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Penalising symmetric stable Lévy paths

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Abstract. Limit theorems for the normalized laws with respect to two kinds of weight functionals are studied for any symmetric stable Lévy process of index $1 < \alpha \leq 2$. The first kind is a function of the local time at the origin, and the second kind is the exponential of an occupation time integral. Special emphasis is put on the role played by a stable Lévy counterpart of the universal σ-finite measure, found in [**9**] and [**10**], which unifies the corresponding limit theorems in the Brownian setup for which $\alpha = 2$.

1. Introduction.

Roynette, Vallois and Yor ([**15**], [**14**] and [**13**] and references therein) have shown the existence of the limit laws for normalized Wiener measures with respect to various weight processes; we call these studies *penalisation problems*. Najnudel, Roynette and Yor (see [**16**], [**8**], [**9**] and [**10**]) have recently discovered that these penalisation problems may be unified with the help of the following "universal" σ-finite measure on the canonical space:

$$\mathscr{W} = \int_0^\infty \frac{\mathrm{d}u}{\sqrt{2\pi u}} W^{(u)} \bullet P_0^{3\mathrm{B}} \tag{1.1}$$

where $W^{(u)}$ stands for the law of the Brownian bridge from 0 to 0 of length u, $P_0^{3\mathrm{B}}$ for that of the symmetrized 3-dimensional Bessel process starting from 0, i.e., $P_0^{3\mathrm{B}} = (P_0^{3\mathrm{B},+} + P_0^{3\mathrm{B},-})/2$, and the symbol $\bullet$ for the concatenation between the laws of these two processes.

The purpose of the present paper is to develop some of these penalisation problems in the case of any symmetric stable Lévy process of index $1 < \alpha \leq 2$. As an analogue of $\mathscr{W}$, we introduce the following σ-finite measure

$$\mathscr{P} = \frac{\Gamma(1/\alpha)}{\alpha\pi} \int_0^\infty \frac{\mathrm{d}u}{u^{1/\alpha}} Q^{(u)} \bullet P_0^h \tag{1.2}$$

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where $Q^{(u)}$ stands for the law of the bridge from 0 to 0 of length u and P_0^h for the h-path process of the killed process with respect to the function $|x|^{\alpha-1}$. We shall put some special emphasis on the role played by the universal σ-finite measure $\mathscr{P}$ which helps to unify our penalisation problems.

Let $\boldsymbol{D}$ denote the canonical space of càdlàg paths $w : [0, \infty) \to \boldsymbol{R}$. Let (X_t) denote the coordinate process, $(\mathscr{F}_t)$ its natural filtration, and $\mathscr{F}_\infty = \vee_{t\geq 0}\mathscr{F}_t$. Let (P_x) denote the law on $\boldsymbol{D}$ of the symmetric stable process of index $1 < \alpha \leq 2$ such that $P_0[\mathrm{e}^{i\lambda X_t}] = \mathrm{e}^{-t|\lambda|^\alpha}$ for $\lambda \in \boldsymbol{R}$. Note that, if $\alpha = 2$, then (X_t) has the same law as $\sqrt{2}$ times the standard Brownian motion.

We say that *a family of measures* $\{\mathscr{M}_t\}_{t\geq 0}$ *on* $\mathscr{F}_\infty$ *converges as* $t \to \infty$ *to a measure* $\mathscr{M}$ *along* $(\mathscr{F}_s)$ if, for each $s > 0$, we have $\mathscr{M}_t[Z_s] \to \mathscr{M}[Z_s]$ as $t \to \infty$ for all bounded $\mathscr{F}_s$-measurable functionals Z_s. For a measure $\mathscr{M}$ on $\mathscr{F}_\infty$ and a functional F measurable with respect to $\mathscr{F}_\infty$, the symbol $F \cdot \mathscr{M}$ stands for the measure $A \mapsto \mathscr{M}[1_A F]$. Let $x \in \boldsymbol{R}$ be fixed. Then penalisation problems are stated as follows:

QUESTION 1. Let $\Gamma = (\Gamma_t : t \geq 0)$ be a given non-negative process such that $P_x[\Gamma_t] \neq 0$ for large enough t.

(Q1) Does there exist a limit probability measure P_x^Γ such that

$$\frac{\Gamma_t \cdot P_x}{P_x[\Gamma_t]} \overset{t\to\infty}{\longrightarrow} P_x^\Gamma \qquad \text{along } (\mathscr{F}_s)? \tag{1.3}$$

(Q2) How can one characterise the limit probability measure P_x^Γ assuming it exists?

For each $x \in \boldsymbol{R}$, let $\mathscr{P}_x$ denote the law of $(x + X_t : t \geq 0)$ under $\mathscr{P}$. We can gain a clear insight into some of these penalisation problems if we answer the following

QUESTION 2. Let Γ as above.

(Q1′) Can one find a positive function $\mu(t)$ and a measurable functional Γ_∞ such that

$$\frac{\Gamma_t \cdot P_x}{\mu(t)} \overset{t\to\infty}{\longrightarrow} \Gamma_\infty \cdot \mathscr{P}_x \qquad \text{along } (\mathscr{F}_s)? \tag{1.4}$$

(Q2′) For any non-negative $\mathscr{P}_x$-integrable functional F, can one find a non-negative $(\mathscr{F}_t, P_x)$-martingale $(M_{t,x}(F) : t \geq 0)$ such that

$$(F \cdot \mathscr{P}_x)|_{\mathscr{F}_t} = M_{t,x}(F) \cdot P_x|_{\mathscr{F}_t}, \qquad t \geq 0? \tag{1.5}$$

If we can find such a function $\mu(t)$ as in (1.4) and if $0 < \mathscr{P}_x[\Gamma_\infty] < \infty$, then we obtain the convergence (1.3) with the limit probability measure

$$P_x^\Gamma = \frac{\Gamma_\infty \cdot \mathscr{P}_x}{\mathscr{P}_x[\Gamma_\infty]}. \tag{1.6}$$

We shall prove in Theorem 5.3 that there exist such martingales $(M_{t,x}(F))$ as in (1.5). We shall call $M_{t,x}(\cdot)$ the *martingale generator* and we shall study its properties in Sections 5 and 9. Then the limit probability measure P_x^Γ is characterised by

$$P_x^\Gamma|_{\mathscr{F}_t} = \frac{M_{t,x}(\Gamma_\infty)}{\mathscr{P}_x[\Gamma_\infty]} \cdot P_x|_{\mathscr{F}_t}, \qquad t \geq 0. \tag{1.7}$$

Therefore, if we answer Question 2, then we have answered Question 1.

Our strategy to answer (Q1′) is as follows. Since the index α is supposed to be in $(1, 2]$, each point of $\boldsymbol{R}$ is regular and recurrent. Hence, associated with the process, there is a jointly continuous local time $(L(t, x))$. We simply write $L_t = L(t, 0)$, and, associated with this local time, there is *Itô's measure* $\boldsymbol{n}$ *of excursions away from the origin* (see Section 3). Let R denote the *lifetime* of an excursion path. For $t > 0$, we define $M^{(t)}$ as the probability measure on $\mathscr{F}_t$ given by

$$M^{(t)} = \frac{1_{\{R>t\}}}{\boldsymbol{n}(R > t)} \cdot \boldsymbol{n}|_{\mathscr{F}_t} \tag{1.8}$$

and here we call $M^{(t)}$ the distribution of the *stable meander*. We remark that our meander distribution (1.8) is definitely different from that of [**4**] etc. where the meander is defined by conditioning on $\{R > t\}$ the excursion process for the *reflected* stable Lévy process $(X_t - \min_{s\leq t} X_s : t \geq 0)$. We shall prove the following formula (Theorem 4.1) of disintegration of $P_0|_{\mathscr{F}_t}$ for each $t > 0$ with respect to last exit time from the origin:

$$\frac{1}{\boldsymbol{n}(R > t)} P_0|_{\mathscr{F}_t} = \frac{\Gamma(1/\alpha)}{\alpha\pi} \int_0^t \left(1 - \frac{u}{t}\right)^{\frac{1}{\alpha}-1} \frac{\mathrm{d}u}{u^{1/\alpha}} Q^{(u)} \bullet M^{(t-u)}. \tag{1.9}$$

As a check, the total masses of both sides agree, as we shall show in Proposition 3.4. Then, we shall establish (in Theorem 4.9) the convergence

$$M^{(t)} \overset{t\to\infty}{\longrightarrow} P_0^h \qquad \text{along } (\mathscr{F}_s). \tag{1.10}$$

Noting that $(1-(u/t))^{(1/\alpha)-1} \to 1$ as $t \to \infty$, we may expect that, in some sense:

$$\int_0^t \left(1-\frac{u}{t}\right)^{\frac{1}{\alpha}-1} \frac{\mathrm{d}u}{u^{1/\alpha}}\, Q^{(u)} \bullet M^{(t-u)} \overset{t\to\infty}{\longrightarrow} \int_0^\infty \frac{\mathrm{d}u}{u^{1/\alpha}}\, Q^{(u)} \bullet P_0^h. \tag{1.11}$$

We shall prove several analytic lemmas which justify the convergence (1.11) and then we shall establish the convergence (1.4) with the function $\mu(t) = \boldsymbol{n}(R > t)$.

In order to answer Question 2 (and in particular (Q2′)), we shall establish the convergence (1.4) and compute the martingale generator by case study. We confine ourselves to the following two kinds of weight functionals:

(i) $\Gamma_t = f(L_t)$ for some non-negative Borel functions f with some integrability property;

(ii) $\Gamma_t = \exp\{-\int L(t,x)V(\mathrm{d}x)\}$ for some non-negative Borel measure V.
We call the problems in such a case the *Feynman-Kac penalisations*.

The organisation of the present paper is as follows. In Section 2 we recall some preliminary facts about symmetric stable Lévy processes. In Section 3 we study Itô's measure of excursions away from the origin relatively to the symmetric stable process. In Section 4 we prove several formulae concerning the stable meander and h-path process, which play important roles in the study of our penalisation problems. In Section 5 we make general observations on the universal σ-finite measure $\mathscr{P}_x$ and the martingale generator $M_{t,x}(\cdot)$. In Section 6 we prove several convergence lemmas which play fundamental roles in the proof of our penalisation problems. Section 7 is devoted to the study of penalisations with a function of the local time at the origin. Section 8 is devoted to the study of Feynman-Kac penalisations. In Section 9 we characterise certain non-negative $(P_0, \mathscr{F}_t)$-martingales in terms of $\mathscr{P}$.

2. Preliminaries about the symmetric stable process of index $1 < \alpha \leq 2$.

Recall that $(X_t, \mathscr{F}_t, P_x)$ is the canonical representation of a one-dimensional symmetric stable Lévy process of index $1 < \alpha \leq 2$ such that

$$P_0[\mathrm{e}^{i\lambda X_t}] = \mathrm{e}^{-t|\lambda|^\alpha} \quad \text{for } \lambda \in \boldsymbol{R}. \tag{2.1}$$

All results presented in this section are well-known; see, e.g., [**1**].

1). (X_t) has a transition density $P_x(X_t \in \mathrm{d}y) = p_t(y-x)\mathrm{d}y$ where $p_t(x)$ is given by

$$p_t(x) = \frac{1}{\pi}\int_0^\infty (\cos x\lambda)\mathrm{e}^{-t\lambda^\alpha}\mathrm{d}\lambda. \tag{2.2}$$

For $q > 0$, we set

$$u_q(x) = \int_0^\infty \mathrm{e}^{-qt}p_t(x)\mathrm{d}t = \frac{1}{\pi}\int_0^\infty \frac{\cos x\lambda}{q+\lambda^\alpha}\,\mathrm{d}\lambda. \tag{2.3}$$

In particular, if we take $x = 0$, we have

$$p_t(0) = p_1(0)t^{-\frac{1}{\alpha}} \quad \text{where } p_1(0) = \frac{\Gamma(1/\alpha)}{\alpha\pi} \tag{2.4}$$

and

$$u_q(0) = u_1(0)q^{\frac{1}{\alpha}-1} \quad \text{where } u_1(0) = \frac{\Gamma(1-1/\alpha)\Gamma(1/\alpha)}{\alpha\pi}. \tag{2.5}$$

2). Let $T_{\{a\}}$ denote the first hitting time of a for the coordinate process (X_t):

$$T_{\{a\}} = \inf\{t > 0 : X_t = a\}. \tag{2.6}$$

Then the Laplace transform of the law of $T_{\{0\}}$ is given by

$$P_x[\mathrm{e}^{-qT_{\{0\}}}] = \frac{u_q(x)}{u_q(0)}, \qquad x \in \boldsymbol{R},\ q > 0 \tag{2.7}$$

(see, e.g., [**1**, p. 64]). For further study of the law of $T_{\{0\}}$, see [**21**].

Since $T_{\{y\}}$ under P_x has the same law as $T_{\{0\}}$ under P_{x-y}, the formula (2.7) implies the following facts:

(i) Each point is a recurrent state, i.e., $P_x(T_{\{y\}} < \infty) = 1$ for any $x, y \in \boldsymbol{R}$ with $x \neq y$;

(ii) Each point is regular for itself, i.e., $P_x(T_{\{x\}} = 0) = 1$ for any $x \in \boldsymbol{R}$.

