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Sheet number and quandle-colored 2-knot

Dedicated to Professor Akio Kawauchi for his 60th birthday

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Abstract. A diagram of a 2-knot consists of a finite number of compact, connected surfaces called sheets. We prove that if a 2-knot admits a non-trivial coloring by some quandle, then any diagram of the 2-knot needs at least four sheets. Moreover, if a 2-knot admits a non-trivial 5- or 7-coloring, then any diagram needs at least five or six sheets, respectively.

1. Introduction.

The crossing number, $\mathrm{cr}(k)$, of a 1-dimensional knot k is one of the fundamental quantities which reflect the knotting of k. For an odd prime p, we consider the minimal number c_p of $\mathrm{cr}(k)$'s for all p-colorable 1-knots k. In other words, if k admits a non-trivial p-coloring, then it holds that $\mathrm{cr}(k) \geq c_p$, and there is a 1-knot k which realizes the equality. By checking the list of 1-knots, we can easily obtain the following table of c_p's for $3 \leq p \leq 61$.

p	3	5	7	11	13	17	19	23–37	41–61	$\cdots$
c_p	3	4	5	6		7		8	9	$\cdots$

A 2-dimensional knot K is an oriented 2-sphere smoothly embedded in $\boldsymbol{R}^4$. A diagram of K is the generic image under a projection of $\boldsymbol{R}^4$ onto $\boldsymbol{R}^3$—which may have double points, isolated triple points, and isolated branch points—equipped with crossing information specified by breaking lower disks near double points in a similar way to the description of a 1-knot diagram.

For a 2-knot K, there are two kinds of quantities analogous to the crossing number. One is the minimal number of triple points for all diagrams of K, which is called the triple point number [**10**], [**11**]. On the other hand, the sheet number of K, denoted by $\mathrm{sh}(K)$, is the minimal number of broken sheets for all diagrams of

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K [**8**]. Here, a diagram is regarded as a disjoint union of a finite number of connected, compact surfaces, each of which is called a broken sheet or a sheet simply [**5**].

Saito and the author [**8**] gave a lower bound of the sheet number by using a coloring for a 2-knot by a particular quandle, and proved that the spun trefoil has the sheet number four. In this paper, we first generalize the technique to a coloring by any quandle, which allows us to show that the sheet numbers of almost all 2-knots are greater than or equal to four. Let s_p denote the minimal number of $\mathrm{sh}(K)$'s for all p-colorable 2-knots K. Since the spun trefoil admits a non-trivial 3-coloring, we have $s_3 = 4$ immediately.

It is easy to see that, if a 1-knot k is p-colorable, then so is the spinning of k. Moreover, it has a diagram consisting of $\mathrm{cr}(k) + 1$ broken sheets. Hence, it holds that $s_p \leq c_p + 1$ for any p. The second aim of this paper is to study the lower bound of the sheet number by using non-trivial 5- and 7-colorings. As a corollary, we see that the equality $s_p = c_p + 1$ holds for $p = 5$ and 7.

Question. Is there an odd prime $p > 7$ such that $s_p < c_p + 1$?

This paper is organized as follows: In Section 2, we review the definitions of a 2-knot diagram, the fundamental quandle, and a coloring by a quandle. In Section 3, we introduce the notion of an "exclusive" sheet and prepare several lemmas. In Section 4, we prove that if the fundamental quandle is non-trivial, then the sheet number is greater than or equal to four (Theorem 4.2). In Section 5, we give several properties of p-colorings associated with the dihedral quandle of order p. Sections 6 and 7 are devoted to studying the cases of $p = 5$ and 7; in particular, we prove that if a 2-knot have a non-trivial 5- or 7-coloring, then the sheet number is greater than or equal to five or six, respectively (Theorems 6.1 and 7.1). In particular, we see that the spun 4_1-knot and 5_2-knot have the sheet numbers five and six, respectively.

2. Preliminaries.

2.1. 2-knot diagrams.

A 2-*knot* is the image of a smooth embedding of an oriented 2-sphere into 4-space $\boldsymbol{R}^4$. We identify $\boldsymbol{R}^4$ with the product $\boldsymbol{R}^3 \times \boldsymbol{R}$. Let $p : \boldsymbol{R}^4 \to \boldsymbol{R}^3$ and $h : \boldsymbol{R}^4 \to \boldsymbol{R}$ be the projections onto the first and second factors, respectively. By a slight perturbation of a 2-knot K if necessary, we may assume that the image $p(K)$ is generic; that is, any point on $p(K)$ is a regular point, a double point, a triple point, or a branch point. Refer to [**5**] for more details.

A *diagram* of K is such a projection image $p(K)$ where we distinguish between upper and lower disks near double points with respect to the height function h. We indicate crossing information by breaking lower disks in a similar way to a 1-knot diagram. This modification extends to neighborhoods of triple points and branch points naturally. In particular, there are three intersecting disks near a triple point of $p(K)$, called *top*, *middle*, and *bottom* with respect to h, and the middle and bottom disks are broken into two and four pieces in a diagram, respectively. Then a diagram is regarded as a disjoint union of connected, compact surfaces, each of which is called a *broken sheet* or a *sheet* simply. We denote by S_D the set of the sheets of a diagram D. The *sheet number* of K is the minimal number of sheets for all diagrams of K, and denoted by $\mathrm{sh}(K)$ (cf. [**8**]).

In Figure 1, we give an example of a diagram D, which is divided into five blocks. It is easy to see that D consists of three sheets.

2.2. Lower graphs.

Let D be a diagram of K. The preimage of a double point by the restriction $p|_K : K \to \boldsymbol{R}^3$ consists of a pair of upper and lower points with respect to h. We denote by U_D and L_D the closures of the sets of upper and lower points, respectively. By identifying K with a 2-sphere S^2, the sets U_D and L_D are regarded as graphs in S^2. Each vertex v of the graphs is of degree 1 such that $p(v)$ is a branch point, or of degree 4 such that $p(v)$ is a triple point. We call U_D and L_D the *upper* and *lower graphs* associated with D, respectively. By definition, there is a correspondence between the sheets of D and the connected regions of the complement $S^2 \setminus L_D$ of the lower graph. Hence, we often identify a sheet $A \in S_D$ with a connected region of $S^2 \setminus L_D$.

In Figure 1, we also illustrate the upper and lower graphs U_D and L_D for D by dotted and solid lines, respectively. We remark that the complement $S^2 \setminus L_D$ consists of three connected regions.

2.3. Types of double points.

Assume that a 2-knot K is oriented, and hence, so is a diagram D. Let $B \in S_D$ be the upper sheet near a double point P of a diagram D. We take a normal vector to B representing the orientation of B. Let A and $C \in S_D$ be the lower sheets divided by B such that the vector points from A to C. We denote the *type* of P by $(A \overset{B}{\to} C)$. The orientation of the corresponding edge of L_D is indicated by a normal vector pointing from A to C. We give the label (B) to the edge so that L_D is regarded as an oriented, labeled graph. See Figure 2. If a double point is of type $(A \overset{B}{\to} C)$ or $(C \overset{B}{\to} A)$, we use the notation $(A \overset{B}{\leftrightarrow} C)$.

2.4. The fundamental quandle.

A non-empty set Q with a binary operation $*$ is a *quandle* [**6**], [**7**] if it satisfies

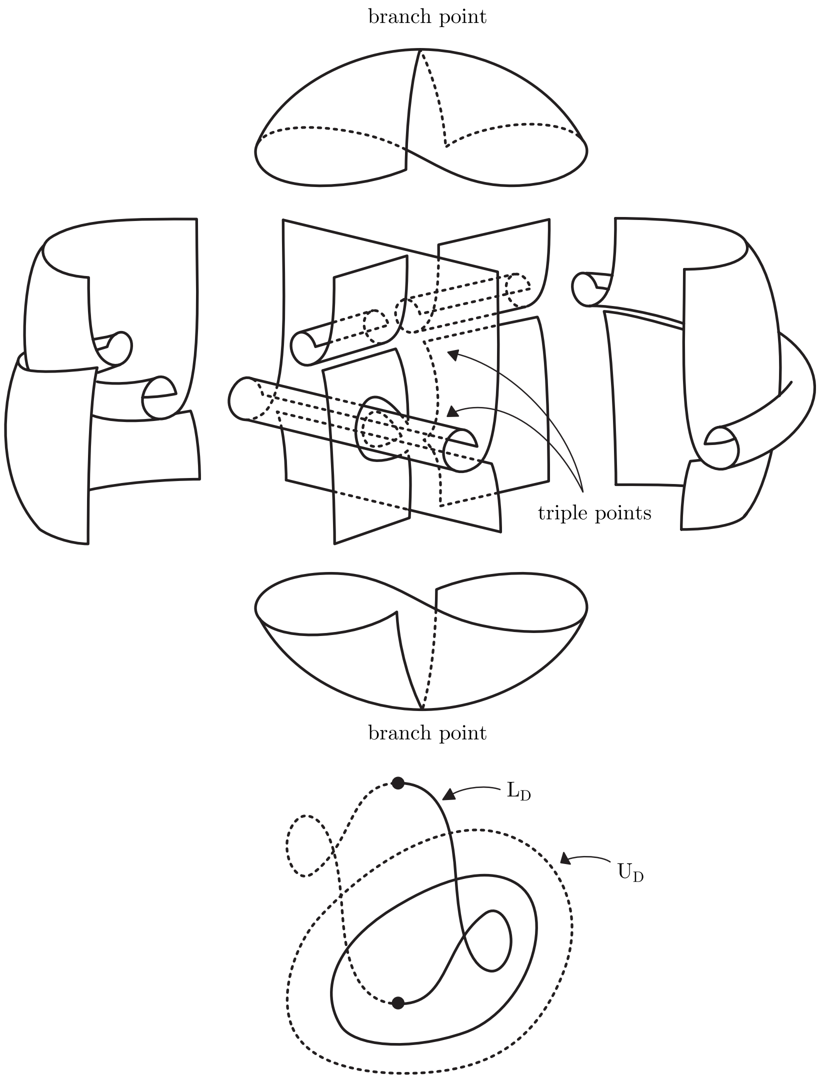

Figure 1. A diagram D with graphs U_D and L_D.

