence of a chain-connected minimal model is an immediate consequence of Lemma 3.2. The unicity of $D_{\min}$ follows if we check that our minimal model $D_{\min}$ enjoys the properties (2) and (3).

We show (2) by induction on the number of the irreducible components of D. Assume that K_Δ is nef. If K_D is nef, then $D_{\min} = D$ and the assertion trivially holds. If K_D is not nef and D contains a $(-1)_D$-curve E, we see that $\Delta \preceq D - E$ for any $(-1)_D$-curve E by Lemma 3.2 (5). Hence we see both Δ and $D_{\min}$ are subcurves of $D - E$, and induction works.

Let Δ be a subcurve of D. Since D is chain-connected, we find a connecting chain $\Delta = D_0, D_1, \ldots, D_s = D$, where $E_i = D_i - D_{i-1}$ is irreducible with $E_i D_{i-1} \geq 1$. We prove (3) by induction on s. If $s = 0$, then $\Delta = D$ and the assertion is trivial. Assume that $s \geq 1$. If $\chi(\Delta, \mathscr{O}_\Delta) = \chi(D, \mathscr{O}_D)$, then Lemma 3.2

shows that E_s is a $(-1)_D$-curve of D, and hence the two curves Δ and $D_{\min}$ are subcurves of $D - E_s$. Then by our induction hypothesis, $\Delta \succeq D_{\min}$. □

EXAMPLE 3.4. In the usual minimal model theory of surfaces, the exceptional locus which should be blown down is always a union of trees of $\boldsymbol{P}^1$'s. It is not the case for our minimal model theory of curves.

Let E_1, E_2, E_3 be three $\boldsymbol{P}^1$'s on a surface X with a triangle configuration and with self-intersection numbers $E_i^2 = -i$. The curve $D = 2E_1 + 2E_2 + 2E_3$ is chain-connected, with a connecting chain E_1, E_1+E_2, $E_1+E_2+E_3$, $2E_1+E_2+E_3$, $2E_1 + 2E_2 + E_3$, $2E_1 + 2E_2 + 2E_3 = D$. We have $DE_i = 4 - 2i$, $K_X E_i = i - 2$, $\deg(K_D|_{E_i}) = 2-i$, and hence D contains a single $(-1)_D$-curve, which is the (-3)-curve E_3. Then E_2 is the $(-1)_{D-E_3}$-curve and E_1 is the $(-1)_{D-E_3-E_2}$-curve. The reduced curve $E_1 + E_2 + E_3 = D - E_1 - E_2 - E_3$ is the minimal model of D, i.e., $D = 2D_{\min}$.

EXAMPLE 3.5. If D is not chain-connected, then there may be more than one minimal models. For instance, let $C \subset X$ be an elliptic curve whose normal bundle is an element of infinite order in $\mathrm{Pic}^0(C)$. Then K_{mC} is nef on C, while the restriction maps $H^i(mC, \mathscr{O}_{mC}) \to H^i(C, \mathscr{O}_C)$ are isomorphisms $(i = 0, 1)$. Hence mC is a minimal model of nC for $1 \le m \le n$.

LEMMA 3.6. *Let D be a chain-connected curve and Δ a non-trivial subcurve of D with $p_a(\Delta) = p_a(D)$. Then $D - \Delta$ decomposes as $D - \Delta = \Gamma_1 + \cdots + \Gamma_n$, where Γ_i is a chain-connected curve, $\mathscr{O}_{\Gamma_j}(-\Gamma_i)$ is numerically trivial for $i < j$, and $\Delta + \Gamma_i$ is a chain-connected curve satisfying $\Delta\Gamma_i = 1 - p_a(\Gamma_i)$ for $i \in \{1, 2, \ldots, n\}$.*

PROOF. We write a CCC decomposition of $D - \Delta$ as $\Gamma_1 + \cdots + \Gamma_n$, where Γ_i is a chain-connected curve and $\mathscr{O}_{\Gamma_j}(-\Gamma_i)$ is nef for $i < j$. We have $p_a(D) = p_a(\Delta) + \sum_{i=1}^n (p_a(\Gamma_i) - 1 + \Delta\Gamma_i) + \sum_{i<j} \Gamma_i\Gamma_j$. Since $p_a(\Delta) = p_a(D)$ and $\mathscr{O}_{\Gamma_j}(-\Gamma_i)$ is nef for $i < j$, we get

$$\sum_{i=1}^n (p_a(\Gamma_i) - 1 + \Delta\Gamma_i) = -\sum_{i<j} \Gamma_i\Gamma_j \ge 0.$$