3). The process admits a jointly continuous local time $L(t,x)$ such that

$$L(t,x) = \lim_{\varepsilon \to 0+} \frac{1}{2\varepsilon} \int_0^t 1_{\{|X_s - x| < \varepsilon\}} \mathrm{d}s \tag{2.8}$$

almost surely. We simply write $L_t = L(t,0)$. Denote the inverse local time at the origin by $\tau_l = \inf\{t > 0 : L_t > l\}$. Then $(\tau_l : l \geq 0)$ is a stable subordinator of index $1 - 1/\alpha$ such that

$$P_0[\mathrm{e}^{-q\tau_l}] = \mathrm{e}^{-l/u_q(0)} \tag{2.9}$$

(see, e.g., [**1**, p. 131]), where $u_q(0)$ is given explicitly by (2.5). Let $\theta_t : \boldsymbol{D} \to \boldsymbol{D}$ stand for the shift operator: $\theta_t(w) = w(t + \cdot)$. Since $\tau_l = T_{\{0\}} + \tau_l \circ \theta_{T_{\{0\}}}$, we have

$$P_x\left[\int_0^\infty \mathrm{e}^{-qt}\mathrm{d}L_t\right] = P_x\left[\int_0^\infty \mathrm{e}^{-q\tau_l}\mathrm{d}l\right] = P_x[\mathrm{e}^{-qT_{\{0\}}}] \int_0^\infty P_0[\mathrm{e}^{-q\tau_l}]\mathrm{d}l \tag{2.10}$$

$$= \frac{u_q(x)}{u_q(0)} \cdot u_q(0) = \int_0^\infty \mathrm{e}^{-qt} p_t(x)\mathrm{d}t \tag{2.11}$$

for all $q > 0$. Hence we see that

$$P_x\left[\int_0^\infty f(t)\mathrm{d}L_t\right] = \int_0^\infty f(t)p_t(x)\mathrm{d}t \tag{2.12}$$

for any non-negative measurable function f on $[0,\infty)$. Consequently, we may write

$$P_x[\mathrm{d}L_t] = p_t(x)\mathrm{d}t, \qquad x \in \boldsymbol{R}. \tag{2.13}$$

3. Itô's measure of excursions away from the origin.

Since the origin is a regular and recurrent state, we can apply Itô's excursion theory ([**7**]; see also [**1**] and [**2**] for details).

We denote by $\boldsymbol{E}$ the set of càdlàg paths $e : [0,\infty) \to \boldsymbol{R} \cup \{\Delta\}$ such that

$$\begin{cases} e(t) \in \boldsymbol{R} \setminus \{0\} & \text{for } 0 < t < R(e), \\ e(t) = \Delta & \text{for } t \geq R(e) \end{cases} \tag{3.1}$$

where

$$R = R(e) = \inf\{t > 0 : e(t) = \Delta\}. \tag{3.2}$$

We call $\boldsymbol{E}$ *the set of excursions* and every element e of $\boldsymbol{E}$ an *excursion path*. For an excursion path $e \in \boldsymbol{E}$, we call $R(e)$ the *lifetime* of e. The point Δ is called the *cemetery*.

We set $D = \{l : \tau_l - \tau_{l-} > 0\}$. For each $l \in D$, we set

$$e_l(t) = \begin{cases} X_{t+\tau_{l-}}, & \text{for } \ 0 \le \ t < \tau_l - \tau_{l-}, \\ \Delta, & \text{for } \ t \ge \ \tau_l - \tau_{l-}. \end{cases} \tag{3.3}$$

Then Itô's fundamental theorem ([**7**]) asserts that the point process $(e_l : l \in D)$ taking values on $\boldsymbol{E}$ is a Poisson point process. Its characteristic measure will be denoted by $\boldsymbol{n}$ and called *Itô's measure of excursions away from the origin*. Itô's measure $\boldsymbol{n}$ is a σ-finite measure on any $\mathscr{F}_t$ which has no mass outside the set

$$\{e \in \boldsymbol{E} : X_0(e) = 0, \ 0 < R(e) < \infty\}. \tag{3.4}$$

For the fact that $\boldsymbol{n}(\{X_0 = 0\}^c) = 0$, see [**20**].

For $x \in \boldsymbol{R} \setminus \{0\}$, we denote by P_x^0 the law of the killed process, i.e., the law on $\boldsymbol{E}$ of the path (X_t^0) under P_x where

$$X_t^0 = \begin{cases} X_t, & 0 \le t < T_{\{0\}}, \\ \Delta, & t \ge T_{\{0\}}. \end{cases} \tag{3.5}$$

We shall utilise the following formulae.

THEOREM 3.1 (Markov property of $\boldsymbol{n}$). *It holds that*

$$\boldsymbol{n}[Z_t F(X \circ \theta_t)] = \int \boldsymbol{n}[Z_t; X_t \in \mathrm{d}x] P_x^0[F(X)] \tag{3.6}$$

for any $t > 0$, any non-negative $\mathscr{F}_t$-measurable functional Z_t and any non-negative measurable functional F on $\boldsymbol{E}$.

THEOREM 3.2 (Compensation formula). *Let $F = F(t, \omega, e)$ be a measurable functional on $[0, \infty) \times \boldsymbol{D} \times \boldsymbol{E}$ such that, for every fixed $e \in \boldsymbol{E}$, the process $(F(t, \cdot, e) : t \ge 0)$ is $(\mathscr{F}_t)$-predictable. Then*[*]

$$P_0\left[\sum_{l \in D} F(\tau_{l-}, X, e_l)\right] = P_0 \otimes \widetilde{\boldsymbol{n}}\left[\int_0^\infty \mathrm{d}L_t F(t, X, \widetilde{X})\right]. \tag{3.7}$$

[*]Here the symbol $\widetilde{}$ means independence.

We omit the proofs of Theorems 3.1 and 3.2. For their proofs, see [**1**], [**2**] and [**12**].

3.1. Entrance law.

In order to characterise the entrance law, we need the following

THEOREM 3.3 ([**5**] and [**6**]). *For any non-negative measurable function f on $\boldsymbol{R}$, it holds that*

$$\int_0^\infty \mathrm{e}^{-qt}\boldsymbol{n}[f(X_t)]\mathrm{d}t = \int f(x)P_x\big[\mathrm{e}^{-qT_{\{0\}}}\big]\mathrm{d}x. \tag{3.8}$$

We remark that the relation (3.8) can be found in Chen-Fukushima-Ying [**5**, Equation (2.8)] and Fitzsimmons-Getoor [**6**, Equation (3.22)] in a fairly general Markovian framework as

$$\int_0^\infty \mathrm{e}^{-qt}\boldsymbol{n}[f(X_t)]\mathrm{d}t = \int f(x)\widehat{P}_x\big[\mathrm{e}^{-qT_{\{0\}}(\widehat{X})}\big]m(\mathrm{d}x) \tag{3.9}$$

where (X_t, P_x) and $(\widehat{X}, \widehat{P}_x)$ are in weak duality with respect to the reference measure m. In our case, $(\widehat{X}, \widehat{P}_x) = (-X_t, P_x)$ and $m(\mathrm{d}x) = \mathrm{d}x$, the Lebesgue measure. Although (3.8) is a special case of (3.9), we give the proof of Theorem 3.3 for completeness of this paper.

PROOF OF THEOREM 3.3. Note that

$$\int_0^\infty \mathrm{e}^{-qt}f(X_t)\mathrm{d}t = \sum_{l\in D}\mathrm{e}^{-q\tau_{l-}}\int_0^{R(e_l)}\mathrm{e}^{-qt}f(e_l(t))\mathrm{d}t. \tag{3.10}$$

By Theorem 3.2, we obtain

$$P_0\bigg[\int_0^\infty \mathrm{e}^{-qt}f(X_t)\mathrm{d}t\bigg] = P_0\bigg[\int_0^\infty \mathrm{e}^{-qt}\mathrm{d}L_t\bigg]\boldsymbol{n}\bigg[\int_0^R \mathrm{e}^{-qt}f(X_t)\mathrm{d}t\bigg]. \tag{3.11}$$

Since $P_0\big[\int_0^\infty \mathrm{e}^{-qt}\mathrm{d}L_t\big] = u_q(0)$, we have

$$\int_0^\infty \mathrm{e}^{-qt}\boldsymbol{n}[f(X_t)]\mathrm{d}t = \int f(x)\,\frac{u_q(x)}{u_q(0)}\,\mathrm{d}x. \tag{3.12}$$

By the identity (2.7), we obtain (3.8). The proof is complete. □

The following formula holds:

PROPOSITION 3.4.

$$\boldsymbol{n}(R > t) = \boldsymbol{n}(R > 1)t^{\frac{1}{\alpha}-1} \tag{3.13}$$

where

$$\boldsymbol{n}(R > 1) = \frac{\alpha\pi}{\Gamma(1 - 1/\alpha)\Gamma(1/\alpha)^2}\,. \tag{3.14}$$

In particular,

$$\frac{\boldsymbol{n}(R > t - s)}{\boldsymbol{n}(R > t)} = \left(1 - \frac{s}{t}\right)^{\frac{1}{\alpha}-1} \qquad \textit{for } 0 < s < t. \tag{3.15}$$

Although it is well-known, we again give the proof for completeness of this paper.

PROOF. Take $f = 1$ in (3.12). Then we have $\boldsymbol{n}[f(X_t)] = \boldsymbol{n}(R > t)$, and the identity (3.12) implies that

$$\int_0^\infty \mathrm{e}^{-qt}\boldsymbol{n}(R > t)\mathrm{d}t = \frac{1}{qu_q(0)} = \frac{1}{u_1(0)}\,q^{-1/\alpha}. \tag{3.16}$$

This completes the proof. □

The following theorem characterises the entrance law.

THEOREM 3.5. *There exists a bi-measurable function $\rho(t, x)$ which is at the same time a space density of the entrance law*

$$\boldsymbol{n}(X_t \in \mathrm{d}x) = \rho(t, x)\mathrm{d}x \tag{3.17}$$

and a time density of the first hitting time

$$P_x(T_{\{0\}} \in \mathrm{d}t) = \rho(t, x)\mathrm{d}t. \tag{3.18}$$

That is,

$$\rho(t,x) = \frac{\boldsymbol{n}(X_t \in \mathrm{d}x)}{\mathrm{d}x} = \frac{P_x(T_{\{0\}} \in \mathrm{d}t)}{\mathrm{d}t}. \tag{3.19}$$

PROOF. Note that $P_x^0(X_t \in \mathrm{d}y) = p_t^0(x,y)\mathrm{d}y$ where

$$p_t^0(x,y) = p_t(y-x) - \int_0^t p_{t-s}(y) P_x(T_{\{0\}} \in \mathrm{d}s). \tag{3.20}$$

Now we set

$$\rho(t,x) = \int \boldsymbol{n}(X_{t/2} \in \mathrm{d}y) p_{t/2}^0(y,x). \tag{3.21}$$

Let f be a non-negative measurable function on $\boldsymbol{R}$. By the Markov property, we see that

$$\boldsymbol{n}[f(X_t)] = \int \boldsymbol{n}(X_{t/2} \in \mathrm{d}y) P_y^0[f(X_{t/2})] = \int f(x)\rho(t,x)\mathrm{d}x. \tag{3.22}$$

Hence we obtain (3.17). Using the formulae (3.22) and (3.8), we see that

$$\int \mathrm{d}x f(x) \int_0^\infty \mathrm{e}^{-qt}\rho(t,x)\mathrm{d}t = \int_0^\infty \mathrm{e}^{-qt}\boldsymbol{n}[f(X_t)]\mathrm{d}t \tag{3.23}$$

$$= \int \mathrm{d}x f(x) P_x[\mathrm{e}^{-qT_{\{0\}}}]. \tag{3.24}$$

Hence we obtain (3.18). □

4. Stable meander and h-path process.

4.1. Disintegration with respect to the last exit time.

For $u > 0$, let $Q^{(u)}$ denote the law of the bridge $P_0(\cdot | X_u = 0)$ considered to be a probability measure on $\mathscr{F}_u$. We denote by $X^{(u)} = (X_t : 0 \le t \le u)$ the coordinate process considered up to time u. We denote the *concatenation* between the two processes $X^{(u)}$ and $\widetilde{X}^{(v)} = (\widetilde{X}_t : 0 \le t \le v)$ by $X^{(u)} \bullet \widetilde{X}^{(v)} = ((X^{(u)} \bullet \widetilde{X}^{(v)})_t : 0 \le t \le u+v)$:

$$\left(X^{(u)} \bullet \widetilde{X}^{(v)}\right)_t = \begin{cases} X_t^{(u)}, & 0 \le t < u, \\ \widetilde{X}_{t-u}^{(v)}, & u \le t \le u+v. \end{cases} \tag{4.1}$$

The measure $Q^{(u)} \bullet M^{(v)}$ is defined as the law of the concatenation $X^{(u)} \bullet \widetilde{X}^{(v)}$ between the two processes $X^{(u)}$ and $\widetilde{X}^{(v)}$ where $(X^{(u)}, \widetilde{X}^{(v)})$ is considered under the product measure $Q^{(u)} \otimes M^{(v)}$. Here and in what follows, we emphasize independence with the symbol $\widetilde{\ }$, unless otherwise stated.