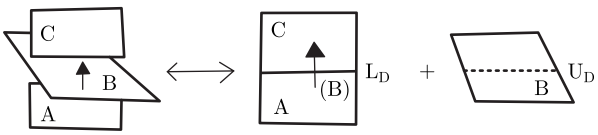

Figure 2. The orientation and label of an edge of L_D.

the following three conditions:

(i) For any $a \in Q$, it holds that $a * a = a$.

(ii) For any a and $b \in Q$, there is a unique element x such that $x * b = a$.

(iii) For any element a, b, and $c \in Q$, it holds that $(a * b) * c = (a * c) * (b * c)$.

The map $\varphi_a : Q \to Q$ defined by $\varphi_a(x) = x * a$ is a quandle isomorphism. We use the notation $\varphi_a^n(x) = x * a^n$ for $n \in \boldsymbol{Z}$ formally. In particular, the element x in the condition (ii) is denoted by $a * b^{-1}$. Then it holds that $a * a^{-1} = a$ and $a * (b * c) = ((a * c^{-1}) * b) * c$.

For an odd prime p, the *dihedral quandle* of order p, denoted by R_p, is the set $\boldsymbol{Z}_p = \{0, 1, \ldots, p-1\}$ equipped with the binary operation $a * b \equiv 2b - a \pmod{p}$. The quandle with a single element is called *trivial*.

The *fundamental quandle* of a 2-knot K, denoted by $Q(K)$, is the quandle with a finite representation such that the generators correspond to the sheets of D, and a relation $a * b = c$ is given at any double point of type $(A \xrightarrow{B} C)$, where a, b, and c are the elements of $Q(K)$ corresponding to A, B, and C, respectively.

Consider the diagram D of a 2-knot K in Figure 1 again. By choosing an orientation of K, we have the oriented, labeled lower graph L_D as shown in Figure 3. Let A, B, and C be the connected regions of $S^2 \setminus L_D$ as in the figure, and a, b, and c the corresponding elements of $Q(K)$. The fundamental quandle $Q(K)$ is generated by a, b, and c, and the relations are given by

$$a * b = c, \;\; a * b = a, \;\; \text{and } a * b = b$$

combined with the trivial relations $a * a = a$ and $b * b = b$. It is easy to see that $a = b = c$, and hence $Q(K)$ is trivial.

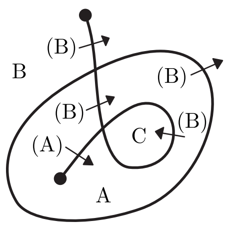

Figure 3. A lower graph L_D.

2.5. Colorings by a quandle.

A *coloring* for a diagram D by a quandle Q_0 is a map

$$f : S_D = \{\text{the sheets of } D\} \to Q_0$$

which satisfies $f(A) * f(B) = f(C)$ at every double point of type $(A \xrightarrow{B} C)$ (cf. [**3**]).

In particular, a coloring by the dihedral quandle R_p is called a *p-coloring*. The element $f(X)$ is called the *color* of a sheet X. We say that a coloring f is *trivial* if it is a constant map. By definition, a coloring $f : S_D \to Q_0$ naturally extends to a quandle homomorphism $\tilde{f} : Q(K) \to Q_0$ such that $\tilde{f}(x) = f(X)$ holds for any generator x of $Q(K)$ corresponding to a sheet X.

Recall that a diagram D is obtained from the projection image $p(K)$ equipped with crossing information. Choose any connected region Δ_0 of $\boldsymbol{R}^3 \setminus p(K)$ and color it by any element $a \in Q_0$. For any connected region Δ of $\boldsymbol{R}^3 \setminus p(K)$, we take a path α from Δ_0 to Δ such that the intersections of α and D are transverse regular points. Let $x_1, x_2, \ldots, x_n$ be the colors of the sheets which α meets in this order, and ε_i the sign of the ith intersection such that $\varepsilon_i = +1$ if and only if the orientation of α matches with that of the ith sheet $(1 \le i \le n)$. Then we color the region Δ by the element

$$\varphi_{x_n}^{\varepsilon_n} \circ \ldots \varphi_{x_2}^{\varepsilon_2} \circ \varphi_{x_1}^{\varepsilon_1}(a) = (\cdots((a * x_1^{\varepsilon_1}) * x_2^{\varepsilon_2}) * \cdots) * x_n^{\varepsilon_n}.$$

See Figure 4, where the colors of regions are surrounded by squares. This coloring for $\boldsymbol{R}^3 \setminus p(K)$ is well-defined independently of a particular choice of a path α, which is called a *shadow coloring* (cf. [**4**]).

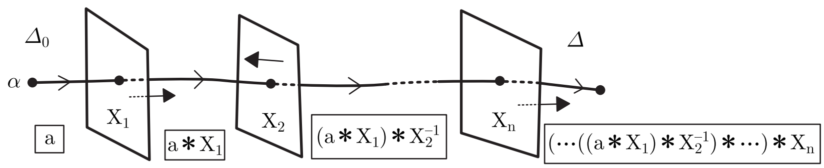

Figure 4. A shadow coloring for $\boldsymbol{R}^3 \setminus p(K)$.

2.6. Reduced lower graphs.

We say that a coloring $f : S_D \to Q_0$ is *separating* if f is injective, that is, any different sheets of D have different colors.

For a (possibly non-separating) coloring f, we denote by $L_D(f)$ the subgraph of L_D such that each edge of $L_D(f)$ is adjacent to the regions of $S^2 \setminus L_D$ with different colors. Equivalently, corresponding double point is of type $(X \overset{Y}{\leftrightarrow} Z)$ with $f(X) \neq f(Z)$. We call $L_D(f)$ the *reduced lower graph* associated with f.

Consider the complement $S^2 \setminus L_D(f)$. Since $L_D(f) \subset L_D$, it holds that

$$\#\{\text{the connected regions of } S^2 \setminus L_D(f)\} \le \# S_D,$$

and in particular, the equality holds if f is separating. We give a color $f(X)$ to the region of $S^2 \setminus L_D(f)$ containing a sheet $X \in S_D$, and a label $(f(Y))$ to an edge of

$L_D(f)$ with a label (Y). The reduced lower graph $L_D(f)$ is often more convenient than L_D.

3. Exclusive sheets with respect to a coloring.

Let $f : S_D \to Q_0$ be a coloring for a diagram D by a quandle Q_0. For a sheet $A \in S_D$, we say that A is *exclusive* with respect to f if any type $(X \overset{A}{\leftrightarrow} Y)$ whose upper sheet is A satisfies $f(X) = f(Y) = f(A)$. We remark that X and Y may be different from A.

For a pair of sheets A and $B \in S_D$, we say that B is A-*exclusive* with respect to f if D has a double point of type $(A \overset{B}{\leftrightarrow} X)$ for some $X \in S_D$ and any type $(Y \overset{B}{\leftrightarrow} Z)$ whose upper sheet is B satisfies $Y = A$, $Z = A$, or $f(Y) = f(Z) = f(B)$.

Lemma 3.1. *Let A be an exclusive sheet with $f(A) = a$. Consider the shadow coloring such that a region Δ_0 of $\boldsymbol{R}^3 \setminus p(K)$ adjacent to A have the color a. Then any region Δ adjacent to A have the same color a.*

Proof. Choose a pair of points P_0 and P on A adjacent to Δ_0 and Δ, respectively. We take a path on A connecting from P_0 to P. By assumption, the regions near P_0 have the color a. Since every double point on α is of type $(X \overset{A}{\leftrightarrow} Y)$ with $f(X) = f(Y) = a$, the color a of Δ_0 extends along α so that the regions near P have the same color a by the equation $a * a^{\pm 1} = a$. Hence, Δ has the color a. $\square$

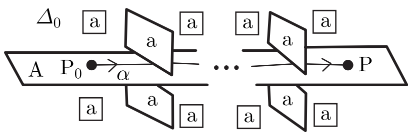

Figure 5. Lemma 3.1.

We call the shadow coloring in Lemma 3.1 *the shadow coloring associated with A*.

Lemma 3.2. *Suppose that D has a pair of sheets A and B such that A is exclusive and B is A-exclusive with $f(A) = a$ and $f(B) = b$. Then, for the shadow coloring associated with A, any region adjacent to B have the color $a * b^n$ for some $n \in \boldsymbol{Z}$.*

Proof. We take a point P on B. It is sufficient to prove that a region near P has the color $a * b^n$ for some $n \in \boldsymbol{Z}$. Since B is A-exclusive, D has a double point P_0 of type $(A \overset{B}{\leftrightarrow} X)$ for some $X \in S_D$. Let α be a path on the sheet B from P_0 to P.

We may assume that every double point on the interior of α is of type $(Y \overset{B}{\leftrightarrow} Z)$ with $f(Y) = f(Z) = b$. In fact, if there are double points of type $(A \overset{B}{\leftrightarrow} X)$ on the interior of α, we take the nearest one from P among them as P_0 again. See Figure 6.

