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Degenerate elliptic boundary value problems with asymmetric nonlinearity

Dedicated to Professor Daisuke Fujiwara on the occasion of his 70th birthday

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Abstract. This paper is devoted to the study of a class of semilinear *degenerate* elliptic boundary value problems with asymmetric nonlinearity which include as particular cases the Dirichlet and Robin problems. The most essential point is how to generalize the classical variational approach to eigenvalue problems with an indefinite weight to the degenerate case. The variational approach here is based on the theory of fractional powers of analytic semigroups. By making use of global inversion theorems with singularities between Banach spaces, we prove very exact results on the number of solutions of our problem. The results extend an earlier theorem due to Ambrosetti and Prodi to the degenerate case.

1. Statement of main results.

Let Ω be a bounded domain of Euclidean space $\boldsymbol{R}^N$, $N \geq 2$, with smooth boundary $\partial\Omega$; its closure $\overline{\Omega} = \Omega \cup \partial\Omega$ is an N dimensional, compact smooth manifold with boundary. Let A be a second-order, elliptic differential operator with real coefficients such that

$$Au = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{N} a^{ij}(x)\frac{\partial u}{\partial x_j}\right) + c(x)u. \tag{1.1}$$

Here:

(1) $a^{ij} \in C^\infty(\overline{\Omega})$ and $a^{ij}(x) = a^{ji}(x)$ for all $x \in \overline{\Omega}$ and $1 \leq i, j \leq N$, and there exists a positive constant a_0 such that

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$$\sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \geq a_0|\xi|^2 \quad \text{for all } (x,\xi) \in \overline{\Omega} \times \boldsymbol{R}^N.$$

(2) $c \in C^\infty(\overline{\Omega})$ and $c(x) \geq 0$ in Ω.

Let B be a first-order, boundary condition with real coefficients such that

$$Bu = a(x')\frac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u. \tag{1.2}$$

Here:

(3) $a \in C^\infty(\partial\Omega)$ and $a(x') \geq 0$ on $\partial\Omega$.
(4) $b \in C^\infty(\partial\Omega)$ and $b(x') \geq 0$ on $\partial\Omega$.
(5) $\partial/\partial\boldsymbol{\nu}$ is the conormal derivative associated with the operator A:

$$\frac{\partial}{\partial \boldsymbol{\nu}} = \sum_{i,j=1}^{N} a^{ij}(x')\, n_j \frac{\partial}{\partial x_i},$$

where $\boldsymbol{n} = (n_1, n_2, \ldots, n_N)$ is the unit exterior normal to the boundary $\partial\Omega$.

Our fundamental hypotheses on the boundary condition B are the following:

(H.1) $a(x') + b(x') > 0$ on $\partial\Omega$.
(H.2) $b(x') \not\equiv 0$ on $\partial\Omega$.

The intuitive meaning of hypotheses (H.1) and (H.2) is that either the reflection phenomenon or the absorption phenomenon does occur at each point of the boundary $\partial\Omega$. More precisely, hypothesis (H.1) implies that the absorption phenomenon occurs at each point of the set $M = \{x' \in \partial\Omega : a(x') = 0\}$, while the reflection phenomenon occurs at each point of the set $\partial\Omega \setminus M = \{x' \in \partial\Omega : a(x') > 0\}$ (see [**15**]). In other words, a Markovian particle moves continuously in the space $\overline{\Omega} \setminus M$ until it dies at the time it reaches the set M where the particle is definitely absorbed (see Figure 1). On the other hand, hypothesis (H.2) implies that the boundary condition B is not equal to the purely Neumann condition.

In this paper we consider the following semilinear elliptic boundary value problem: Let $p(\xi)$ be a function defined on $\boldsymbol{R}$. Given a function $h(x)$ in Ω, find a function $u(x)$ in Ω such that

$$\begin{cases} -Au + p(u) = h & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

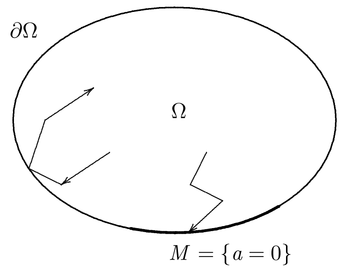


Figure 1.

It should be emphasized that problem (1.3) becomes a *degenerate* boundary value problem from an analytical point of view. This is due to the fact that the so-called Lopatinskii–Shapiro complementary condition is violated at each point of the set M (see [**9**]). Amann [**2**] studied the non-degenerate case; more precisely, he assumes that the boundary $\partial\Omega$ is the disjoint union of the two closed subsets M and $\partial\Omega \setminus M$, each of which is an $(N-1)$ dimensional, compact smooth manifold.

In order to study problem (1.3) in the framework of Hölder spaces, we consider the linear elliptic boundary value problem

$$\begin{cases} Au = g & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega \end{cases} \tag{1.4}$$

in the framework of the Hilbert space $L^2(\Omega)$. We associate with problem (1.4) a densely defined, closed linear operator

$$\mathfrak{A} : L^2(\Omega) \longrightarrow L^2(\Omega)$$

as follows:

(1) $D(\mathfrak{A}) = \{u \in W^{2,2}(\Omega) : Bu = 0 \text{ on } \partial\Omega\}$.
(2) $\mathfrak{A}u = Au$ for all $u \in D(\mathfrak{A})$.

Here and in the following the Sobolev space $W^{k,p}(\Omega)$ for $k \in \boldsymbol{N}$ and $1 < p < \infty$ is defined as follows:

$$\begin{aligned} W^{k,p}(\Omega) = &\text{the space of functions } u \in L^p(\Omega) \text{ whose derivatives } D^\alpha u, \\ &|\alpha| \le k, \text{ in the sense of distributions are in } L^p(\Omega), \end{aligned}$$

and its norm $\| \cdot \|_{W^{k,p}(\Omega)}$ is given by the formula

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \le k} \int_\Omega |D^\alpha u(x)|^p \, dx \right)^{1/p}.$$

Then we have the following fundamental spectral results (i), (ii) and (iii) of the operator $\mathfrak{A}$ (see [**16**, Theorem 5.1]):

(i) The operator $\mathfrak{A}$ is positive and selfadjoint in $L^2(\Omega)$.

(ii) The first eigenvalue λ_1 of $\mathfrak{A}$ is positive and *algebraically simple*, and its corresponding eigenfunction $\phi_1 \in C^{2+\alpha}(\overline{\Omega})$, with exponent $0 < \alpha < 1$, may be chosen to be *strictly positive* in Ω. Namely, we have the assertions

$$\begin{cases} A\phi_1 = \lambda_1 \phi_1 & \text{in } \Omega, \\ \phi_1 > 0 & \text{in } \Omega, \\ B\phi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

(iii) *No* other eigenvalues λ_j, $j \ge 2$, have positive eigenfunctions.

Now we impose the following four conditions (P.1) through (P.4) on the nonlinear term $p(\xi)$:

(P.1) The function $p(\xi)$ is real-valued and is of class C^2 on $\boldsymbol{R}$, and $p(0) = 0$.

(P.2) $p''(\xi) > 0$ on $\boldsymbol{R}$.

(P.3) The limit $\gamma' = \lim_{\xi \to -\infty} p'(\xi)$ exists and satisfies the condition

$$0 < \gamma' < \lambda_1.$$

(P.4) The limit $\gamma'' = \lim_{\xi \to +\infty} p'(\xi)$ exists and satisfies the condition

$$\lambda_1 < \gamma'' < \lambda_2.$$

The purpose of this paper is to prove very exact results on the number of solutions of problem (1.3). In order to study problem (1.3), we introduce a nonlinear map F in the framework of Hölder spaces, and make use of the global inversion theorem with singularities between Banach spaces due to Ambrosetti–Prodi [**3**]. It is worthwhile pointing out here that the method of Leray–Schauder degree gives no useful result for problem (1.3), since the topological degree of the nonlinear map F is equal to *zero* (cf. [**6**], [**12**]).

The next theorem is a generalization of Ambrosetti–Prodi [**3**, Theorem 3.1] and [**4**, Chapter 4, Theorem 2.4] to the degenerate case (see also [**12**, Theorem 3.7.5], [**13**, Section 6.6]):

Theorem 1.1. *Assume that the nonlinear term $p(\xi)$ satisfies conditions* (P.1) *through* (P.4)*. Then there exist a C^1 manifold Y_1 of codimension one in the Hölder space $Y = C^\alpha(\overline{\Omega})$ and two disjoint open subsets Y_0 and Y_2 of Y such that $Y = Y_0 \cup Y_1 \cup Y_2$ with the following two properties*:

(i) *Problem* (1.3) *has a unique solution $u \in C^{2+\alpha}(\overline{\Omega})$ for any function $h \in Y_1$.*
(ii) *Problem* (1.3) *has exactly two solutions u_1, $u_2 \in C^{2+\alpha}(\overline{\Omega})$ for any function $h \in Y_2$, while problem* (1.3) *has no solution for any function $h \in Y_0$.*

The next corollary is a generalization of Berger–Podolak [**5**, Theorem 3] to the degenerate case, and may be proved just as in Nirenberg [**12**, Section 3.7, Exercise]:

Corollary 1.2. *If the nonlinear term $p(\xi)$ satisfies conditions* (P.1) *through* (P.4)*, then the semilinear problem*

$$\begin{cases} -Au + p(u) = 0 & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega \end{cases}$$

has exactly one non-trivial solution $u \in C^{2+\alpha}(\overline{\Omega})$ if and only if $p'(0) \neq \lambda_1$.