For $t > 0$, we denote last exit time from the origin before t by

$$g_t = \sup\{s \leq t : X_s = 0\}. \tag{4.2}$$

The following formula describes disintegration of $P_0|_{\mathscr{F}_t}$ with respect to g_t:

THEOREM 4.1. *For each $t > 0$, it holds that*

$$P_0|_{\mathscr{F}_t} = \int_0^t \boldsymbol{n}(R > t-u) P_0[\mathrm{d}L_u] Q^{(u)} \bullet M^{(t-u)}. \tag{4.3}$$

In other words, the following statements hold:

(i) *The distribution of g_t is given by $P_0(g_t \in \mathrm{d}u) = \boldsymbol{n}(R > t-u)P_0[\mathrm{d}L_u]$;*

(ii) *Given $g_t = u$, $(X_t : t \in [0,u])$ and $(X_{u+t} : t \in [0, t-u])$ are independent under P_0;*

(iii) *Given $g_t = u$, $(X_t : t \in [0,u])$ under P_0 is distributed as the bridge $Q^{(u)}$;*

(iv) *Given $g_t = u$, $(X_{u+t} : t \in [0, t-u])$ under P_0 is distributed as the meander $M^{(t-u)}$.*

REMARK 4.2. We note that the formula (4.3) is the counterpart of Salminen [**17**, Proposition 4] in his study of last exit decomposition for linear diffusions.

REMARK 4.3. We remark that (i) implies

$$P_0(g_t \in \mathrm{d}u) = \frac{(t-u)^{\frac{1}{\alpha}-1} u^{-\frac{1}{\alpha}} \mathrm{d}u}{\Gamma(1-1/\alpha)\Gamma(1/\alpha)} \tag{4.4}$$

for some constant C, which shows that $(1/t)g_t$ has the $\mathrm{Beta}(1-(1/\alpha),(1/\alpha))$ distribution. For further discussions, see [**21**].

PROOF OF THEOREM 4.1. Let us prove

$$P_0|_{\mathscr{F}_t} = \int_0^t P_0[\mathrm{d}L_u] Q^{(u)} \bullet (\boldsymbol{n}|_{\mathscr{F}_{t-u}}), \tag{4.5}$$

which is equivalent to (4.3). Let $F(t,w)$ be a non-negative continuous functional

on $[0,\infty)\times \boldsymbol{D}$. For each $t\geq 0$, we define a measurable functional F_t on $\boldsymbol{D}([0,t];\boldsymbol{R})$ by $F_t(X^{(t)})=F(t,X_{t\wedge\cdot})$. Then we have

$$\int_0^\infty \mathrm{d}tF_t(X^{(t)})=\sum_{l\in D}\int_0^{R(e_l)}\mathrm{d}rF_{\tau_{l-}+r}\Big(X^{(\tau_{l-})}\bullet e_l\Big). \tag{4.6}$$

Now we appeal to Theorem 3.2 and we obtain

$$\int_0^\infty P_0[F_t(X^{(t)})]\mathrm{d}t=(P_0\otimes\widetilde{\boldsymbol{n}})\bigg[\int_0^\infty \mathrm{d}L_t\int_0^\infty \mathrm{d}r1_{\{\widetilde{R}>r\}}F_{t+r}\Big(X^{(t)}\bullet\widetilde{X}^{(r)}\Big)\bigg]. \tag{4.7}$$

Since $P_0[\int_0^\infty G(X^{(u)})\mathrm{d}L_u]=\int_0^\infty P_0[\mathrm{d}L_u]Q^{(u)}[G(X^{(u)})]$, we obtain

$$\int_0^\infty P_0[F_t(X^{(t)})]\mathrm{d}t=\int_0^\infty P_0[\mathrm{d}L_u]\Big(Q^{(u)}\otimes\widetilde{\boldsymbol{n}}\Big)\bigg[\int_0^\infty \mathrm{d}r1_{\{\widetilde{R}>r\}}F_{u+r}\Big(X^{(u)}\bullet\widetilde{X}^{(r)}\Big)\bigg]. \tag{4.8}$$

Changing variables to $t=r+u$ and the order of integrations, we have

$$\int_0^\infty P_0[F_t(X^{(t)})]\mathrm{d}t=\int_0^\infty \mathrm{d}t\int_0^t P_0[\mathrm{d}L_u]\Big(Q^{(u)}\bullet(\boldsymbol{n}|_{\mathscr{F}_{t-u}})\Big)\Big[1_{\{R>t-u\}}F_t\Big(X^{(t)}\Big)\Big]. \tag{4.9}$$

Since the identity (4.9) holds with F_t replaced by $\mathrm{e}^{-qt}F_t$ for any $q>0$, we obtain

$$P_0[F_t(X^{(t)})]=\int_0^t P_0[\mathrm{d}L_u]\Big(Q^{(u)}\bullet(\boldsymbol{n}|_{\mathscr{F}_{t-u}})\Big)\Big[1_{\{R>t-u\}}F_t(X^{(t)})\Big]. \tag{4.10}$$

This completes the proof. □

REMARK 4.4. In the above argument, we have proven the following formulae:

$$\int_0^\infty P_0^{(t)}\mathrm{d}t=\int_0^\infty P_0^{(\tau_l)}\mathrm{d}l\bullet\int_0^\infty \boldsymbol{n}(R>r)M^{(r)}\mathrm{d}r \tag{4.11}$$

$$=\int_0^\infty P_0[\mathrm{d}L_u]Q^{(u)}\bullet\int_0^\infty \boldsymbol{n}(R>r)M^{(r)}\mathrm{d}r. \tag{4.12}$$

Here we adopt the notations $P_0^{(t)}$ and $P_0^{(\tau_l)}$ which are found in [**12**], but we do not go into details.

4.2. Harmonicity of the function $|x|^{\alpha-1}$.
Set

$$h(x) = \lim_{q\to 0+}\{u_q(0) - u_q(x)\} = \frac{1}{\pi}\int_0^\infty \frac{1-\cos x\lambda}{\lambda^\alpha}\,\mathrm{d}\lambda. \tag{4.13}$$

Then we have

$$h(x) = h(1)|x|^{\alpha-1} \tag{4.14}$$

where $h(1)$ is given as follows (see [**21**, Appendix]):

$$h(1) = \frac{1}{2\Gamma(\alpha)\sin\frac{\pi(\alpha-1)}{2}}. \tag{4.15}$$

THEOREM 4.5. *The function $h(x) = h(1)|x|^{\alpha-1}$ is harmonic for the killed process, i.e.,*

$$P_x^0[h(X_t)] = P_x[h(X_t); T_{\{0\}} > t] = h(x), \qquad x \in \boldsymbol{R}\setminus\{0\},\ t > 0. \tag{4.16}$$

Equivalently, $(h(X_{t\wedge T_{\{0\}}}))$ is a $(P_x, \mathscr{F}_t)$-martingale.

We omit the proof, because Theorem 4.5 follows immediately from the

THEOREM 4.6 (Salminen-Yor [**18**]). *For $x \in \boldsymbol{R}$, there exist a square-integrable martingale N_t^x and some constant C such that*

$$|X_t|^{\alpha-1} = |x|^{\alpha-1} + N_t^x + CL(t,x) \quad \textit{under } P_x. \tag{4.17}$$

THEOREM 4.7. *It holds that*

$$\boldsymbol{n}[h(X_t)] = 1, \qquad t > 0. \tag{4.18}$$

PROOF OF THEOREM 4.7. Theorem 3.5 and the identity (2.7) imply that

$$\int_0^\infty \mathrm{e}^{-qt}\boldsymbol{n}[h(X_t)]\mathrm{d}t = \int h(x)P_x[\mathrm{e}^{-qT_{\{0\}}}]\mathrm{d}x = \int h(x)\,\frac{u_q(x)}{u_q(0)}\,\mathrm{d}x. \tag{4.19}$$

Hence it suffices to prove that

$$\int u_q(x)h(x)\mathrm{d}x = \frac{u_q(0)}{q}, \qquad x \in \boldsymbol{R}. \tag{4.20}$$

Let r be such that $0 < r < q$. By the resolvent equation $U_qU_r = (U_r - U_q)/(q-r)$, we have

$$\int u_q(x-y)u_r(y-z)\mathrm{d}y = \frac{1}{q-r}\big\{u_r(x-z) - u_q(x-z)\big\}. \tag{4.21}$$

Letting $x = z = 0$ and using the symmetry $u_q(-y) = u_q(y)$, we have

$$\int u_q(y)u_r(y)\mathrm{d}y = \frac{1}{q-r}\big\{u_r(0) - u_q(0)\big\}. \tag{4.22}$$

Now we have

$$\int u_q(y)\{u_r(0) - u_r(y)\}\mathrm{d}y = \frac{u_q(0)}{q-r} - \frac{ru_r(0)}{q(q-r)}. \tag{4.23}$$

If we let r decrease to 0, then we see that

$$u_r(0) - u_r(x) = \frac{1}{\pi}\int_0^\infty \frac{1-\cos x\lambda}{r+\lambda^\alpha}\,\mathrm{d}\lambda \tag{4.24}$$

increases to $h(x)$, and that $ru_r(0) \to 0$. Hence we obtain (4.20) by the monotone convergence theorem. □

REMARK 4.8. For generalisations of Theorems 4.5 and 4.7 for symmetric Lévy processes, see [**20**].

4.3. Convergence of the stable meander to the *h*-path process. Let us introduce the *h-path process* $(P_x^h : x \in \boldsymbol{R})$ as

$$P_x^h|_{\mathscr{F}_t} = \frac{h(X_t)}{h(x)} \cdot P_x^0|_{\mathscr{F}_t}, \qquad x \in \boldsymbol{R} \setminus \{0\}, \tag{4.25}$$

$$P_0^h|_{\mathscr{F}_t} = h(X_t) \cdot \boldsymbol{n}|_{\mathscr{F}_t}. \tag{4.26}$$

From Theorem 4.5 and the Markov properties of P_x^0 and $\boldsymbol{n}$, it follows that such a process exists uniquely. Remark that, when $\alpha = 2$, the h-path process coincides up

to some scale transform with the symmetrization of three-dimensional Bessel process; consequently, the identity (4.26) is nothing but the *Imhof relation* (see, e.g., [**12**, Exercise XII.4.18]).

The following result asserts that the meander converges to the h-path process.

THEOREM 4.9. *It holds that*

$$M^{(t)} \overset{t\to\infty}{\longrightarrow} P_0^h \qquad \text{along } (\mathscr{F}_s). \tag{4.27}$$

In order to prove Theorem 4.9, we need the

LEMMA 4.10. *For $t > 0$ and $x \neq 0$, set*

$$Y(t,x) = \frac{P_x(T_{\{0\}} > t)}{h(x)\boldsymbol{n}(R > t)}. \tag{4.28}$$

Then it holds that $Y(t,x) \to 1$ as $t \to \infty$ for any fixed $x \neq 0$, and that $Y(t,x)$ is bounded in $t > 0$ and $x \neq 0$.

PROOF OF LEMMA 4.10. Using (2.7), we have

$$\int_0^\infty \mathrm{e}^{-qt} P_x(T_{\{0\}} > t)\mathrm{d}t = \frac{u_q(0) - u_q(x)}{q u_q(0)} \sim h(x)\frac{q^{-1/\alpha}}{u_1(0)} \qquad \text{as } q \to 0+. \tag{4.29}$$

Hence we may apply a Tauberian theorem. By Proposition 3.4, we obtain

$$P_x(T_{\{0\}} > t) \sim h(x)\boldsymbol{n}(R > t) \qquad \text{as } t \to \infty. \tag{4.30}$$

This shows the first assertion.

Since the function $t \mapsto Y(t,1)$ is continuous and $Y(t,1) \to 1$ as $t \to \infty$, we see that $Y(t,1)$ is bounded in $t > 0$. By scaling property $P_x(T_{\{0\}} > t) = P_1(T_{\{0\}} > |x|^{-\alpha}t)$, we have $Y(t,x) = Y(|x|^{-\alpha}t, 1)$. This proves the second assertion. □

Now let us proceed to prove Theorem 4.9.

PROOF OF THEOREM 4.9. Let $s > 0$ be fixed and let Z_s be a bounded $\mathscr{F}_s$-measurable functional. By the Markov property of $\boldsymbol{n}$, we have

$$\boldsymbol{n}\big[Z_s 1_{\{R>t\}}\big] = \boldsymbol{n}\big[Z_s 1_{\{R>s\}} P_{X_s}(T_{\{0\}} > t - s)\big]. \tag{4.31}$$

By the Imhof relation (4.26) and by (4.28), we have

$$\boldsymbol{n}[Z_s 1_{\{R>t\}}] = P_0^h\left[Z_s \frac{P_{X_s}(T_{\{0\}} > t-s)}{h(X_s)}\right] \tag{4.32}$$

$$= P_0^h[Z_s Y(t-s, X_s)] \cdot \boldsymbol{n}(R > t-s). \tag{4.33}$$

Dividing both sides by $\boldsymbol{n}(R>t)$, using Proposition 3.4, and then applying the bounded convergence theorem, we obtain

$$M^{(t)}[Z_s] = P_0^h[Z_s Y(t-s, X_s)] \cdot \left(1 - \frac{s}{t}\right)^{\frac{1}{\alpha}-1} \to P_0^h[Z_s] \tag{4.34}$$

as $t \to \infty$. This completes the proof. □

4.4. Convergence of the meander weighed by a multiplicative functional.

Let $(\mathscr{E}_t : t \geq 0)$ be an $(\mathscr{F}_t)$-adapted process which satisfies $0 \leq \mathscr{E}_t \leq 1$ and enjoys the multiplicativity property:

$$\mathscr{E}_{t+s} = \mathscr{E}_t \cdot (\mathscr{E}_s \circ \theta_t). \tag{4.35}$$

Such a process is called a *multiplicative functional*; see, e.g., [**3**]. Then it necessarily follows that $t \mapsto \mathscr{E}_t$ is non-increasing.