Since A is exclusive, the regions adjacent to A near P_0 have the color a. By extending it along α, it is easy to see that the regions near P have the colors $a * b^n$ and $a * b^{n+1}$ for some $n \in \boldsymbol{Z}$. □

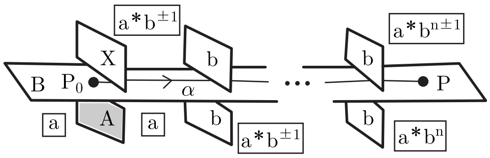

Figure 6. Lemma 3.2.

The following is a generalization of the property used in [**8**].

Lemma 3.3. *Suppose that D has a pair of sheets A and B such that A is exclusive and B is A-exclusive with $f(A) = a$ and $f(B) = b$. If $a \neq b$, then D has no double point of type $(A \overset{X}{\leftrightarrow} B)$ for any $X \in S_D$.*

Proof. Assume that D has a double point of type $(A \overset{X}{\leftrightarrow} B)$ for some $X \in S_D$. The double point gives the relation $a * x^{\pm 1} = b$, where $x = f(X)$. On the other hand, consider the shadow coloring associated with A. Then, by Lemmas 3.1 and 3.2, we have the relation $a * x^{\pm 1} = a * b^n$ for some $n \in \boldsymbol{Z}$. Hence, we obtain $a * b^n = b$, or equivalently $a = b$, which contradicts to the assumption. □

For a triplet of sheets A_1, A_2, and $B \in S_D$, we say that B is $A_1 \cup A_2$-*exclusive* associated with f if D has a double point of type $(A_1 \overset{B}{\leftrightarrow} X)$ or $(A_2 \overset{B}{\leftrightarrow} X)$ for some $X \in S_D$ and any type $(Y \overset{B}{\leftrightarrow} Z)$ whose upper sheet is B satisfies $Y = A_1$, $Y = A_2$, $Z = A_1$, $Z = A_2$, or $f(Y) = f(Z) = f(B)$.

Lemma 3.4. *Suppose that D has a triplet of sheets A_1, A_2, and $B \in S_D$ such that A_1 and A_2 are exclusive and B is $A_1 \cup A_2$-exclusive with $f(A_1) = f(A_2) = a$ and $f(B) = b$. If A_1 and A_2 are adjacent along L_D and $a \neq b$, then D has no double point of type $(A_1 \overset{X}{\leftrightarrow} B)$ nor $(A_2 \overset{X}{\leftrightarrow} B)$ for any $X \in S_D$.*

Proof. Since A_1 and A_2 are adjacent, there is a double point of type $(A_1 \overset{Y}{\leftrightarrow} A_2)$ for some $Y \in S_D$. Hence, the shadow coloring associated with A_1 is coincident with that with A_2. Then any region adjacent to B have the color $a * b^n$ for some $n \in \boldsymbol{Z}$ similarly to Lemma 3.2. Moreover, if D has a double point of type $(A_1 \overset{X}{\leftrightarrow} B)$ or $(A_2 \overset{X}{\leftrightarrow} B)$, then the relations around the double point give $a = b$ similarly to Lemma 3.3, which contradicts to the assumption. □

LEMMA 3.5. *Suppose that D has a triplet of sheets A, B, and C such that A is exclusive, B is A-exclusive, and C is B-exclusive with $f(A) = a$, $f(B) = b$, and $f(C) = c$. If $a * b^n \neq c$ for any $n \in \boldsymbol{Z}$, then D has no double point of type $(A \overset{X}{\leftrightarrow} C)$ for any $X \in S_D$.*

PROOF. Assume that D has a double point P_0 of type $(A \overset{X}{\leftrightarrow} C)$ for some $X \in S_D$. Since C is B-exclusive, D has also a double point P of type $(B \overset{C}{\leftrightarrow} Y)$ for some $Y \in S_D$. Let α be a path on C from P_0 to P. We may assume that every double point on the interior of α is of type $(Z \overset{C}{\leftrightarrow} W)$ with $f(Z) = f(W) = c$. If not, we take the nearest double point of type $(B \overset{C}{\leftrightarrow} Y)$ from P_0 as P again. See Figure 7.

Since A is exclusive, by Lemma 3.1, the regions adjacent to A near P_0 have the color a, and hence, the regions adjacent to C near P_0 have the color c. By extending it along α, we have $a * b^n = c$ near P for some $n \in \boldsymbol{Z}$ by Lemma 3.2, which contradicts to the assumption. □

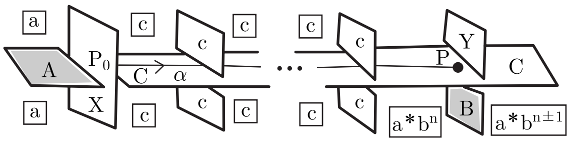

Figure 7. Lemma 3.5.

The following lemma will be used in the last of this paper.

LEMMA 3.6. *Suppose that D has a triplet of sheets A, B, and C such that A is exclusive, B is A-exclusive, and C is B-exclusive with $f(A) = a$, $f(B) = b$, and $f(C) = c$. If there is an element $x \in Q_0$ such that $a * b^l \neq (a * x) * c^m$ for any $l, m \in \boldsymbol{Z}$, then D has no double point of type $(B \overset{X}{\rightarrow} C)$ with $f(X) = x$. Similarly, if there is an element $x \in Q_0$ such that $a * b^l \neq (a * x^{-1}) * c^m$ for any $l, m \in \boldsymbol{Z}$, then D has no double point of type $(C \overset{X}{\rightarrow} B)$ with $f(X) = x$.*

PROOF. Assume that D has a double point P_0 of type $(B \overset{X}{\rightarrow} C)$ with $f(X) = x$. Similarly to the proof of Lemma 3.5, we take a double point P of type

$(B \overset{C}{\leftrightarrow} Y)$ and a path α from P_0 to P. We may assume that every double point on the interior of α is of type $(Z \overset{C}{\leftrightarrow} W)$ with $f(Z) = f(W) = c$.

Since B is A-exclusive, by Lemma 3.2, a region adjacent to B near P_0 has the color $a * b^n$ for some $n \in \boldsymbol{Z}$, and hence, a region adjacent to C near P_0 has the color $(a * b^n) * x$. It is not difficult to see that, by using the relation $b * x = c$ given at P_0, we have $(a * b^n) * x = (a * x) * c^n$. By extending it along α, we have $a * b^l = (a * x) * c^m$ near P for some $l, m \in \boldsymbol{Z}$ by Lemma 3.2, which contradicts to the assumption. The case $(C \overset{X}{\rightarrow} B)$ is similarly proved. $\square$

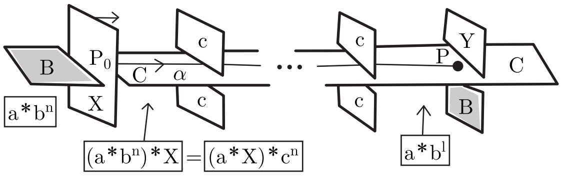

Figure 8. Lemma 3.6.

4. A diagram with three sheets.

Let D be a diagram of an oriented 2-knot K. We first consider the case that D consists of one or two sheets as follows.

Lemma 4.1 ([**8**]). *If D consists of at most two sheets, then $Q(K)$ is trivial.*

Proof. If D consists of a sheet, then $Q(K)$ is generated by the single element corresponding to the sheet. Hence, $Q(K)$ is trivial.

Assume that D consists of two sheets A and B. Let a and $b \in Q(K)$ be the elements corresponding to A and B, respectively. Since $S^2 \setminus L_D = A \cup B$ for the lower graph L_D, there is an edge of L_D whose adjacent regions are A and B. It corresponds to a double point of D of type $(A \overset{X}{\leftrightarrow} B)$ for some $X \in S_D = \{A, B\}$. Then we have the relation $a * a^{\pm 1} = b$ or $a * b^{\pm 1} = b$ with respect to $X = A$ or $X = B$, each of which is equivalent to $a = b$. Hence, $Q(K)$ is trivial. $\square$

The aim of this section is to prove the following.

Theorem 4.2. *If $Q(K)$ is non-trivial, then* $\mathrm{sh}(K) \geq 4$.

Theorem 4.2 follows immediately from Lemma 4.1 and Proposition 4.3 as below.

Proposition 4.3. *If D consists of three sheets, then $Q(K)$ is trivial.*

In the rest of this section, we assume that D consists of three sheets A, B, and C, and denote by a, b, and c the corresponding elements of $Q(K)$, respectively.

LEMMA 4.4. *If at least two of a, b, and c are the same in $Q(K)$, then $Q(K)$ is trivial.*

PROOF. We may assume that $b = c$. Since $S^2 \setminus L_D = A \cup B \cup C$, there is an edge of L_D whose adjacent regions are A and B, or A and C. It corresponds to a double point of type $(A \overset{X}{\leftrightarrow} B)$ or $(A \overset{X}{\leftrightarrow} C)$ for some $X \in S_D = \{A, B, C\}$. Then we have the relation $a * a^{\pm 1} = b$ or $a * b^{\pm 1} = b$, each of which is equivalent to $a = b$. Hence, $Q(K)$ is trivial. $\square$

LEMMA 4.5. *If D has a double point of type $(X \overset{X}{\leftrightarrow} Y)$ for some $X, Y \in S_D$ with $X \neq Y$, then $Q(K)$ is trivial.*

PROOF. We may assume that $X = A$ and $Y = B$. Then we have the relation $a * a^{\pm 1} = b$, which is equivalent to $a = b$. Hence, $Q(K)$ is trivial by Lemma 4.4. $\square$

LEMMA 4.6. *If D has a pair of double points of type*

$$(X \overset{Y}{\leftrightarrow} X) \text{ and } (X \overset{Y}{\leftrightarrow} Z)$$

for some $X, Y, Z \in S_D$ with $\{X, Y, Z\} = S_D$, then $Q(K)$ is trivial.