For each i, we have $p_a(\Delta) \le \dim H^1(\Delta, \mathscr{O}_\Delta) \le \dim H^1(\Delta + \Gamma_i, \mathscr{O}) \le \dim H^1(D, \mathscr{O}_D) = p_a(D)$ from which we get $\dim H^1(\Delta + \Gamma_i, \mathscr{O}) = p_a(\Delta)$. Then $\dim H^1(\Delta+\Gamma_i, \mathscr{O}) \ge p_a(\Delta+\Gamma_i) = p_a(\Delta)+p_a(\Gamma_i)-1+\Delta\Gamma_i$ yields $p_a(\Gamma_i)-1+\Delta\Gamma_i \le 0$. From this and the above (in)equality, we get $\Delta\Gamma_i = 1 - p_a(\Gamma_i)$ for any i and see that $\mathscr{O}_{\Gamma_j}(-\Gamma_i)$ is numerically trivial for $i < j$. Furthermore, the equality $p_a(D) = p_a(\Delta + \Gamma_i)$ is sufficient to imply that $\Delta + \Gamma_i$ is chain-connected, by Lemma 3.2 (6). □

4. Canonical cycles.

Let (V, o) be (a germ of) a normal surface singularity and $\pi : X \to V$ the minimal resolution. We denote by Z the fundamental cycle on the exceptional set $\pi^{-1}(o)$. We have three different genera for (V, o) (see [**15**]):

$$p_f(V, o) := p_a(Z) \quad \text{(fundamental genus)}$$

$$p_a(V, o) := \max\{p_a(\Gamma) \mid 0 \prec \Gamma,\ \mathrm{Supp}(\Gamma) \subset \pi^{-1}(o)\} \quad \text{(arithmetic genus)}$$

$$p_g(V, o) := \dim_{\boldsymbol{C}}(R^1\pi_*\mathscr{O}_X)_o \quad \text{(geometric genus).}$$

We have $p_f(V, o) \leq p_a(V, o) \leq p_g(V, o)$. When $p_a(V, o) = 1$, (V, o) is called a *weakly elliptic* singularity. It is known that $p_f(V, o) = p_a(V, o)$ if $p_f(V, o) \leq 1$ (see, [**1**], [**2**], [**8**] and [**15**]).

(V, o) is *numerically Gorenstein* if there exists a (possibly zero) curve Z_K with support $\subseteq \pi^{-1}(o)$ such that $-Z_K$ is numerically equivalent to K_X on $\pi^{-1}(o)$. Such Z_K is called the *canonical cycle*. We have $Z_K = 0$ if and only if (V, o) is a rational double point. We tacitly neglect such a trivial case in what follows. Note that the dualizing sheaf ω_{Z_K} is numerically trivial by the adjunction formula. We have $p_g(V, o) = \dim H^1(Z_K, \mathscr{O}_{Z_K}) = \dim H^0(Z_K, \omega_{Z_K})$ (see, e.g. [**13**]).

When (V, o) is a weakly elliptic, numerically Gorenstein singularity, S. S. T. Yau [**16**] introduced a decreasing sequence of fundamental cycles starting from Z, called the *elliptic sequence*, in order to compute Z_K. Furthermore, he gave a bound on $p_g(V, o)$ by the length of the sequence. On the other hand, Tomaru [**14**] considered the case where Z_K is sum $Z + E$ of the fundamental cycle Z and its minimal model E for singularities with $p_f(V, o) > 0$, and showed that $p_g(V, o) \leq p_f(V, o) + 1$ holds. These results of Yau and Tomaru can be generalized as follows:

THEOREM 4.1. *Let (V, o) be a numerically Gorenstein surface singularity with $p_f(V, o) > 0$, $\pi : X \to V$ the minimal resolution and let $Z_K = m_1D_1 + \cdots + m_rD_r$ be a CCC decomposition of the canonical cycle Z_K on $\pi^{-1}(o)$. Put $I = \{i \mid D_i$ is a minimal member of $\{D_j\}_{j=1}^r\}$ and $\nu = \#I$. Then the following hold*:

(1) *D_1 is the fundamental cycle on $\pi^{-1}(o)$.*
(2) *K_{D_i} is nef for $i \in I$.*
(3) *$p_a(D_i) > 0$ for every i and $p_f(V, o) \geq \sum_{i \in I} p_a(D_i) \geq \nu$.*
(4) *Assume that $n = \sum_{i=1}^r m_i \geq 2$. If $m_1 = 1$, then $D_2 = \gcd(D_1, Z_K - D_1)$, $p_a(D_2) = p_f(V, o)$ and $\mathrm{Supp}(D_1 - D_2) \cap \mathrm{Supp}(Z_K - D_1 - D_2) = \emptyset$. In particular, $m_2 = 1$ if $m_1 = 1$.*

Furthermore,

$$(5)\ p_g(V,o) \le \sum_{i=1}^{r} m_i p_a(D_i) - \sum_{i\in I}(p_a(D_i)-1) \le (n-\nu)p_f(V,o)+\nu.$$

If $p_g(V,o)$ attains the bound in the first inequality, then (V,o) is a Gorenstein singularity.

Proof.

(1) Since π is the minimal resolution, $K_X \equiv -Z_K$ is nef on $\pi^{-1}(o)$, where the symbol $\equiv$ means the numerical equivalence. It follows from Proposition 1.5 (1) that $Z \preceq Z_K$ and hence $D_1 = Z$, being the biggest chain-connected curve with support $\pi^{-1}(o)$.

