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The Lévy-Itô decomposition of sample paths of Lévy processes with values in the space of probability measures

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Abstract. A definition of Lévy processes with values in the space of probability measures was introduced by Shiga and Tanaka (Electronic J. Prob. 11 (2006)). It is shown that the Lévy process with values in the space of probability measures in law has a modification satisfying a certain condition. The modification is a Lévy process in the sense of Shiga and Tanaka. The Lévy-Itô decomposition of sample paths of the Lévy process satisfying the condition is derived.

1. Introduction.

Let $\mathscr{P}(\boldsymbol{R})$ be the set of all probability distributions on $\boldsymbol{R}$. We denote by $\hat{\mu}(z)$ the characteristic function of a probability measure $\mu \in \mathscr{P}(\boldsymbol{R})$. We define the metric on $\mathscr{P}(\boldsymbol{R})$ by

$$d(\mu_1, \mu_2) = \sum_{m=1}^{\infty} 2^{-m} \sup_{|z| \leq m} |\hat{\mu}_1(z) - \hat{\mu}_2(z)|$$

for μ_1 and $\mu_2 \in \mathscr{P}(\boldsymbol{R})$. Then $(\mathscr{P}(\boldsymbol{R}), d)$ is a complete separable metric space. Shiga and Tanaka [**9**] introduced a Lévy process on $\mathscr{P}(\boldsymbol{R})$ in law as follows:

Definition. Let $\{\Xi_t : t \geq 0\}$ be a $\mathscr{P}(\boldsymbol{R})$-valued stochastic process on a probability space $(\Omega, \mathscr{F}, P)$. $\{\Xi_t : t \geq 0\}$ is called a Lévy process on $\mathscr{P}(\boldsymbol{R})$ in law, if

(D.1) $\Xi_0 = \delta_0$,
(D.2) $\{\Xi_t : t \geq 0\}$ is stochastically continuous,
(D.3) there exists a family of $\mathscr{P}(\boldsymbol{R})$-valued random variables $\{\Xi_{s,t}\}_{0 \leq s < t}$ satisfying the following conditions.
  (D.3.1) For any $n \geq 2$ and $0 = t_0 < t_1 < \cdots < t_n$,

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$$\Xi_{t_n} = \Xi_{t_0,t_1} * \Xi_{t_1,t_2} * \cdots * \Xi_{t_{n-1},t_n} \qquad \text{a.s.}$$

(D.3.2) For any $n \geq 2$ and $0 = t_0 < t_1 < \cdots < t_n$, $\{\Xi_{t_{i-1},t_i}\}_{1 \leq i \leq n}$ are independent.

(D.3.3) For $0 < s < t$

$$\Xi_{s,t} \overset{(d)}{=} \Xi_{t-s},$$

where $\overset{(d)}{=}$ means equality in distribution.

Shiga and Tanaka [**9**] called $\{\Xi_t : t \geq 0\}$ a Lévy process on $\mathscr{P}(\boldsymbol{R})$, if it is a Lévy process on $\mathscr{P}(\boldsymbol{R})$ in law and $t \in [0, \infty) \to \Xi_t \in \mathscr{P}(\boldsymbol{R})$ is right continuous in $t \geq 0$ with left limits in $t > 0$ a.s., that is, $\{\Xi_t : t \geq 0\}$ is càdlàg. Now we call a stochastic process $\{\widetilde{\Xi}_t\}$ a modification of a stochastic process $\{\Xi_t\}$, if $P(\Xi_t = \widetilde{\Xi}_t) = 1$ for $t \in [0, \infty)$. Shiga and Tanaka did not prove the existence of a càdlàg modification, nevertheless they make such claim and do not give the argument. They concretely construct a Lévy process (see Theorem 6.6 in [**9**]).

First of all, we show that any Lévy process on $\mathscr{P}(\boldsymbol{R})$ in law has a modification satisfying a certain condition. The proof is similar to that in Sato [**8**] and is omitted. The proof in Sato [**8**] is used a result of Dynkin [**2**] and Kinney [**5**].

THEOREM 1.1. *A Lévy process on $\mathscr{P}(\boldsymbol{R})$ in law $\{\Xi_t\}$ has a modification satisfying the following*:

(D.4) *There exists a family of $\mathscr{P}(\boldsymbol{R})$-valued random variables $\{\Xi^{\omega}_{s,t}\}_{0 \leq s < t}$ satisfying the following conditions. There is $\Omega_0 \in \mathscr{F}$ with $P(\Omega_0) = 1$ such that, for every $\omega \in \Omega_0$,*

(D.4.1) $\Xi^{\omega}_{s,t}$ *is right continuous in t with $t \geq s$ and has left limits in t with $t > s$,*

(D.4.2) $\Xi^{\omega}_{s,t}$ *is right continuous in s with $t > s \geq 0$ and has left limits in s with $t \geq s > 0$,*

(D.4.3) *for $t > s \geq 0$,*

$$\Xi^{\omega}_t = \Xi^{\omega}_s * \Xi^{\omega}_{s,t}.$$

(D.4.4) *For any $n \geq 2$ and $0 = t_0 < t_1 < \cdots < t_n$, $\{\Xi_{t_{i-1},t_i}\}_{1 \leq i \leq n}$ are independent.*

(D.4.5) *For $0 < s < t$*

$$\Xi_{s,t} \overset{(d)}{=} \Xi_{t-s}.$$

REMARK.

(i) If a Lévy process on $\mathscr{P}(\boldsymbol{R})$ in law $\{\Xi_t\}$ satisfies (D.4), then it is càdlàg: Indeed, since (D.1) and (D.4.3) are satisfied, we have $\Xi_t = \Xi_{0,t}$ a.s. It follows from (D.4.1) that $\{\Xi_t\}$ is càdlàg.

(ii) If a $\mathscr{P}(\boldsymbol{R})$-valued stochastic process $\{\Xi_t\}$ satisfies (D.1), (D.2) and (D.4), it is a Lévy process on $\mathscr{P}(\boldsymbol{R})$. Indeed, using (D.4.3) repeatedly, we get (D.3.1).

One of the basic results for classical Lévy processes on $\boldsymbol{R}$ is that a Lévy process has the Lévy-Itô decomposition of sample paths. The decomposition was conceived by Lévy (see [**6**], [**7**]). It consists of two independent parts. One part is continuous and another part is a compensated sum of independent jumps. And it was formulated and proved by Itô [**4**]. The proof is found also in the books of Itô [**4**] and Sato [**8**].

Now, for the sample path Ξ_t, the jump size μ should be defined by

$$\Xi_t = \Xi_{t-} * \mu.$$

Under the definition of Lévy processes in the sense of Shiga and Tanaka [**9**], existence of such μ does not follow automatically and we cannot show the Lévy-Itô decomposition of sample paths. Indeed, we have $\widehat{\Xi}_t(z) = \widehat{\Xi}_{t-}(z)\widehat{\mu}(z)$. Then, if $\widehat{\Xi}_t(z_0) = \widehat{\Xi}_{t-}(z_0) = 0$ for some z_0, we cannot decide $\widehat{\mu}(z_0)$. Hence the jump size μ is not exactly caught. But, for a Lévy process satisfying (D.4), the jump size is exactly caught as $\Xi_{t-,t}$.