PROOF. We may assume that $X = A$, $Y = B$, and $Z = C$. The first double point gives the relation $a * b = a$, or equivalently, $a * b^{-1} = a$. The second double point gives the relation $a * b = c$ or $a * b^{-1} = c$. Hence, we have $a = c$, and $Q(K)$ is trivial by Lemma 4.4. $\square$

LEMMA 4.7. *If D has a triplet of double points of type*

$$(X \overset{Y}{\leftrightarrow} Z), \ (Y \overset{X}{\leftrightarrow} Z), \text{ and } (V \overset{W}{\leftrightarrow} V)$$

for some $X, Y, Z \in S_D$ with $\{X, Y, Z\} = S_D$ and some $V, W \in S_D$ with $V \neq W$, then $Q(K)$ is trivial.

PROOF. We may assume that $X = A$, $Y = B$, and $Z = C$. Then the types $(A \overset{B}{\leftrightarrow} C)$ and $(B \overset{A}{\leftrightarrow} C)$ give the relations $a * b = c$ or $a * b^{-1} = c$, and $b * a = c$ or $b * a^{-1} = c$, respectively. By Lemma 4.6, $Q(K)$ is trivial for

$$(V \overset{W}{\leftrightarrow} V) = (A \overset{B}{\leftrightarrow} A), \ (C \overset{B}{\leftrightarrow} C), \ (B \overset{A}{\leftrightarrow} B), \text{ or } (C \overset{A}{\leftrightarrow} C).$$

If $(V \overset{W}{\leftrightarrow} V) = (A \overset{C}{\leftrightarrow} A)$, then we have the relation $a * c = a$. Since $c = b * a^{\pm 1}$, it holds that

$$\begin{aligned} a * (b * a^{\pm 1}) = a &\Leftrightarrow (a * a^{\pm 1}) * (b * a^{\pm 1}) = a \\ &\Leftrightarrow (a * b) * a^{\pm 1} = a \\ &\Leftrightarrow a * b = a \Leftrightarrow a * b^{-1} = a. \end{aligned}$$

Since $a * b^{\pm 1} = c$, we have $a = c$. Hence, $Q(K)$ is trivial by Lemma 4.4. Similarly, if $(V \overset{W}{\leftrightarrow} V) = (B \overset{C}{\leftrightarrow} B)$, then we have $b = c$, and hence, $Q(K)$ is trivial. $\square$

Each vertex of the lower graph L_D is of degree 1 or 4 corresponding to a branch point or a triple point, respectively. In particular, the regions of $S^2 \setminus L_D$ near a vertex of degree 4 correspond to the bottom sheets. We draw a picture of a neighborhood of such a vertex instead of that of a triple point as shown in Figure 9. Here, we remove small segments from a diagonal pair of edges connecting to the vertex which correspond to the middle sheets.

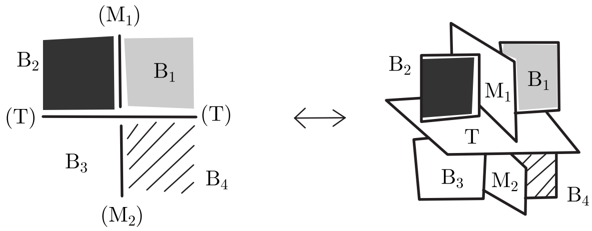

Figure 9. A vertex of degree 4 and a triple point.

Consider a triple points of D whose bottom sheets are not all the same. We divide such triple points into seven types as shown in Figure 10, where $\{X, Y, Z\} = S_D$. The meaning of "degrees" in the figure will be explained in the proof of Proposition 4.3.

(i) Each pair of the bottom sheets on the same side of the top sheet is the same.
(ii) Each pair of the bottom sheets on the same side of the middle sheets is the same.
(iii) Three of the bottom sheets are the same, and the other is different.
(iv) A pair of the bottom sheets on the same side of the middle sheets is the same, and the other two are different.
(v) A pair of the bottom sheets on the same side of the top sheet is the same, and the other two are different.

(vi) Each diagonal pair of the bottom sheets is the same.
(vii) A diagonal pair of the bottom sheets is the same, and the other two are different.

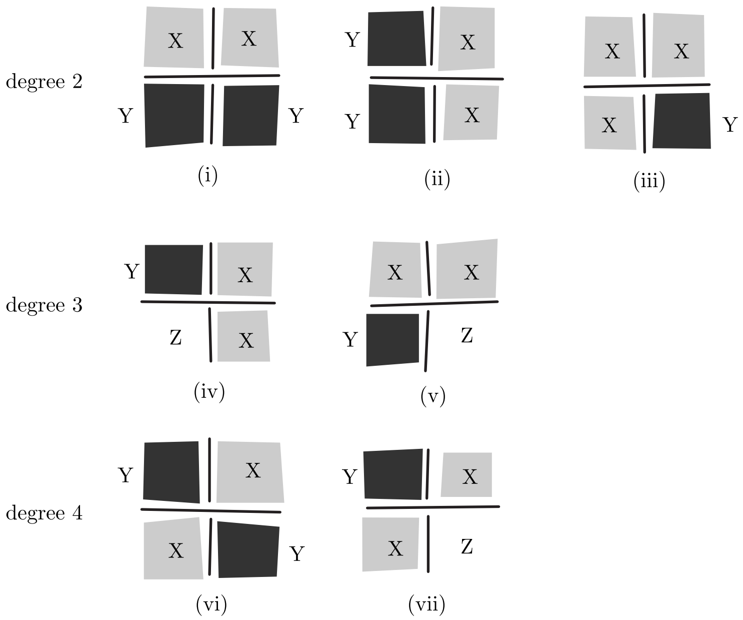

Figure 10. Vertices of degree 2, 3, and 4 of $L_D(f_0)$.

LEMMA 4.8. *If D has a triple point of type* (iii)*,* (v)*, or* (vii)*, then $Q(K)$ is trivial.*

PROOF. (iii) If the top sheet is X or Y, then there is a double point of type $(X \overset{X}{\leftrightarrow} Y)$ or $(X \overset{Y}{\leftrightarrow} Y)$, respectively. Hence, $Q(K)$ is trivial by Lemma 4.5. If the top sheet is Z, then there is a pair of double points of types $(X \overset{Z}{\leftrightarrow} X)$ and $(X \overset{Z}{\leftrightarrow} Y)$. Hence, $Q(K)$ is trivial by Lemma 4.6.

(v) If the top sheet is X, Y, or Z, then there is a double point of type $(X \overset{X}{\leftrightarrow} Y)$, $(X \overset{Y}{\leftrightarrow} Y)$, or $(X \overset{Z}{\leftrightarrow} Z)$, respectively. Hence, $Q(K)$ is trivial by Lemma 4.5.

(vii) The proof is similar to (v). □

LEMMA 4.9. *D has no triple point of type* (vi)*.*

PROOF. Assume that D has a triple point of type (vi). Then it is easy to find a pair of simple closed circles on S^2 with a single intersection. More precisely, the circles are contained in the regions X and Y, respectively, except the intersection at the vertex of degree 4 corresponding to the triple point. Hence, we have a contradiction. $\square$

We are ready to prove Proposition 4.3.

Proof of Proposition 4.3. Assume that $Q(K)$ is non-trivial. We take the map

$$f_0 : S_D = \{A, B, C\} \to Q(K)$$

defined by $f_0(A) = a$, $f_0(B) = b$, and $f_0(C) = c$. By definition, f_0 is a coloring for D by $Q(K)$. Moreover, it is separating by Lemma 4.4.

Let us consider the reduced lower graph $L_D(f_0)$ associated with f_0. Since the edge of L_D connecting to a vertex of degree 1 is adjacent to a single region, it is eliminated in $L_D(f_0)$. Similarly, a vertex of degree 4 of L_D, where the adjacent four regions are the same, is also eliminated. Hence, each vertex of $L_D(f_0)$ is of degree 2, 3, or 4, which corresponds to a triple point of type (i)–(iii), (iv) or (v), or (vi) or (vii), respectively. See Figure 10 again.

Since $Q(K)$ is non-trivial, it follows by Lemmas 4.8 and 4.9 that each vertex of $L_D(f_0)$ is of degree 2 corresponding to type (i) or (ii), or of degree 3 corresponding to (iv). In particular, the pair of edges connecting to a vertex of degree 2 have the same orientation and label. Hence, we can ignore such a vertex by identifying the pair of edges with a single one so that $L_D(f_0)$ is a graph whose vertices are of degree 3. We remark that $L_D(f_0)$ may have circle components each of which is regarded as an edge with no vertex.

Since f_0 is separating, the complement $S^2 \setminus L_D(f_0)$ consists of three regions. Hence, $L_D(f_0)$ is one of the graphs (a) and (b) as shown in Figure 11, where the regions A, B, and C of $S^2 \setminus L_D(f_0)$ are colored by a, b, and c, respectively. We remark that, by Lemma 4.5, the edges of $L_D(f_0)$ between A and B, B and C, and C and A are colored by c, a, and b, respectively.

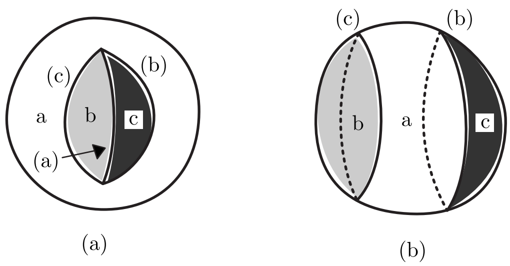

Figure 11. Reduced graphs (a) and (b).