For later use, we need the following result which asserts that the convergence of the meander to the h-path process is still valid with an extra weighing by a multiplicative functional.

THEOREM 4.11.

$$\mathscr{E}_t \cdot M^{(t)} \overset{t\to\infty}{\longrightarrow} \mathscr{E}_\infty \cdot P_0^h \qquad \textit{along } (\mathscr{F}_s). \tag{4.36}$$

To prove Theorem 4.11, we need the following two lemmas.

LEMMA 4.12. *For any $x \in \boldsymbol{R}$, it holds that*

$$\begin{aligned} &\Big((X_t : t \geq 0), (\lambda^{-1/\alpha} X_{\lambda t} : t \geq 0)\Big) \textit{ under } P_x^h \\ \overset{\text{law}}{\longrightarrow}\ &\Big((X_t : t \geq 0), (\widetilde{X}_t : t \geq 0)\Big) \textit{ under } P_x^h \otimes \widetilde{P} \end{aligned} \tag{4.37}$$

as $\lambda \to \infty$ where

$$\tilde{P} = \begin{cases} P_0^h & \text{if } 1 < \alpha < 2 \text{ or if } x = 0, \\ P_0^{3\mathrm{B},+} & \text{if } \alpha = 2 \text{ and } x > 0, \\ P_0^{3\mathrm{B},-} & \text{if } \alpha = 2 \text{ and } x < 0. \end{cases} \tag{4.38}$$

PROOF. We prove the claim only in the case $1 < \alpha < 2$; in fact, almost the same argument works in the other cases. Set $X_t^{(\lambda)} = \lambda^{-1/\alpha} X_{\lambda t}$. Let us apply the convergence theorem of [**11**, Theorem VI.16].

First, let $t \geq 0$ be fixed and let $f : \boldsymbol{R} \to \boldsymbol{R}$ be a continuous function such that $\lim_{|x|\to\infty} f(x) = 0$. Then we have

$$\lim_{\lambda\to\infty} P_x^h[f(X_t^{(\lambda)})] = \lim_{\lambda\to\infty} P_{\lambda^{-1/\alpha}x}^h[f(X_t)] = P_0^h[f(X_t)]. \tag{4.39}$$

In fact, the first identity follows from the scaling property and the second follows from the Feller property of the h-path process, which is proved in [**20**]. Hence we obtain

$$X_t^{(\lambda)} \text{ under } P_x^h \quad \overset{\text{law}}{\longrightarrow} \quad X_t \text{ under } P_0^h \tag{4.40}$$

as $\lambda \to \infty$. By a standard argument involving the Markov property, we see that the convergence (4.37) holds in the sense of finite dimensional distributions.

Second, for any sequence $\{\lambda_n\}$ with $\lambda_n \to \infty$, let us check the *Aldous condition*: For a sequence of positive constants $\{\delta_n\}$ converging to zero and for a bounded sequence of stopping times $\{\rho_n\}$,

$$\left|X_{\rho_n+\delta_n} - X_{\rho_n}\right| + \left|X_{\rho_n+\delta_n}^{(\lambda_n)} - X_{\rho_n}^{(\lambda_n)}\right| \overset{n\to\infty}{\longrightarrow} 0 \qquad \text{in } P_x^h\text{-probability.} \tag{4.41}$$

The convergence (4.41) is equivalent to

$$X_{\rho_n+\delta_n}^{(\lambda_n)} - X_{\rho_n}^{(\lambda_n)} \overset{n\to\infty}{\longrightarrow} 0 \qquad \text{in } P_x^h\text{-probability.} \tag{4.42}$$

To prove (4.42), it suffices to prove that

$$P_x^h\left[\left|X_{\rho_n+\delta_n}^{(\lambda_n)} - X_{\rho_n}^{(\lambda_n)}\right| \wedge 1\right] \overset{n\to\infty}{\longrightarrow} 0. \tag{4.43}$$

By the strong Markov property and by the scaling property, we have

$$P_x^h\left[\left|X^{(\lambda_n)}_{\rho_n+\delta_n} - X^{(\lambda_n)}_{\rho_n}\right| \wedge 1\right] = P_x^h\left[P^h_{\lambda^{-1/\alpha}X_{\rho_n}}[|X_{\delta_n} - X_0| \wedge 1]\right]. \tag{4.44}$$

Hence we can easily obtain the convergence (4.43) by the Feller property of the h-path process. □

LEMMA 4.13. *For any $x \neq 0$, it holds that*

$$\frac{P_x[\mathscr{E}_t; T_{\{0\}} > t]}{h(x)\boldsymbol{n}(R > t)} \to P_x^h[\mathscr{E}_\infty] \qquad \text{as } t \to \infty. \tag{4.45}$$

PROOF OF LEMMA 4.13. For $t > s > 0$, we have $\mathscr{E}_t \leq \mathscr{E}_s$, and hence we have

$$\frac{P_x[\mathscr{E}_t; T_{\{0\}} > t]}{h(x)\boldsymbol{n}(R > t)} \leq \frac{P_x[\mathscr{E}_s; T_{\{0\}} > t]}{h(x)\boldsymbol{n}(R > t)} \tag{4.46}$$

$$= P_x^h\left[\mathscr{E}_s \frac{P_{X_s}(T_{\{0\}} > t - s)}{h(X_s)\boldsymbol{n}(R > t)}\right] \tag{4.47}$$

$$= P_x^h[\mathscr{E}_s Y(t - s, X_s)] \cdot \left(1 - \frac{s}{t}\right)^{\frac{1}{\alpha}-1}. \tag{4.48}$$

By Lemma 4.10 and by the bounded convergence theorem, we have

$$\limsup_{t\to\infty} \frac{P_x[\mathscr{E}_t; T_{\{0\}} > t]}{h(x)\boldsymbol{n}(R > t)} \leq P_x^h[\mathscr{E}_s]. \tag{4.49}$$

Since $P_x^h[\mathscr{E}_s] \to P_x^h[\mathscr{E}_\infty]$ as $s \to \infty$, we obtain the upper estimate:

$$\limsup_{t\to\infty} \frac{P_x[\mathscr{E}_t; T_{\{0\}} > t]}{h(x)\boldsymbol{n}(R > t)} \leq P_x^h[\mathscr{E}_\infty]. \tag{4.50}$$

By Lemma 4.12, we have

$$\left(\mathscr{E}_t, t^{-1/\alpha}|X_t|\right) \text{ under } P_x^h \xrightarrow{\text{law}} (\mathscr{E}_\infty, |\widetilde{X}_1|) \text{ under } P_x^h \otimes P_0^h. \tag{4.51}$$

Hence, by Fatou's lemma, we have

$$\liminf_{t\to\infty}\frac{P_x^h[\mathscr{E}_t/h(X_t)]}{\boldsymbol{n}(R>t)}=\liminf_{t\to\infty}\frac{P_x^h[\mathscr{E}_t/h(t^{-1/\alpha}|X_t|)]}{\boldsymbol{n}(R>1)} \tag{4.52}$$

$$\geq\frac{P_x^h[\mathscr{E}_\infty]P_0^h[1/h(|X_1|)]}{\boldsymbol{n}(R>1)}=P_x^h[\mathscr{E}_\infty]. \tag{4.53}$$

Thus we obtain the lower estimate:

$$\liminf_{t\to\infty}\frac{P_x[\mathscr{E}_t;T_{\{0\}}>t]}{h(x)\boldsymbol{n}(R>t)}=\liminf_{t\to\infty}\frac{P_x^h[\mathscr{E}_t/h(X_t)]}{\boldsymbol{n}(R>t)}\geq P_x^h[\mathscr{E}_\infty]. \tag{4.54}$$

Therefore the proof is now completed. □

Now we prove Theorem 4.11.

PROOF OF THEOREM 4.11. For a bounded $\mathscr{F}_s$-measurable functional Z_s and for $t>s>0$, we have

$$M^{(t)}[Z_s\mathscr{E}_t]=P_0^h\left[Z_s\mathscr{E}_s\,\frac{P_{X_s}[\mathscr{E}_{t-s};T_{\{0\}}>t-s]}{h(X_s)\boldsymbol{n}(R>t-s)}\right]\cdot\left(1-\frac{s}{t}\right)^{\frac{1}{\alpha}-1}. \tag{4.55}$$

Note that

$$\frac{P_x[\mathscr{E}_t;T_{\{0\}}>r]}{h(x)\boldsymbol{n}(R>r)}\leq\frac{P_x(T_{\{0\}}>r)}{h(x)\boldsymbol{n}(R>r)}=Y(r,x), \tag{4.56}$$

which is uniformly bounded in $r>0$ and $x\neq 0$ by Lemma 4.10. Note also that

$$\frac{P_x[\mathscr{E}_t;T_{\{0\}}>r]}{h(x)\boldsymbol{n}(R>r)}\ \overset{r\to\infty}{\longrightarrow}\ P_x^h[\mathscr{E}_\infty],\qquad x\neq 0 \tag{4.57}$$

by Lemma 4.13. Hence we apply bounded convergence theorem and obtain

$$M^{(t)}[Z_s\mathscr{E}_t]\ \overset{t\to\infty}{\longrightarrow}\ P_0^h\Big[Z_s\mathscr{E}_sP_{X_s}^h[\mathscr{E}_\infty]\Big]=P_0^h[Z_s\mathscr{E}_\infty]. \tag{4.58}$$

This completes the proof. □

5. General observations on the σ-finite measure unifying our penalisation problems and the martingale generator.

Following [**9**] and [**10**], we make general observations on the measure $\mathscr{P}$.

5.1. The σ-finite measure unifying our penalisation problems.

Recall the definition of $\mathscr{P}$:

$$\mathscr{P} = \int_0^\infty P_0[\mathrm{d}L_u]Q^{(u)} \bullet P_0^h \tag{5.1}$$

where

$$P_0[\mathrm{d}L_u] = \frac{\Gamma(1/\alpha)}{\alpha\pi}\frac{\mathrm{d}u}{u^{1/\alpha}} \tag{5.2}$$

and where P_0^h is defined by

$$P_0^h|_{\mathscr{F}_t} = h(X_t)\cdot \boldsymbol{n}|_{\mathscr{F}_t}, \qquad t>0. \tag{5.3}$$

Denote

$$g = \sup\{t \geq 0 : X_t = 0\}. \tag{5.4}$$

THEOREM 5.1. *The following statements hold*:
(i) $\mathscr{P}(g \in \mathrm{d}u) = P_0[\mathrm{d}L_u]$;
(ii) $\mathscr{P}$ *is a* σ*-finite measure on* $\mathscr{F}_\infty$;
(iii) $\mathscr{P}$ *is singular with respect to* P_0 *on* $\mathscr{F}_\infty$;
(iv) *For each* $t>0$ *and for* $A \in \mathscr{F}_t$, *one has*

$$\mathscr{P}(A) = 0 \qquad \text{if } P_0(A) = 0, \tag{5.5}$$

$$\mathscr{P}(A) = \infty \qquad \text{if } P_0(A) > 0. \tag{5.6}$$

REMARK 5.2. For each $t>0$, (5.5) asserts that $\mathscr{P}$ is equivalent to P_0 on $\mathscr{F}_t$, but (5.6) asserts that $\mathscr{P}$ is *never* σ-finite on $\mathscr{F}_t$. We insist that, since $\mathscr{P}$ is not σ-finite on $\mathscr{F}_t$, (5.5) does not imply the existence of an $\mathscr{F}_t$-measurable Radon-Nikodym density.

PROOF OF THEOREM 5.1.

(i) Since P_0^h is locally equivalent to $\boldsymbol{n}$, we see that $P_0^h(X_s \neq 0 \text{ for any } s \leq t) = 1$ for any $t > 0$. This shows that $P_0^h(X_t \neq 0 \text{ for any } t > 0) = 1$. Hence we see, by the definition (5.1) of $\mathscr{P}$, that $g = u$ under the measure $Q^{(u)} \bullet P_0^h$. Thus we obtain the desired result.

(ii) It is obvious by (i) that $\mathscr{P}(g < u)$ is finite for each $u > 0$.

(iii) On one hand, we have $\mathscr{P}(g = \infty) = 0$. On the other hand, since the origin for (X_t, P_0) is recurrent, we have $P_0(g < \infty) = 0$. This implies that $\mathscr{P}$ is singular to P_0 on $\mathscr{F}_\infty$.

(iv) Let $A \in \mathscr{F}_t$ and suppose that $P_0(A) = 0$. For $T > t$, we have

$$\int_0^T P_0[\mathrm{d}L_u]\Big(Q^{(u)} \bullet P_0^h\Big)(A) = \int_0^t P_0[\mathrm{d}L_u]\Big(Q^{(u)} \bullet P_0^h\Big)(A) + \int_t^T P_0[\mathrm{d}L_u]Q^{(u)}(A). \quad (5.7)$$

For $0 < u < t$, we have $(Q^{(u)} \bullet P_0^h)(A) = (Q^{(u)} \bullet \boldsymbol{n})[1_A h(X_t)]$, and hence we obtain

$$\int_0^t P_0[\mathrm{d}L_u]\Big(Q^{(u)} \bullet P_0^h\Big)(A) = P_0[1_A h(X_t)] = 0. \quad (5.8)$$

For $t < u < T$, we have

$$\int_t^T P_0[\mathrm{d}L_u]Q^{(u)}(A) = P_0[1_A(L_T - L_t)] = 0. \quad (5.9)$$

Letting $T \to \infty$, we obtain $\int_t^\infty P_0[\mathrm{d}L_u]Q^{(u)}(A) = 0$. Therefore we obtain $\mathscr{P}(A) = 0$.