(2) On D_i, K_{D_i} is numerically equivalent to $-Z_K + D_i = -\sum_{j<i} m_j D_j - \sum_{j>i} m_j D_j - (m_i-1)D_i$. If $i \in I$, then $\mathrm{Supp}(D_j) \cap \mathrm{Supp}(D_i) = \emptyset$ for $j > i$ and we have $K_{D_i} \equiv -(\sum_{j<i} m_j D_j + (m_i-1)D_i)|_{D_i}$. Hence K_{D_i} is nef and $p_a(D_i) > 0$ for $i \in I$.

(3) By (2), we have $p_a(D_i) > 0$ for any i. We remark that $D_i \prec D_1$ for any $i \ge 2$, since D_i is a chain-connected curve, $\mathscr{O}_{D_i}(-D_1)$ is nef by (1) and $\mathrm{Supp}(D_i) \subset \pi^{-1}(o)$. Since any two distinct members in $\{D_i\}_{i\in I}$ do not intersect, we have $\sum_{i\in I} D_i \preceq D_1$. Then $\sum_{i\in I} p_a(D_i) = \sum_{i\in I} \dim H^1(D_i, \mathscr{O}_{D_i}) = \dim H^1(\sum_{i\in I} D_i, \mathscr{O}) \le \dim H^1(D_1, \mathscr{O}_{D_1}) = p_f(V,o)$.

(4) Assume that $m_1 = 1$ and put $G = \gcd(D_1, Z_K - D_1)$. Then $D_2 \preceq G$. We have $2p_a(G) - 2 = -G(Z_K - G) = -D_1(Z_K - D_1) + (D_1 - G)(Z_K - D_1 - G) = 2p_a(D_1) - 2 + (D_1 - G)(Z_K - D_1 - G)$. By the choice of G, $D_1 - G$ has no components in common with $Z_K - D_1 - G$ and hence $(D_1-G)(Z_K-D_1-G) \ge 0$. Then $p_a(G) \ge p_a(D_1)$. On the other hand, clearly $p_a(G) \le p_a(D_1)$. Hence $p_a(G) = p_a(D_1)$ and $\mathrm{Supp}(D_1 - G) \cap \mathrm{Supp}(Z_K - D_1 - G) = \emptyset$. In view of Lemma 3.2 (6), the former is sufficient to imply that G is chain-connected. Thus $G = D_2$, being a chain-connected component of $Z_K - D_1$. The latter assertion for supports (with $G = D_2$) shows $m_2 = 1$, because $D_1 - D_2$ has an irreducible component meeting D_2 by the chain-connectivity of D_1.

(5) Recall that $\omega_{Z_K} = \mathscr{O}_{Z_K}(K_X + Z_X)$ is numerically trivial. We get $p_g(V,o) = \dim H^0(Z_K, \omega_{Z_K}) \le \sum_{i=1}^{r} \sum_{l=0}^{m_i-1} \dim H^0(D_i, K_X + Z_K - \sum_{j<i} m_j D_j - lD_i)$ as in the proof of Corollary 2.2. We have $Z_K - \sum_{j<i} m_j D_j - lD_i = (m_i - l)D_i + \sum_{j>i} m_j D_j$. Hence, for $i \in I$ and $l = m_i - 1$, we have $(Z_K - \sum_{j<i} m_j D_j - (m_i - 1)D_i)|_{D_i} = D_i|_{D_i}$ and it follows $\mathscr{O}_{D_i}(K_X + Z_K - \sum_{j<i} m_j D_j - (m_i-1)D_i) \simeq \omega_{D_i}$, $-D_i(\sum_{j<i} m_j D_j + (m_i-1)D_i) = \deg \omega_{D_i} = 2p_a(D_i) - 2$ and $\dim H^0(D_i, K_X + Z_K - \sum_{j<i} m_j D_j - (m_i-1)D_i) = p_a(D_i) = 1 - D_i(\sum_{j<i} m_j D_j + (m_i-1)D_i) - (p_a(D_i) - 1)$. For the other pairs (i,l), we have $\dim H^0(D_i, K_X + Z_K - \sum_{j<i} m_j D_j - lD_i) \le 1 - D_i(\sum_{j<i} m_j D_j + lD_i)$ by Theorem 2.1. Summing up, we get

$$p_g(V,o) \leq \sum_{i=1}^{r} m_i - \frac{1}{2}(Z_K^2 - \sum_{i=1}^{r} m_i D_i^2) - \sum_{i \in I}(p_a(D_i) - 1).$$

We have $\sum_{i=1}^r m_i p_a(D_i) = \sum_{i=1}^r m_i - (1/2)(Z_K^2 - \sum_{i=1}^r m_i D_i^2)$ by $0 = p_a(Z_K) - 1 = \sum_{i=1}^r m_i(p_a(D_i) - 1) + (1/2)(Z_K^2 - \sum_{i=1}^r m_i D_i^2)$. Hence we get the first inequality in (5). If the bound is attained, then the restriction $H^0(Z_K, K_X + Z_K) \to H^0(D_1, K_X + Z_K)$ is surjective and $\dim H^0(D_1, K_X + Z_K) = 1$. This implies that $\omega_{Z_K} \simeq \mathcal{O}_{Z_K}$, i.e., (V,o) is Gorenstein. The second inequality in (5) follows from the obvious fact: $p_a(D_i) \leq p_f(V,o)$. $\square$

We confirm that a CCC decomposition of Z_K induces Yau's elliptic sequence, when (V,o) is a weakly elliptic singularity.