Graph (a): We consider the case that $L_D(f_0)$ is a θ-graph embedded in S^2. Recall that a vertex of degree 3 corresponds to a triple point of type (iv). We may assume that two bottom sheets of such a triple point are colored by a. Then the top sheet is also colored by a. See Figure 12. It is easy to find a pair of double points of type $(B \overset{A}{\to} C)$ and $(C \overset{A}{\to} B)$. Hence, there are at least two edges of $L_D(f_0)$ each of which is adjacent to B and C. This is a contradiction.

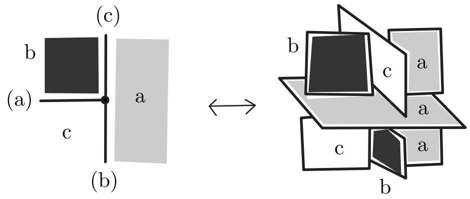

Figure 12. A trivalent vertex and a triple point of type (iv).

Graph (b): We consider the case that $L_D(f_0)$ is a disjoint union of a pair of circles. We may assume that the regions B and C are adjacent to A as shown in Figure 11. It is easy to see that D have a pair of double points of type $(A \overset{B}{\leftrightarrow} C)$ and $(A \overset{C}{\leftrightarrow} B)$, and never have a double point of type $(B \overset{A}{\leftrightarrow} C)$.

On the other hand, by Lemma 4.7, D has no double point of type $(X \overset{Y}{\to} X)$ for any $X, Y \in S_D$ with $X \neq Y$. Hence, any double point of D is of type

$$(A \overset{B}{\leftrightarrow} C),\ (A \overset{C}{\leftrightarrow} B),\ (A \overset{A}{\leftrightarrow} A),\ (B \overset{B}{\leftrightarrow} B),\ \text{or } (C \overset{C}{\leftrightarrow} C).$$

In particular, A is exclusive and B is A-exclusive. Since D has a double point of type $(A \overset{C}{\leftrightarrow} B)$, we have a contradiction to Lemma 3.3. $\square$

5. Properties of p-colorings.

Let D be a diagram of a 2-knot K and p an odd prime. Recall that the dihedral quandle R_p is the set $\boldsymbol{Z}_p = \{0, 1, \ldots, p-1\}$ with the binary operation $a * b = 2b - a$. Hence, a p-coloring is regarded as a map

$$f : S_D = \{\text{the sheets of } D\} \to \boldsymbol{Z}_p$$

such that $f(X) + f(Z) = 2f(Y)$ holds at every double point of type $(X \overset{Y}{\leftrightarrow} Z)$. We say that a 2-knot K is *p-colorable* if a diagram D of K has a non-trivial p-coloring. For $a, b \in \boldsymbol{Z}_p$, the map $X \mapsto af(X) + b$ for $X \in S_D$ is also a p-coloring, which is denoted by $af + b$.

Lemma 5.1. *Let X and $Y \in S_D$ be a pair of sheets of D. If there is a p-coloring f with $f(X) \neq f(Y)$, then, for any different colors a and $b \in \boldsymbol{Z}_p$, there is a p-coloring f' with $f'(X) = a$ and $f'(Y) = b$.*

Proof. Put $x = f(X)$ and $y = f(Y)$. Then the map $f' = \frac{b-a}{y-x}(f - x) + a$ satisfies $f'(X) = a$ and $f'(Y) = b$. $\square$

Let $\mathrm{Im}(f)$ be the image of f, that is, the set of the colors of the sheets of D. We construct a graph $P_D(f)$ associated with f such that

(i) the set of the vertices is $\mathrm{Im}(f)$, and

(ii) two different vertices x and y are connected by an edge if and only if there is a double point of type $(X \overset{A}{\leftrightarrow} Y)$ with $x = f(X)$ and $y = f(Y)$.

We call $P_D(f)$ the *pallet graph* of f. We label the edge in (ii) by $\frac{x+y}{2} \in \boldsymbol{Z}_p$.

Lemma 5.2. *Let f be a p-coloring for D.*

(i) *The pallet graph $P_D(f)$ is connected.*

(ii) *If x and y are connected by an edge, then $\frac{x+y}{2} \in \mathrm{Im}(f)$.*

(iii) *x, y, and $\frac{x+y}{2}$ are mutually different in* (ii).

Proof.

(i) Recall that S_D is identified with the set of the complementary regions of $S^2 \setminus L_D$. By definition, $x \neq y \in \mathrm{Im}(f)$ are connected if and only if there are adjacent regions colored by x and y. Since S^2 is connected, so is $P_D(f)$.

(ii) There is a double point of type $(X \overset{A}{\leftrightarrow} Y)$ with $x = f(X)$ and $y = f(Y)$. Hence, we have $\frac{x+y}{2} = f(A) \in \mathrm{Im}(f)$.

(iii) Put $a = \frac{x+y}{2}$. Then $2a = x + y$ in $\boldsymbol{Z}_p$. If two of x, y, and a are the same, then it follows $x = y = a$. $\square$

Lemma 5.3. *If f is a non-trivial p-coloring for $p \geq 5$, then $\#\mathrm{Im}(f) \geq 4$.*

Proof. If $\#\mathrm{Im}(f) = 1$, then f is trivial. If $\#\mathrm{Im}(f) = 2$, then we may assume that $\mathrm{Im}(f) = \{0, 1\}$ by Lemma 5.1. Since 0 and 1 are connected by an edge by Lemma 5.2(i), we have $\frac{0+1}{2} \in \mathrm{Im}(f)$ by Lemma 5.2(ii). It contradicts to Lemma 5.2(iii).

Assume that $\#\mathrm{Im}(f) = 3$. Also, we may assume that $0, 1 \in \mathrm{Im}(f)$ and they are connected by an edge. Then we have $\mathrm{Im}(f) = \{0, 1, \frac{1}{2}\}$. Since $P_D(f)$ is connected, there is an edge between 0 and $\frac{1}{2}$, or 1 and $\frac{1}{2}$. If the vertex $\frac{1}{2}$ is connected by 0, then we have $\frac{1}{2}(0 + \frac{1}{2}) = \frac{1}{4} = 1$, which implies that $p = 3$. Similarly, if the $\frac{1}{2}$ is connected by 1, then we have $\frac{1}{2}(1 + \frac{1}{2}) = \frac{3}{4} = 0$, which also implies that $p = 3$. $\square$

Recall that each edge of the lower graph L_D and its reduced lower graph

$L_D(f)$ is labeled by (X) where X is the upper sheet of the corresponding double point. The *color of the edge* is $f(X) \in \boldsymbol{Z}_p$. By definition, each vertex of $L_D(f)$ is of degree 2, 3, or 4. In Figure 13, we illustrate eight types of triple points which give the vertices of $L_D(f)$, where x, y, z, and $w \in \boldsymbol{Z}_p$ are mutually different colors of bottom sheets.

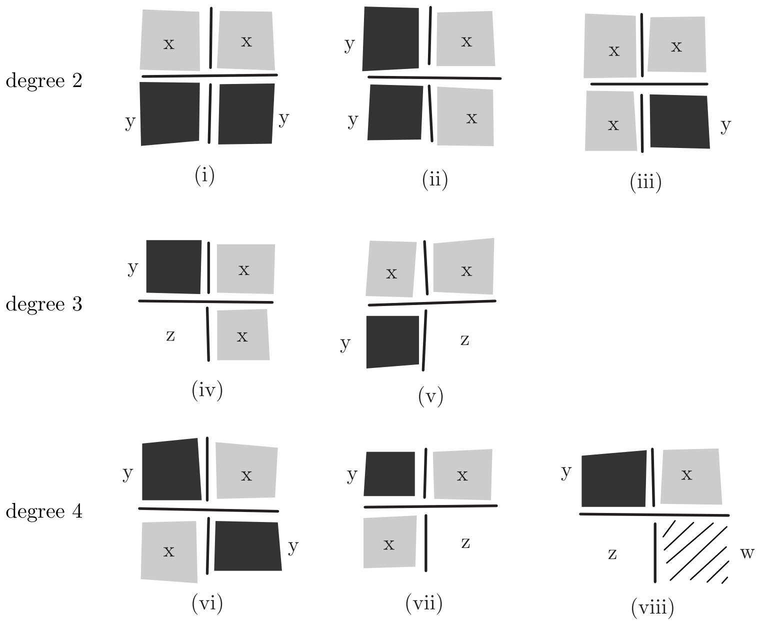

Figure 13. Vertices of degree 2, 3, and 4 of $L_D(f)$.

LEMMA 5.4. *D has no triple point of type* (ii), (iii), (v), *nor* (vii).

PROOF. Let a be the color of the top sheet.
(ii) Since $2x = 2y = a$, we have $x = y$.
(iii) Since $2x = x + y = 2a$, we have $x = y$.
(v) Since $x + y = x + z = 2a$, we have $y = z$.
(vii) Since $z + x = x + y = 2a$, we have $y = z$.
In each case, we have a contradiction. □

At each vertex of degree 2 of $L_D(f)$, which is of type (i) by Lemma 5.4, the edges have the same color and orientation corresponding to the top sheet. Hence, we identify the edges with a single one so that the degree of any vertex of $L_D(f)$ is assumed to be 3 or 4. We remark that, among the edges connecting to a vertex of degree 4, each diagonal pair has the same orientation.