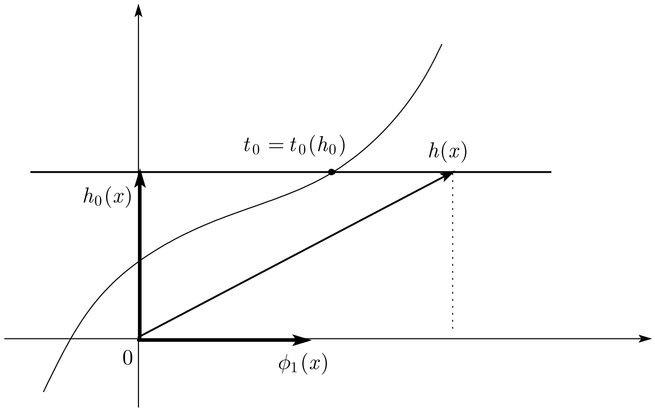
The most essential point in the proof of Theorem 1.1 is how to generalize the classical variational approach to eigenvalue problems with an indefinite weight to the degenerate case. Our variational approach is based on the theory of fractional powers of analytic semigroups (see [**14**]). It should be noticed that our Hilbert space $\mathscr{H} = D(\mathfrak{A}^{1/2})$ is the right space for the variational approach (see formula (3.2) and Remark 3.1).

We can interpret Theorem 1.1 as follows (see [**5**, Theorem 1], [**12**, Section 3.7]): Given a function $h(x)$ defined in Ω, we decompose it as an orthogonal sum in the Hilbert space $L^2(\Omega)$

$$h(x) = h_0(x) + t\,\phi_1(x), \quad t \in \boldsymbol{R},$$

and regard the nonlinear term $p(u)$ as fixed. In this setting, Theorem 1.1 asserts that there exists a number $t_0 = t_0(h_0) \in \boldsymbol{R}$, depending on h_0, such that the semilinear problem (1.3) has no solution for $t > t_0$, exactly one solution for $t = t_0$, and exactly two solutions for $t < t_0$ (see Figure 2).

The rest of this paper is organized as follows. Section 2 deals with local and global inversions of mappings between Banach spaces which go back to Hadamard in the finite dimensional case and to Cacciopoli and Lévy for general Banach spaces. Moreover, we study mappings that possess singularities and are not global



$h(x) = h_0(x) + t\phi_1(x)$

Figure 2.

homeomorphisms (Theorem 2.3). The presentation here is taken from Ambrosetti–Prodi [**4**] and Nirenberg [**12**]. Section 3 is devoted to the study of eigenvalue problems with an indefinite weight. In this section we describe the eigenvalues and eigenfunctions of the eigenvalue problems with an indefinite weight, generalizing the classical results to the degenerate case. This section is the heart of the subject. The crucial point in our variational approach is how to use the theory of fractional powers of analytic semigroups developed in Taira [**14**], which is an essential step in the study of the semilinear problem (1.3) (Theorem 3.1). Furthermore, we make use of a new Kreĭn–Rutman theory for problem (3.1) adapted to the degenerate case (Theorem 3.8). In Section 4 we prove Theorem 1.1, by using global inversion theorems with singularities, just as in Ambrosetti–Prodi [**3**, Theorem 3.1]. To do this, we have only to verify all the conditions of Theorem 2.3. Our proof of Theorem 1.1 is based on the extensive use of the ideas and techniques characteristic of the recent developments in the theory of degenerate elliptic boundary value problems ([**16**]).

2. Local and global inversion theorems.

This section deals with local and global inversions of mappings between Banach spaces which go back to Hadamard in the finite dimensional case and to Cacciopoli and Lévy for general Banach spaces. Moreover, we study mappings that possess singularities and are not global homeomorphisms (Theorem 2.3). The presentation here is taken from Ambrosetti–Prodi [**4**] and Nirenberg [**12**].

2.1. Local inversion theorem.

Let X and Y be Banach spaces and let $F : X \to Y$ be a C^1 map. Namely, the map F is differentiable in X and the Fréchet derivative DF is continuous as a map of X into the space $B(X, Y)$ of bounded (continuous) linear operators on X into Y.

Definition 2.1. A continuous map $F : X \to Y$ is said to be *locally invertible* at a point u^* of X if there exist an open neighborhood U of u^*, an open neighborhood V of $F(u^*)$ and a continuous map $G : V \to U$ such that

$$\begin{cases} G(F(u)) = u & \text{for all } u \in U, \\ F(G(v)) = v & \text{for all } v \in V. \end{cases}$$

The map G is called the *local inverse* of F, and will be denoted by F^{-1}.

The local inversion theorem reads as follows (see [**4**, Chapter 2, Theorem 1.2]):

Theorem 2.1 (the local inversion theorem). *Let F be a C^1 map of a Banach space X into a Banach space Y. Assume that the Fréchet derivative $DF(u^*) : X \to Y$ is continuous and invertible at a point $u^* \in X$. Then it follows that F is* locally invertible *at u^* with C^1 inverse F^{-1}. More precisely, there exist an open neighborhood U of u^* and an open neighborhood V of $F(u^*)$ such that the inverse $F^{-1} : V \to U$ is a C^1 map and that*

$$D(F^{-1})(v) = (DF(u))^{-1} \quad \textit{for all } v = F(u) \textit{ with } v \in U.$$

2.2. Global inversion theorem.

Let M and N be metric spaces and let $F : M \to N$ be a continuous map. The map $F : M \to N$ is said to be *proper* if the preimage $F^{-1}(K)$ is compact in M for any compact set K in N. We remark that if F is proper, then it maps closed sets in M into closed sets in N.

A topological space T is said to be *simply connected* if it is arcwise connected and if every closed path σ in T is homotopic to a constant. Namely, for any given map $\sigma \in C([0, 1], T)$ with $\sigma(0) = \sigma(1)$ there exist a map $h \in C([0, 1] \times [0, 1], T)$ and a point $v \in T$ such that

$$\begin{cases} h(s, 0) = \sigma(s) & \text{for } 0 \le s \le 1, \\ h(s, 1) = v & \text{for } 0 \le s \le 1, \\ h(0, t) = h(1, t) & \text{for } 0 \le t \le 1. \end{cases}$$

Now we are in position to state the global inversion theorem (see [**4**, Chapter 3, Theorem 1.8]):

Theorem 2.2 (the global inversion theorem). *Let M be an arcwise connected metric space and let N be a simply connected metric space. Assume that a continuous map $F : M \to N$ is proper and locally invertible on all of M. Then it follows that F is a* homeomorphism *of M onto N.*

2.3. Global inversion theorem with singularities.

Let X and Y be real Banach spaces and let $F : X \to Y$ be a C^2 map of an open set in X into Y. In this subsection we study the equation $F(u) = h$. The problem is to find for which elements $h \in Y$ there exist solutions $u \in X$, and how many there are.

To do this, we introduce the following:

Definition 2.2. We define the singular set

$$\Sigma' = \{u \in X : \text{the Fréchet derivative } DF(u) \text{ at } u \text{ is not invertible}\}.$$

A point u of Σ' is called an *ordinary singular point* if it satisfies the following three conditions:

(a) The null space $N(DF(u))$ of $DF(u)$ is one dimensional, and is spanned by some element $\phi \in X$, i.e., $N(DF(u)) = \operatorname{span}[\phi]$.
(b) The range $R(DF(u))$ of $DF(u)$ is closed and has codimension one in Y, i.e., $\operatorname{codim} R(DF(u)) = \dim Y/R(DF(u)) = 1$.
(c) $D^2F(u)[\phi, \phi] \notin R(DF(u))$ where $D^2F(u)$ is the second Fréchet derivative at u.

Definition 2.3. Let X be a real Banach space. A subset M of X is called a C^1 *manifold of codimension one* in X if, for every point $u^* \in M$ there exist a positive number δ and a functional $\Gamma : B(u^*, \delta) \to \boldsymbol{R}$ of class C^1 such that

$$\begin{cases} M \cap B(u^*, \delta) = \{u \in B(u^*, \delta) : \Gamma(u) = 0\}, \\ D\Gamma(u^*) \neq 0. \end{cases}$$

If $u \in \Sigma'$ is an ordinary singular point, then we can compute locally the number of solutions of the equation $F(u) = h$. More precisely, we can obtain the following global inversion theorem with singularities (see [**4**, Chapter 3, Theorem 2.6], [**12**, Theorem 3.7.5]):

Theorem 2.3 (the global inversion theorem with singularities). *Let X and*

Y be Banach spaces and let $F : X \to Y$ be a C^2 map. Assume that the following four conditions are satisfied:

(S.1) *The mapping $F : X \to Y$ is proper.*
(S.2) *Every point $u \in \Sigma'$ is an ordinary singular point.*
(S.3) *For every $h \in F(\Sigma')$, the equation $F(u) = h$ has a unique solution $u \in X$.*
(S.4) *The singular set Σ' is connected.*

Then it follows that $Y_1 = F(\Sigma')$ is a C^1 manifold of codimension one in Y and further that there exist two disjoint open subsets Y_0 and Y_2 of Y such that

(i) *$Y = Y_0 \cup Y_1 \cup Y_2$.*
(ii) *The number $[h]$ of solutions of the equation $F(u) = h$ is given as follows:*

$$[h] = \begin{cases} 0 & \text{if } h \in Y_0, \\ 1 & \text{if } h \in Y_1, \\ 2 & \text{if } h \in Y_2. \end{cases}$$

3. Eigenvalue problems with an indefinite weight.

This section is devoted to the study of the following eigenvalue problem with an indefinite weight: Given a weight function $m(x)$ defined in Ω, find a function $u(x)$ in Ω and a number λ such that

$$\begin{cases} Au = \lambda\, m(x) u & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

It should be emphasized that the weight function $m(x)$ may change sign in Ω. In this section we describe the eigenvalues and eigenfunctions of problem (3.1), generalizing the classical results of de Figueiredo [**7**] to the degenerate case. The crucial point in our variational approach is how to use the theory of fractional powers of analytic semigroups developed in Taira [**14**], which is an essential step in the study of the semilinear problem (1.3) (Theorem 3.1).