COROLLARY 4.2. *Let (V,o) be a weakly elliptic, numerically Gorenstein singularity and Z_K the canonical cycle on its minimal resolution. Then Z_K has a unique CCC decomposition of the form $D_1 + \cdots + D_n$, where*

(1) $D_n \prec D_{n-1} \prec \cdots \prec D_1$,
(2) *each D_i is the fundamental cycle on its support and $p_a(D_i) = 1$; $D_1 = Z$ and D_n is a minimally elliptic cycle* [**8**],
(3) $\mathcal{O}_{D_j}(-D_i)$ *is numerically trivial when $i < j$,*
(4) $\mathrm{Supp}(D_i - D_j) \cap \mathrm{Supp}(D_k) = \emptyset$ *for $i < j < k$,*
(5) $p_g(V,o) \leq n$ *with equality holding only if (V,o) is Gorenstein.*

In other words, the sequence $D_n \prec D_{n-1} \prec \cdots \prec D_1$ coincides with Yau's elliptic sequence.

PROOF. Let $Z_K = m_1 D_1 + \cdots + m_n D_n$ be a CCC decomposition. We know that D_1 is the fundamental cycle Z by Theorem 4.1 (1). Since everything is clear when $Z_K = D_1$, we assume $n(D) = \sum_{i=1}^n m_i \geq 2$.

We first remark that $-D_1$ is numerically trivial on $Z_K - D_1$. If not, then $D_1(Z_K - D_1) < 0$ and we would get $p_a(Z_K - D_1) > 1$ by $p_a(Z_K) = p_a(D_1) + p_a(Z_K - D_1) - 1 + D_1(Z_K - D_1)$ and $p_a(Z_K) = p_a(D_1) = 1$, which is impossible by $p_a(V,o) = 1$. In particular, this implies $m_1 = 1$, because we would have $D_1(Z_K - D_1) = D_1^2 + D_1(Z_K - 2D_1) \leq D_1^2 < 0$ if $m_1 > 1$. Then, by Theorem 4.1 (4), we have $m_2 = 1$, $p_a(D_2) = 1$, $D_2 = \gcd(D_1, Z_K - D_1)$ and $\mathrm{Supp}(D_1 - D_2) \cap \mathrm{Supp}(Z_K - D_1 - D_2) = \emptyset$. Note that $Z_K - D_1$ is the canonical cycle on its support with a CCC decomposition $Z_K - D_1 = D_2 + m_3 D_3 + \cdots + m_n D_n$, since $-D_1$ is numerically trivial on $Z_K - D_1$. Therefore, an obvious induction using Theorem 4.1 (4) gives us $m_i = 1$, $p_a(D_i) = 1$, $D_i = \gcd(D_{i-1}, Z_K - \sum_{j<i} D_j)$ and $\mathrm{Supp}(D_{i-1} - D_i) \cap \mathrm{Supp}(Z_K - \sum_{j \leq i} D_j) = \emptyset$. Now, all the assertions are clear from this and Theorem 4.1, except the statement for D_n in (2).

It follows from Theorem 4.1 (2) that ω_{D_n} is nef and, hence, numerically trivial by $p_a(D_n) = 1$. Since D_n is chain-connected and $\dim H^0(D_n, \omega_{D_n}) = 1$, we get $\omega_{D_n} \simeq \mathscr{O}_{D_n}$. This is sufficient to imply that D_n is 2-connected, and hence it is a minimally elliptic cycle. $\square$

The last inequality for $p_g(V, o)$ in Theorem 4.1 tells us that the geometric genus, which is an analytic invariant, is bounded from above by topological data determined by the resolution dual graph. However, the bound seems rather crude.

Example 4.3. We borrow an example from [**14**, p. 293] and consider $(V, o) = \{x_0^2 + x_1^8 + x_2^{12+8t} = 0\}$. We follow [**14**] for the numbering of irreducible components A_i of the exceptional set. The canonical cycle is given by

$$Z_K = (6t+8)A_0 + (3t+4)(A_1 + A_2) + 3\sum_{i=0}^{t}(t+1-i)(A_{3,i} + A_{4,i}).$$

For $0 \leq j \leq t$, we put

$$D_{j+1} = 2A_0 + A_1 + A_2 + \sum_{i=0}^{t-j}(A_{3,i} + A_{4,i}).$$

Then D_1 is the fundamental cycle and D_{t+1} is its minimal model. We further put $D_{t+2} = A_0 + A_1 + A_2$ and $D_{t+3} = A_0$. Then $Z_K = 3\sum_{j=1}^{t+1} D_j + D_{t+2} + D_{t+3}$ is the CCC decomposition. We know that D_{t+3} is the smallest member of $\{D_j\}$. Furthermore, $p_f(V, o) = p_a(D_1) = \cdots = p_a(D_{t+1}) = 3$ and $p_a(D_{t+2}) = p_a(D_{t+3}) = 1$. Hence the bound given in Theorem 4.1 becomes $p_g(V, o) \leq 3 \times 3(t+1) + 1 + 1 = 9t + 11$, while it is known that $p_g(V, o) = 6t + 8$.