LEMMA 5.5. *If $L_D(f)$ has a vertex T of degree 3, then $P_D(f)$ has a cycle of length 3. More precisely, if the regions around T are colored by x, y, and z, then the vertices of the cycle are x, y, and z. In addition, if $p \geq 5$, then just one of x, y, and z appears as a color of the edge connecting to T, which corresponds to that of the top sheet.*

PROOF. By Lemma 5.4, T is of type (iv). We may assume that a pair of the bottom sheets are colored by x, and so is the top sheet. Since $\frac{y+z}{2} = x$, we have

$$\frac{x+y}{2} - z = \frac{3(y-x)}{2} \neq 0 \text{ and } \frac{x+z}{2} - y = \frac{3(z-x)}{2} \neq 0$$

for $p \geq 5$. Hence, we have the conclusion. □

LEMMA 5.6. *Suppose that $L_D(f)$ has a vertex T of degree 4.*

(i) *If f is separating, then the four regions near T have mutually different colors, and $P_D(f)$ has a cycle of length 4. More precisely, if the regions around T are colored by x, y, z, and w in cyclic order, then the vertices of the cycle are x, y, z, and w. In addition, for the labels of the edges of the cycle, one of the equations $\frac{x+y}{2} = \frac{z+w}{2}$ and $\frac{x+w}{2} = \frac{y+z}{2}$ holds.*

(ii) *If $P_D(f)$ has no cycle of length 4, then each diagonal pair of the four regions near T have the same color, and f is non-separating.*

PROOF. By Lemma 5.4, T is of type (vi) or (viii).

(i) If f is separating, then T never be of type (vi). In fact, if T is of type (vi), then, similarly to the proof of Lemma 4.9, there is a pair of simple closed circles on S^2 with a single intersection, which is a contradiction. Hence, T is of type (viii) and we have the conclusion. We remark that the diagonal pair of the edges corresponding to the top sheet have the same color.

(ii) It follows by (i) that T is of type (vi). In particular, a diagonal pair of the four regions belongs to different sheets. Hence, f is non-separating. □

For the reduced lower graph $L_D(f)$, let v_i be the number of vertices of degree i ($i = 3, 4$), ℓ the number of circle components, n the number of connected components except circles, and r the number of the connected regions of $S^2 \setminus L_D(f)$. The following is easily obtained by the calculation of the Euler characteristic of S^2.

LEMMA 5.7. $r = \frac{1}{2}v_3 + v_4 + n + \ell + 1.$

6. A 5-colored diagram with four sheets.

In this section, we consider a non-trivial 5-coloring for a diagram of a 2-knot. We use the notations introduced in the previous sections. The aim of this section is to prove the following.

THEOREM 6.1. *If a 2-knot K is 5-colorable, then* $\mathrm{sh}(K) \geq 5$.

Theorem 6.1 follows immediately from Propositions 4.3 and 6.2 as below. In fact, if D consists of at most three sheets, then $Q(K)$ is trivial, and hence, so is any coloring $\tilde{f} : Q(K) \to R_5$ by the dihedral quandle R_5.

PROPOSITION 6.2. *If D consists of four sheets, then any 5-coloring for D is trivial.*

Let P_1 and P_2 be the graphs as shown in Figure 14.

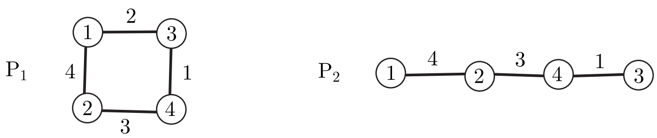

Figure 14. Pallet graphs with four colors for $p = 5$.

LEMMA 6.3. *If D has a non-trivial 5-coloring f with $\#\,\mathrm{Im}(f) = 4$, then it has a non-trivial 5-coloring f' such that*
(i) $\mathrm{Im}(f') = \{1, 2, 3, 4\}$, *and*
(ii) $P_D(f') = P_1$ *or* P_2.

PROOF. By assumption, it holds that $\mathrm{Im}(f) = \boldsymbol{Z}_5 \setminus \{a\}$ for some $a \in \boldsymbol{Z}_5$. Then the 5-coloring $g = f - a$ satisfies $\mathrm{Im}(g) = \{1, 2, 3, 4\}$. Since $\frac{1+4}{2} = \frac{2+3}{2} = 0 \notin \mathrm{Im}(g)$, $P_D(g)$ is P_1 or one of the four graphs obtained from P_1 by deleting an edge. For the latter cases, it is easy to see that the 5-colorings g, $2g$, $3g$, and $4g$ satisfy the condition (i), and just one of them has the reduced graph P_2. $\square$

PROOF OF PROPOSITION 6.2. Assume that D has a non-trivial 5-coloring f. Since D consists of four sheets and $\#\,\mathrm{Im}(f) \geq 4$ by Lemma 5.3, we have $\#\,\mathrm{Im}(f) = 4$ and f is separating. We may assume that $\mathrm{Im}(f) = \{1, 2, 3, 4\}$ and $P_D(f) = P_1$ or P_2 by Lemma 6.3.

Since both P_1 and P_2 have no cycle of length 3, the reduced lower graph $L_D(f)$ has no vertex of degree 3 by Lemma 5.5. Moreover, it follows by Lemma 5.6(i) that $L_D(f)$ has no vertex of degree 4. Hence, it follows by Lemma 5.7 that

$(v_3, v_4, n, \ell) = (0, 0, 0, 3)$, that is, $L_D(f)$ is the union of concentric three circles with $P_D(f) = P_2$.

Let $H_i \in S_D$ denote the sheet whose color is i ($i = 1, 2, 3, 4$). By observing the graph P_2, it is easy to see that any double point of D is of type

$$(H_1 \overset{H_4}{\leftrightarrow} H_2),\ (H_2 \overset{H_3}{\leftrightarrow} H_4),\ (H_4 \overset{H_1}{\leftrightarrow} H_3),\ \text{or } (H_i \overset{H_i}{\leftrightarrow} H_i) \text{ for } i = 1, 2, 3, 4,$$

and the double points of first three types must exist in D. In particular, H_2 is exclusive and H_4 is H_2-exclusive. Since D has a double point of type $(H_2 \overset{H_3}{\leftrightarrow} H_4)$, we have a contradiction to Lemma 3.3. $\square$

For an odd prime p, let s_p denote the minimal number of $\mathrm{sh}(K)$'s for all p-colorable 2-knots K. Since the spun trefoil has a 3-colorable diagram with four sheets, we have $s_3 = 4$ by Theorem 4.2. On the other hand, Theorem 6.1 implies that $s_5 \geq 5$.

Proposition 6.4. *It holds that $s_5 = 5$. In particular, the spun 4_1-knot has the sheet number five.*

Proof. It is sufficient to prove that the spun 4_1-knot K is 5-colorable and has a diagram with five sheets. We take a tangle diagram of the 4_1-knot with four crossings in the upper half plane, and rotate it about the boundary axis so that we obtain a diagram D of K (cf. [**2**]). A non-trivial 5-coloring for the tangle diagram induces that for D naturally. Since the tangle diagram consists of five arcs, D consists of five sheets such that two of them are disks and the other three are annuli. $\square$

7. A 7-colored diagram with four or five sheets.

In this section, we consider a non-trivial 7-coloring for a diagram of a 2-knot. The argument is parallel to that for a 5-coloring in the previous section, however, we need to consider the case of non-separating coloring. The aim of this section is to prove the following.

Theorem 7.1. *If a 2-knot K is 7-colorable, then $\mathrm{sh}(K) \geq 6$.*

Theorem 7.1 follows immediately from Propositions 4.3, 7.2, and 7.3 as below.

Proposition 7.2. *If D consists of four sheets, then any 7-coloring for D is trivial.*

PROPOSITION 7.3. *If D consists of five sheets, then any 7-coloring for D is trivial.*

Let P_3 be the graph as shown in Figure 15.

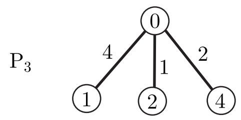

Figure 15. A pallet graph with four colors for $p = 7$.

LEMMA 7.4. *If D has a non-trvial 7-coloring f with $\#\operatorname{Im}(f) = 4$, then it has a non-trivial 7-coloring f' such that*
(i) $\operatorname{Im}(f') = \{0, 1, 2, 4\}$, *and*
(ii) $P_D(f') = P_3$.

PROOF. By Lemma 5.1, we may assume that $0, 1 \in \operatorname{Im}(f)$ and they are connected by an edge of $P_D(f)$. Since $\frac{0+1}{2} = 4$, we have $\operatorname{Im}(f) = \{0, 1, 4, a\}$ for some $a \neq 0, 1, 4$.

If $a = 2$, then we have $\operatorname{Im}(f) = \{0, 1, 2, 4\}$. Since $\frac{1+2}{2} = 5$, $\frac{2+4}{2} = 3$, and $\frac{1+4}{2} = 6 \notin \operatorname{Im}(f)$, we have $P_D(f) = P_3$ by Lemma 5.2. If $a = 3$, then we have $\frac{0+4}{2} = \frac{1+3}{2} = 2$, $\frac{0+3}{2} = 5$, and $\frac{1+4}{2} = 6 \notin \operatorname{Im}(f)$, and hence, $P_D(f)$ is disconnected. If $a = 5$, then we have $V_D(f + 3) = \{0, 1, 3, 4\}$, which is reduced to the case $a = 3$. If $a = 6$, then $\operatorname{Im}(1 - f) = \{0, 1, 2, 4\}$, which is reduced to the case $a = 2$. □

PROOF OF PROPOSITION 7.2. Assume that D has a non-trivial 7-coloring f. Since D consists of four sheets and $\#\operatorname{Im}(f) \geq 4$ by Lemma 5.3, we have $\#\operatorname{Im}(f) = 4$ and f is separating. We may assume that $\operatorname{Im}(f) = \{0, 1, 2, 4\}$ and $P_D(f) = P_3$ by Lemma 7.4.