Since the operator $\mathfrak{A}$ is positive and selfadjoint in $L^2(\Omega)$, we can define its square root $\mathscr{C} = \mathfrak{A}^{1/2}$, and introduce a Hilbert space $\mathscr{H}$ as follows:

$$\begin{aligned} &\mathscr{H} = \text{the domain } D(\mathscr{C}) \text{ with the inner product} \\ &\qquad (u,v)_{\mathscr{H}} = (\mathscr{C}u, \mathscr{C}v)_{L^2(\Omega)} \quad \text{for all } u,\, v \in D(\mathscr{C}). \end{aligned} \tag{3.2}$$

Here it is worthwhile pointing out (see [**14**, Theorem 1.10]) that the explicit formula for the fractional power $\mathscr{C} = \mathfrak{A}^{1/2}$ on the domain $D(\mathfrak{A})$ is given by the formula

$$\mathscr{C}u = -\frac{1}{\pi}\int_0^\infty s^{-1/2}(sI+\mathfrak{A})^{-1}\mathfrak{A}u\,ds \quad \text{for all } u \in D(\mathfrak{A}).$$

The next theorem gives a more concrete characterization of the Hilbert space $\mathscr{H}$:

THEOREM 3.1. *The Hilbert space $\mathscr{H}$ coincides with the* completion *of the domain $D(\mathfrak{A}) = \{u \in W^{2,2}(\Omega) : Bu = 0$ on $\partial\Omega\}$ with respect to the inner product*

$$\begin{aligned}(u,v)_{\mathscr{H}} &= (\mathfrak{A}u, v)_{L^2(\Omega)} \\ &= \sum_{i,j=1}^N \int_\Omega a^{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\,dx + \int_\Omega c(x)u\cdot v\,dx \\ &\quad + \int_{\{a(x')\neq 0\}} \frac{b(x')}{a(x')}\,u\cdot v\,d\sigma \quad \textit{for all } u,\, v \in D(\mathfrak{A}). \qquad (3.3)\end{aligned}$$

Here the last term on the right-hand side is an inner product of the Hilbert space $L^2(\partial\Omega)$.

PROOF. We have only to show that the domain $D(\mathfrak{A})$ is dense in the domain $D(\mathfrak{A}^{1/2}) = D(\mathscr{C})$.

To do this, we remark (see [**14**, Section 1.2]) that the operators

$$\mathscr{C}^{-1} : L^2(\Omega) \longrightarrow D(\mathscr{C})$$

and

$$\mathscr{C}^{-1} : D(\mathfrak{A}^{1/2}) \longrightarrow D(\mathfrak{A}^{3/2})$$

are algebraic and topological isomorphisms, and further that

$$D(\mathfrak{A}^{3/2}) \subset D(\mathfrak{A}).$$

Therefore, we obtain that the domain $D(\mathfrak{A})$ is dense in the domain $D(\mathscr{C})$, since $D(\mathfrak{A})$ is dense in $L^2(\Omega)$. The situation can be visualized in the following diagram:

$$
\begin{tikzcd}
L^2(\Omega) \arrow[r, "\mathscr{C}^{-1}"] & D(\mathscr{C}) = \mathscr{H} \\
D(\mathscr{C}) \arrow[u] \arrow[r, "\mathscr{C}^{-1}"] & D(\mathscr{C}^2) = D(\mathfrak{A}) \arrow[u] \\
D(\mathfrak{A}) \arrow[u] \arrow[r, "\mathscr{C}^{-1}"'] & D(\mathfrak{A}^{3/2}) \arrow[u]
\end{tikzcd}
$$

The proof of Theorem 3.1 is complete. □

COROLLARY 3.2. *We have the inclusions*

$$D(\mathfrak{A}) \subset \mathscr{H} \subset W^{1,2}(\Omega) \tag{3.4}$$

with continuous injections.

PROOF. It suffices to show that the inclusion $\mathscr{H} \subset W^{1,2}(\Omega)$ is continuous.

First, it follows from an application of the Rayleigh principle that we have, for all $u \in D(\mathfrak{A})$,

$$\|u\|_{\mathscr{H}}^2 = \|\mathscr{C}u\|_{L^2(\Omega)}^2 = (\mathscr{C}^2 u, u)_{L^2(\Omega)} = (\mathfrak{A}u, u)_{L^2(\Omega)} \geq \lambda_1 \|u\|_{L^2(\Omega)}^2. \tag{3.5}$$

Moreover, it follows from formula (3.3) that

$$\begin{aligned}
\|u\|_{\mathscr{H}}^2 &= \sum_{i,j=1}^N \int_\Omega a^{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx + \int_\Omega c(x)|u|^2 dx + \int_{\{a(x')\neq 0\}} \frac{b(x')}{a(x')} |u|^2 d\sigma \\
&\geq a_0 \sum_{i=1}^N \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^2 dx = a_0 \|\nabla u\|_{L^2(\Omega)}^2.
\end{aligned} \tag{3.6}$$

Therefore, by combining inequalities (3.5) and (3.6) we obtain that, for all $u \in D(\mathfrak{A})$,

$$2\|u\|_{\mathscr{H}}^2 \geq \lambda_1 \|u\|_{L^2(\Omega)}^2 + a_0 \|\nabla u\|_{L^2(\Omega)}^2 \geq \min(\lambda_1, a_0) \|u\|_{W^{1,2}(\Omega)}^2.$$

This proves that the injection $\mathscr{H} \to W^{1,2}(\Omega)$ is continuous.

The proof of Corollary 3.2 is complete. □

REMARK 3.1. It should be noticed that the Hilbert space $\mathscr{H} = D(\mathfrak{A}^{1/2})$ is the right space for the variational approach. In fact, it is known (see [**8**]) that we have the assertion

$$\mathscr{H} = \begin{cases} W_0^{1,2}(\Omega) & \text{if } a(x') \equiv 0 \text{ on } \partial\Omega \text{ (the Dirichlet case)}, \\ W^{1,2}(\Omega) & \text{if } a(x') > 0 \text{ on } \partial\Omega \text{ (the Robin case)}, \end{cases}$$

where

$$W_0^{1,2}(\Omega) = \{u \in W^{1,2}(\Omega) : u = 0 \text{ on } \partial\Omega\}.$$

We associate with problem (3.1) a linear operator

$$\mathscr{T} : \mathscr{H} \longrightarrow \mathscr{H}$$

as follows:

(1) $D(\mathscr{T}) = \mathscr{H}$.
(2) $\mathscr{T}u = \mathfrak{A}^{-1}(m(x)u)$ for all $u \in D(\mathscr{T})$.

The next proposition asserts that the eigenvalue value problem (3.1) with an indefinite weight can be reduced to the study of an operator equation for $\mathscr{T}$:

PROPOSITION 3.3. *Let $m(x)$ be a weight function in $L^r(\Omega)$ with $r > N/2$. Then we have the following two assertions*:

(i) *The eigenvalue problem* (3.1) *has a non-trivial solution $u \in W^{2,2}(\Omega)$ if and only if $\lambda \neq 0$ and the operator equation*

$$\mathscr{T}u = \frac{1}{\lambda}u \tag{3.7}$$

has a non-trivial solution $u \in \mathscr{H}$.

(ii) *The operator $\mathscr{T} : \mathscr{H} \to \mathscr{H}$ is symmetric and completely continuous.*

PROOF. (i) We remark that if $\lambda = 0$, then we have, by [**16**, Theorem 1.1] with $p := 2$,

$$\begin{cases} Au = 0 & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega \end{cases} \Longrightarrow u = 0 \quad \text{in } \Omega.$$

Hence we have only to consider the case where $\lambda \neq 0$. If the operator equation (3.7) has a non-trivial solution $u \in \mathscr{H}$, then it follows that

$$u = \lambda \mathscr{T} u = \lambda \mathfrak{A}^{-1}(m(x)u) \in D(\mathfrak{A}) \subset W^{2,2}(\Omega),$$

so that

$$Au = \lambda \mathfrak{A}(\mathscr{T} u) = \lambda m(x) u \quad \text{in } \Omega.$$

This proves that $u \in W^{2,2}(\Omega)$ is a non-trivial solution of the eigenvalue problem (3.1).