5. Subcurves of a 1-connected curve.

We study subcurves of a 1-connected curve by means of CCC decompositions. Almost all results in this section can be shown also by using the 0-maximality argument as in [**9**] and [**7**].

Theorem 5.1. *Let Δ be a non-trivial subcurve of a* 1*-connected curve D and L a line bundle on Δ which is numerically trivial. If $\Delta = \Gamma_1 + \cdots + \Gamma_n$ denotes a CCC decomposition, then* $\dim H^0(\Delta, L) \leq \Delta(D - \Delta) + \sum_{i<j} \Gamma_i \Gamma_j \leq \Delta(D - \Delta)$. *Furthermore, if* $\dim H^0(\Delta, L) = \Delta(D - \Delta)$*, then the following hold.*

(1) $\dim H^0(\Delta, L) = n$ *and* $\mathscr{O}_\Delta(L) \simeq \mathscr{O}_\Delta$,
(2) $\mathscr{O}_{\Gamma_i + \cdots + \Gamma_n}(-\Gamma_{i-1}) \simeq \mathscr{O}_{\Gamma_i + \cdots + \Gamma_n}$ *for* $2 \leq i \leq n$,

(3) Γ_i *and* $D-\Delta$ *are* 1*-connected curves with* $(D-\Gamma_i)\Gamma_i = (D-\Delta)\Gamma_i = 1$ *for* $1 \le i \le n$,

(4) $(D-\Delta)\Delta \le p_a(D) - p_a(D-\Delta)$ *holds, when* K_D *is nef on* Δ.

PROOF. Let $\Delta = \Gamma_1 + \cdots + \Gamma_n$ be a CCC decomposition, where Γ_i is a chain-connected curve and $\mathscr{O}_{\Gamma_j}(-\Gamma_i)$ is nef for $i<j$. We put $a = a(\Delta) = -\sum_{i<j}\Gamma_i\Gamma_j$. Then $a \ge 0$. Since D is 1-connected, we have

$$1 \le (D-\Gamma_i)\Gamma_i = (D-\Delta)\Gamma_i + \Gamma_i \sum_{j\ne i}\Gamma_j \tag{5.1}$$

for each i. Summing up, we get $n \le (D-\Delta)\Delta + 2\sum_{i<j}\Gamma_i\Gamma_j$, that is, $n+2a \le (D-\Delta)\Delta$. On the other hand, we have $\dim H^0(\Delta, L) \le n - \sum_{i<j}\Gamma_i\Gamma_j = n+a$ by Corollary 2.2. Therefore, $\dim H^0(\Delta, L) \le n+a \le (D-\Delta)\Delta - a \le (D-\Delta)\Delta$, which is what we want.

Assume now that $\dim H^0(\Delta, L) = (D-\Delta)\Delta$. Then we have equality signs everywhere in the inequalities appeared in the above discussion. In particular, $a=0$ and $\dim H^0(\Delta, L) = n$. The assertions (1), (2) follow from Corollary 2.2. We show (3). Since $(D-\Gamma_i)\Gamma_i = 1$ by (5.1), we see that Γ_i and $D-\Gamma_i$ are 1-connected. We have $(D-\Delta)\Gamma_i = 1$. Since $\Gamma_i\Gamma_j = 0$ when $i \ne j$, starting from $D-\Gamma_1$, we can inductively show that $D-\Gamma_1-\cdots-\Gamma_i$ is 1-connected. In particular, so is $D-\Delta = D-\sum_{i=1}^n \Gamma_i$. Finally, we show (4). We have $a(\Delta)=0$ and $(D-\Gamma_i)\Gamma_i = 1$ for any i. Then

$$p_a(D) = p_a(D-\Delta) + (D-\Delta)\Delta - n + \sum_{i=1}^n p_a(\Gamma_i).$$

Since K_D is nef on Δ, we have $0 \le \deg K_D|_{\Gamma_i} = \deg K_{\Gamma_i} + (D-\Gamma_i)\Gamma_i = \deg K_{\Gamma_i}+1$. It follows $p_a(\Gamma_i) > 0$ for each i. Hence $p_a(D) \ge p_a(D-\Delta) + (D-\Delta)\Delta$. □

Quite similarly, one can show the following two corollaries.