Let $H_i \in S_D$ denote the sheet whose color is i ($i = 0, 1, 2, 4$). By observing the graph P_3, it is easy to see that any double point of D is of type

$$(H_0 \overset{H_4}{\leftrightarrow} H_1),\ (H_0 \overset{H_1}{\leftrightarrow} H_2),\ (H_0 \overset{H_2}{\leftrightarrow} H_4),\ \text{or}\ (H_i \overset{H_i}{\leftrightarrow} H_i)\ \text{for}\ i = 0, 1, 2, 4,$$

and the double points of first three types must exist in D. In particular, H_0 is exclusive and H_1 is H_0-exclusive. Since D has a double point of type $(H_0 \overset{H_4}{\leftrightarrow} H_1)$, we have a contradiction to Lemma 3.3. □

Since $\#\operatorname{Im}(f) \geq 4$, Proposition 7.3 is divided into two parts as follows.

LEMMA 7.5. *If D consists of five sheets, then there is no non-trivial 7-coloring f with $\#\operatorname{Im}(f) = 4$.*

Lemma 7.6. *If D consists of five sheets, then there is no non-trivial 7-coloring f with $\#\operatorname{Im}(f)=5$.*

Proof of Lemma 7.5. Assume that D has a non-trivial 7-coloring f with $\#\operatorname{Im}(f)=4$. By Lemma 7.4, we may assume that $\operatorname{Im}(f)=\{0,1,2,4\}$ and $P_D(f)=P_3$. Since D has five sheets, f is non-separating and two of the sheets have the same color $k\in\operatorname{Im}(f)$. Let H_k and H'_k be the sheets colored by k, and H_i the sheet colored by $i\in\operatorname{Im}(f)\setminus\{k\}$.

We first consider the case $k\neq 0$. By taking $2f$ or $4f$ instead of f if necessary, we may assume that $k=4$, that is, $S_D=\{H_0,H_1,H_2,H_4,H'_4\}$. Similarly to the proof of Proposition 7.2, we see that H_0 is exclusive and H_1 is H_0-exclusive. Since D has a double point of type $(H_0\overset{H_4}{\leftrightarrow}H_1)$ or $(H_0\overset{H'_4}{\leftrightarrow}H_1)$, we have a contradiction to Lemma 3.3.

Next, we consider the case $k=0$, that is, $S_D=\{H_0,H'_0,H_1,H_2,H_4\}$. By observing the graph P_3, we see that H_0 and H'_0 are both exclusive. We divide the argument into two cases whether H_0 and H'_0 are adjacent or not.

Assume that H_0 and H'_0 are adjacent along L_D. Since H_1 is $H_0\cup H'_0$-exclusive and D has a double point of type $(H_0\overset{H_4}{\leftrightarrow}H_1)$ or $(H'_0\overset{H_4}{\leftrightarrow}H_1)$, we have a contradiction to Lemma 3.4.

Assume that H_0 and H'_0 are not adjacent, that is, $S^2\setminus L_D(f)$ consists of five connected components. Since P_1 has no cycle of length 3, $L_D(f)$ has no vertex of degree 3 by Lemma 5.5. It follows by Lemma 5.7 that

$$(v_4,n,\ell)=(3,1,0),\ (2,2,0),\ (2,1,1),\ (1,1,2),\ \text{and } (0,0,4).$$

By Lemma 5.6(ii), the regions near each vertex of degree 4 have the colors 0 and a for some $a\in\{1,2,4\}$.

$(v_4,n,\ell)=(3,1,0)$: Since $n=1$, there is an element $a\in\{1,2,4\}$ such that the regions have the colors 0 and a at any vertex. It implies that $\operatorname{Im}(f)=\{0,a\}$, which is a contradiction.

$(v_4,n,\ell)=(2,2,0)$: There are elements a and $b\in\{1,2,4\}$ such that the regions have the colors 0 and a at one vertex, and 0 and b at the other. It implies that $\operatorname{Im}(f)=\{0,a,b\}$, which is a contradiction.

$(v_4,n,\ell)=(2,1,1)$: Since $n=1$, there is an element $a\in\{1,2,4\}$ such that the regions have the colors 0 and a at any vertex. Also, since $\ell=1$, there is an element $b\in\{1,2,4\}$ such that the regions along the circle component have the colors 0 and b. It implies that $\operatorname{Im}(f)=\{0,a,b\}$, which is a contradiction.

$(v_4, n, \ell) = (1, 1, 2)$: The reduced lower graph $L_D(f)$ is a disjoint union of a 2-bouquet "∞" and a pair of circles. Since $P_D(f) = P_3$, by taking $2f$ or $4f$ instead of f if necessary, it is sufficient to consider the graphs as shown in Figure 16. In each case, since H_1 is H_0-exclusive and D has a double point of type $(H_0 \overset{H_4}{\leftrightarrow} H_1)$, we have a contradiction to Lemma 3.3.

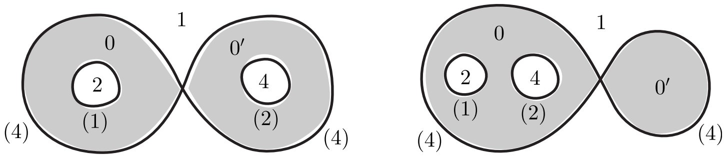

Figure 16. $(v_4, n, \ell) = (1, 1, 2)$.

$(v_4, n, \ell) = (0, 0, 4)$: There are three ways to arrange four circles on S^2. Since $P_D(f) = P_3$, the regions along each circle component have the colors 0 and a for some $a \in \{1, 2, 4\}$.

If $L_D(f)$ is the boundary of disjoint four disks in S^2, then the interiors of the disks are colored by 1, 2, or 4, and the exterior is colored by 0. This contradicts to the assumption that there are two regions colored by 0.

If $L_D(f)$ is a split union of a pair of concentric circles and a pair of split circles, then it is sufficient to consider the case as shown in the left of Figure 17, where H_0 is adjacent to H_1, H_2, and H_4, and H_0' is adjacent to H_1 only. Since H_1 is H_0-exclusive and D has a double point of type $(H_0 \overset{H_4}{\leftrightarrow} H_1)$, we have a contradiction to Lemma 3.3.

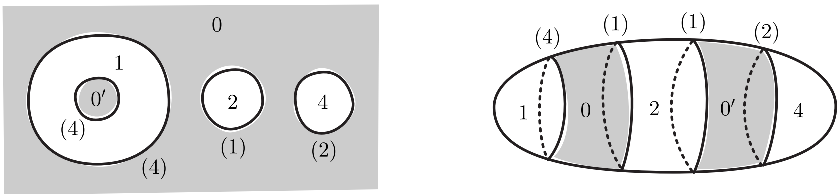

Figure 17. $(v_4, n, \ell) = (0, 0, 4)$.

If $L_D(f)$ is the union of concentric four circles, then it is sufficient to consider the case as shown in the right of Figure 17, where H_0 is adjacent to H_1 and H_2, and H_0' is adjacent to H_2 and H_4. Hence, H_4 is H_0-exclusive and H_2 is H_4-exclusive. On the other hand, we have $0 * 4^n = 0$ (n: even) or 1 (n: odd) in R_7, which is not equal to 2. Since D has a double point of type $(H_0 \overset{H_1}{\leftrightarrow} H_2)$, we have a contradiction to Lemma 3.5. □

Let P_4 be the graphs as shown in Figure 18.

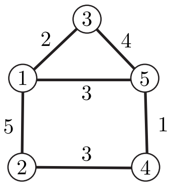

Figure 18. A pallet graph with five colors for $p = 7$.

Lemma 7.7. *If D has a non-trvial 7-coloring f with $\# \mathrm{Im}(f) = 5$, then it has a non-trivial 7-coloring f' such that*
(i) $\mathrm{Im}(f') = \{1, 2, 3, 4, 5\}$, *and*
(ii) $P_D(f')$ *is a connected subgraph of* P_4.

Proof. Assume that $\mathrm{Im}(f) = \boldsymbol{Z}_7 \setminus \{x, y\}$. We consider a 7-coloring $f' = \frac{6-0}{y-x}(f - x) + 0$ similarly to the proof of Lemma 5.1. Then we have $V_D(f') = \boldsymbol{Z}_7 \setminus \{0, 6\}$. Since $\frac{2+5}{2} = \frac{3+4}{2} = 0$ and $\frac{1+4}{2} = \frac{2+3}{2} = 6 \notin V_D(f')$, the coloring f' satisfies (ii) by Lemma 5.2. □

Proof of Lemma 7.6. Assume that D has a non-trivial 7-coloring f with $\# \mathrm{Im}(f) = 5$. Since D consists of five sheets, f is separating. By Lemma 7.7, we may assume that $\mathrm{Im}(f) = \{1, 2, 3, 4, 5\}$ and $P_D(f)$ is a connected subgraph of P_4.