Conversely, we assume that the eigenvalue problem (3.1) has a non-trivial solution $u \in W^{2,2}(\Omega)$. Then we have, by Corollary 3.2,

$$u \in D(\mathfrak{A}) \subset \mathscr{H},$$

and also

$$u = \lambda \mathfrak{A}^{-1}(m(x)u) = \lambda \mathscr{T} u.$$

This proves that $u \in \mathscr{H}$ is a non-trivial solution of the operator equation (3.7).

(ii) First, we prove that $\mathscr{T} : \mathscr{H} \to \mathscr{H}$ is symmetric:

$$(\mathscr{T} u, v)_{\mathscr{H}} = (u, \mathscr{T} v)_{\mathscr{H}} \quad \text{for all } u,\, v \in \mathscr{H}. \tag{3.8}$$

Indeed, we have, by the definition of $\mathscr{T}$,

$$\begin{aligned}
(\mathscr{T} u, v)_{\mathscr{H}} &= (\mathfrak{A}^{-1}(m(x)u), v)_{\mathscr{H}} \\
&= (\mathscr{C}\mathfrak{A}^{-1}(m(x)u), \mathscr{C} v)_{L^2(\Omega)} \\
&= (\mathscr{C}^2\mathfrak{A}^{-1}(m(x)u), v)_{L^2(\Omega)} \\
&= (m(x)u, v)_{L^2(\Omega)} \\
&= \int_\Omega m(x) u \cdot v \, dx,
\end{aligned} \tag{3.9}$$

and also

$$\begin{aligned}
(u, \mathscr{T} v)_{\mathscr{H}} &= (u, \mathfrak{A}^{-1}(m(x)v))_{\mathscr{H}} \\
&= (\mathscr{C} u, \mathscr{C} \mathfrak{A}^{-1}(m(x)v))_{L^2(\Omega)} \\
&= (u, \mathscr{C}^2 \mathfrak{A}^{-1}(m(x)v))_{L^2(\Omega)} \\
&= (u, m(x)v)_{L^2(\Omega)} \\
&= \int_\Omega m(x) u \cdot v \, dx.
\end{aligned} \tag{3.10}$$

Therefore, the desired formula (3.8) follows by combining formulas (3.9) and (3.10).

Secondly, we prove that $\mathscr{T} : \mathscr{H} \to \mathscr{H}$ is completely continuous. To do this, we assume that $\{u_n\}$ is a bounded sequence in the space $\mathscr{H}$. Then it follows from assertion (3.4) that $\{u_n\}$ is bounded in the Sobolev space $W^{1,2}(\Omega)$. By applying the Rellich–Kondrachov theorem (see [**1**, Theorem 6.3]), we can find a subsequence $\{u_{n'}\}$ which converges to some function u in the space $L^2(\Omega)$ as $n' \to \infty$. Moreover, we recall that the operator $\mathfrak{A}^{-1} : L^2(\Omega) \to L^2(\Omega)$ is continuous. Summing up, we have the following two assertions:

(a) $m(x)u_{n'} \to m(x)u$ in $L^2(\Omega)$ as $n' \to \infty$.
(b) $\mathfrak{A}^{-1}(m(x)u_{n'}) \to \mathfrak{A}^{-1}(m(x)u)$ in $L^2(\Omega)$ as $n' \to \infty$.

Therefore, in view of Schwarz's inequality it follows that

$$\begin{aligned}
&\|\mathscr{T} u_{n'} - \mathscr{T} u\|_{\mathscr{H}}^2 \\
&\quad = \|\mathfrak{A}^{-1}(m(x)u_{n'}) - \mathfrak{A}^{-1}(m(x)u)\|_{\mathscr{H}}^2 \\
&\quad = \big(\mathfrak{A}(\mathfrak{A}^{-1}(m(x)u_{n'}) - \mathfrak{A}^{-1}(m(x)u)), \mathfrak{A}^{-1}(m(x)u_{n'}) - \mathfrak{A}^{-1}(m(x)u)\big)_{L^2(\Omega)} \\
&\quad = \big(m(x)u_{n'} - m(x)u, \mathfrak{A}^{-1}(m(x)u_{n'}) - \mathfrak{A}^{-1}(m(x)u)\big)_{L^2(\Omega)} \\
&\quad \leq \|m(x)u_{n'} - m(x)u\|_{L^2(\Omega)} \cdot \big\|\mathfrak{A}^{-1}(m(x)u_{n'}) - \mathfrak{A}^{-1}(m(x)u)\big\|_{L^2(\Omega)} \\
&\longrightarrow 0 \quad \text{as } n' \to \infty.
\end{aligned}$$

This proves that the subsequence $\{\mathscr{T} u_{n'}\}$ converges to the function $\mathscr{T} u$ in $\mathscr{H}$ as $n' \to \infty$.

The proof of Proposition 3.3 is complete. □

The next proposition gives the variational characterization of eigenvalues of problem (3.1) (cf. [**7**, Proposition 1.10]):

PROPOSITION 3.4. *Let $m(x)$ be a weight function in $L^r(\Omega)$ with $r > N/2$.*

Then the eigenvalue problem (3.1) *with an indefinite weight has a double sequence of eigenvalues*

$$0 < \lambda_1(m) < \lambda_2(m) \leq \ldots \leq \lambda_n(m) \leq \ldots,$$
$$\ldots \leq \lambda_{-n}(m) \leq \ldots \leq \lambda_{-2}(m) < \lambda_{-1}(m) < 0.$$

Moreover, we have the following two assertions:

(i) *The eigenvalues* $\lambda_n(m)$ *and* $\lambda_{-n}(m)$ *are characterized respectively as follows*:

$$\frac{1}{\lambda_n(m)} = \sup_{F_n} \inf \left\{ \int_\Omega m(x)|u|^2 \, dx : \|u\|_{\mathscr{H}} = 1, \ u \in F_n \right\},$$
$$\frac{1}{\lambda_{-n}(m)} = \inf_{F_n} \sup \left\{ \int_\Omega m(x)|u|^2 \, dx : \|u\|_{\mathscr{H}} = 1, \ u \in F_n \right\},$$

where F_n *varies over all* n*-dimensional subspaces of* $\mathscr{H}$*.*

(ii) *The corresponding orthonormal eigenfunctions* $\varphi_n(x)$ *and* $\varphi_{-n}(x)$ *in* $\mathscr{H}$ *are characterized respectively as follows*:

$$\begin{aligned}(\varphi_n, v)_{\mathscr{H}} &= \sum_{i,j=1}^N \int_\Omega a^{ij}(x) \frac{\partial \varphi_n}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx + \int_\Omega c(x) \varphi_n \cdot v \, dx \\ &\quad + \int_{\{a(x') \neq 0\}} \frac{b(x')}{a(x')} \varphi_n \cdot v \, d\sigma \\ &= \lambda_n(m) \int_\Omega m(x) \varphi_n \cdot v \, dx \quad \text{for all } v \in \mathscr{H}. \\ \frac{1}{\lambda_n(m)} &= \int_\Omega m(x) |\varphi_n|^2 dx.\end{aligned}$$

$$\begin{aligned}(\varphi_{-n}, v)_{\mathscr{H}} &= \sum_{i,j=1}^N \int_\Omega a^{ij}(x) \frac{\partial \varphi_{-n}}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx + \int_\Omega c(x) \varphi_{-n} \cdot v \, dx \\ &\quad + \int_{\{a(x') \neq 0\}} \frac{b(x')}{a(x')} \varphi_{-n} \cdot v \, d\sigma \\ &= \lambda_{-n}(m) \int_\Omega m(x) \varphi_{-n} \cdot v \, dx \quad \text{for all } v \in \mathscr{H}. \\ \frac{1}{\lambda_{-n}(m)} &= \int_\Omega m(x) |\varphi_{-n}|^2 dx.\end{aligned}$$

Proof. Indeed, it suffices to note that we have, for all u, $v \in \mathscr{H}$,

$$\begin{aligned}(\mathscr{T}u, v)_{\mathscr{H}} &= (\mathscr{C}\mathscr{T}u, \mathscr{C}v)_{L^2(\Omega)} \\ &= (\mathscr{C}^2\mathfrak{A}^{-1}(m(x)u), v)_{L^2(\Omega)} \\ &= (m(x)u, v)_{L^2(\Omega)} \\ &= \int_\Omega m(x)u \cdot v\, dx,\end{aligned}$$

and further that we have, for all $v \in \mathscr{H}$,

$$\begin{aligned}(\mathscr{T}\varphi_n, v)_{\mathscr{H}} &= \left(\frac{1}{\lambda_n(m)}\,\varphi_n, v\right)_{\mathscr{H}} = \frac{1}{\lambda_n(m)}(\varphi_n, v)_{\mathscr{H}}, \\ (\mathscr{T}\varphi_{-n}, v)_{\mathscr{H}} &= \left(\frac{1}{\lambda_{-n}(m)}\,\varphi_{-n}, v\right)_{\mathscr{H}} = \frac{1}{\lambda_{-n}(m)}(\varphi_{-n}, v)_{\mathscr{H}}.\end{aligned}$$