Conversely, let $A \in \mathscr{F}_t$ and suppose that $P_0(A) > 0$. Then

$$\mathscr{P}(A) \geq \int_t^\infty P_0[\mathrm{d}L_u]Q^{(u)}(A) = P_0[1_A(L_\infty - L_t)]. \quad (5.10)$$

Note that the last quantity is ∞ since $P_0(L_\infty = \infty) = 1$. Hence we obtain $\mathscr{P}(A) = \infty$. □

5.2. The martingale generator.

THEOREM 5.3. *For each $x \in \boldsymbol{R}$, $t \geq 0$ and for each non-negative measurable or $\mathscr{P}_x$-integrable functional F, there exists a unique $\mathscr{F}_t$-measurable functional $M_{t,x}(F)$ (possibly taking infinite values) such that*

$$(F \cdot \mathscr{P}_x)|_{\mathscr{F}_t} = M_{t,x}(F) \cdot P_x|_{\mathscr{F}_t}. \quad (5.11)$$

In particular, if F is $\mathscr{P}_x$-integrable, then the process $(M_{t,x}(F) : t \geq 0)$ is an $(\mathscr{F}_t, P_x)$-martingale such that

$$M_{0,x}(F) = \mathscr{P}_x[F] \tag{5.12}$$

and that

$$\lim_{t\to\infty} M_{t,x}(F) = 0 \qquad P_x\text{-almost surely.} \tag{5.13}$$

In the case $x = 0$, we write $M_t(F)$ for $M_{t,0}(F)$. For each $x \in \boldsymbol{R}$, we call the operator $L^1(\mathscr{P}_x) \ni F \mapsto (M_{t,x}(F) : t \geq 0)$ the *martingale generator*.

PROOF. It is obvious that the uniqueness holds in the sense that, if $F = G$ $\mathscr{P}$-almost everywhere, then $M_{t,x}(F) = M_{t,x}(G)$ P_0-almost surely. Without loss of generality, we may suppose that $x = 0$ and that F is non-negative.

Let n be a positive integer and set $F_n = F \cdot 1_{\{g<n\}}$. By (ii) and (iv) of Theorem 5.1, we see that $(F_n \cdot \mathscr{P})|_{\mathscr{F}_t}$ is a finite measure and is absolutely continuous with respect to $P_0|_{\mathscr{F}_t}$. Hence we may apply the Radon-Nikodym theorem to obtain the desired functional $M_t(F_n)$ as the Radon-Nikodym derivative. Hence the desired functional $M_t(F)$ is obtained as the increasing limit $\lim_{n\to\infty} M_t(F_n)$ by the monotone convergence theorem.

Suppose that F is $\mathscr{P}$-integrable. For $s \leq t$, we have

$$P_0[Z_s M_t(F)] = \mathscr{P}[Z_s F] = P_0[Z_s M_s(F)]. \tag{5.14}$$

Hence $(M_t(F) : t \geq 0)$ is a $(\mathscr{F}_t, P_0)$-martingale. It is obvious that $M_0(F) = \mathscr{P}[F]$.

Since $(M_t(F) : t \geq 0)$ is a non-negative martingale, $M_t(F)$ converges P_0-almost surely to a non-negative $\mathscr{F}_\infty$-measurable functional $M_\infty(F)$. For $0 < s < t \leq \infty$, set $A(s,t) = \{g_t \geq s\} \in \mathscr{F}_t$. Note that $P_0(A(s,\infty)) = P_0(g \geq s) = 1$. Applying Fatou's lemma and then applying the dominated convergence theorem, we obtain

$$P_0[M_\infty(F)] = P_0[1_{A(s,\infty)} M_\infty(F)] \leq \liminf_{t\to\infty} P_0[1_{A(s,t)} M_t(F)] \tag{5.15}$$

$$= \liminf_{t\to\infty} \mathscr{P}[1_{A(s,t)} F] = \mathscr{P}[1_{A(s,\infty)} F]. \tag{5.16}$$

Since $\mathscr{P}(g = \infty) = 0$, we have $\lim_{s\to\infty} \mathscr{P}[1_{A(s,\infty)} F] = 0$. Hence we obtain $P_0[M_\infty(F)] = 0$, which implies that $P_0(M_\infty(F) = 0) = 1$. Therefore the proof is completed. □

6. Convergence lemmas.

Let $0 < \gamma < 1$. For integrable functions $\psi_t(u)$ such that $\psi_t(u) \to \exists\psi(u)$ as $t \to \infty$, we may expect that

$$\int_0^t \left(1 - \frac{u}{t}\right)^{\gamma-1} \psi_t(u)\mathrm{d}u \to \int_0^\infty \psi(u)\mathrm{d}u \qquad \text{as } t \to \infty. \tag{6.1}$$

We need this convergence for several functions ψ_t in order to solve our penalisation problems, as we have seen roughly in (1.11). In fact, we shall see that we must be careful in dealing with the convergence (6.1). In this section we give some sufficient conditions for the convergence (6.1) as well as a counterexample.

If ψ_t's satisfy

$$\int_0^t \psi_t(u)\mathrm{d}u \to \int_0^\infty \psi(u)\mathrm{d}u \qquad \text{as } t \to \infty, \tag{6.2}$$

then the convergence (6.1) is equivalent to

$$I(\psi_t, t) \to 0 \qquad \text{as } t \to \infty \tag{6.3}$$

where

$$I(\psi, t) = \int_0^t \left\{ \left(1 - \frac{u}{t}\right)^{\gamma-1} - 1 \right\} \psi(u)\mathrm{d}u. \tag{6.4}$$

First, we present the following counterexample.

EXAMPLE 6.1. The convergence (6.1) *fails* if

$$\psi_t(u) \equiv \psi(u) = \sum_{n=1}^\infty n^{\frac{2+\gamma}{1-\gamma}} 1_{\left(n - n^{-\frac{4-\gamma}{1-\gamma}}, n\right)}(u). \tag{6.5}$$

PROOF. ψ is integrable since $\int_0^\infty \psi(u)\mathrm{d}u = \sum_{n=1}^\infty n^{-2} < \infty$. But $\limsup_t I(\psi, t) = \infty$ because

$$I(\psi,n) \geq n^{\frac{2+\gamma}{1-\gamma}} \cdot n^{1-\gamma} \int_{n-n^{-\frac{4-\gamma}{1-\gamma}}}^{n} (n-u)^{\gamma-1} \mathrm{d}u - n^{-2} \tag{6.6}$$

$$= n^{\frac{2+\gamma}{1-\gamma}} \cdot n^{1-\gamma} \cdot \gamma^{-1} n^{-\frac{(4-\gamma)\gamma}{1-\gamma}} - n^{-2} \tag{6.7}$$

$$= \gamma^{-1} n^{3-2\gamma} - n^{-2} \to \infty \qquad \text{as } n \to \infty. \tag{6.8}$$

This prevents the convergence (6.1).

On the other hand, we give three sufficient conditions for the convergence (6.1); the first one is rather theoretical, but the second and third ones can be readily applied. □

LEMMA 6.2 (Dominated convergence). *Suppose that ψ_t's are integrable functions such that $\int_0^\infty \psi_t(u)\mathrm{d}u \to \int_0^\infty \psi(u)\mathrm{d}u$ for some integrable function ψ. Suppose, in addition, that $|\psi_t| \leq \widetilde{\psi}_t$ for some integrable function $\widetilde{\psi}_t$ such that $\lim_{t\to\infty} I(\widetilde{\psi}_t, t) = 0$. Then*

$$\int_0^t \left(1 - \frac{u}{t}\right)^{\gamma-1} \psi_t(u)\mathrm{d}u \to \int_0^\infty \psi(u)\mathrm{d}u \qquad \text{as } t \to \infty \tag{6.9}$$

holds.

PROOF. This is obvious by $|I(\psi_t, t)| \leq I(|\psi_t|, t) \leq I(\widetilde{\psi}_t, t) \to 0$ as $t \to \infty$. □

LEMMA 6.3. *Suppose that ψ is a non-negative integrable function and satisfies*

$$\lim_{t\to\infty} \left\{ t \sup_{u>t} \psi(u) \right\} = 0. \tag{6.10}$$

Then $\lim_{t\to\infty} I(\psi, t) = 0$.

PROOF. Let $0 < \varepsilon < 1$ be fixed. We split $I(\psi, t)$ into a sum $I(\psi_1, t) + I(\psi_2, t)$ where $\psi_1 = \psi 1_{(\varepsilon t, \infty)}$ and $\psi_2 = \psi 1_{(0, \varepsilon t)}$.

By the definition of $I(\psi_1, t)$ and changing variables to $v = ut$, we have

$$I(\psi_1, t) = \int_{\varepsilon t}^{t} \left\{ \left(1 - \frac{u}{t}\right)^{\gamma-1} - 1 \right\} \psi(u)\mathrm{d}u \tag{6.11}$$

$$\leq t \sup_{u>\varepsilon t} \psi(u) \int_{\varepsilon t}^{t} \left\{ \left(1 - \frac{u}{t}\right)^{\gamma-1} - 1 \right\} \frac{\mathrm{d}u}{t} \tag{6.12}$$

$$= \frac{1}{\varepsilon} \left\{ \varepsilon t \sup_{u>\varepsilon t} \psi(u) \right\} \int_{\varepsilon}^{1} \left\{ (1-v)^{\gamma-1} - 1 \right\} \mathrm{d}v. \tag{6.13}$$

By the assumption (6.10), we obtain $\lim_{t\to\infty} I(\psi_1, t) = 0$ for any fixed $\varepsilon > 0$.

By the definition of $I(\psi_2, t)$, we have

$$I(\psi_2, t) \leq \left\{ \frac{1}{(1-\varepsilon)^{1-\gamma}} - 1 \right\} \int_0^\infty \psi(u)\mathrm{d}u. \tag{6.14}$$

Hence we have $\limsup_{t\to\infty} I(\psi_2, t)$ vanishes as $\varepsilon \to 0+$. Now the proof is completed. □

LEMMA 6.4. *Suppose that $\psi_t(u) = \psi_1(u)\psi_2(t-u)$ where ψ_1 is integrable and ψ_2 is bounded measurable with $\lim_{u\to\infty} \psi_2(u) = \psi_2(\infty) > 0$. Suppose, in addition, that the function $t \mapsto \int_0^t (t-u)^{\gamma-1}\psi_t(u)\mathrm{d}u$ is ultimately non-increasing as t increases. Then $\lim_{t\to\infty} I(\psi_t, t) = 0$.*

PROOF. Taking the Laplace transform, we have

$$\int_0^\infty \mathrm{d}t\mathrm{e}^{-qt} \int_0^t (t-u)^{\gamma-1}\psi_t(u)\mathrm{d}u = \int_0^\infty \mathrm{e}^{-qu}\psi_1(u)\mathrm{d}u \int_0^\infty \mathrm{e}^{-qt} t^{\gamma-1}\psi_2(t)\mathrm{d}t \tag{6.15}$$

$$\sim \Gamma(\gamma) q^{-\gamma} \psi_2(\infty) \int_0^\infty \psi_1(u)\mathrm{d}u \quad \text{as } q \to 0+. \tag{6.16}$$

Hence we may apply the Tauberian theorem. By the monotonicity assumption, we obtain

$$\int_0^t (t-u)^{\gamma-1}\psi_t(u)\mathrm{d}u \sim t^{\gamma-1}\psi_2(\infty) \int_0^\infty \psi_1(u)\mathrm{d}u \qquad \text{as } t \to \infty. \tag{6.17}$$

On the other hand, we have

$$\int_0^t \psi_t(u)\mathrm{d}u = \int_0^t \psi_1(u)\psi_2(t-u)\mathrm{d}u \to \psi_2(\infty) \int_0^\infty \psi_1(u)\mathrm{d}u \qquad \text{as } t \to \infty. \tag{6.18}$$

Therefore we obtain $\lim_{t\to\infty} I(\psi_t, t) = 0$. □

7. Penalisation with a function of the local time at the origin.

7.1. Results.

THEOREM 7.1. *Let f be a non-negative function on $[0, \infty)$. Then it holds that*

$$M_t(f(L_\infty)) = h(X_t)f(L_t) + \int_{L_t}^\infty f(l)\mathrm{d}l, \qquad t \geq 0. \tag{7.1}$$

Consequently, it holds that

$$\mathscr{P}[f(L_\infty)] = M_0(f(L_\infty)) = \int_0^\infty f(l)\mathrm{d}l. \tag{7.2}$$

REMARK 7.2. As an outcome of (7.1), we have established that its right hand side is a $(P_0, \mathscr{F}_t)$-martingale, a well-known fact for $\alpha = 2$ (see [**12**, Proposition VI.4.5]).