We show that a Lévy process satisfying (D.4) has the Lévy-Itô decomposition of sample paths. Let Ξ be a random probability distribution (shortly, RPD) with the distribution $Q \in \mathscr{P}(\mathscr{P}(\boldsymbol{R}))$. And let $\boldsymbol{R}^{\boldsymbol{N}} = \boldsymbol{R} \times \boldsymbol{R} \times \cdots$ and let $\mu^{\otimes \boldsymbol{N}} = \mu \otimes \mu \otimes \cdots$ for $\mu \in \mathscr{P}(\boldsymbol{R})$, where $\boldsymbol{N}$ is the set of positive integers. Denote by $\ell_0(\boldsymbol{N})$ the totality of $\boldsymbol{z} = \{z_j\} \in \boldsymbol{R}^{\boldsymbol{N}}$ with $z_j = 0$ except for finitely many $j \in \boldsymbol{N}$. Then the characteristic function of a RPD Ξ with distribution Q is defined by

$$\Phi_\Xi(\boldsymbol{z}) = \int_{\mathscr{P}(\boldsymbol{R})} \prod_{j \in \boldsymbol{N}} \hat{\mu}(z_j) Q(d\mu)$$

for $\boldsymbol{z} \in \ell_0(\boldsymbol{N})$.

Here we prepare some notations:

$$\langle \boldsymbol{z}, \boldsymbol{x} \rangle = \sum_{j \in \boldsymbol{N}} z_j x_j$$

for $\boldsymbol{x} \in \boldsymbol{R}^{\boldsymbol{N}}$ and $\boldsymbol{z} \in \ell_0(\boldsymbol{N})$, and

$$\langle \mu, F\rangle = \int_{\boldsymbol{R}} \mu(dx)F(x), \qquad \langle \mu^{\otimes \boldsymbol{N}}, G\rangle = \int_{\boldsymbol{R}^{\otimes \boldsymbol{N}}} \mu^{\otimes \boldsymbol{N}}(d\boldsymbol{x})G(\boldsymbol{x})$$

for $\mu \in \mathscr{P}(\boldsymbol{R})$ and bounded measurable functions F and G.

Let $\{\Xi_t\}$ be a Lévy process on $\mathscr{P}(\boldsymbol{R})$. The distribution of Ξ_t at each time t is infinitely divisible and the characteristic function of Ξ_1 is represented as

$$\begin{aligned}&\Phi_{\Xi_1}(\boldsymbol{z})\\ &= \exp\biggl[\sum_{j\in\boldsymbol{N}} \Bigl(-2^{-1}\alpha z_j^2 + i\gamma z_j + \int_{\boldsymbol{R}\setminus\{0\}} (e^{iz_j x} - 1 - iz_j x I_{[|x|\le 1]}(x))\rho(dx)\Bigr)\\ &\qquad - \frac{\beta}{2}\Bigl(\sum_{j\in\boldsymbol{N}} z_j\Bigr)^2 + \int_{\mathscr{P}_*(\boldsymbol{R})} \Bigl\langle \mu^{\otimes\boldsymbol{N}}, e^{i\langle \boldsymbol{z},\boldsymbol{x}\rangle} - 1 - i\Bigl(\sum_{j\in\boldsymbol{N}} z_j x_j I_{[|x_j|\le 1]}\Bigr)\Bigr\rangle m(d\mu)\biggr]\end{aligned} \tag{1.1}$$

for $\boldsymbol{z} \in \ell_0(\boldsymbol{N})$ (see [**9**]). Here $\alpha \geq 0$, $\beta \geq 0$ and $\gamma \in \boldsymbol{R}$, ρ is a σ-finite measure satisfying $\int_{\boldsymbol{R}\setminus\{0\}}(x^2 \wedge 1)\rho(dx) < \infty$, and m is a σ-finite measure on $\mathscr{P}_*(\boldsymbol{R}) = \mathscr{P}(\boldsymbol{R})\setminus\{\delta_0\}$ satisfying

$$\int_{\mathscr{P}_*(\boldsymbol{R})} \langle \mu, x^2 \wedge 1\rangle m(d\mu) < \infty.$$

We call m the Lévy measure of Ξ_1. The characteristic function Φ_{Ξ_1} characterizes the law of $\{\Xi_t\}$.

Furthermore, we define a shift operator θ_b ($b \in \boldsymbol{R}$) on $\mathscr{P}(\boldsymbol{R})$ by

$$\theta_b \cdot \mu(A) = \mu(A + b) \quad (A \in \mathscr{B}(\boldsymbol{R})).$$

And we define

$$\int \mu * \Pi(d\mu) = \mu_1 * \mu_2 * \cdots$$

for $\Pi = \sum_j \delta_{\mu_j}$. For $\epsilon > 0$ let

$$A^{(\epsilon)} = \{\mu \in \mathscr{P}_*(\boldsymbol{R}) : \langle \mu, x^2 \wedge 1\rangle > \epsilon\}$$

and

$$a_\epsilon = \int_{A^{(\epsilon)}} \langle \mu, xI_{[|x|\le 1]} \rangle m(d\mu).$$

Now we show the Lévy-Itô decomposition for Lévy processes on $\mathscr{P}(\boldsymbol{R})$ satisfying the condition (D.4).

Theorem 1.2. *Let $\{\Xi_t\}$ be a Lévy process on $\mathscr{P}(\boldsymbol{R})$ satisfying the condition* (D.4)*. Denote by $\sharp A$ be the number of elements of a set A. Let $\epsilon > 0$. For $B \in A^{(\epsilon)}$ we define*

$$\begin{aligned} &N(B;0,\omega) = 0, \\ &N(B;t,\omega) = \begin{cases} \sharp\{\tau \in (0,t] : \Xi^{\omega}_{\tau-,\tau} \in B\} & \text{for } \omega \in \Omega_0, \\ 0 & \text{for } \omega \notin \Omega_0 \end{cases} \end{aligned}$$

for $t > 0$. Set

$$m(U) = E[N(U;1,\omega)]$$

for $U \in \mathscr{B}(A^{(\epsilon)})$. Then the following holds:

(i) *If $U \in \mathscr{B}(A^{(\epsilon)})$, $N(U;t,\omega)$ is a Poisson process with parameter $m(U)$. If $U_1, U_2, \ldots, U_n \in \mathscr{B}(A^{(\epsilon)})$ and they are pairwise disjoint, then $N(U_1;t,\omega)$, $\ldots, N(U_n;t,\omega)$ are independent. $N(U;t,\omega)$ is a finite measure in $U \in A^{(\epsilon)}$, and*

$$\int_{\mathscr{P}_*(\boldsymbol{R})} \langle \mu, x^2 \wedge 1 \rangle m(d\mu) < \infty.$$

(ii) *Set*

$$S_t^{(\epsilon)} = \theta_{ta_\epsilon} \cdot \int_{A^{(\epsilon)}} \xi * N(d\xi;t,\omega).$$

There exists a Lévy process $\{S_t\}$ on $\mathscr{P}(\boldsymbol{R})$ such that

$$\lim_{\epsilon\to 0} \sup_{0\le t\le T} d(S_t^{(\epsilon)}, S_t) = 0 \quad (\forall T > 0) \quad a.s. \tag{1.2}$$

The characteristic function of S_t is represented as

$$\Phi_{S_t}(\boldsymbol{z}) = \exp\left[t\int_{\mathscr{P}_*(\boldsymbol{R})}\left\langle \mu^{\otimes \boldsymbol{N}}, e^{i\langle \boldsymbol{z},\boldsymbol{x}\rangle} - 1 - i\biggl(\sum_{j\in\boldsymbol{N}} z_j x_j I_{[|x_j|\le 1]}\biggr)\right\rangle m(d\mu)\right] \tag{1.3}$$

for $\boldsymbol{z} \in \ell_0(\boldsymbol{N})$.