COROLLARY 5.2. *Let* Δ *and* D *be as in Theorem* 5.1. *Let* $\Delta = \Gamma_1+\cdots+\Gamma_n$ *be a CCC decomposition and put* $a = a(\Delta) = -\sum_{i<j}\Gamma_i\Gamma_j$. *If* L *is a nef line bundle on* Δ *satisfying* $\deg L \le a$, *then* $\dim H^0(\Delta, L) \le \Delta(D-\Delta)$ *holds. If the equality holds here, then* $\deg L = a$, $\dim H^0(\Delta, L) = n+2a$ *and each* Γ_i *is a* 1*-connected curve satisfying* $(D-\Gamma_i)\Gamma_i = 1$.

COROLLARY 5.3. *Let* Δ *be a non-trivial subcurve of a* 2*-connected curve* D. *If* L *is a numerically trivial line bundle on* Δ, *then* $2\dim H^0(\Delta, L) \le \Delta(D-\Delta)$.

Theorem 5.4. *Let L be a line bundle on a* 1*-connected curve D which is numerically equivalent to K_D, and let Z be a non-trivial subcurve of D such that the restriction map $H^0(D,L) \to H^0(Z,L)$ is the zero map. Then*

$$p_a(Z) \leq \begin{cases} 0, & \text{if } L = K_D, \\ 1, & \text{otherwise.} \end{cases}$$

If $p_a(Z)$ attains the bound, then Z is 1*-connected and D decomposes as*

$$D = Z + \Gamma_1 + \cdots + \Gamma_n,$$

where $n = Z(D-Z)$, $\mathscr{O}_{D-Z}(L) \simeq \mathscr{O}_{D-Z}(K_D)$, each Γ_i is a 1*-connected curve with $(D-\Gamma_i)\Gamma_i = Z\Gamma_i = 1$, $\mathscr{O}_{\Gamma_j+\cdots+\Gamma_n}(-\Gamma_{j-1}) \simeq \mathscr{O}_{\Gamma_j+\cdots+\Gamma_n}$ for $2 \leq j \leq n$ and, either $\Gamma_j \preceq \Gamma_i$ or $\mathrm{Supp}(\Gamma_i) \cap \mathrm{Supp}(\Gamma_j) = \emptyset$ when $i < j$.*

Proof. By the assumption, we have $H^1(D,L) = 0$ unless $L = K_D$. It follows from the cohomology long exact sequence for

$$0 \to \mathscr{O}_{D-Z}(L-Z) \to \mathscr{O}_D(L) \to \mathscr{O}_Z(L) \to 0$$

that $\dim H^1(D-Z, L-Z) = \dim H^0(Z,L) + \dim H^1(D,L)$. By the Riemann-Roch theorem and the adjunction formula, we have $\dim H^0(Z,L) = \deg L|_Z + 1 - p_a(Z) = \deg(L-K_D)|_Z + \deg K_Z + Z(D-Z) + 1 - p_a(Z) = p_a(Z) + Z(D-Z) - 1$. Hence

$$\dim H^0(D-Z, K_D - L) = \begin{cases} p_a(Z) + Z(D-Z), & \text{if } L = K_D, \\ p_a(Z) + Z(D-Z) - 1, & \text{otherwise.} \end{cases}$$

Since $K_D - L$ is numerically trivial, we get $\dim H^0(D-Z, K_D - L) \leq Z(D-Z) - a(D-Z)$ by Theorem 5.1 applied to $\Delta = D - Z$. Hence

$$p_a(Z) \leq p_a(Z) + a(D-Z) \leq \begin{cases} 0, & \text{if } L = K_D, \\ 1, & \text{otherwise.} \end{cases}$$

The rest follows from Theorem 5.1. $\square$

Corollary 5.5 ([**7**]). *Let D be a* 1*-connected curve with $p_a(D) > 0$ and Z the fixed part of $|K_D|$, that is, the biggest subcurve such that the restriction $H^0(D,K_D) \to H^0(Z,K_D)$ is zero. Then $H^1(Z, \mathscr{O}_Z) = 0$.*

Proof. Since $p_a(D) > 0$, we see that Z is a non-trivial subcurve of D. Furthermore, the restriction $H^0(D, K_D) \to H^0(Z', K_D)$ is zero for any subcurve $Z' \preceq Z$. By Theorem 5.4, we have $p_a(Z') \leq 0$. Hence $H^1(Z, \mathscr{O}_Z) = 0$ by [**7**, Proposition 1.7]. □

As Theorem 5.4 suggests, it is worth studying curves D such that $p_a(D') \leq 1$ holds for any subcurve $D' \preceq D$. For such, we have the following:

Lemma 5.6. *Let D be a curve such that $p_a(D') \leq 1$ for any $0 \prec D' \preceq D$. Assume that $p_a(D) = 1$. Then D is 0-connected and decomposes as $D = \Gamma_1 + \cdots + \Gamma_n$, where each Γ_i is a chain-connected curve with $p_a(\Gamma_i) = 1$ and $\mathscr{O}_{\Gamma_i+\cdots+\Gamma_n}(-\Gamma_{i-1})$ is numerically trivial. In particular, $\Gamma_i\Gamma_j = 0$ and, ether $\Gamma_j \preceq \Gamma_i$ or $\mathrm{Supp}(\Gamma_i) \cap \mathrm{Supp}(\Gamma_j) = \emptyset$ for $i < j$. Furthermore, $\dim H^0(D, \mathscr{O}_D) \leq n$ with equality holding only when $\mathscr{O}_{\Gamma_i+\cdots+\Gamma_n}(-\Gamma_{i-1}) \simeq \mathscr{O}_{\Gamma_i+\cdots+\Gamma_n}$ for $2 \leq i \leq n$. If $\mathrm{Supp}(D)$ is connected, then $\mathrm{Supp}(D) = \mathrm{Supp}(\Gamma_1)$ and $\Gamma_n \preceq \Gamma_{n-1} \preceq \cdots \preceq \Gamma_1$.*