The proof of Proposition 3.4 is complete. $\square$

By using Proposition 3.4, we can describe how the eigenvalues $\lambda_n(m)$ vary as a function of $m(x)$. First, we prove the following comparison property of eigenvalues $\lambda_n(m)$ (cf. [**7**, Proposition 1.12A]):

Proposition 3.5. *Let $m(x)$ and $\widehat{m}(x)$ be two weight functions in $L^r(\Omega)$ with $r > N/2$ such that*

$$m(x) \leq \widehat{m}(x) \quad \text{almost everywhere in } \Omega.$$

If the Lebesgue measure of the set

$$\Omega_+ = \{x \in \Omega : m(x) > 0\}$$

is positive, then we have, for all $n \in \boldsymbol{N}$,

$$\lambda_n(m) \geq \lambda_n(\widehat{m}).$$

Moreover, if $m(x) < \widehat{m}(x)$ on a subset of positive measure in Ω, then we have, for all $n \in \boldsymbol{N}$,

$$\lambda_n(m) > \lambda_n(\widehat{m}).$$

Proof. Indeed, it follows from an application of Proposition 3.4 that

$$\begin{aligned}\frac{1}{\lambda_n(m)} &= \sup_{F_n} \inf \left\{ \int_\Omega m(x)|u|^2 dx : \|u\|_{\mathscr{H}} = 1,\ u \in F_n \right\} \\ &\leq \sup_{F_n} \inf \left\{ \int_\Omega \widehat{m}(x)|u|^2 dx : \|u\|_{\mathscr{H}} = 1,\ u \in F_n \right\} \\ &= \frac{1}{\lambda_n(\widehat{m})},\end{aligned}$$

so that

$$\lambda_n(m) \geq \lambda_n(\widehat{m}).$$

It is easy to see that if $m(x) < \widehat{m}(x)$ on a subset of positive measure in Ω, then we have the strict inequality

$$\lambda_n(m) > \lambda_n(\widehat{m}).$$

The proof of Proposition 3.5 is complete. □

Remark 3.2. If $m(x) \equiv 1$ in Ω, then we simply write

$$\lambda_j = \lambda_j(1), \quad j = 1, 2, \ldots.$$

It is easy to see that we have, for all $\alpha > 0$,

$$\lambda_j(\alpha) = \frac{\lambda_j}{\alpha}, \quad j = 1, 2, \ldots.$$

Indeed, it suffices to note that, for all $\alpha > 0$,

$$\begin{aligned}\frac{1}{\lambda_j(\alpha)} &= \sup_{F_j} \inf \left\{ \int_\Omega \alpha|u|^2 dx : \|u\|_{\mathscr{H}} = 1,\ u \in F_j \right\} \\ &= \alpha \times \sup_{F_j} \inf \left\{ \int_\Omega |u|^2 dx : \|u\|_{\mathscr{H}} = 1,\ u \in F_n \right\} \\ &= \alpha \times \frac{1}{\lambda_j(1)} = \frac{\alpha}{\lambda_j}.\end{aligned}$$

Secondly, we prove the following comparison property of eigenvalues $\lambda_n(m)$ and $\lambda_n = \lambda_n(1)$:

Corollary 3.6. *Let $m(x)$ be a weight function in $L^r(\Omega)$ with $r > N/2$ such that we have, for all $n \in \boldsymbol{N}$,*

$$m(x) < \lambda_n = \lambda_n(1) \quad \textit{almost everywhere in } \Omega.$$

Then we have, for all $n \in \boldsymbol{N}$,

$$\lambda_n(m) > 1.$$

Proof. Indeed, by combining Propositions 3.4 and 3.5 we obtain that

$$\begin{aligned}
\frac{1}{\lambda_n(m)} &= \sup_{F_n} \inf \left\{ \int_\Omega m(x)|u|^2 dx : \|u\|_{\mathscr{H}} = 1,\ u \in F_n \right\} \\
&< \sup_{F_n} \inf \left\{ \lambda_n(1) \int_\Omega |u|^2 dx : \|u\|_{\mathscr{H}} = 1,\ u \in F_n \right\} \\
&= \lambda_n(1) \times \sup_{F_n} \inf \left\{ \int_\Omega |u|^2 dx : \|u\|_{\mathscr{H}} = 1,\ u \in F_n \right\} \\
&= \lambda_n(1) \times \frac{1}{\lambda_n(1)} \\
&= 1,
\end{aligned}$$

so that

$$\lambda_n(m) > 1.$$

The proof of Corollary 3.6 is complete. $\square$

Thirdly, the next proposition proves the continuity property of eigenvalues $\lambda_n(m)$ in the framework of L^p spaces (cf. [**7**, Proposition 1.12B]):

Proposition 3.7. *The eigenvalues $\lambda_n(m)$ depend continuously on the weight function $m(x)$ in the $L^{N/2}$ topology. More precisely, if $m_j(x)$ and $m(x)$ are functions in $L^r(\Omega)$ with $r > N/2$ such that*

$$m_j(x) \longrightarrow m(x) \quad \textit{in } L^{N/2}(\Omega) \textit{ as } j \to \infty,$$

then we have, for each $n \in \boldsymbol{N}$,

$$\lambda_n(m_j) \longrightarrow \lambda_n(m) \quad \textit{as } j \to \infty.$$

PROOF. First, we have, for all u, $v \in \mathscr{H}$,

$$\begin{aligned}(\mathscr{T}u, v)_{\mathscr{H}} &= (\mathscr{C}\mathscr{T}u, \mathscr{C}v)_{L^2(\Omega)} \\ &= (\mathscr{C}^2\mathfrak{A}^{-1}(m(x)u), v)_{L^2(\Omega)} \\ &= (m(x)u, v)_{L^2(\Omega)} \\ &= \int_\Omega m(x)u \cdot v\, dx. \end{aligned} \tag{3.11}$$

However, by applying the generalized Hölder inequality (see [**1**, Corollary 2.6]) we obtain that

$$\left| \int_\Omega m(x)u \cdot v\, dx \right| \leq \|m\|_{L^{N/2}(\Omega)} \|u\|_{L^{2^*}(\Omega)} \|v\|_{L^{2^*}(\Omega)}, \tag{3.12}$$

where

$$2^* = \frac{2N}{N-2}.$$

Indeed, it suffices to note that

$$\frac{1}{N/2} + \frac{1}{2^*} + \frac{1}{2^*} = 1.$$

Moreover, it follows from an application of Sobolev's imbedding theorem (see [**1**, Theorem 4.12, Part I]) that we have the imbeddings

$$W^{1,2}(\Omega) \subset L^q(\Omega) \quad \text{if} \quad \begin{cases} N \geq 3, & 2 < q \leq 2^* = \dfrac{2N}{N-2}, \\ N = 2, & 2 < q < \infty. \end{cases} \tag{3.13}$$

Therefore, by using inequality (3.13) and inclusion (3.4) we obtain from formula (3.11) and inequality (3.12) that

$$
\begin{aligned}
|(\mathscr{T}u, v)_{\mathscr{H}}| &= \left| \int_\Omega m(x) u \cdot v \, dx \right| \\
&\le \|m\|_{L^{N/2}(\Omega)} \|u\|_{L^{2^*}(\Omega)} \|v\|_{L^{2^*}(\Omega)} \\
&\le C \|m\|_{L^{N/2}(\Omega)} \|u\|_{W^{1,2}(\Omega)} \|v\|_{W^{1,2}(\Omega)} \\
&\le C' \|m\|_{L^{N/2}(\Omega)} \|u\|_{\mathscr{H}} \|v\|_{\mathscr{H}}.
\end{aligned}
$$

In view of Proposition 3.4, this inequality proves that the eigenvalues $\lambda_n(m)$ depend continuously on the weight function $m(x)$ in the $L^{N/2}$ topology.

The proof of Proposition 3.7 is complete. □

Finally, we recall a theorem of the Kreĭn and Rutman type for problem (3.1) proved in Taira [**16**, Theorem 1.2] (cf. [**10**], [**7**, Theorem 1.13]):

Theorem 3.8. *Assume that the weight function $m(x)$ is in the space $L^\infty(\Omega)$, and takes a positive value in a subset of positive measure in Ω. Then the first eigenvalue $\lambda_1(m)$ of problem* (3.1) *is positive and* algebraically simple*, and its corresponding eigenfunction $\phi_1 \in W^{2,p}(\Omega)$, $N < p < \infty$, may be chosen to be* strictly positive *in Ω. Moreover,* no *other eigenvalues, $\lambda_j(m)$, $j \ge 2$, have positive eigenfunctions.*

Remark 3.3. If $m(x)$ is Hölder continuous on $\overline{\Omega}$ with exponent $0 < \alpha < 1$, then we find from the proof of [**15**, Theorem 9.1] that $\phi_1 \in C^{2+\alpha}(\overline{\Omega})$. In particular, it follows that the eigenfunction $\phi_1(x)$ of the operator $\mathfrak{A}$ belongs to $C^{2+\alpha}(\overline{\Omega})$, by taking $m(x) \equiv 1$ on $\overline{\Omega}$.

It should be emphasized that Theorem 3.8 goes back to Manes–Micheletti [**11**] in the Dirichlet case.