THEOREM 7.3. *Let f be a non-negative function on $[0,\infty)$ such that*

$$I(f) := \int_0^\infty f(l)\mathrm{d}l \in (0,\infty). \tag{7.3}$$

Then it holds that

$$\frac{f(L_t)\cdot P_0}{\boldsymbol{n}(R>t)} \overset{t\to\infty}{\longrightarrow} f(L_\infty)\cdot\mathscr{P} \qquad \text{along } (\mathscr{F}_s). \tag{7.4}$$

Consequently, the penalisation with the weight functional $\Gamma_t = f(L_t)$ is given as

$$\frac{f(L_t)\cdot P_0}{P_0[f(L_t)]} \overset{t\to\infty}{\longrightarrow} \frac{f(L_\infty)\cdot\mathscr{P}}{I(f)} \qquad \text{along } (\mathscr{F}_s). \tag{7.5}$$

7.2. Proofs.

PROOF OF THEOREM 7.1. Let $t>0$ be fixed and Z_t a non-negative $\mathscr{F}_t$-measurable functional. On the one hand, since $L_\infty = L_t$ on $\{g\le t\}$, we have

$$\begin{aligned}
\mathscr{P}[Z_t f(L_\infty)1_{\{g\le t\}}] =& \mathscr{P}[Z_t f(L_t)1_{\{g\le t\}}] && (7.6)\\
=& \int_0^t P_0[\mathrm{d}L_u]\Big(Q^{(u)}\bullet P_0^h\Big)[Z_t f(L_t)] && (7.7)\\
=& \int_0^t P_0[\mathrm{d}L_u]\Big(Q^{(u)}\bullet \boldsymbol{n}\Big)[Z_t f(L_t)h(X_t)] && (7.8)\\
=& \int_0^t \boldsymbol{n}(R>t-u)P_0[\mathrm{d}L_u]\Big(Q^{(u)}\bullet M^{(t-u)}\Big)[Z_t f(L_t)h(X_t)] && (7.9)\\
=& P_0[Z_t f(L_t)h(X_t)]. && (7.10)
\end{aligned}$$

On the other hand, we have

$$\mathscr{P}[Z_t f(L_\infty)1_{\{g>t\}}] = \int_t^\infty P_0[\mathrm{d}L_u]\Big(Q^{(u)} \bullet P_0^h\Big)[Z_t f(L_u)] \tag{7.11}$$

$$= \int_t^\infty P_0[\mathrm{d}L_u]Q^{(u)}[Z_t f(L_u)] \tag{7.12}$$

$$= P_0\left[Z_t \int_t^\infty f(L_u)\mathrm{d}L_u\right] \tag{7.13}$$

$$= P_0\left[Z_t \int_{L_t}^\infty f(l)\mathrm{d}l\right]. \tag{7.14}$$

Hence we obtain

$$\mathscr{P}[Z_t f(L_\infty)] = P_0\left[Z_t\left\{f(L_t)h(X_t) + \int_{L_t}^\infty f(l)\mathrm{d}l\right\}\right]. \tag{7.15}$$

Therefore we have completed the proof. □

Proof of Theorem 7.3. We need only to prove the first assertion that

$$\frac{f(L_t)\cdot P_0}{\boldsymbol{n}(R>t)} \overset{t\to\infty}{\longrightarrow} f(L_\infty)\cdot\mathscr{P} \qquad \text{along } (\mathscr{F}_s). \tag{7.16}$$

Set $\psi(u) = p_u(0)Q^{(u)}[f(L_u)]$. We will prove in Lemma 7.4 below that ψ satisfies the assumption of Lemma 6.3. Now we apply Lemma 6.3 for the function ψ and we obtain

$$\int_0^t \left(1-\frac{u}{t}\right)^{\frac{1}{\alpha}-1} P_0[\mathrm{d}L_u]Q^{(u)}[f(L_u)] \to \int_0^\infty P_0[\mathrm{d}L_u]Q^{(u)}[f(L_u)] \tag{7.17}$$

as $t\to\infty$. Let $s>0$ be fixed and let Z_s be a bounded $\mathscr{F}_s$-measurable functional. Then

$$\int_0^t P_0[\mathrm{d}L_u]\Big(Q^{(u)} \bullet M^{(t-u)}\Big)[Z_s f(L_u)] \to \int_0^\infty P_0[\mathrm{d}L_u]\Big(Q^{(u)} \bullet P_0^h\Big)[Z_s f(L_u)] \tag{7.18}$$

as $t\to\infty$ by Lebesgue's convergence theorem. Hence we can apply Lemma 6.2, and we obtain

$$\frac{P_0[Z_s f(L_t)]}{\boldsymbol{n}(R>t)} = \int_0^t \left(1-\frac{u}{t}\right)^{\frac{1}{\alpha}-1} P_0[\mathrm{d}L_u]\Big(Q^{(u)} \bullet M^{(t-u)}\Big)[Z_s f(L_u)] \tag{7.19}$$

$$\to \int_0^\infty P_0[\mathrm{d}L_u]\Big(Q^{(u)} \bullet P_0^h\Big)[Z_s f(L_u)] \qquad (\text{as } t\to\infty) \tag{7.20}$$

$$= \mathscr{P}[Z_s f(L_\infty)] \tag{7.21}$$

as $t \to \infty$. This completes the proof. □

LEMMA 7.4. *Set*

$$\psi(u) = p_u(0) Q^{(u)}[f(L_u)]. \tag{7.22}$$

Then the function $\psi(u)$ is continuous and $u\psi(u) \to 0$ as $u \to \infty$. In particular, the function ψ satisfies the assumption (6.10) *of Lemma* 6.3.

PROOF. For any non-negative Borel function ϕ, we have

$$\int_0^\infty \phi(u)\psi(u)\mathrm{d}u = P_0\left[\int_0^\infty \phi(u) f(L_u)\mathrm{d}L_u\right] \tag{7.23}$$

$$= \int_0^\infty P_0[\phi(\tau_s)] f(s)\mathrm{d}s \tag{7.24}$$

$$= \int_0^\infty P_0[\phi(s^{1/\beta}\tau_1)] f(s)\mathrm{d}s \tag{7.25}$$

where $\beta = 1 - 1/\alpha$. If we denote $\rho^{(\beta)}(v) = P_0(\tau_1 \in \mathrm{d}v)/\mathrm{d}v$, we have

$$= \int_0^\infty \mathrm{d}s f(s) \int_0^\infty \phi(s^{1/\beta}v)\rho^{(\beta)}(v)\mathrm{d}v \tag{7.26}$$

$$= \int_0^\infty \mathrm{d}u \phi(u) \int_0^\infty s^{-1/\beta}\rho^{(\beta)}(s^{-1/\beta}u) f(s)\mathrm{d}s. \tag{7.27}$$

Hence we obtain

$$\psi(u) = \int_0^\infty s^{-1/\beta}\rho^{(\beta)}(s^{-1/\beta}u) f(s)\mathrm{d}s. \tag{7.28}$$

Since the function $\rho^{(\beta)}(v)$ is unimodal (see, e.g., Sato [**19**]), we see that $v\rho^{(\beta)}(v)$ is bounded in $v > 0$ and that $v\rho^{(\beta)}(v) \to 0$ as $v \to \infty$. Therefore, by the assumption

that $\int_0^\infty f(s)\mathrm{d}s < \infty$, we obtain the desired result. □

REMARK 7.5. In the Brownian case $\alpha = 2$, the corresponding β equals $1/2$ and

$$\rho^{(1/2)}(v) = \frac{1}{2\sqrt{\pi v^3}}\,\mathrm{e}^{-\frac{1}{4v}}. \tag{7.29}$$

8. Feynman-Kac penalisations.

8.1. Results.

Recall that our Feynman-Kac penalisation is the penalisation with the weight functional

$$\mathscr{E}_t^V = \exp\left\{-\int L(t,x)V(\mathrm{d}x)\right\}, \qquad t \geq 0 \tag{8.1}$$

for a non-negative measure $V(\mathrm{d}x)$ on $\boldsymbol{R}$.

THEOREM 8.1. *Let V be a non-negative measure on $\boldsymbol{R}$ such that*

$$0 < \int (1 + |y|^{\alpha-1})V(\mathrm{d}y) < \infty. \tag{8.2}$$

Let $x \in \boldsymbol{R}$. Then it holds that

$$0 < \mathscr{P}_x[\mathscr{E}_\infty^V] < \infty \tag{8.3}$$

and that

$$\frac{(\mathscr{E}_t^V 1_{\{T_{\{0\}}>t\}}) \cdot P_x}{\boldsymbol{n}(R>t)} \overset{t\to\infty}{\longrightarrow} (\mathscr{E}_\infty^V 1_{\{T_{\{0\}}=\infty\}}) \cdot \mathscr{P}_x \qquad \textit{along } (\mathscr{F}_s), \tag{8.4}$$

$$\frac{(\mathscr{E}_t^V 1_{\{T_{\{0\}}\leq t\}}) \cdot P_x}{\boldsymbol{n}(R>t)} \overset{t\to\infty}{\longrightarrow} (\mathscr{E}_\infty^V 1_{\{T_{\{0\}}<\infty\}}) \cdot \mathscr{P}_x \qquad \textit{along } (\mathscr{F}_s) \tag{8.5}$$

and

$$\frac{\mathscr{E}_t^V \cdot P_x}{\boldsymbol{n}(R>t)} \overset{t\to\infty}{\longrightarrow} \mathscr{E}_\infty^V \cdot \mathscr{P}_x \qquad \textit{along } (\mathscr{F}_s). \tag{8.6}$$

COROLLARY 8.2. *Let V be a non-negative measure on $\boldsymbol{R}$ such that* (8.2) *holds. Then the penalisation with the weight functional $\Gamma_t = \mathscr{E}_t^V$ is given as*

$$\frac{\mathscr{E}_t^V \cdot P_x}{P_x[\mathscr{E}_t^V]} \overset{t\to\infty}{\longrightarrow} \frac{\mathscr{E}_\infty^V \cdot \mathscr{P}_x}{\mathscr{P}_x[\mathscr{E}_\infty^V]} \qquad \text{along } (\mathscr{F}_s). \tag{8.7}$$

THEOREM 8.3. *Let V be a non-negative measure on $\boldsymbol{R}$ such that* (8.2) *holds. Set*

$$C_V = \mathscr{P}[\mathscr{E}_\infty^V]. \tag{8.8}$$

Let $x \in \boldsymbol{R}$. Then it holds that

$$\varphi_V^1(x) := \lim_{t\to\infty} \frac{P_x[\mathscr{E}_t^V; T_{\{0\}} > t]}{\boldsymbol{n}(R > t)} = \mathscr{P}_x[\mathscr{E}_\infty^V; T_{\{0\}} = \infty] = h(x) P_x^h[\mathscr{E}_\infty^V], \tag{8.9}$$

$$\varphi_V^2(x) := \lim_{t\to\infty} \frac{P_x[\mathscr{E}_t^V; T_{\{0\}} \le t]}{\boldsymbol{n}(R > t)} = \mathscr{P}_x[\mathscr{E}_\infty^V; T_{\{0\}} < \infty] = C_V P_x[\mathscr{E}_{T_{\{0\}}}^V] \tag{8.10}$$

and

$$\varphi_V(x) := \lim_{t\to\infty} \frac{P_x[\mathscr{E}_t^V]}{\boldsymbol{n}(R > t)} = \mathscr{P}_x[\mathscr{E}_\infty^V] = h(x) P_x^h[\mathscr{E}_\infty^V] + C_V P_x[\mathscr{E}_{T_{\{0\}}}^V] \tag{8.11}$$

$$\equiv \varphi_V^1(x) + \varphi_V^2(x). \tag{8.12}$$

Moreover, for $t \ge 0$, it holds that

$$M_{t,x}(\mathscr{E}_\infty^V 1_{\{T_{\{0\}}=\infty\}}) = \varphi_V^1(X_t) 1_{\{T_{\{0\}}>t\}} \mathscr{E}_t^V, \tag{8.13}$$

$$M_{t,x}(\mathscr{E}_\infty^V 1_{\{T_{\{0\}}<\infty\}}) = \Big\{ \varphi_V^1(X_t) 1_{\{T_{\{0\}}\le t\}} + \varphi_V^2(X_t) \Big\} \mathscr{E}_t^V, \tag{8.14}$$

and

$$M_{t,x}(\mathscr{E}_\infty^V) = \varphi_V(X_t) \mathscr{E}_t^V. \tag{8.15}$$

We divide the proofs of Theorems 8.1 and 8.3 into several steps in the following subsections.

REMARK 8.4. *For the Feynman-Kac penalisations (Theorems 8.1 and 8.3)

*Tildes in this remark have nothing to do with our previous notation's for independence.

in the Brownian case, Roynette-Vallois-Yor ([**15**], [**14**], [**13**]) have given more characterisations of the limit measure than the contents of Theorem 8.3. For convenience, we consider the Wiener measures $(W_x : x \in \boldsymbol{R})$ normalized with the weight functional

$$\widetilde{\mathscr{E}}_t^V = \exp\left\{-\frac{1}{2}\int L(t,x)V(\mathrm{d}x)\right\}. \tag{8.16}$$

For each $x \in \boldsymbol{R}$, let $\mathscr{W}_x$ denote the law of $(x + X_t : t \geq 0)$ under $\mathscr{W}$. Then the function

$$\widetilde{\varphi}_V(x) = \lim_{t\to\infty} \sqrt{t} W_x[\widetilde{\mathscr{E}}_t^V] = \mathscr{W}_x[\widetilde{\mathscr{E}}_\infty^V] \tag{8.17}$$

is the unique solution of the Sturm-Liouville differential equation

$$\mathrm{d}\widetilde{\varphi}'_V(x) = \widetilde{\varphi}_V(x)V(\mathrm{d}x) \tag{8.18}$$

subject to the boundary conditions

$$\lim_{x\to\infty} \widetilde{\varphi}'_V(x) = \sqrt{\frac{2}{\pi}} \qquad \text{and} \qquad \lim_{x\to-\infty} \widetilde{\varphi}'_V(x) = -\sqrt{\frac{2}{\pi}}. \tag{8.19}$$

Moreover, the limit measure W_x^V (instead of P_x^V) is the law of the unique solution of the stochastic differential equation

$$\mathrm{d}X_t = \mathrm{d}B_t + \frac{\widetilde{\varphi}'_V(X_t)}{\widetilde{\varphi}_V(X_t)}\,\mathrm{d}t, \qquad X_0 = x. \tag{8.20}$$

We do not know how to develop these arguments in the stable Lévy case, for which it would be interesting to obtain counterparts of (8.18) and (8.20).