(iii) *There is* $\Omega_1 \in \mathscr{F}$ *with* $P(\Omega_1) = 1$ *such that, for* $\omega \in \Omega_1$, *there exists a Lévy process* $\{V_t\}$ *on* $\mathscr{P}(\boldsymbol{R})$ *satisfying*

$$\Xi_t^\omega = S_t^\omega * V_t^\omega \tag{1.4}$$

for all $t \ge 0$ *and* V_t^ω *is continuous in* t. *And* V_t *has the following characteristic function*:

$$\begin{aligned}\Phi_{V_t}(\boldsymbol{z})&\\ = \exp\Biggl[t\biggl(\sum_{j\in\boldsymbol{N}}\biggl(&-\frac{1}{2}\alpha z_j^2 + i\gamma z_j + \int_{\boldsymbol{R}\setminus\{0\}}(e^{iz_jx} - 1 - iz_jxI_{[|x|\le 1]}(x))\rho(dx)\biggr)\\ &-\frac{\beta}{2}\biggl(\sum_{j\in\boldsymbol{N}} z_j\biggr)^2\biggr)\Biggr]\end{aligned} \tag{1.5}$$

for $\boldsymbol{z} \in \ell_0(\boldsymbol{N})$.

(iv) *The processes* $\{S_t\}$ *and* $\{V_t\}$ *are independent.*

2. Construction of Lévy processes on $\mathscr{P}(\boldsymbol{R})$.

Any finite dimensional distribution of Lévy process on $\mathscr{P}(\boldsymbol{R})$ is uniquely determined by the characteristic function at time 1. The representation of the characteristic function is given by Shiga and Tanaka [**9**]. In this section we construct a Lévy process corresponding to the given characteristic function. We need the construction to prove Theorem 1.2. In [**9**] we can find a similar result.

Let m be a σ-finite measure on $\mathscr{P}_*(\boldsymbol{R}) = \mathscr{P}(\boldsymbol{R})\setminus\{\delta_0\}$ satisfying that

$$\int_{\mathscr{P}_*(\boldsymbol{R})}\langle \mu, x^2 \wedge 1\rangle m(d\mu) < \infty.$$

For $\epsilon > 0$ let

$$A^{(\epsilon)} = \{\mu \in \mathscr{P}_*(\boldsymbol{R}) : \langle \mu, x^2 \wedge 1\rangle > \epsilon\}.$$

There exists, on a probability space $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \widetilde{P})$, a Poisson random measure $\Pi_{\lambda\times m}$

on $(0,\infty)\times\mathscr{P}_*(\boldsymbol{R})$ with intensity measure $\lambda\times m$. It satifies the following: For any $\epsilon>0$ and $t>0$, $\Pi_{\lambda\times m}|_{(0,t]\times A^{(\epsilon)}}$ is supported on a finite number of points. Here λ stands for the Lebesgue measure on $[0,\infty)$ and $\Pi_{\lambda\times m}|_{(0,t]\times A^{(\epsilon)}}$ is the restriction of $\Pi_{\lambda\times m}$ to $(0,t]\times A^{(\epsilon)}$. Then we have $m(A^{(\epsilon)})<\infty$. Indeed, $m(A^{(\epsilon)})\le\epsilon^{-1}\int_{A^{(\epsilon)}}\langle\mu,x^2\wedge 1\rangle m(d\mu)<\infty$. Furthermore, we have $a_\epsilon=\int_{A^{(\epsilon)}}\langle\mu,xI_{[|x|\le 1]}\rangle m(d\mu)\le m(A^{(\epsilon)})<\infty$. Hence we can define

$$\Xi_{s,t}^{(\epsilon)}=\theta_{a_\epsilon(t-s)}\cdot\int_{(s,t]\times A^{(\epsilon)}}\mu*\Pi_{\lambda\times m}(dsd\mu)$$

for $t>s\ge 0$. If $t=s$, we define $\Xi_{s,s}^{(\epsilon)}=\delta_0$. And we define $\Xi_t^{(\epsilon)}=\Xi_{0,t}^{(\epsilon)}$. We remark that Shiga and Tanaka [**9**] takes $A^{(\epsilon)}=\{\mu\in\mathscr{P}_*(\boldsymbol{R}):\langle\mu,|x|\wedge 1\rangle>\epsilon\}$ in place of our $A^{(\epsilon)}$.

First of all, we mention two theorems. The proofs are given in the latter half.

Theorem 2.1. *There exists a Lévy process $\{\Xi_t\}$ on $\mathscr{P}(\boldsymbol{R})$ with the condition* (D.4) *such that*

$$\lim_{\epsilon\to 0}\sup_{0\le t\le T}d(\Xi_t^{(\epsilon)},\Xi_t)=0\quad(\forall T>0)\quad a.s.$$

Furthermore, the characteristic function of Ξ_1 is represented as

$$\Phi_{\Xi_1}(\boldsymbol{z})=\exp\left[\int_{\mathscr{P}_*(\boldsymbol{R})}\left\langle\mu^{\otimes\boldsymbol{N}},e^{i\langle\boldsymbol{z},\boldsymbol{x}\rangle}-1-i\Big(\sum_{j\in\boldsymbol{N}}z_jx_jI_{[|x_j|\le 1]}\Big)\right\rangle m(d\mu)\right]\quad(2.1)$$

for $\boldsymbol{z}\in\ell_0(\boldsymbol{N})$.

Remark 2.2.

(i) The limit Ξ_t in Theorem 2.1 is denoted by

$$\Xi_t=\int_{(0,t]\times\mathscr{P}_*(\boldsymbol{R})}\mu*\Pi_{\lambda\times m}^{reno}(dsd\mu).$$

(ii) Let $m\neq 0$ and let $T>0$. The probability that Ξ_t has a discontinuous point on $[0,T]$ is positive. The reason is as follows: Since $\lim_{\epsilon\to 0}\sup_{0\le t\le T}d(\Xi_t^{(\epsilon)},\Xi_t)=0$ a.s. from Theorem 2.1, we have $\lim_{\epsilon\to 0}d(\Xi_{t-}^{(\epsilon)},\Xi_{t-})=0$ a.s. Hence we have $d(\Xi_t,\Xi_{t-})=\lim_{\epsilon\to 0}d(\Xi_t^{(\epsilon)},\Xi_{t-}^{(\epsilon)})$ a.s. If $m\neq 0$, the probability that $\Xi_t^{(\epsilon)}$ has a discontinuous point is positive for sufficiently small ϵ. Let $0 < \epsilon < \epsilon_0$. If $\Xi_t^{(\epsilon_0)}$ is not continuous at t_0, then $\Xi_t^{(\epsilon)}$ is not continuous at t_0 and Ξ_t is not continuous at t_0. Hence the probability that Ξ_t has a discontinuous point is positive.

Furthermore, let $\{B_t\}$ be a standard Brownian motion defined on $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \widetilde{P})$, where $\{B_t\}$ and $\Pi_{\lambda\times m}$ are independent. For $t > 0$ we set

$$\Xi_t = \nu_t * \delta_{\beta^{1/2}B_t} * \left(\int_{(0,t]\times\mathscr{P}_*(\boldsymbol{R})} \mu * \Pi^{reno}_{\lambda\times m}(dsd\mu)\right). \tag{2.2}$$

Here ν_t is an infinitely divisible distribution on $\boldsymbol{R}$ with characteristic function

$$\hat{\nu}_t(z) = \exp\left[t\left\{ -2^{-1}\alpha z^2 + iz\gamma + \int_{\boldsymbol{R}\setminus\{0\}} (e^{izx} - 1 - izxI_{[|x|\le 1]}(x))\rho(dx)\right\}\right].$$

Shiga and Tanaka proved that, for their $\{\Xi_t\}$, the characteristic function at time 1 is represented as (1.1) and a similar result holds in our case.

THEOREM 2.3. *Let $\{\Xi_t\}$ be defined by* (2.2). *Then $\{\Xi_t\}$ is a Lévy process on $\mathscr{P}(\boldsymbol{R})$ with the condition* (D.4) *such that the characteristic function of Ξ_1 coincides with* (1.1).