Proof. Let D' be any non-trivial subcurve of D. We have $p_a(D') \leq 1$ and $p_a(D - D') \leq 1$ by the assumption. Then $1 = p_a(D) = p_a(D') + p_a(D - D') - 1 + (D - D')D' \leq 1 + (D - D')D'$. Hence $(D - D')D' \geq 0$ and D is 0-connected. Let $D = \Gamma_1 + \cdots + \Gamma_n$ be a CCC decomposition.

Since $p_a(D) = 1$ and

$$p_a(D) - 1 = \sum_{i=1}^{n}(p_a(\Gamma_i) - 1) + \sum_{i<j}\Gamma_i\Gamma_j \leq \sum_{i=1}^{n}(p_a(\Gamma_i) - 1) \leq 0,$$

we see that $p_a(\Gamma_i) = 1$ and $\mathscr{O}_{\Gamma_i+\cdots+\Gamma_n}(-\Gamma_{i-1})$ is numerically trivial for each i. Then $\dim H^0(D, \mathscr{O}_D) \leq n$ by Corollary 2.2.

If D has connected support, then $\mathrm{Supp}(\Gamma_1) = \mathrm{Supp}(D)$ by Proposition 1.2 (3). Hence $\Gamma_i \preceq \Gamma_1$. Since we have $p_a(\Gamma_i) = p_a(\Gamma_1)$, Corollary 3.3 implies that every Γ_i contains the minimal model of Γ_1 as a common subcurve. Therefore, $\Gamma_j \preceq \Gamma_i$ for $i < j$. □

6. Subcurves of a multiple fibre.

In this section, F denotes a fibre in a fibred surface of genus $g > 0$. We know that the intersection form is negative semi-definite on $\mathrm{Supp}(F)$ by Zariski's lemma. Let D be the numerical cycle on $\mathrm{Supp}(F)$. Then it is 1-connected and there exists a positive integer m such that $F = mD$. We have $g - 1 = m(p_a(D) - 1)$. When $m > 1$, F is called a multiple fibre and $\mathscr{O}_D(D)$ is a torsion element of order m in $\mathrm{Pic}(D)$.

The following is an analogue of Theorem 5.1.

THEOREM 6.1. *Let $F = mD$ be a multiple fibre. Then, for a given curve Δ with $0 \prec \Delta \preceq F$, the inequality $\dim H^0(\Delta, \mathscr{O}_\Delta) \le -\Delta^2 + 1$ holds. If the upper bound is attained, then Δ has a CCC decomposition of the form $\Delta = kD + \Gamma_1 + \cdots + \Gamma_n$ ($n = -\Delta^2$), where*

(1) $1 \le k \le m$,
(2) Γ_i *is a* 1*-connected curve with* $\Gamma_i^2 = -1$ *for* $1 \le i \le n$,
(3) $\mathscr{O}_{\Gamma_j + \cdots + \Gamma_n}(-\Gamma_{j-1}) \simeq \mathscr{O}_{\Gamma_j + \cdots + \Gamma_n}$ *for* $1 \le j \le n$, *where* $\Gamma_0 = kD$.

PROOF. Let kD be the maximal multiple of D such that $kD \preceq \Delta$ and put $A = \Delta - kD$. Then $0 \le k \le m$ and $\Delta^2 = A^2$.

Assume that $A = 0$. Then $k > 0$. We consider the exact sequence

$$0 \to H^0(D, -(i-1)D) \to H^0(iD, \mathscr{O}_{iD}) \to H^0((i-1)D, \mathscr{O}_{(i-1)D})$$

for $2 \le i \le m$. Since $\mathscr{O}_D(-(i-1)D) \not\simeq \mathscr{O}_D$ and D is 1-connected, we have $\dim H^0(D, -(i-1)D) = 0$. Hence $\dim H^0(iD, \mathscr{O}_{iD}) \le \dim H^0((i-1)D, \mathscr{O}_{(i-1)D})$. By induction, we get $\dim H^0(iD, \mathscr{O}_{iD}) \le \dim H^0(D, \mathscr{O}_D) = 1$. In particular, $\dim H^0(\Delta, \mathscr{O}_\Delta) = 1$.