8.2. Penalisation weighed by a general multiplicative functional.

In this subsection, we make a general study. Let $\mathscr{E} = (\mathscr{E}_t : t \geq 0)$ be an $(\mathscr{F}_t)$-adapted process which satisfies $0 \leq \mathscr{E}_t \leq 1$, $t \geq 0$ and is a multiplicative functional:

$$\mathscr{E}_{t+s} = \mathscr{E}_t \cdot (\mathscr{E}_s \circ \theta_t), \qquad t, s \geq 0. \tag{8.21}$$

Note that the process $t \mapsto \mathscr{E}_t$ is necessarily non-increasing; in fact, $\mathscr{E}_{t+s} =$

$\mathscr{E}_t \cdot (\mathscr{E}_s \circ \theta_t) \leq \mathscr{E}_t$ for any $t, s \geq 0$.

THEOREM 8.5. *Suppose that*

$$\int_0^\infty P_0[\mathrm{d}L_u] Q^{(u)}[\mathscr{E}_u] < \infty. \tag{8.22}$$

Then it holds that

$$\frac{\mathscr{E}_t \cdot P_0}{\boldsymbol{n}(R > t)} \overset{t\to\infty}{\longrightarrow} \mathscr{E}_\infty \cdot \mathscr{P} \qquad \textit{along } (\mathscr{F}_s). \tag{8.23}$$

PROOF. By the multiplicativity property, we have

$$\mathscr{P}[\mathscr{E}_\infty] = \int_0^\infty P_0[\mathrm{d}L_u] \Big(Q^{(u)} \bullet P_0^h \Big)[\mathscr{E}_\infty] \tag{8.24}$$

$$= \left\{ \int_0^\infty P_0[\mathrm{d}L_u] Q^{(u)}[\mathscr{E}_u] \right\} P_0^h[\mathscr{E}_\infty]. \tag{8.25}$$

Set

$$\psi_t(u) = p_u(0)(Q^{(u)} \bullet M^{(t-u)})[\mathscr{E}_t]. \tag{8.26}$$

Then we have $\psi_t(u) = \psi_1(u)\psi_2(t-u)$ where $\psi_1(u) = p_u(0)Q^{(u)}[\mathscr{E}_u]$ and $\psi_2(t) = M^{(t)}[\mathscr{E}_t]$. Let us check that all the assumptions of Lemma 6.4 are satisfied for $\psi_t(u)$. Note that

$$\int_0^\infty \psi_1(u)\mathrm{d}u = \int_0^\infty P_0[\mathrm{d}L_u] Q^{(u)}[\mathscr{E}_u] \tag{8.27}$$

and it is finite by the assumption (8.22). Note also that ψ_2 is bounded and that $\lim_{t\to\infty} \psi_2(t) = P_0^h[\mathscr{E}_\infty]$ by Theorem 4.11. Recall the following identity:

$$P_0[\mathscr{E}_t] = \int_0^t \boldsymbol{n}(R > t-u) P_0[\mathrm{d}L_u] \Big(Q^{(u)} \bullet M^{(t-u)} \Big)[\mathscr{E}_t]. \tag{8.28}$$

Since the left-hand side is non-increasing as t increases, we see that the function $t \mapsto \int_0^t (t-u)^{\frac{1}{\alpha}-1} \psi_t(u)\mathrm{d}u$ is non-increasing as t increases. Hence we have verified all the assumptions of Lemma 6.4, and we obtain $\lim_{t\to\infty} I(\psi_t, t) = 0$. The

remainder of the proof follows from Lemma 6.2. □

8.3. Non-degeneracy condition.

Now we return to the case where $\mathscr{E}_t = \mathscr{E}_t^V$. By the multiplicativity of $(\mathscr{E}_t^V)$, we have

$$C_V = \mathscr{P}[\mathscr{E}_\infty^V] = \int_0^\infty P_0[\mathrm{d}L_u]\Big(Q^{(u)} \bullet P_0^h\Big)[\mathscr{E}_\infty^V] \tag{8.29}$$

$$= \int_0^\infty P_0[\mathrm{d}L_u]Q^{(u)}[\mathscr{E}_u^V]P_0^h[\mathscr{E}_\infty^V] \tag{8.30}$$

$$= \left\{\int_0^\infty P_0[\mathscr{E}_{\tau_s}^V]\mathrm{d}s\right\}P_0^h[\mathscr{E}_\infty^V]. \tag{8.31}$$

THEOREM 8.6. *The following assertions hold*:
(i) *If* $V \neq 0$, *then* $\int_0^\infty P_0[\mathscr{E}_{\tau_s}^V]\mathrm{d}s < \infty$;
(ii) *If* $V((-\varepsilon,\varepsilon)) < \infty$ *for some* $\varepsilon > 0$, *then* $\int_0^\infty P_0[\mathscr{E}_{\tau_s}^V]\mathrm{d}s > 0$;
(iii) *If* $\int h(x)V(\mathrm{d}x) < \infty$, *then* $P_0^h[\mathscr{E}_\infty^V] > 0$;
(iv) *If* $0 < \int\{1 + h(x)\}V(\mathrm{d}x) < \infty$, *then* $0 < C_V < \infty$.

For the proof of Theorem 8.6, we need the following

LEMMA 8.7. *The following statements hold*:
(i) $\boldsymbol{n}[L(R,x)] = 1$ *for any* $x \in \boldsymbol{R} \setminus \{0\}$;
(ii) $P_0^h[L(t,x)] = h(x)P_x(T_{\{0\}} < t)$ *for any* $t \geq 0$ *and any* $x \in \boldsymbol{R}$;
(iii) $P_0^h[L(\infty,x)] = h(x)$ *for any* $x \in \boldsymbol{R}$.

Remark that $\boldsymbol{n}[L(R,0)] = 0$; in fact, $L(R,0) = 0$ $\boldsymbol{n}$-almost everywhere.

PROOF.
(i) For a non-negative Borel function f, we have

$$\int \mathrm{d}x f(x)\boldsymbol{n}[L(R,x)] = \int_0^\infty \boldsymbol{n}[f(X_t)]\mathrm{d}t = \int \mathrm{d}x f(x)\int_0^\infty \rho(t,x)\mathrm{d}t = \int \mathrm{d}x f(x). \tag{8.32}$$

Hence we obtain $\boldsymbol{n}[L(R,x)] = 1$ for almost every $x \in \boldsymbol{R}$. By the scaling property, we obtain the desired conclusion.

(ii) Let $t \geq 0$ be fixed. For a non-negative Borel function f, we have

$$\int \mathrm{d}x f(x) P_0^h[L(t,x)] = \int_0^t P_0^h[f(X_s)]\mathrm{d}s = \int_0^t \boldsymbol{n}[f(X_s)h(X_s)]\mathrm{d}s \tag{8.33}$$

$$= \int \mathrm{d}x f(x)h(x) \int_0^t \rho(s,x)\mathrm{d}s \tag{8.34}$$

$$= \int \mathrm{d}x f(x)h(x) P_x(T_{\{0\}} < t). \tag{8.35}$$

Hence we see that $P_0^h[L(t,x)] = h(x)P_x(T_{\{0\}} < t)$ for almost every $x \in \boldsymbol{R}$. Since $t \mapsto P_0^h[L(t,1)]$ is continuous by the monotone convergence theorem, we see, by the scaling property, that $\boldsymbol{R} \setminus \{0\} \ni x \mapsto P_0^h[L(t,x)]$ is continuous. Noting that $L(t,0) = 0$ P_0^h-almost surely, we complete the proof.

(iii) Letting $t \to \infty$ in (ii), we obtain $P_0^h[L(\infty,x)] = h(x)$ by the monotone convergence theorem. □

Now we prove Theorem 8.6.

PROOF OF THEOREM 8.6. Note that

$$\mathscr{E}_{\tau_s}^V = \exp\left\{ -sV(\{0\}) - \sum_{l \in D,\, l \le s} \int_{\{x \neq 0\}} L(R,x)[e_l]V(\mathrm{d}x) \right\} \tag{8.36}$$

where $L(R,x)[e_l]$ is the local time at x of the excursion e_l up to its lifetime. Hence we have $P_0[\mathscr{E}_{\tau_s}^V] = \exp\{-sK_V\}$ where

$$K_V = V(\{0\}) + \boldsymbol{n}\left[1 - \exp\left\{ -\int_{\{x \neq 0\}} L(R,x)V(\mathrm{d}x) \right\} \right]. \tag{8.37}$$

Consequently we have

$$\int_0^\infty P_0[\mathscr{E}_{\tau_s}]\mathrm{d}s = \frac{1}{K_V}. \tag{8.38}$$

(i) If $\int_0^\infty P_0[\mathscr{E}_{\tau_s}^V]\mathrm{d}s = \infty$, then we have $K_V = 0$, which implies that $V = 0$. Hence the assertion is proved by contraposition.

(ii) Suppose that $V((-\varepsilon,\varepsilon)) < \infty$ for $\varepsilon > 0$. Then, by Lemma 8.7, we have

$$\boldsymbol{n}\left[\int_{(-\varepsilon,\varepsilon)} L(R,x)V(\mathrm{d}x) \right] = \int_{(-\varepsilon,\varepsilon)} \boldsymbol{n}[L(R,x)]V(\mathrm{d}x) = V((-\varepsilon,\varepsilon)) < \infty. \tag{8.39}$$

Now we obtain

$$\boldsymbol{n}\left[1-\exp\left\{-\int L(R,x)V(\mathrm{d}x)\right\};\sup_{t\geq 0}|X(t)|<\varepsilon\right] \tag{8.40}$$

$$=\boldsymbol{n}\left[1-\exp\left\{-\int_{(-\varepsilon,\varepsilon)} L(R,x)V(\mathrm{d}x)\right\};\sup_{t\geq 0}|X(t)|<\varepsilon\right] \tag{8.41}$$

$$\leq\boldsymbol{n}\left[\int_{(-\varepsilon,\varepsilon)} L(R,x)V(\mathrm{d}x)\right]<\infty. \tag{8.42}$$

Since $\boldsymbol{n}(\sup_{t\geq 0}|X(t)|\geq\varepsilon)<\infty$, we obtain $K_V<\infty$. Hence the assertion is proved.

(iii) By Lemma 8.7, we obtain

$$P_0^h\left[\int L(\infty,x)V(\mathrm{d}x)\right]=\int h(x)V(\mathrm{d}x)<\infty. \tag{8.43}$$

This implies that

$$P_0^h\left(\int L(\infty,x)V(\mathrm{d}x)<\infty\right)=P_0^h(\mathscr{E}_\infty^V>0)=1, \tag{8.44}$$

which proves $P_0^h[\mathscr{E}_\infty^V]>0$.

(iv) Suppose that $0<\int\{1+h(x)\}V(\mathrm{d}x)<\infty$. Then the assumptions of (i)–(iii) are all satisfied. Noting that $\mathscr{E}_\infty^V\leq 1$, we obtain $0<C_V<\infty$. □

8.4. Proof of Theorems.

PROOF OF THEOREM 8.1. Note that $(\mathscr{E}_t^V)$ and $(\mathscr{E}_t^V 1_{\{T_{\{0\}}>t\}})$ are multiplicative functionals which take values in $[0,1]$. By Theorem 8.6, we may apply Theorem 8.5 (after translating x to the origin) to obtain (8.6) and (8.4). Subtracting both sides of (8.4) from (8.6), we obtain (8.5). □

PROOF OF THEOREM 8.3. The second equalities of (8.9), (8.10) and (8.12) are obvious by Theorem 8.1. The last equality of (8.9) is obvious by Lemma 4.13. The last equality of (8.10) is obtained as follows:

$$\frac{P_x[\mathscr{E}_t^V 1_{\{T_{\{0\}}\leq t\}}]}{\boldsymbol{n}(R>t)}=\int_0^t P_x[\mathscr{E}_s^V;T_{\{0\}}\in\mathrm{d}s]\,\frac{P_0[\mathscr{E}_{t-s}^V]}{\boldsymbol{n}(R>t)} \tag{8.45}$$

$$\overset{t\to\infty}{\longrightarrow}\int_0^\infty P_x[\mathscr{E}_s^V;T_{\{0\}}\in\mathrm{d}s]\mathscr{P}[\mathscr{E}_\infty^V]=C_V P_x[\mathscr{E}_{T_{\{0\}}}^V]. \tag{8.46}$$

Now we obtain the last equality of (8.12) by adding (8.9) and (8.10).