We prepare some lemmas for the proofs of Theorems 2.1 and 2.3. Let $C_b(\boldsymbol{R})$ be the totality of bounded real functions on $\boldsymbol{R}$ and let $C_b^n(\boldsymbol{R})$ be the set of $f \in C_b(\boldsymbol{R})$ such that f is n times differentiable and the derivatives with order $\le n$ belong to $C_b(\boldsymbol{R})$. Let $U = \{U^\omega_{s,t}\}_{0\le s<t}$ and $V = \{V^\omega_{s,t}\}_{0\le s<t}$ be two families of $\mathscr{P}(\boldsymbol{R})$-valued random variables satisfying (D.4.1) and (D.4.2). Then we set

$$d_T(U,V) = \sup_{0\le s<t\le T} d(U_{s,t}, V_{s,t}). \tag{2.3}$$

Furthermore, for $\epsilon_0 > \epsilon > 0$ let

$$A^{(\epsilon,\epsilon_0)} = \{\mu \in \mathscr{P}(\boldsymbol{R}) : \epsilon < \langle \mu, x^2 \wedge 1\rangle \le \epsilon_0\}$$

for $T > 0$. Then we have $m(A^{(\epsilon,\epsilon_0)}) < \infty$ and we can set

$$\Xi^{(\epsilon,\epsilon_0)}_{s,t} = \theta_{a_{\epsilon,\epsilon_0}(t-s)} \cdot \int_{(s,t]\times A^{(\epsilon,\epsilon_0)}} \mu * \Pi_{\lambda\times m}(dud\mu),$$

where

$$a_{\epsilon,\epsilon_0} = \int_{A^{(\epsilon,\epsilon_0)}} \langle \mu, xI_{[|x|\leq 1]}\rangle m(d\mu).$$

In particular, we set $\Xi_t^{(\epsilon,\epsilon_0)} = \Xi_{0,t}^{(\epsilon,\epsilon_0)}$.

Under the above definition, assertions similar to Lemmas 6.2, 6.3, 6.4 and 6.5 in [**9**] remain true. The proofs are the same as those in [**9**], where, in order to prove Lemma 6.4(ii), we need to use the following in place of the maximal inequality for martingales: Let $\widetilde{\Pi}_{\lambda\times m} = \Pi_{\lambda\times m} - \lambda \times m$. For any nonnegative bounded measurable function H, we have

$$E\biggl(\int H(u,\mu)\widetilde{\Pi}_{\lambda\times m}(dud\mu)\biggr)^2 \leq \int H(u,\mu)^2 dum(d\mu).$$

Furthermore, we prepare to prove Theorems 2.1 and 2.3.

Lemma 2.4. *Let $\Xi_1^n, \Xi_2^n, \ldots, \Xi_m^n$ be independent $\mathscr{P}(\boldsymbol{R})$-valued random variables. If $\lim_{n\to\infty} d(\Xi_j^n, \Xi_j) = 0$ a.s. for each j, then $\{\Xi_j\}_{1\leq j\leq m}$ are independent.*

Proof. Let $\{\nu_j : j \in \boldsymbol{N}\}$ be a dense subset of $\mathscr{P}(\boldsymbol{R})$. And we take

$$U_\epsilon(\nu_j) = \{\mu \in \mathscr{P}(\boldsymbol{R}) : d(\nu_j,\mu) < \epsilon\},\ j \in \boldsymbol{N},\ \epsilon > 0,$$

as a neighborhood system. We denote by $\mathscr{O}$ the π-system generated by this neighborhood system. We denote by $\bar{A}$ the closure of a set A. From the independence of $\{\Xi_j^n\}_{1\leq j\leq m}$, we obtain that

$$\begin{aligned} P(\Xi_1 \in \bar{A}_1, \ldots, \Xi_m \in \bar{A}_m) &\geq P(\Xi_1 \in \bar{A}_1)\cdots P(\Xi_m \in \bar{A}_m) \\ &\geq P(\Xi_1 \in A_1)\cdots P(\Xi_m \in A_m) \end{aligned}$$

for $A_1, \ldots, A_m \in \mathscr{O}$. Since ϵ of $U_\epsilon(\nu_j)$ is free to be taken, we have

$$P(\Xi_1 \in A_1, \ldots, \Xi_m \in A_m) \geq P(\Xi_1 \in A_1)\cdots P(\Xi_m \in A_m)$$

for $A_1, \ldots, A_m \in \mathscr{O}$. The converse inequality is obtained in the same way as above. Hence

$$P(\Xi_1 \in A_1, \ldots, \Xi_m \in A_m) = P(\Xi_1 \in A_1)\cdots P(\Xi_m \in A_m) \tag{2.4}$$

Here we use mathematical induction in m. By using the π-λ theorem in [**1**, p. 42], (2.4) holds for $A_1, \dots, A_m \in \sigma(\mathscr{O})$. Since $\mathscr{P}(\boldsymbol{R})$ is separable, we have $\sigma(\mathscr{O}) \supset \mathscr{B}(\mathscr{P}(\boldsymbol{R}))$. We have completed the proof. □

LEMMA 2.5. *The characteristic function of* $\Xi_1^{(\epsilon)}$ *is represented as*

$$\Phi_{\Xi_1^{(\epsilon)}}(\boldsymbol{z}) = \exp\left[\int_{A^{(\epsilon)}} \left\langle \mu^{\otimes \boldsymbol{N}}, e^{i\langle \boldsymbol{z}, \boldsymbol{x}\rangle} - 1 - i\left(\sum_{j \in \boldsymbol{N}} z_j x_j I_{[|x_j| \le 1]}\right)\right\rangle m(d\mu)\right]. \quad (2.5)$$

PROOF. Now $\Pi((0,1] \times B)$ has a Poisson distribution with mean $m(B)$. Recall that $m(A^{(\epsilon)}) < \infty$. Hence, Lemma 3.1 in [**9**] is applied to $\Xi = \int_{\mathscr{P}(\boldsymbol{R})} \mu * \Pi_{m|_{A^{(\epsilon)}}}(d\mu)$ and we obtain (2.5). Here we denote by $m|_{A^{(\epsilon)}}$ the restriction of m to $A^{(\epsilon)}$. □

Now we prove Theorems 2.1 and 2.3.

PROOF OF THEOREMS 2.1 AND 2.3. Let $\Xi^{(\epsilon)} = \{\Xi_{s,t}^{(\epsilon)}\}_{0 \le s < t}$. As mentioned before, assertions similar to Lemmas 6.3 and 6.5 in [**9**] remain true. Thus it follows that for every $\eta > 0$,

$$\lim_{\epsilon_0 \to 0} P\Big(\sup_{0 < \epsilon < \epsilon_0} d_T(\Xi^{(\epsilon)}, \Xi^{(\epsilon_0)}) \ge 3\eta\Big) = 0.$$

Hence we have

$$P\Big(\lim_{\epsilon_0 \to 0} \sup_{0 < \epsilon, \epsilon' < \epsilon_0} d_T(\Xi^{(\epsilon)}, \Xi^{(\epsilon')}) \ge 6\eta\Big) = 0,$$

that is,

$$\lim_{\epsilon_0 \to 0} \sup_{0 < \epsilon, \epsilon' < \epsilon_0} d_T(\Xi^{(\epsilon)}, \Xi^{(\epsilon')}) = 0 \quad \text{a.s.}$$

Then there is $\Omega_0 \in \mathscr{F}$ with probability 1 such that the above limit holds. Therefore, there exists $\Xi_{s,t}$ such that $\lim_{\epsilon \to 0} d(\Xi_{s,t}^{(\epsilon)}, \Xi_{s,t}) = 0$ on Ω_0, and $\Xi_{s,t}$ satisfies the conditions (D.4.1) and (D.4.2). In particular, if $s = 0$, then there exists Ξ_t such that $\lim_{\epsilon \to 0} \sup_{0 \le t \le T} d(\Xi_t^{(\epsilon)}, \Xi_t) = 0$ $(\forall T > 0)$ a.s. Then, since $\Xi_0^{(\epsilon)} = \delta_0$, we have $\Xi_0 = \delta_0$. Furthermore, $\{\Xi_t^{(\epsilon)}\}$ satisfies the condition (D.2), and so does $\{\Xi_t\}$.