4. Proof of Theorem 1.1.

This section is devoted to the proof of Theorem 1.1 which is inspired by Ambrosetti–Prodi [**3**, Theorem 3.1]. The crucial point in the proof is how to verify all the conditions (S.1) through (S.4) of the global inversion theorem with singularities (Theorem 2.3). The approach here is based on the extensive use of the ideas and techniques characteristic of the recent developments in the theory of semilinear degenerate elliptic boundary value problems ([**16**]). The proof is divided into three steps.

Step I: We let

$$X = C_B^{2+\alpha}(\overline{\Omega}) = \{u \in C^{2+\alpha}(\overline{\Omega}) : Bu = 0 \quad \text{on } \partial\Omega\},$$
$$Y = C^{\alpha}(\overline{\Omega}),$$

and introduce a nonlinear map

$$F : X \longrightarrow Y$$

by the formula

$$F(u) = -Au + p(u) \quad \text{for all } u \in X.$$

Then it is easy to verify the following three assertions:

(1) $DF(u)v = (-A + p'(u))v$ for all $v \in X$.
(2) $D^2F(u)[v, w] = p''(u)vw$ for all v, $w \in X$.
(3) $F \in C^2(X, Y)$.

First, the next lemma verifies condition (S.1) of Theorem 2.3:

LEMMA 4.1. *The mapping $F : X \to Y$ is proper.*

PROOF. The proof is divided into two steps.

Step 1: Let $\{h_n\}$ be an arbitrary bounded sequence in the space Y such that $F(u_n) = h_n$ with $u_n \in X$, that is,

$$F(u_n) = -Au_n + p(u_n) = h_n. \tag{4.1}$$

We show that the sequence $\{u_n\}$ is bounded in the space Y.

Assume, to the contrary, that

$$\|u_n\|_Y \longrightarrow \infty \quad \text{as } n \to \infty. \tag{4.2}$$

Then, by letting

$$z_n(x) = \frac{u_n(x)}{\|u_n\|_Y} \quad \text{for all } x \in \overline{\Omega},$$

we obtain from equation (4.1) that

$$\begin{cases} -Az_n + \varphi(u_n)z_n = \dfrac{h_n}{\|u_n\|_Y} & \text{in } \Omega, \\ Bz_n = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\varphi(\xi)$ is a continuous function on $\boldsymbol{R}$ defined by the formula

$$\varphi(\xi) = \begin{cases} \dfrac{p(\xi)}{\xi} & \text{if } \xi \neq 0, \\ p'(0) & \text{if } \xi = 0. \end{cases}$$

Since $\varphi(\xi)$ is bounded and since $\{h_n\}$ and $\{z_n\}$ are bounded in $Y = C^\alpha(\overline{\Omega})$, it follows from condition (4.2) that the two functions

$$\varphi(u_n(x))z_n(x), \quad \frac{h_n(x)}{\|u_n\|_Y}$$

are bounded in the space $C(\overline{\Omega})$. By applying [**16**, Theorem 1.1] for $p > N/(1-\alpha)$, we obtain from the equation

$$Az_n = \varphi(u_n)z_n - \frac{h_n}{\|u_n\|_Y} \quad \text{in } \Omega \tag{4.3}$$

that the sequence $\{z_n\}$ is bounded in the Hölder space $C^{1+\alpha}(\overline{\Omega})$. Namely, we have, for some positive constant C,

$$\|z_n\|_{C^{1+\alpha}(\overline{\Omega})} \leq C.$$

Indeed, it follows from an application of Sobolev's imbedding theorem (see [**1**, Theorem 4.12, Part II]) that we have the imbedding

$$W^{2,p}(\Omega) \subset C^{2-N/p}(\overline{\Omega}) \subset C^{1+\alpha}(\overline{\Omega}),$$

for $p > N/(1-\alpha)$.

Therefore, by the Ascoli–Arzelà theorem we may assume that the sequence $\{z_n\}$ itself converges to some function z^* in the space $C^1(\overline{\Omega})$ as $n \to \infty$:

$$z_n \longrightarrow z^* \quad \text{in } C^1(\overline{\Omega}) \text{ as } n \to \infty.$$

We remark here that the limit function $z^*(x)$ satisfies the condition

$$\|z^*\|_Y = \lim_{n\to\infty} \|z_n\|_Y = 1, \tag{4.4}$$

and also the boundary condition

$$Bz^* = \lim_{n\to\infty} Bz_n = 0 \quad \text{on } \partial\Omega. \tag{4.5}$$

On the other hand, by multiplying equation (4.3) by an arbitrary test function $w \in C_0^\infty(\Omega)$ and integrating over Ω we obtain that

$$\begin{aligned} &-\sum_{i,j=1}^N \int_\Omega a^{ij}(x) \frac{\partial z_n}{\partial x_i}\frac{\partial w}{\partial x_j}\, dx - \int_\Omega c(x) z_n w \, dx + \int_\Omega \varphi(u_n) z_n w\, dx \\ &= \int_\Omega \frac{h_n}{\|u_n\|_Y} w\, dx. \end{aligned} \tag{4.6}$$

However, we have, by conditions (4.2), (P.3) and (P.4),

$$\begin{cases} (a)\ z^*(x) < 0 \implies u_n(x) = z_n(x)\|u_n\|_Y \longrightarrow -\infty \\ \quad \implies \lim_{n\to\infty} \varphi(u_n(x)) = \lim_{n\to\infty} \dfrac{p(u_n(x))}{u_n(x)} = \gamma'; \\ (b)\ z^*(x) > 0 \implies u_n(x) = z_n(x)\|u_n\|_Y \longrightarrow +\infty \\ \quad \implies \lim_{n\to\infty} \varphi(u_n(x)) = \lim_{n\to\infty} \dfrac{p(u_n(x))}{u_n(x)} = \gamma''; \\ (c)\ z^*(x) = 0 \implies \lim_{n\to\infty} z_n(x)\varphi(u_n(x)) = 0. \end{cases}$$

Hence, if we define a function $m(x)$ by the formula

$$m(x) = \begin{cases} \gamma' & \text{if } z^*(x) < 0, \\ \gamma'' & \text{if } z^*(x) > 0, \\ p'(0) & \text{if } z^*(x) = 0, \end{cases} \tag{4.7}$$

then it follows that

$$\varphi(u_n(x)) z_n(x) \longrightarrow m(x) z^*(x) \quad \text{in } \Omega \text{ as } n \to \infty.$$

Therefore, by applying the Lebesgue bounded convergence theorem we obtain from equation (4.6) and condition (4.2) that

$$\begin{aligned}
&- (Az^*, w)_{L^2(\Omega)} + (m(x)z^*, w)_{L^2(\Omega)} \\
&= - \sum_{i,j=1}^{N} \int_\Omega a^{ij}(x) \frac{\partial z^*}{\partial x_i} \frac{\partial w}{\partial x_j}\, dx - \int_\Omega c(x) z^* w\, dx + \int_\Omega m(x) z^* w\, dx \\
&= 0 \quad \text{for all } w \in C_0^\infty(\Omega). \qquad (4.8)
\end{aligned}$$

By combining three formulas (4.4), (4.5) and (4.8), we have proved that the non-trivial function $z^*(x)$ satisfies the conditions

$$\begin{cases} Az^* = m(x)\, z^* & \text{in } \Omega, \\ Bz^* = 0 & \text{on } \partial\Omega. \end{cases} \qquad (4.9)$$

This proves that

$$\lambda_k(m) = 1 \quad \text{for some } k \geq 1.$$

Since we have, by condition (P.4),

$$m(x) \leq \gamma'' < \lambda_2 \quad \text{for almost all } x \in \Omega,$$

it follows from an application of Corollary 3.6 for $n = 2$ that

$$\lambda_2(m) > 1.$$

Hence we have the assertion

$$\lambda_1(m) = 1.$$

By Theorem 3.8, this implies that the corresponding eigenfunction $z^*(x)$ does not change sign in Ω. Hence we obtain from the definition (4.7) of the function $m(x)$ that

$$m(x) \equiv \begin{cases} \gamma' & \text{if } z^*(x) < 0, \\ \gamma'' & \text{if } z^*(x) > 0. \end{cases}$$

Therefore, by conditions (4.9) it follows that we have the assertion

$$\begin{cases} Az^* = \gamma' z^* & \text{in } \Omega, \\ z^*(x) < 0 & \text{in } \Omega, \\ Bz^* = 0 & \text{on } \partial\Omega, \end{cases}$$

or the assertion

$$\begin{cases} Az^* = \gamma'' z^* & \text{in } \Omega, \\ z^*(x) > 0 & \text{in } \Omega, \\ Bz^* = 0 & \text{on } \partial\Omega. \end{cases}$$

This is a contradiction, since both γ' and γ'' are not eigenvalues of the operator $\mathfrak{A}$.

Summing up, we have proved that the sequence $\{u_n\}$ is bounded in the space Y.

Step 2: Secondly, we show that if $\{u_n\}$ is a sequence in X such that the sequence

$$h_n = F(u_n) = -Au_n + p(u_n) \tag{4.10}$$

converges to some function h in X as $n \to \infty$, then the sequence $\{u_n\}$ contains a convergent subsequence in X. This proves that the mapping $F : X \to Y$ is proper.