Assume that $A \neq 0$. We have $\dim H^0(kD, \mathscr{O}_{kD}) = 1$ when $k \neq 0$, as shown above. Let $A = \Gamma_1 + \cdots + \Gamma_n$ be a CCC decomposition of A. Since Γ_i is chain-connected and $\mathscr{O}_{\Gamma_i}(-D)$ is nef, we have $\Gamma_i \prec D$ by Proposition 1.5 (1). Then $\Gamma_i^2 \le -1$ and it follows $A^2 = \sum_{i=1}^n \Gamma_i^2 + 2\sum_{i<j} \Gamma_i\Gamma_j \le -n + 2\sum_{i<j}\Gamma_i\Gamma_j$. Since $\mathscr{O}_A(-kD)$ is numerically trivial, we have $\dim H^0(A, -kD) \le n - \sum_{i<j}\Gamma_i\Gamma_j$ by Corollary 2.2. Hence $\dim H^0(A, -kD) \le n - \sum_{i<j}\Gamma_i\Gamma_j \le -A^2 + \sum_{i<j}\Gamma_i\Gamma_j \le -A^2$. By the cohomology long exact sequence for

$$0 \to \mathscr{O}_A(-kD) \to \mathscr{O}_\Delta \to \mathscr{O}_{kD} \to 0,$$

we get $\dim H^0(\Delta, \mathscr{O}_\Delta) \le \dim H^0(A, -kD) + \dim H^0(kD, \mathscr{O}_{kD}) \le \dim H^0(A, -kD) + 1 \le -A^2 + 1 = -\Delta^2 + 1$. If $\dim H^0(\Delta, \mathscr{O}_\Delta) = -\Delta^2 + 1$, then k is positive, $a(A) = -\sum_{i<j}\Gamma_i\Gamma_j = 0$ and $\dim H^0(A, -kD) = n$. Hence we get (3) by Corollary 2.2. Furthermore, $\Gamma_i^2 = -1$ for $1 \le i \le n$. Since $\Gamma_i \prec D$ and D is 1-connected, Γ_i is also 1-connected. □

COROLLARY 6.2. *Let F be a multiple fibre and Z a subcurve of F such that $H^0(F, K_F) \to H^0(Z, K_F)$ is zero. Then $p_a(Z) \le 1$. If $p_a(Z) = 1$, then Z is* 0*-connected and F decomposes as*

$$F = Z + \Gamma_0 + \Gamma_1 + \cdots + \Gamma_n, \qquad (n = -Z^2)$$

where

(1) *for* $1 \le i \le n$, Γ_i *is a* 1*-connected curve with* $\Gamma_i^2 = -1$, $Z\Gamma_i = 1$, *and* $\mathscr{O}_{\Gamma_i}(-(\Gamma_0 + \cdots + \Gamma_{i-1})) \simeq \mathscr{O}_{\Gamma_i}$, $\mathscr{O}_{\Gamma_j}(-\Gamma_i) \equiv 0$ *when* $i < j$.
(2) Γ_0 *is a positive multiple of the numerical cycle* D.

Proof. If the restriction map $H^0(F, K_F) \to H^0(Z, K_F)$ is zero, then the cohomology long exact sequence for

$$0 \to \mathscr{O}_{F-Z}(K_{F-Z}) \to \mathscr{O}_F(K_F) \to \mathscr{O}_Z(K_F) \to 0$$

yields $\dim H^0(F - Z, \mathscr{O}_{F-Z}) = -Z^2 + p_a(Z)$. Since $\dim H^0(F - Z, \mathscr{O}_{F-Z}) \le -Z^2 + 1$ by Theorem 6.1, we get $p_a(Z) \le 1$. Note that we also have $p_a(Z') \le 1$ for any subcurve $Z' \preceq Z$. If $p_a(Z) = 1$, then Z is 0-connected by Lemma 5.6. The rest follows from Theorem 6.1. □

Finally, we remark that the following holds:

Proposition 6.3. *Let* F *be a fibre in a relatively minimal fibred surface of genus* $g \ge 1$ *and* E *a chain-connected curve contained in the fixed part of* $|K_F|$. *Then the following hold.*

(1) *If* F *is a non-multiple fibre, then* $p_a(E) = 0$ *and* $-E^2 \le g$.
(2) *If* F *is a multiple fibre of multiplicity* $m \ge 2$, *then* $p_a(E) \le 1$. *Furthermore,* $-E^2 \le (g-1)/m$ *when* $p_a(E) = 1$, *and* $-E^2 \le (g-1)/m + 2$ *when* $p_a(E) = 0$.

Proof. Let D be the numerical cycle on $\mathrm{Supp}(F)$. Since E is chain-connected, we have $E \preceq D$ by Proposition 1.5 (1). It is easy to see that the restriction map $H^0(F, K_F) \to H^0(D, K_F)$ is surjective and hence $H^0(D, K_F) \to H^0(E, K_F)$ is zero. By the assumption, $K_F|_D$ is a nef line bundle numerically equivalent to K_D. Hence we get the assertion for $p_a(E)$ by Theorem 5.4. The assertion for E^2 follows from Theorem 5.1 (4) applied to $\Delta = D - E$, except in the case (2), $p_a(E) = 0$. For this exceptional case, one can show $-E^2 = (D-E)E \le p_a(D) + 1$ in a similar way. □

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