We consider the reduced lower graph $L_D(f)$. Let $H_i \in S_D$ denote the sheet whose color is i ($i = 1, 2, \ldots, 5$). By Lemma 5.5, the three regions near each vertex of degree 3 are H_1, H_3, and H_5. Also, by Lemma 5.6(i), the four regions near each vertex of degree 4 are H_1, H_2, H_4, and H_5 in cyclic order. We remark that the edges of $L_D(f)$ between H_2 and H_4 have the endpoints on the vertices of degree 4. Hence, $L_D(f)$ satisfies the following:
(i) $\frac{v_3}{2} + v_4 + n + \ell = 4$ with $\frac{v_3}{2} + v_4 \geq n$.
(ii) v_3 and v_4 are even.
(iii) If $v_3 > 0$ and $v_4 > 0$, then $\ell = 0$.
(iv) If $v_3 = 0$ and $v_4 > 0$, then $\ell = 1$.
(v) If $v_3 > 0$ and $v_4 = 0$, then $\ell = 2$.
By these properties, we have the following four cases.

$$(v_3, v_4, n, \ell) = (2, 2, 1, 0),\ (0, 2, 1, 1),\ (2, 0, 1, 2),\ \text{and}\ (0, 0, 0, 4).$$

$(v_3, v_4, n, \ell) = (2, 2, 1, 0)$: Since H_2, H_3, and H_4 are bigons, it is easy to see that

$L_D(f)$ is uniquely determined as shown in the left of Figure 19. Around the triple point corresponding to a vertex of degree 4, we can find a pair of double points of type $(H_1 \overset{H_3}{\to} H_5)$ and $(H_5 \overset{H_3}{\to} H_1)$ both. See the right of Figure 17. However, since each diagonal pair of edges have the same orientation near a vertex of degree 4, the edges between H_1 and H_5 have the same orientation. This is a contradiction.

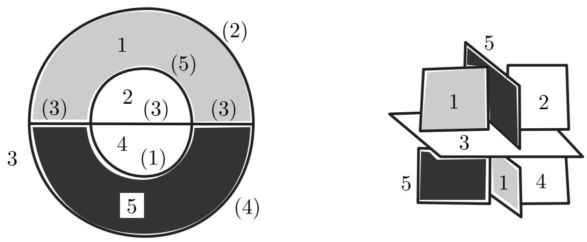

Figure 19. $(v_3, v_4, n, \ell) = (2, 2, 1, 0)$.

$(v_3, v_4, n, \ell) = (0, 2, 1, 1)$: Since $v_4 = 2$, there is a unique edge of $L_D(f)$ where H_1 and H_5 are adjacent. On the other hand, there are at least two such edges by the same reason as in the case $(2, 2, 1, 0)$, which is a contradiction.

$(v_3, v_4, n, \ell) = (2, 0, 1, 2)$: The lower graph $L_D(f)$ is a disjoint union of a θ-graph and a pair of circles. By Lemma 5.5, the triple point corresponding to each vertex of degree 3 is as shown in the left of Figure 20, where the top sheet is H_3. Hence, H_2 and H_4 are adjacent. By taking $6 - f$ instead of f if necessary, we may assume that $L_D(f)$ is the graph as shown in the right of the figure. Since H_1 is exclusive, H_2 is H_1-exclusive, and D has a double point of type $(H_1 \overset{H_5}{\leftrightarrow} H_2)$, we have a contradiction to Lemma 3.3.

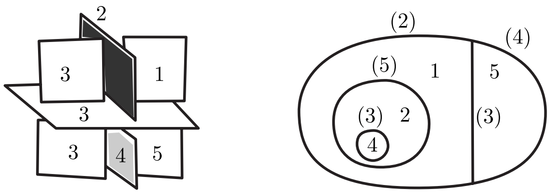

Figure 20. $(v_3, v_4, n, \ell) = (2, 0, 1, 2)$.

$(v_3, v_4, n, \ell) = (0, 0, 0, 4)$: There are three ways to arrange four circles on S^2. If $L_D(f)$ is the boundary of disjoint four disks, then $P_D(f)$ is the graph of "X"-shape with one vertex of degree 4 and four vertices of degree 1. Since it is not a subgraph of P_4, we have a contradiction.

If $L_D(f)$ is the union of concentric four circles, then $P_D(f)$ is the graph of "I"-shape with three vertices of degree 2 and two vertices of degree 1. Since it is a subgraph of P_4, by taking $6-f$ instead of f if necessary, we may assume that $P_D(f)$ is P_5, P_6, P_7, or P_8 as shown in Figure 21.

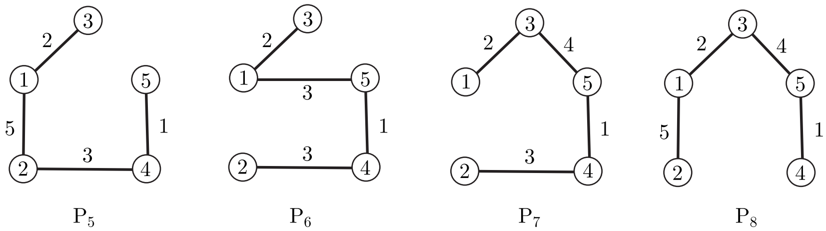

Figure 21. Pallet graphs of "I"-shape.

If $P_D(f) = P_5$, then H_4 is exclusive, H_1 is H_4-exclusive, and H_2 is H_1-exclusive. We have $4 * 1^n = 4$ (n: even) or 5 (n: odd) in R_7, which is not equal to 2. Since D has a double point of type $(H_4 \overset{H_3}{\leftrightarrow} H_2)$, we have a contradiction to Lemma 3.5.

If $P_D(f) = P_6$, then H_5 is exclusive, and H_1 is H_5-exclusive. Since D has a double point of type $(H_5 \overset{H_3}{\leftrightarrow} H_1)$, we have a contradiction to Lemma 3.3.

If $P_D(f) = P_7$, then H_5 is exclusive, and H_4 is H_5-exclusive. Since D has a double point of type $(H_5 \overset{H_1}{\leftrightarrow} H_4)$, we have a contradiction to Lemma 3.3.

If $P_D(f) = P_8$, then H_3 is exclusive, and H_2 is H_3-exclusive, and H_5 is H_2-exclusive. We have $3 * 2^n = 3$ (n: even) or 1 (n: odd) in R_7, which is not equal to 5. Since D has a double point of type $(H_3 \overset{H_4}{\leftrightarrow} H_5)$, we have a contradiction to Lemma 3.5.

If $L_D(f)$ is a split union of a pair of concentric circles and a pair of split circles, then $P_D(f)$ is the graph of "Y"-shape with one vertex of degree 3, one vertex of degree 2, and three vertices of degree 1. By taking $6-f$ instead of f if necessary, we may assume that $P_D(f)$ is P_9 or P_{10} as shown in Figure 22.

P9

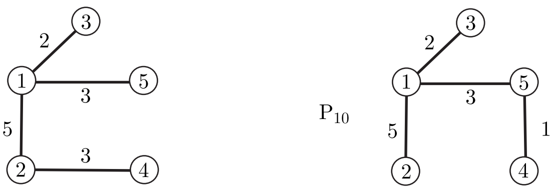

Figure 22. Pallet graphs of "Y"-shape.

If $P_D(f) = P_9$, then H_1 is exclusive, and H_2 is H_1-exclusive. Since D has a double point of type $(H_1 \overset{H_5}{\leftrightarrow} H_2)$, we have a contradiction to Lemma 3.3.

If $P_D(f) = P_{10}$, then H_4 is exclusive, H_1 is H_4-exclusive, and H_5 is H_1-exclusive. For integers $l, m \in \boldsymbol{Z}$, we have $4 * 1^l = 4$ (l: even) or 5 (l: odd), and $(4 * 3^{\pm 1}) * 5^m = 2 * 5^m = 2$ (m: even) or 1 (m: odd), that is, $4 * 1^l \neq (4 * 3^{\pm 1}) * 5^m$. Since D has a double point of type $(H_1 \overset{H_3}{\leftrightarrow} H_5)$ with $f(H_3) = 3$, we have a contradiction to Lemma 3.6. □

PROPOSITION 7.8. *It holds that* $s_7 = 6$. *In particular, the spun* 5_2*-knot has the sheet number six.*

PROOF. The 5_2-knot is 7-colorable and the crossing number is equal to five. Similarly to Proposition 6.4, the proposition follows from Theorem 7.1 immediately. □

EXAMPLE 7.9. We have another example of a 2-knot whose sheet number is equal to six. We take a diagram of a trivial 2-string tangle in the upper half plane as shown in the left of Figure 23, where the arcs are colored by $1, 2, \ldots, 6 \in \boldsymbol{Z}_7$. We rotate it about the axis, and do surgery along a 1-handle connecting between the sheets colored by 5 and 6 under the sheet colored by 2, so that we obtain a diagram of a 2-knot. See the right of the figure. Since the diagram consists of six sheets with the induced non-trivial 7-coloring, the sheet number is equal to six by Theorem 7.1. The Alexander polynomial of the 2-knot is $t^3 - 3t^2 + 2t - 1$ which is not symmetric, and hence, the 2-knot is not a spun knot. We remark that this 2-knot can be found in the table of ribbon 2-knots due to Aiso [**1**].

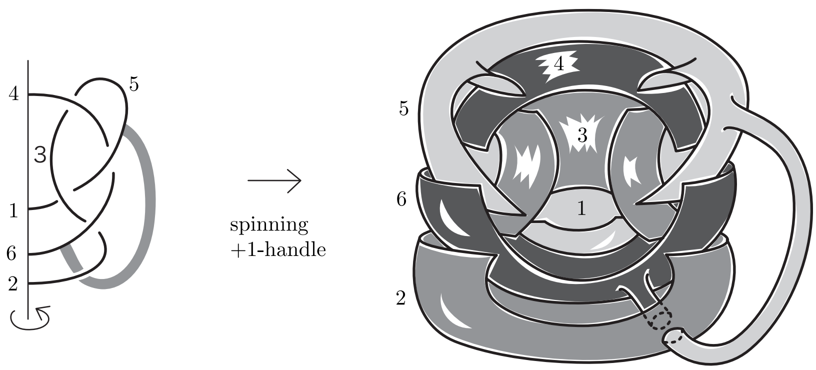

Figure 23. A 2-knot of sheet number six.

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