Since $\{\Xi_{s,t}^{(\epsilon)}\}$ satisfies the condition (D.4.5), we have $\Phi_{\Xi_{s,t}^{(\epsilon)}}(\boldsymbol{z}) = \Phi_{\Xi_{t-s}^{(\epsilon)}}(\boldsymbol{z})$. Letting $\epsilon \to 0$, we obtain that $\Xi_{s,t}$ satisfies the condition (D.4.5).

We have, almost surely, $d(\Xi_t, \Xi_{0,s} * \Xi_{s,t}) = \lim_{\epsilon \to 0} d(\Xi_t^{(\epsilon)}, \Xi_{0,s} * \Xi_{s,t})$ and

$$\begin{aligned} d\big(\Xi_t^{(\epsilon)}, \Xi_{0,s} * \Xi_{s,t}\big) &= d\big(\Xi_{0,s}^{(\epsilon)} * \Xi_{s,t}^{(\epsilon)}, \Xi_{0,s} * \Xi_{s,t}\big) \\ &\le d\big(\Xi_{0,s}^{(\epsilon)} * \Xi_{s,t}^{(\epsilon)}, \Xi_{0,s}^{(\epsilon)} * \Xi_{s,t}\big) + d\big(\Xi_{0,s}^{(\epsilon)} * \Xi_{s,t}, \Xi_{0,s} * \Xi_{s,t}\big) \\ &\le d\big(\Xi_{s,t}^{(\epsilon)}, \Xi_{s,t}\big) + d\big(\Xi_{0,s}^{(\epsilon)}, \Xi_{0,s}\big). \end{aligned}$$

Hence, letting $\epsilon \to 0$, we have the condition (D.4.3).

For any $n \ge 2$ and $0 = t_0 < t_1 < \cdots < t_n$, $\Xi_{t_0,t_1}^{(\epsilon)}, \Xi_{t_1,t_2}^{(\epsilon)}, \ldots, \Xi_{t_{n-1},t_n}^{(\epsilon)}$ are independent. From Lemma 2.4, the condition (D.4.4) is satisfied.

Finally, we get (2.1) as $\epsilon \to 0$ in (2.4). We have completed the proof of Theorem 2.1. Theorem 2.3 is obtained from Theorem 2.1. □

3. Proof of Theorem 1.2.

We denote by $\mathscr{F}_0$ the collection of $\Omega_0 \cap A$, $A \in \mathscr{F}$. In the proofs from Lemma 3.1 to Lemma 3.13, we take a probability space $(\Omega_0, \mathscr{F}_0, P)$ in place of $(\Omega, \mathscr{F}, P)$. So a Lévy process $\{\Xi_t\}$ with the condition (D.4) is defined on $(\Omega_0, \mathscr{F}_0, P)$ in their lemmas. Then we denote by $\mathscr{F}_{s,t}$ the σ-algebra generated by $\{\Xi_{\tau_1,\tau_2}^{\omega} : s \le \tau_1 < \tau_2 \le t\}$.

Lemma 3.1. *Let $T > 0$. The number of jumping times $t \in [0,T]$ satisfying $\Xi_{t-,t} \in A^{(\epsilon)}$ is finite.*

Proof. Suppose that $N(A^{(\epsilon)}; T, \omega) = \infty$. Then the jumping times has a limit point t_0 satisfying $t_j \uparrow t_0$ or $t_j \downarrow t_0$, where $\{t_j\}$ are jumping times satisfying $\Xi_{t_j-,t_j} \in A^{(\epsilon)}$.

First of all, we suppose that $t_j \uparrow t_0$. Let $1 > \delta > 0$. There is η with $0 < \eta < 1$ such that, for sufficiently large j, $|\hat{\Xi}_{t_j,t_0}(z)| \ge \delta$ for any z with $|z| \le \eta$. Set $\mu_j = \Xi_{t_j-,t_j}$. For any z with $|z| \le \eta$ and for sufficiently large j,

$$d\big(\Xi_{t_j-,t_0}, \Xi_{t_j,t_0}\big) \ge 2^{-1}\big|1 - \hat{\mu}_j(z)\big|\big|\widehat{\Xi}_{t_j,t_0}(z)\big| \ge 2^{-1}\delta\big|1 - \hat{\mu}_j(z)\big|. \tag{3.1}$$

Here we have, for any z with $|z| \le \eta$,

$$\begin{aligned} 2^{-1}\big|1 - \hat{\mu}_j(z)\big| &\ge 2^{-1} \int (1 - \cos zx)\mu_j(dx) \\ &\ge 2^{-2}z^2 \int_{|x|\le 1} \left(\frac{\sin 2^{-1}xz}{2^{-1}xz}\right)^2 x^2 \mu_j(dx) \ge c_1 z^2 \int_{|x|\le 1} x^2 \mu_j(dx), \end{aligned}$$

where c_1 is some positive constant. Hence we have, for any z with $|z| \le \eta$,

$$d\big(\Xi_{t_j-,t_0},\Xi_{t_j,t_0}\big) \geq c_1\delta z^2\big\langle \mu_j, x^2 I_{[|x|\leq 1]}\big\rangle. \tag{3.2}$$

And we have

$$\begin{aligned}\int_{-1}^{1} 2^{-1}\big|1-\hat{\mu}_j(z)\big|dz &\geq 2^{-1}\int_{-1}^{1}\int (1-\cos zx)\mu_j(dx)dz \\ &= \int\left(1-\frac{\sin x}{x}\right)\mu_j(dx) \geq c_2\mu_j(|x|>1),\end{aligned}$$

where c_2 is some positive constant. Integrating both sides in (3.1), we have

$$d\big(\Xi_{t_j-,t_0},\Xi_{t_j,t_0}\big) \geq 2^{-1}c_2\delta\mu_j(|x|>1). \tag{3.3}$$

From (3.2) and (3.3) we have, for any z with $0<|z|\leq\eta$ and sufficiently large j,

$$\begin{aligned}d\big(\Xi_{t_j-,t_0},\Xi_{t_j,t_0}\big) &\geq 2^{-1}c_1\delta z^2\big\langle\mu_j, x^2 I_{[|x|\leq 1]}\big\rangle + 2^{-2}c_2\delta\mu_j(|x|>1) \\ &\geq \min\{2^{-1}c_1 z^2, 2^{-2}c_2\}\delta\langle\mu_j, |x|^2\wedge 1\rangle \\ &\geq \min\{2^{-1}c_1\eta z^2, 2^{-2}c_2\}\delta\epsilon.\end{aligned}$$

Then the left-hand side goes to zero as $j\to\infty$ and this is a contradict.