Since $\{u_n\}$ is bounded in the space Y as is shown in Step 1, it follows that the sequence

$$\{p(u_n) - h_n\} = \{Au_n\}$$

is bounded in the space $Y = C^\alpha(\overline{\Omega})$. Hence, by applying [**15**, Theorem 9.1] with $\varphi := 0$ we obtain from equation (4.10) that the sequence $\{u_n\}$ is bounded in the space $X = C_B^{2+\alpha}(\overline{\Omega})$. Namely, we have, for some positive constant C,

$$\|u_n\|_{C^{2+\alpha}(\overline{\Omega})} \leq C.$$

By the Ascoli–Arzelà theorem, we may assume that the sequence $\{u_n\}$ itself converges to some function u^* in the space $C^2(\overline{\Omega})$ as $n \to \infty$:

$$u_n \longrightarrow u^* \quad \text{in } C^2(\overline{\Omega}) \text{ as } n \to \infty. \tag{4.11}$$

We remark here that the limit function $u^*(x)$ satisfies the boundary condition

$$Bu^* = \lim_{n\to\infty} Bu_n = 0 \quad \text{on } \partial\Omega.$$

Moreover, since we have the assertion

$$Au_n = p(u_n) - h_n \longrightarrow p(u^*) - h \quad \text{in } Y = C^{\alpha}(\overline{\Omega}) \text{ as } n \to \infty,$$

it follows from an application of [**15**, Theorem 9.1] with $\varphi := 0$ that

$$u_n = A^{-1}(p(u_n) - h_n) \longrightarrow A^{-1}(p(u^*) - h) \quad \text{in } X = C_B^{2+\alpha}(\overline{\Omega}) \text{ as } n \to \infty.$$

In view of assertion (4.11), this proves that

$$u^* = A^{-1}(p(u^*) - h) \in X,$$

and further that

$$u_n \longrightarrow u^* \quad \text{in } X \text{ as } n \to \infty.$$

Now the proof of Lemma 4.1 is complete. □

Step II: Secondly, we shall study the singular set

$$\Sigma' = \{u \in X : \text{the Fréchet derivative } DF(u) \text{ at } u \text{ is not invertible}\}.$$

First, we prove the Fredholm alternative theorem for the Fréchet derivative $DF(u)$ at a point $u \in X$:

Claim 4.1. *The index of $DF(u) = -A + p'(u) : X \to Y$ is equal to zero*:

$$\operatorname{ind} DF(u) = \dim N(DF(u)) - \operatorname{codim} R(DF(u)) = 0.$$

Proof. If we associate with the linear elliptic boundary value problem

$$\begin{cases} Av = f & \text{in } \Omega, \\ Bv = 0 & \text{on } \partial\Omega \end{cases}$$

a continuous linear operator $\mathscr{A}$ by the formula

$$\mathscr{A} = A : X \longrightarrow Y,$$

then it follows from an application of [**15**, Theorem 9.1] with $\varphi := 0$ that the operator $\mathscr{A}$ is a Fredholm operator with index zero:

$$\operatorname{ind} \mathscr{A} = \dim N(\mathscr{A}) - \operatorname{codim} R(\mathscr{A}) = 0. \tag{4.12}$$

Moreover, if we let

$$P(u)v = p'(u)v \quad \text{for all } v \in X,$$

then we obtain from the Ascoli–Arzelà theorem that the operator

$$P(u) : C^{2+\alpha}(\overline{\Omega}) \longrightarrow C^{\alpha}(\overline{\Omega})$$

is *compact*. Therefore, we find that the operator

$$DF(u) = -\mathscr{A} + P(u) : X \longrightarrow Y$$

is a Fredholm operator with index zero, since we have, by assertion (4.12),

$$\operatorname{ind} DF(u) = \operatorname{ind}(-\mathscr{A}) = 0.$$

The proof of Claim 4.1 is complete. $\square$

By virtue of Claim 4.1, it is easy to see that a point u of X belongs to Σ' if and only if the linear eigenvalue problem with a weight function

$$\begin{cases} Av = \lambda\, p'(u)v & \text{in } \Omega, \\ Bv = 0 & \text{on } \partial\Omega \end{cases}$$

has a non-trivial solution $v \in X$ for $\lambda = 1$. In other words, we have the assertion

$$u \in \Sigma' \iff \lambda_k(p'(u)) = 1 \quad \text{for some } k \geq 1.$$

However, since we have, by condition (P.4),

$$p'(u(x)) < \gamma'' < \lambda_2 \quad \text{for all } x \in \overline{\Omega},$$

it follows from an application of Corollary 3.6 with $m(x) := p'(u(x))$ and $n = 2$ that

$$\lambda_2(p'(u)) > 1.$$

Hence we have the assertion

$$\lambda_1(p'(u)) = 1.$$

Therefore, we have proved the following:

Claim 4.2. *A point $u \in X$ belongs to Σ' if and only if $\lambda_1(p'(u)) = 1$.*

Moreover, the next lemma verifies conditions (S.2) and (S.4) of Theorem 2.3:

Lemma 4.2.

(1) *The singular set*

$$\Sigma' = \{u \in X : DF(u) : X \to Y \text{ is not invertible}\}$$

is not empty, closed and connected.

(2) *Every point $u \in \Sigma'$ is an ordinary singular point.*

Proof. The proof is divided into two steps.

Step 1: In order to prove that the singular set Σ' is not empty and connected, it suffices to show that the set Σ' has a Cartesian representation on a closed linear subspace W of X of codimension one. More precisely, by taking a positive eigenfunction $\phi_1(x) \in X$ of the operator $\mathfrak{A}$ we let

$$W = \left\{u \in X : \int_\Omega u(x)\phi_1(x)\, dx = 0\right\},$$

and show that (see Figure 3)

$$\Sigma' = \{u = \sigma(w)\phi_1 + w : w \in W\},$$

where $\sigma : W \to \boldsymbol{R}$ is a continuous function.

If $\sigma \in \boldsymbol{R}$ and $w(x)$ is a function of W, we define a continuous function $m_\sigma(x)$ by the formula

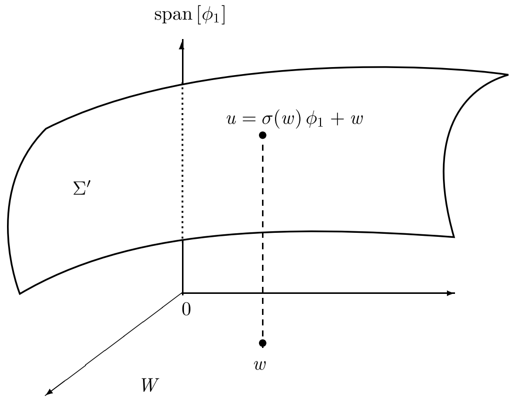


Figure 3.

$$m_\sigma(x) = p'(\sigma\phi_1(x) + w(x)) \quad \text{for all } x \in \overline{\Omega},$$

and consider the following linear eigenvalue problem with a weight function:

$$\begin{cases} Av = \lambda\, m_\sigma(x) v & \text{in } \Omega, \\ Bv = 0 & \text{on } \partial\Omega. \end{cases}$$

(a) First, by Claim 4.2 we know that the point $u = \sigma\phi_1 + w$ of X belongs to the singular set Σ' if and only if $\lambda_1(m_\sigma) = 1$.

(b) Secondly, since $\phi_1(x) > 0$ in Ω and since $p''(\xi) > 0$ on $\boldsymbol{R}$, it follows from an application of Proposition 3.5 that

$$\begin{aligned} &\sigma > \mu \\ &\quad \Longrightarrow m_\sigma(x) = p'(\sigma\phi_1(x) + w(x)) > m_\mu(x) = p'(\mu\phi_1(x) + w(x)) \text{ for all } x \in \overline{\Omega}, \\ &\quad \Longrightarrow \lambda_1(m_\sigma) < \lambda_1(m_\mu). \end{aligned}$$

This implies that the first eigenvalue $\lambda_1(m_\sigma)$ is a strictly decreasing function of σ.

(c) Thirdly, it follows from an application of Proposition 3.7 that the first eigenvalue $\lambda_1(m_\sigma)$ is a continuous function of σ. Indeed, it suffices to note that we have, by the Lebesgue bounded convergence theorem,

$$\|m_\sigma - m_\mu\|_{L^{N/2}(\Omega)} = \|p'(\sigma\phi_1 + w) - p'(\mu\phi_1 + w)\|_{L^{N/2}(\Omega)} \longrightarrow 0 \quad \text{as } \sigma \to \mu,$$

since $p'(\xi)$ is bounded and continuous on $\boldsymbol{R}$.