Let $0 \leq s < t$. By the Markov property of (P_x) and by Theorem 8.1, we have

$$P_x\left[\mathscr{E}_s^V 1_{\{T_{\{0\}}>s\}} \frac{P_{X_s}[\mathscr{E}_{t-s}^V; T_{\{0\}} > t-s]}{\boldsymbol{n}(R>t)}\right] = \frac{P_x[\mathscr{E}_t^V; T_{\{0\}} > t]}{\boldsymbol{n}(R>t)} \tag{8.47}$$

$$\overset{t\to\infty}{\longrightarrow} \mathscr{P}_x[\mathscr{E}_\infty^V 1_{\{T_{\{0\}}=\infty\}}] \tag{8.48}$$

$$= h(x) P_x^h[\mathscr{E}_\infty^V]. \tag{8.49}$$

By the Markov property of (P_x^h), we have

$$h(x)P_x^h[\mathscr{E}_\infty^V] = h(x)P_x^h[\mathscr{E}_s^V P_{X_s}^h[\mathscr{E}_\infty^V]] = P_x[\mathscr{E}_s^V 1_{\{T_{\{0\}}>s\}} \varphi_V^1(X_s)]. \tag{8.50}$$

Hence, by Scheffé's lemma, we obtain

$$\frac{P_{X_s}[\mathscr{E}_{t-s}^V; T_{\{0\}} > t-s]}{\boldsymbol{n}(R>t)} \overset{t\to\infty}{\longrightarrow} \varphi_V^1(X_s) \qquad \text{in } L^1(\mathscr{E}_s^V 1_{\{T_{\{0\}}>s\}} \cdot P_x). \tag{8.51}$$

Therefore, for any bounded $\mathscr{F}_s$-measurable functional Z_s, we have

$$\frac{P_x[Z_s\mathscr{E}_t^V; T_{\{0\}} > t]}{\boldsymbol{n}(R>t)} = P_x\left[Z_s\mathscr{E}_s^V 1_{\{T_{\{0\}}>s\}} \frac{P_{X_s}[\mathscr{E}_{t-s}^V; T_{\{0\}} > t-s]}{\boldsymbol{n}(R>t)}\right] \tag{8.52}$$

$$\overset{t\to\infty}{\longrightarrow} P_x[Z_s\mathscr{E}_s^V 1_{\{T_{\{0\}}>s\}} \varphi_V^1(X_s)]. \tag{8.53}$$

Combining this with (8.4), we obtain

$$\mathscr{P}_x[Z_s\mathscr{E}_\infty^V 1_{\{T_{\{0\}}=\infty\}}] = P_x[Z_s\mathscr{E}_s^V 1_{\{T_{\{0\}}>s\}} \varphi_V^1(X_s)]. \tag{8.54}$$

This implies the identity (8.13).

By similar arguments, we have

$$\frac{P_x[Z_s\mathscr{E}_t^V 1_{\{T_{\{0\}}\leq s\}}]}{\boldsymbol{n}(R>t)} = P_x\left[Z_s\mathscr{E}_s^V 1_{\{T_{\{0\}}\leq s\}} \frac{P_{X_s}[\mathscr{E}_{t-s}^V]}{\boldsymbol{n}(R>t)}\right] \tag{8.55}$$

$$\overset{t\to\infty}{\longrightarrow} P_x\Big[Z_s\mathscr{E}_s^V 1_{\{T_{\{0\}}\leq s\}} \varphi_V(X_s)\Big] \tag{8.56}$$

and

$$\frac{P_x[Z_s\mathscr{E}^V_t 1_{\{s<T_{\{0\}}\le t\}}]}{\boldsymbol{n}(R>t)} = P_x\left[Z_s\mathscr{E}^V_s 1_{\{T_{\{0\}}>s\}}\,\frac{P_{X_s}[\mathscr{E}^V_{t-s};T_{\{0\}}\le t-s]}{\boldsymbol{n}(R>t)}\right] \tag{8.57}$$

$$\overset{t\to\infty}{\longrightarrow} P_x\Big[Z_s\mathscr{E}^V_s 1_{\{T_{\{0\}}>s\}}\varphi^2_V(X_s)\Big]. \tag{8.58}$$

Combining these two limits together with (8.5), we obtain (8.14).

The remainder of the proof is now obvious. □

9. Characterisation of non-negative martingales.

For a non-negative $\mathscr{P}$-integrable functional G such that $\mathscr{P}[G]>0$, we define the probability measure P^G on $\mathscr{F}_\infty$ as

$$P^G = \frac{G\cdot\mathscr{P}}{\mathscr{P}[G]}. \tag{9.1}$$

We say that a statement holds *$\mathscr{P}$-almost surely* if it holds P^G-almost surely for some $\mathscr{P}$-integrable functional G such that $G>0$ $\mathscr{P}$-almost everywhere. By the Radon-Nikodym theorem, $\mathscr{P}$-almost sure statement does not depend on the particular choice of such a functional G.

The following theorem is the stable Lévy version of [**9**, Corollary 1.2.6].

THEOREM 9.1. *Let (N_t) be a non-negative $(\mathscr{F}_t,P_0)$-martingale. Then (N_t) is represented as $N_t=M_t(F)$ for some $F\in L^1(\mathscr{P})$ if and only if it holds that*

$$\frac{N_t}{1+h(X_t)} \overset{t\to\infty}{\longrightarrow} F \ \mathscr{P}\text{-almost surely} \quad \text{and} \quad \mathscr{P}[F]=N_0. \tag{9.2}$$

Although it is completely parallel to that of [**9**], we give the proof for completeness of the paper.

LEMMA 9.2. *Let F and G be a non-negative $\mathscr{P}$-integrable functional and suppose that $G>0$ $\mathscr{P}$-almost everywhere. Then it holds that*

$$\frac{M_t(F)}{M_t(G)} = P^G\left[\frac{F}{G}\,\middle|\,\mathscr{F}_t\right]. \tag{9.3}$$

Consequently, it holds that

$$\frac{M_t(F)}{M_t(G)} \overset{t\to\infty}{\longrightarrow} \frac{F}{G} \qquad P^G\text{-almost surely.} \tag{9.4}$$

PROOF. Let Z_t be a non-negative $\mathscr{F}_t$-measurable functional. On the one hand, we have

$$P^G\left[Z_t \cdot \frac{F}{G}\right] = \mathscr{P}[Z_t F] = P_0[Z_t M_t(F)]. \tag{9.5}$$

On the other hand, we have

$$P^G\left[Z_t \cdot \frac{M_t(F)}{M_t(G)}\right] = \mathscr{P}\left[Z_t \cdot \frac{M_t(F)}{M_t(G)} \cdot G\right] = P_0[Z_t M_t(F)]. \tag{9.6}$$

Hence we obtain $P^G[Z_t F/G] = P^G[Z_t M_t(F)/M_t(G)]$, which completes the proof. □

LEMMA 9.3. *Let F be a non-negative $\mathscr{P}$-integrable functional. Then*

$$\frac{M_t(F)}{1+h(X_t)} \overset{t\to\infty}{\longrightarrow} F \qquad \mathscr{P}\text{-almost surely.} \tag{9.7}$$

PROOF. We apply Theorem 7.1 with $f(l) = \mathrm{e}^{-l}$ to see that $G = \mathrm{e}^{-L_\infty}$ is a positive $\mathscr{P}$-integrable functional such that

$$M_t(G) = (1+h(X_t))\mathrm{e}^{-L_t}. \tag{9.8}$$

Hence we obtain

$$\frac{M_t(G)}{1+h(X_t)} \overset{t\to\infty}{\longrightarrow} G \qquad P^G\text{-almost surely.} \tag{9.9}$$

Hence, by Lemma 9.2, we obtain

$$\frac{M_t(F)}{1+h(X_t)} = \frac{M_t(G)}{1+h(X_t)} \cdot \frac{M_t(F)}{M_t(G)} \overset{t\to\infty}{\longrightarrow} G \cdot \frac{F}{G} = F \qquad P^G\text{-almost surely.} \tag{9.10}$$

This completes the proof. □

The following proposition states an interesting representation of any non-negative $(P_0, \mathscr{F}_t)$-supermartingale, a component of which is a certain $(M_t(F))$ martingale.

PROPOSITION 9.4. *Let (N_t) a non-negative $(\mathscr{F}_t, P_0)$-supermartingale.*
(i) *There exists a non-negative $\mathscr{P}$-integrable functional F such that*

$$\frac{N_t}{1+h(X_t)} \overset{t\to\infty}{\longrightarrow} F \qquad \mathscr{P}\text{-almost surely}; \tag{9.11}$$

(ii) *Denote the P_0-almost sure limit of (N_t) as $t \to \infty$ by N_∞. Then (N_t) decomposes uniquely in the following form*:

$$N_t = M_t(F) + P_0[N_\infty|\mathscr{F}_t] + \xi_t \tag{9.12}$$

where:
(iia) *$(M_t(F))$ is a non-negative $(\mathscr{F}_t, P_0)$-martingale such that*

$$M_t(F) \overset{t\to\infty}{\longrightarrow} 0 \ (P_0\text{-a.s.}) \qquad \text{and} \qquad \frac{M_t(F)}{1+h(X_t)} \overset{t\to\infty}{\longrightarrow} F \ (\mathscr{P}\text{-a.s.}); \tag{9.13}$$

(iib) *$(P_0[N_\infty|\mathscr{F}_t])$ is a non-negative uniformly-integrable $(\mathscr{F}_t, P_0)$-martingale with P_0-integrable terminal value N_∞ such that*

$$P_0[N_\infty|\mathscr{F}_t] \overset{t\to\infty}{\longrightarrow} N_\infty \ (P_0\text{-a.s.}) \qquad \text{and} \qquad \frac{P_0[N_\infty|\mathscr{F}_t]}{1+h(X_t)} \overset{t\to\infty}{\longrightarrow} 0 \ (\mathscr{P}\text{-a.s.}); \tag{9.14}$$

(iic) *(ξ_t) is a non-negative $(\mathscr{F}_t, P_0)$-supermartingale such that*

$$\xi_t \overset{t\to\infty}{\longrightarrow} 0 \ (P_0\text{-a.s.}) \qquad \text{and} \qquad \frac{\xi_t}{1+h(X_t)} \overset{t\to\infty}{\longrightarrow} 0 \ (\mathscr{P}\text{-a.s.}). \tag{9.15}$$

PROOF.
(i) Let $G = \mathrm{e}^{-L_\infty}$. For any non-negative $\mathscr{F}_s$-measurable functional Z_s, we see that

$$P^G\left[Z_s \cdot \frac{N_t}{M_t(G)}\right] = \mathscr{P}\left[Z_s \cdot \frac{N_t}{M_t(G)} \cdot G\right] = P_0[Z_s N_t] \le P_0[Z_s N_s]. \tag{9.16}$$

Hence we conclude that $(N_t/M_t(G))$ is a non-negative $(\mathscr{F}_t, P^G)$-supermartingale.

Thus there exists a non-negative $\mathscr{F}_\infty$-measurable functional ζ such that $N_t/M_t(G) \to \zeta$ P^G-almost surely. By Lemma 9.3, we see that

$$\frac{N_t}{1+h(X_t)} = \frac{N_t}{M_t(G)} \cdot \frac{M_t(G)}{1+h(X_t)} \overset{t\to\infty}{\longrightarrow} \zeta G =: F \qquad P^G\text{-almost surely.} \tag{9.17}$$

(ii) For any non-negative $\mathscr{F}_t$-measurable functional Z_t, we have

$$P_0[Z_t M_t(F)] = \mathscr{P}[Z_t F] = P^G[Z_t \zeta]. \tag{9.18}$$

By Fatou's lemma, the last expectation is dominated by

$$\liminf_{u\to\infty} P^G\left[Z_t \cdot \frac{N_u}{M_u(G)}\right] = \liminf_{u\to\infty} P_0[Z_t N_u] \le P_0[Z_t N_t]. \tag{9.19}$$

This proves that $M_t(F) \le N_t$ P_0-almost surely. Now we see that $(\overline{N}_t := N_t - M_t(F))$ is a non-negative $(\mathscr{F}_t, P_0)$-supermartingale. Since $M_t(G) \to 0$ P_0-almost surely as $t \to \infty$, we see that

$$\lim_{t\to\infty} \overline{N}_t = \lim_{t\to\infty} N_t = N_\infty \qquad P_0\text{-almost surely.} \tag{9.20}$$

For any non-negative $\mathscr{F}_t$-measurable functional Z_t, we have

$$P_0[Z_t N_\infty] \le \liminf_{u\to\infty} P_0[Z_t \overline{N}_u] \le P_0[Z_t \overline{N}_t], \tag{9.21}$$

we see that $P_0[N_\infty] < \infty$ and that $(\xi_t := \overline{N}_t - P_0[N_\infty|\mathscr{F}_t])$ is still a non-negative $(\mathscr{F}_t, P_0)$-supermartingale. Now the proof is completed by (i) and by Lemma 9.3. □

Finally, we proceed to prove Theorem 9.1.

PROOF OF THEOREM 9.1. The necessity is immediate from Lemma 9.3. Let us prove the sufficiency.

Let (N_t) be a non-negative $(\mathscr{F}_t, P_0)$-martingale. Then, by Proposition 9.4, we have the decomposition (9.12). Letting $t = 0$, we have

$$N_0 = M_0(F) + P_0[N_\infty] + \xi_0. \tag{9.22}$$

Since $N_0 = M_0(F) = \mathscr{P}[F]$ by the assumption, we have

$$P_0[N_\infty] = \xi_0 = 0. \tag{9.23}$$

This proves that $N_t = M_t(F)$ P_0-almost surely, which completes the proof. □

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