In the case where $t_j \downarrow t_0$, there is η with $0<\eta<1$ such that, for sufficiently large j, $|\widehat{\Xi}_{t_j}(z)|\geq\delta$ for any z with $|z|\leq\eta$. And, for any z with $0<|z|\leq\eta$ and for sufficiently large j,

$$d\big(\Xi_{t_j-},\Xi_{t_j}\big) \geq 2^{-1}\big|1-\hat{\mu}_j(z)\big|\big|\widehat{\Xi}_{t_j-}(z)\big| \geq 2^{-1}\big|1-\hat{\mu}_j(z)\big|\big|\widehat{\Xi}_{t_j}(z)\big| \geq 2^{-1}\delta\big|1-\hat{\mu}_j(z)\big|.$$

Hence, in the same way as above, this is a contradiction as $j\to\infty$. □

Lemma 3.2. *Let $t>s\geq 0$. For $U\in\mathscr{B}(A^{(\epsilon)})$, $N(U;t,\omega)-N(U;s,\omega)$ is $\mathscr{F}_{s,t}$-measurable. In particular, $N(U;t,\omega)$ is $\mathscr{F}_{0,t}$-measurable.*

Proof. Let $\{\nu_j : j\in\boldsymbol{N}\}$ be a dense set of $A^{(\epsilon)}$. We take

$$U_\epsilon(\nu_j) = \{\mu\in\mathscr{P}(\boldsymbol{R}) : d(\nu_j,\mu)\leq\epsilon\},\quad j\in\boldsymbol{N},\ \epsilon>0,$$

as a neighborhood system in $A^{(\epsilon)}$. We denote by $\mathscr{O}$ the π-system generated by this neighborhood system. First of all, we deal with the case of $U = U_{\epsilon_1}(\widetilde{\nu}_1)\cap\cdots\cap U_{\epsilon_m}(\widetilde{\nu}_m)$. Here $\epsilon_j>0$ for any j and $\{\widetilde{\nu}_j\}$ is a subset of $\{\nu_j\}$. Set $N^U_{s,t} =$

$N(U;t,\omega) - N(U;s,\omega)$. Then $N^U_{s,t}$ takes values in $\{0,1,2,\dots\}$. Thus it suffices to prove that $\{N^U_{s,t} \geq k\}$ is $\mathscr{F}_{s,t}$-measurable for $k \in \boldsymbol{N}$. Now we have

$$\{\omega \in \Omega_0 : N^U_{s,t}(\omega) \geq 1\} = \bigcap_{n=1}^{\infty} \bigcup_{\substack{s<r<r'\leq t \\ r'-r<n^{-1} \\ r,r' \in \boldsymbol{Q}\cup\{t\}}} \{\omega \in \Omega_0 : \Xi^\omega_{r,r'} \in U_n\} \in \mathscr{F}_{s,t}, \tag{3.4}$$

where $U_n = U_{\epsilon_1+n^{-1}}(\widetilde{\nu}_1) \cap \cdots \cap U_{\epsilon_m+n^{-1}}(\widetilde{\nu}_m)$. If $\{\omega \in \Omega_0 : N^U_{s,t}(\omega) \geq k\}$ is $\mathscr{F}_{s,t}$-measurable,

$$\begin{aligned}&\{\omega \in \Omega_0 : N^U_{s,t}(\omega) \geq k+1\} \\ &\quad = \bigcup_{\substack{s<r<t \\ r\in\boldsymbol{Q}}} \{\omega \in \Omega_0 : N^U_{s,r}(\omega) \geq k\} \cap \{\omega \in \Omega_0 : N^U_{r,t}(\omega) \geq 1\} \in \mathscr{F}_{s,t}.\end{aligned}$$

Hence $N^U_{s,t}$ is $\mathscr{F}_{s,t}$-measurable.

Set $\mathscr{A} = \{U \subseteq A^{(\epsilon)} : N^U_{s,t}$ is $\mathscr{F}_{s,t}$-measurable$\}$. Then we have $\mathscr{A} \supset \mathscr{O}$. Let $0 < \epsilon < \epsilon'$. Denote by $\bar{A}^{(\epsilon')}$ the closure of $A^{(\epsilon')}$. In the same way as above, we can show that $N^{\bar{A}^{(\epsilon')}}_{s,t}$ is $\mathscr{F}_{s,t}$-measurable. Letting $\epsilon' \downarrow \epsilon$, we obtain that $N^{A^{(\epsilon)}}_{s,t}$ is $\mathscr{F}_{s,t}$-measurable. We notice this fact and we see that $\mathscr{A}$ is a λ-system. Since $\mathscr{A}$ is a λ-system and $\mathscr{O}$ is a π-system, we have $\sigma(\mathscr{O}) \subset \mathscr{A}$. (For example, see Theorem 3.2 in [**1**].) Therefore $N^U_{s,t}$ is $\mathscr{F}_{s,t}$-measurable for any $U \in \mathscr{B}(A^{(\epsilon)})$. In particular, since $N(U;0,\omega) = 0$, $N(U;t,\omega)$ is $\mathscr{F}_{0,t}$-measurable. We have completed the proof. □

Let $t_{nj} = s + (t-s)jn^{-1}, j = 0,1,\dots,n$. For each $\omega \in \Omega_0$, we set $N^U_{n,i}(\omega) = N(U;t_{ni},\omega) - N(U;t_{n(i-1)},\omega)$ and $\Xi^\omega_{n,i} = \Xi^\omega_{t_{n(i-1)},t_{ni}}$. Furthermore we set, for each $\omega \in \Omega_0$,

$$S_{t_{n(i-1)},t_{ni}} = \begin{cases} \Xi^\omega_{n,i} & \text{if } N^U_{n,i}(\omega) \geq 1, \\ \delta_0 & \text{if } N^U_{n,i}(\omega) = 0, \end{cases} \qquad V_{t_{n(i-1)},t_{ni}} = \begin{cases} \delta_0 & \text{if } N^U_{n,i}(\omega) \geq 1, \\ \Xi^\omega_{n,i} & \text{if } N^U_{n,i}(\omega) = 0. \end{cases}$$

Set $S_{ni} = S_{t_{n(i-1)},t_{ni}}$ and $V_{ni} = V_{t_{n(i-1)},t_{ni}}$ for short. The number of discontinuous points in $(s,t]$ of $N(U;u,\omega)$ is finite. Thus there exist limits

$$S_{s,t}(U) = \lim_{n\to\infty} \mathop{*}_{i=1}^{n} S_{ni} \quad \text{and} \quad V_{s,t}(U) = \lim_{n\to\infty} \mathop{*}_{i=1}^{n} V_{ni}.$$

In particular, we set $S_t(U) = S_{0,t}(U)$ and $V_t(U) = V_{0,t}(U)$ for short. Then we have

$$\Xi_t = S_t(U) * V_t(U)$$

for each $\omega \in \Omega_0$. Let $\epsilon_0 > \epsilon > 0$. We remark that $V_{s,t}(A^{(\epsilon_0)}) = S_{s,t}(A^{(\epsilon,\epsilon_0)}) * V_{s,t}(A^{(\epsilon)})$, where $A^{(\epsilon,\epsilon_0)} = \{\mu \in \mathscr{P}(\boldsymbol{R}) : \epsilon < \langle \mu, x^2 \wedge 1\rangle \le \epsilon_0\}$.