(d) Since $\phi_1(x) > 0$ in Ω, it follows that we have, for each $x \in \Omega$,

$$m_\sigma(x) = p'(\sigma\phi_1(x) + w(x)) \longrightarrow \begin{cases} \lim_{\xi\to-\infty} p'(\xi) = \gamma' & \text{as } \sigma \to -\infty, \\ \lim_{\xi\to+\infty} p'(\xi) = \gamma'' & \text{as } \sigma \to +\infty, \end{cases}$$

and further from an application of the Lebesgue bounded convergence theorem that we have, for all $r > 1$,

$$\begin{cases} \|m_\sigma - \gamma'\|_{L^r(\Omega)} \longrightarrow 0 & \text{as } \sigma \to -\infty, \\ \|m_\sigma - \gamma''\|_{L^r(\Omega)} \longrightarrow 0 & \text{as } \sigma \to +\infty. \end{cases}$$

Therefore, by using the continuity of the eigenvalues (see Proposition 3.7) we obtain that

$$\begin{cases} \lambda_1(m_\sigma) \longrightarrow \lambda_1(\gamma') = \dfrac{\lambda_1}{\gamma'} > 1 & \text{as } \sigma \to -\infty, \\ \lambda_1(m_\sigma) \longrightarrow \lambda_1(\gamma'') = \dfrac{\lambda_1}{\gamma''} < 1 & \text{as } \sigma \to +\infty. \end{cases} \tag{4.13}$$

Since the function $\lambda_1(m_\sigma)$ is a strictly decreasing function of σ, it follows from assertion (4.13) that there exists a unique value $\sigma^* \in \boldsymbol{R}$ satisfying the condition (see Figure 4)

$$\lambda_1(m_{\sigma^*}) = \lambda_1(p'(\sigma^*\phi_1 + w)) = 1.$$

It should be emphasized here that every straight line $\sigma \mapsto \sigma\phi_1 + w$ meets the space W in a unique way and further that this point depends continuously on W (see Figure 3).

Summing up, by letting $\sigma^* = \sigma(w)$ we have proved that

$$\begin{aligned} \Sigma' &= \{u = \sigma(w)\phi_1 + w : w \in W\} \\ &= \{u = \sigma\phi_1 + w : \sigma \in \boldsymbol{R},\ w \in W,\ \lambda_1(p'(\sigma\phi_1 + w)) = 1\}. \end{aligned}$$

This representation formula proves the desired assertion (1).

Step 2: In order to prove assertion (2), let $u \in \Sigma'$. Then, by Claim 4.2 it follows that $\lambda_1(p'(u)) = 1$. Therefore, by applying Theorem 3.8 with $m(x) :=$

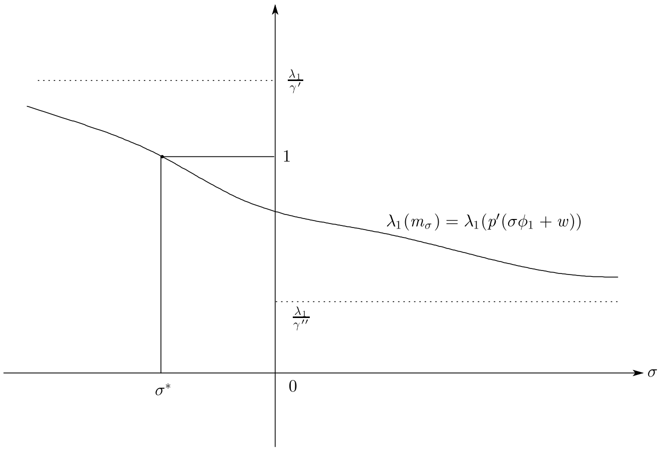


Figure 4.

$p'(u(x))$ and Remark 3.3 we obtain that the null space

$$N(DF(u)) = \left\{v \in C_B^{2+\alpha}(\overline{\Omega}) : DF(u)v = 0\right\}$$

is spanned by a strictly positive function $\phi_1 \in X = C_B^{2+\alpha}(\overline{\Omega})$. This verifies condition (a) of Subsection 2.3 with $\phi := \phi_1$.

Moreover, we remark the following orthogonal decomposition of the Hölder space $Y = C^\alpha(\overline{\Omega})$ in the Hilbert space $L^2(\Omega)$:

$$\begin{aligned} C^\alpha(\overline{\Omega}) &= \left\{v \in C_B^{2+\alpha}(\overline{\Omega}) : DF(u)v = 0\right\} \oplus \left\{DF(u)w : w \in C_B^{2+\alpha}(\overline{\Omega})\right\} \\ &= \operatorname{span}[\phi_1] \oplus R(DF(u)). \end{aligned} \tag{4.14}$$

Indeed, it suffices to note that

$$DF(u) = -\mathfrak{A} + p'(u(x))$$

and further that the operator $\mathfrak{A}$ is selfadjoint and $p'(u) \in C^1(\overline{\Omega})$. Therefore, we have, for any $h \in C^\alpha(\overline{\Omega})$,

$$h \in R(DF(u)) \iff \int_\Omega h(x)\phi_1(x)\,dx = 0.$$

This verifies condition (b) of Subsection 2.3 with $\phi := \phi_1$.

In order to verify condition (c) of Subsection 2.3, we define a (projection) functional $\Psi : X \to \boldsymbol{R}$ by the formula

$$\Psi : h \longmapsto \int_\Omega \phi_1(x) h(x)\, dx.$$

Then we have, by the orthogonal decomposition (4.14),

$$h \in R(DF(u)) \iff \langle \Psi, h \rangle = \int_\Omega h(x)\phi_1(x)\, dx = 0. \tag{4.15}$$

Since we have the formula

$$D^2F(u)[v,w] = p''(u)vw \quad \text{for all } v,\, w \in X,$$

it follows that

$$\left\langle \Psi, D^2F(u)[\phi_1,\phi_1] \right\rangle = \int_\Omega p''(u(x))\phi_1(x)^3\, dx. \tag{4.16}$$

However, since $p''(\xi) > 0$ on $\boldsymbol{R}$ and since $\phi_1(x) > 0$ in Ω, it follows that

$$\int_\Omega p''(u(x))\phi_1(x)^3\, dx > 0.$$

By formula (4.16), this proves that

$$\left\langle \Psi, D^2F(u)[\phi_1,\phi_1] \right\rangle \neq 0.$$

Hence we have, by assertion (4.15),

$$D^2F(u)[\phi_1,\phi_1] \notin R(DF(u)).$$

This verifies condition (c) of Subsection 2.3 with $\phi := \phi_1$.

Summing up, we have proved the desired assertion (2).

The proof of Lemma 4.2 is complete. □

Step III: Finally, the next lemma verifies condition (S.3) of Theorem 2.3:

LEMMA 4.3. *For every $h \in F(\Sigma')$, the equation $F(u) = h$ has a unique solution $u \in \Sigma'$.*

PROOF. Let u be a solution in Σ' such that $F(u) = h$. We assume, to the contrary, that there exists another solution w in Σ' satisfying the equation

$$F(w) = h.$$

Then we define a continuous function $\omega(x)$ by the formula

$$\omega(x) = \begin{cases} \dfrac{p(w(x)) - p(u(x))}{w(x) - u(x)} & \text{if } w(x) \neq u(x), \\ p'(u(x)) & \text{if } w(x) = u(x). \end{cases}$$

Since we have the formula

$$\begin{cases} -Au + p(u) = -Aw + p(w) & \text{in } \Omega, \\ Bu = Bw = 0 & \text{on } \partial\Omega, \end{cases}$$

we obtain that the function

$$v(x) = w(x) - u(x)$$

is a non-trivial solution of the linear eigenvalue problem with a weight function

$$\begin{cases} Av = \omega(x)v & \text{in } \Omega, \\ Bv = 0 & \text{on } \partial\Omega. \end{cases}$$

This proves that

$$\lambda_k(\omega) = 1 \quad \text{for some } k \geq 1.$$

However, since we have, by conditions (P.1) through (P.4),

$$\gamma' < \omega(x) < \gamma'' < \lambda_2 \quad \text{for all } x \in \overline{\Omega},$$

it follows from an application of Corollary 3.6 with $m(x) := \omega(x)$ and $n = 2$ that

$$\lambda_2(\omega) > 1.$$

Hence we have the assertion

$$\lambda_1(\omega) = 1.$$

By Theorem 3.8 with $m(x) := \omega(x)$, this implies that the corresponding eigenfunction $v(x) = w(x) - u(x)$ does not change sign in Ω.

(1) First, we consider the case where $v(x) = w(x) - u(x) > 0$ in Ω: Since $p''(\xi) > 0$ on $\boldsymbol{R}$ and $w(x) - u(x) > 0$ in Ω, it follows that

$$\omega(x) = \frac{p(w(x)) - p(u(x))}{w(x) - u(x)} > p'(u(x)) \quad \text{in } \Omega,$$

so that, by Proposition 3.5 for $n = 1$,

$$1 = \lambda_1(\omega) < \lambda_1(p'(u)).$$

However, we have, by Claim 4.2,

$$\lambda_1(p'(u)) = 1 \quad \text{for all } u \in \Sigma'.$$

This is a contradiction.

(2) Secondly, we consider the case where $v(x) = w(x) - u(x) < 0$ in Ω: Since $p''(\xi) > 0$ on $\boldsymbol{R}$ and $w(x) - u(x) < 0$ in Ω, it follows that

$$\omega(x) = \frac{p(w(x)) - p(u(x))}{w(x) - u(x)} < p'(u(x)) \quad \text{in } \Omega,$$

so that, by Proposition 3.5 for $n = 1$,

$$1 = \lambda_1(\omega) > \lambda_1(p'(u)).$$

However, we have, by Claim 4.2,

$$\lambda_1(p'(u)) = 1 \quad \text{for all } u \in \Sigma'.$$

This is also a contradiction.

Summing up, we have proved that the equation $F(u) = h$ has a unique solution for every $h \in F(\Sigma')$.

The proof of Lemma 4.3 is complete. □

Now the proof of Theorem 1.1 is complete.

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