LEMMA 3.3. *For $U \in \mathscr{B}(A^{(\epsilon)})$, $S_{s,t}(U)$ and $V_{s,t}(U)$ are $\mathscr{F}_{s,t}$-measurable.*

PROOF. S_{ni} and V_{ni} are $\mathscr{F}_{s,t}$-measurable. Hence the lemma holds. $\square$

LEMMA 3.4. *Suppose that $\{\nu_k\}_{k\ge 1} \subset \mathscr{P}(\boldsymbol{R})$. If Ξ_1 and Ξ_2 are random probability distributions, then, for $\delta_k > 0$ and $\epsilon_k > 0$, $k = 1, 2, \ldots, l$,*

$$\begin{aligned}
&\left|P\Big(A \cap \bigcap_{k=1}^{l}\{d(\Xi_1,\nu_k) < \delta_k\}\Big) - P\Big(A \cap \bigcap_{k=1}^{l}\{d(\Xi_2,\nu_k) < \delta_k\}\Big)\right| \\
&\quad \le \sum_{k=1}^{l}\big[P(A \cap \{|d(\Xi_1,\nu_k) - \delta_k| \le \epsilon_k\}) + P(A \cap \{|d(\Xi_1,\nu_k) - d(\Xi_2,\nu_k)| \ge \epsilon_k\})\big].
\end{aligned}$$

PROOF. If $d(\Xi_1,\nu_k) < \delta_k$ and $d(\Xi_2,\nu_k) \ge \delta_k$, then we have

$$d(\Xi_1,\nu_k) - d(\Xi_2,\nu_k) \le -\epsilon_k \quad \text{or} \quad 0 > d(\Xi_1,\nu_k) - \delta_k > -\epsilon_k.$$

And, if $d(\Xi_2,\nu_k) < \delta_k$ and $d(\Xi_1,\nu_k) \ge \delta_k$, then we have

$$d(\Xi_1,\nu_k) - d(\Xi_2,\nu_k) \ge \epsilon_k \quad \text{or} \quad 0 \le d(\Xi_1,\nu_k) - \delta_k < \epsilon_k.$$

Hence

$$\begin{aligned}
&\left|P\Big(A \cap \bigcap_{k=1}^{l}\{d(\Xi_1,\nu_k) < \delta_k\}\Big) - P\Big(A \cap \bigcap_{k=1}^{l}\{d(\Xi_2,\nu_k) < \delta_k\}\Big)\right| \\
&\quad \le \sum_{k=1}^{l}\big[P(A \cap \{d(\Xi_1,\nu_k) \ge \delta_k,\ d(\Xi_2,\nu_k) < \delta_k\}) \\
&\qquad\qquad + P(A \cap \{d(\Xi_1,\nu_k) < \delta_k,\ d(\Xi_2,\nu_k) \ge \delta_k\})\big] \\
&\quad \le \sum_{k=1}^{l}\big[P(A \cap \{|d(\Xi_1,\nu_k) - \delta_k| \le \epsilon_k\}) + P(A \cap \{|d(\Xi_1,\nu_k) - d(\Xi_2,\nu_k)| \ge \epsilon_k\})\big].
\end{aligned}$$

We have completed the proof. □

LEMMA 3.5. *For $U \in \mathscr{B}(A^{(\epsilon)})$, $N(U;t,\omega) - N(U;s,\omega)$ and $V_{s,t}(U)$ are independent.*

PROOF. Let $\{\nu_j\}$ be a dense set in U. Set $N = \sum_{i=1}^n N_{n,i}^U$. It suffices to prove that, for any $\delta_j > 0,\ 1 \le j \le l$, and for any $k \in \boldsymbol{N}$,

$$\begin{aligned} &P(N = k,\ d(\widetilde{\nu}_j, V_{s,t}(U)) < \delta_j\ (1 \le j \le l)) \\ &\quad = P(N = k)P(d(\widetilde{\nu}_j, V_{s,t}(U)) < \delta_j\ (1 \le j \le l)), \end{aligned} \tag{3.5}$$

where $\{\widetilde{\nu}_j\}_{1 \le j \le l}$ is a subset of $\{\nu_j\}$. As this probability is left continuous in each δ_j, we only deal with the case where each δ_j is a continuous point of the distribution of $d(\widetilde{\nu}_j, V_{s,t}(U))$.

Let $k \ge 1$. First of all, we prove that

$$\begin{aligned} &P(N = k,\ d(V_{s,t}(U), \widetilde{\nu}_j) < \delta_j\ (1 \le j \le l)) \\ &\quad = P(N = k)\frac{P(N = 0,\ d(V_{s,t}(U), \widetilde{\nu}_j) < \delta_j\ (1 \le j \le l))}{P(N = 0)}. \end{aligned} \tag{3.6}$$

If $N = k$, then we have

$$\begin{aligned} &N_{n,i(1)}^U = m_1,\ N_{n,i(2)}^U = m_2, \dots, N_{n,i(p)}^U = m_p, \\ &N_{n,j(1)}^U = N_{n,j(2)}^U = \cdots = N_{n,j(n-p)}^U = 0, \end{aligned} \tag{3.7}$$

where $1 \le p \le k$ and $m_1, \dots, m_p > 0$, $m_1 + \cdots + m_p = k$, $1 \le i(1) < i(2) < \cdots < i(p) \le n$, $\{j(1), \dots, j(n-p)\}$ do not intersect $\{i(1), \dots, i(p)\}$, and $j(1) < j(2) < \cdots < j(n-p)$.

Since $N_{n,i}^U$ and $\Xi_{n,i}$ are $\mathscr{F}_{t_{n(i-1)},t_{ni}}$-measurable and $\mathscr{F}_{t_{n0},t_{n1}}, \dots, \mathscr{F}_{t_{n(n-1)},t_{nn}}$ are independent, we obtain that $\{N_{ni(1)}^U, \dots, N_{ni(p)}^U\}$ and $\{N_{nj(1)}^U = \cdots = N_{nj(n-p)}^U = 0,\ *_{\sigma=1}^{n-p}\, \Xi_{n,j(\sigma)}\}$ are independent. Hence we have

$$\begin{aligned} &P\Big(N = k,\ d\Big(\mathop{*}_{i=1}^{n} V_{ni}, \widetilde{\nu}_j\Big) < \delta_j\ (1 \le j \le l)\Big) \\ &\quad = \sum P\Big(N_{n,i(1)}^U = m_1, \dots, N_{n,i(p)}^U = m_p, N_{n,j(1)}^U = \cdots = N_{n,j(n-p)}^U = 0, \\ &\qquad\qquad d\Big(\mathop{*}_{i=1}^{n} V_{ni}, \widetilde{\nu}_j\Big) < \delta_j\ (1 \le j \le l)\Big) \end{aligned}$$

$$
\begin{aligned}
&= \sum P\big(N^U_{n,i(1)} = m_1, \dots, N^U_{n,i(p)} = m_p\big) \\
&\quad \times \frac{P\big(N^U_{n,j(1)} = \cdots = N^U_{n,j(n-p)} = 0\big) P\big(N^U_{n,i(1)} = \cdots = N^U_{n,i(p)} = 0\big)}{P\big(N^U_{n,i(1)} = \cdots = N^U_{n,i(p)} = 0\big) P\big(N^U_{n,j(1)} = \cdots = N^U_{n,j(n-p)} = 0\big)} \\
&\quad \times P\Big(N^U_{n,j(1)} = \cdots = N^U_{n,j(n-p)} = 0,\ d\Big(\mathop{*}_{\sigma=1}^{n-p} \Xi_{n,j(\sigma)}, \widetilde{\nu}_j\Big) < \delta_j\ (1 \le j \le l)\Big) \\
&= \sum P\big(N^U_{n,i(1)} = m_1, \dots, N^U_{n,i(p)} = m_p, N^U_{n,j(1)} = \cdots = N^U_{n,j(n-p)} = 0\big) \\
&\quad \times \frac{P\big(N = 0,\ d\big(*_{\sigma=1}^{n-p} \Xi_{n,j(\sigma)}, \widetilde{\nu}_j\big) < \delta_j\ (1 \le j \le l)\big)}{P(N=0)},
\end{aligned}
\tag{3.8}
$$

where the summation runs through the condition (3.7). From Lemma 3.4, we obtain that

$$
\begin{aligned}
&\Big| P\Big(N = 0,\ d\Big(\mathop{*}_{\sigma=1}^{n-p} \Xi_{n,j(\sigma)}, \widetilde{\nu}_j\Big) < \delta_j\ (1 \le j \le l)\Big) \\
&\qquad - P\big(N = 0,\ d(V_{s,t}(U), \widetilde{\nu}_j) < \delta_j\ (1 \le j \le l)\big)\Big| \\
&\le \sum_{j=1}^{l} \Big[P(|d(V_{s,t}(U), \widetilde{\nu}_j) - \delta_j| \le \epsilon_j) \\
&\qquad + P\Big(N = 0,\ |d(V_{s,t}(U), \widetilde{\nu}_j) - d\Big(\mathop{*}_{\sigma=1}^{n-p} \Xi_{n,j(\sigma)}, \widetilde{\nu}_j\Big)| \ge \epsilon_j\Big)\Big] \\
&\le \sum_{j=1}^{l} \Big[P(|d(V_{s,t}(U), \widetilde{\nu}_j) - \delta_j| \le \epsilon_j) + P\Big(N = 0,\ d\Big(V_{s,t}(U), \mathop{*}_{\sigma=1}^{n-p} \Xi_{n,j(\sigma)}\Big) \ge \epsilon_j\Big)\Big].
\end{aligned}
\tag{3.9}
$$

Let $\epsilon > 0$. Taking a sufficiently small ϵ_j, we have

$$
P(|d(V_{s,t}(U), \widetilde{\nu}_j) - \delta_j| \le \epsilon_j) < \epsilon.
$$

It is because δ_j is a continuous point of the distribution of $d(V_{s,t}(U), \widetilde{\nu}_j)$. And, if $N = 0$, then $V_{s,t}(U) = \Xi_{s,t}$. Hence we have

$$
\begin{aligned}
&P\Big(N = 0,\ d\Big(V_{s,t}(U), \mathop{*}_{\sigma=1}^{n-p} \Xi_{n,j(\sigma)}\Big) \ge \epsilon_j\Big) \\
&\quad \le P\Big(d\Big(\Xi_{s,t}, \mathop{*}_{\sigma=1}^{n-p} \Xi_{n,j(\sigma)}\Big) \ge \epsilon_j\Big) \le P\Big(d\Big(\delta_0, \mathop{*}_{\pi=1}^{p} \Xi_{n,i(\pi)}\Big) \ge \epsilon_j\Big) \\
&\quad \le \epsilon_j^{-1} \sum_{\pi=1}^{p} Ed(\delta_0, \Xi_{n,i(\pi)}) = \epsilon_j^{-1} p \times Ed(\delta_0, \Xi_{(t-s)n^{-1}}).
\end{aligned}
$$

As $n \to \infty$, the last probability converges uniformly to 0 under the condition (3.7) and so does (3.9). From Lemma 3.4, we obtain that

$$
\begin{aligned}
&\Big| P\Big(N = k,\ d\Big(\overset{n}{\underset{i=1}{*}} V_{ni}, \widetilde{\nu}_j\Big) < \delta_j\ (1 \le j \le l)\Big) \\
&\qquad - P(N = k,\ d(V_{s,t}(U), \widetilde{\nu}_j) < \delta_j\ (1 \le j \le l))\Big| \\
&\quad \le \sum_{j=1}^{l} \Big[P(|d(V_{s,t}(U), \widetilde{\nu}_j) - \delta_j)| \le \epsilon_j) \\
&\qquad\qquad + P\Big(\Big| d\Big(\overset{n}{\underset{i=1}{*}} V_{ni}, \widetilde{\nu}_j\Big) - d(V_{s,t}(U), \widetilde{\nu}_j)\Big| \ge \epsilon_j\Big)\Big] \\
&\quad \le \sum_{j=1}^{l} \Big[P(|d(V_{s,t}(U), \widetilde{\nu}_j) - \delta_j)| \le \epsilon_j) + P\Big(d\Big(\overset{n}{\underset{i=1}{*}} V_{ni}, V_{s,t}(U)\Big) \ge \epsilon_j\Big)\Big]. \qquad (3.10)
\end{aligned}
$$

Let $\epsilon > 0$. Taking a sufficiently small ϵ_j, we have $P(|d(V_{s,t}(U), \widetilde{\nu}_j) - \delta_j| \le \epsilon_j) < \epsilon$. And we have $\lim_{n\to\infty} P(d(*_{i=1}^{n} V_{ni}, V_{s,t}(U)) \ge \epsilon_j) = 0$. Hence (3.10) goes to 0 as $n \to \infty$.

Consequently, we obtain (3.6) as $n \to \infty$ in (3.8). It is obvious that (3.6) holds for $k = 0$. Hence, from (3.6), we have

$$
\begin{aligned}
&P(d(V_{s,t}(U), \widetilde{\nu}_j) < \delta_j\ (1 \le j \le l)) \\
&\quad = \sum_{k=0}^{\infty} P(N = k) \frac{P(N = 0,\ d(V_{s,t}(U), \widetilde{\nu}_j) < \delta_j\ (1 \le j \le l))}{P(N = 0)} \\
&\quad = \frac{P(N = 0,\ d(V_{s,t}(U), \widetilde{\nu}_j) < \delta_j\ (1 \le j \le l))}{P(N = 0)}.
\end{aligned}
$$

We substitute the above equation into (3.6) and (3.5) is obtained. We have completed the proof. □

Let $\{\Xi'_t\}$ be a $\mathscr{P}(\boldsymbol{R})$-valued stochastic process. In this section, the process $\{\Xi'_t\}$ is called an additive process on $\mathscr{P}(\boldsymbol{R})$, if it satisfies (D.1), (D.2), (D.4.1), (D.4.2), (D.4.3), and (D.4.4), except (D.4.5).

Lemma 3.6. *For $U \in \mathscr{B}(A^{(\epsilon)})$, $\{V_t(U)\}$ is an additive process on $\mathscr{P}(\boldsymbol{R})$.*

Proof. As $\{\Xi_t\}$ is stochastically continuous, we have

$$
P(V_{t-}(U) \ne V_t(U)) = P(V_{t-,t}(U) \ne \delta_0) \le P(\Xi_{t-,t} \ne \delta_0) = 0.
$$

Hence $V_t(U)$ is stochastically continous.

Let $0 = t_0 < t_1 < \cdots < t_n$. From Lemma 3.3, $V_{t_{j-1},t_j}(U)$ is $\mathscr{F}_{t_{j-1},t_j}$-measurable for j with $1 \le j \le n$. Hence, $\{V_{t_{j-1},t_j}(U)\}_{1\le j\le n}$ are independent.

The conditions (D.4.1), (D.4.2), and (D.4.3) are satisfied from the way of making $V_{s,t}(U)$. □

Lemma 3.7. *Let $U \in \mathscr{B}(A^{(\epsilon)})$. $\{N(U;t,\omega) : t \ge 0\}$ and $\{V_{s,t}(U) : 0 \le s < t\}$ are independent.*

Proof. Let $0 \le s_1 < s_2 < \cdots < s_m$ and $0 \le \sigma_1 < \sigma_2 < \cdots < \sigma_\mu$. It suffices to prove $\mathscr{R} = (N_i = N(U; s_i, \omega) : i = 1,2,\dots,m)$ and $\mathscr{G} = (V_i = V_{\sigma_i,\sigma_{i+1}}(U) : i = 1,2,\dots,\mu-1)$ are independent.

We put $s_1,\dots,s_m,\sigma_1,\dots,\sigma_\mu$ in order as follows:

$$0 \le t_0 < t_1 < t_2 < \cdots < t_n,\ (n \le m + \mu - 1).$$

Set $N_i' = N(U;t_i,\omega) - N(U;t_{i-1},\omega)$ and $V_i' = V_{t_{i-1},t_i}(U)$. Furthermore, it suffices to prove that $\mathscr{R}' = (N_i' : i = 1,2,\dots,n)$ and $\mathscr{G}' = (V_i' : i = 1,2,\dots,n)$ are independent. From Lemmas 3.2 and 3.3, N_i' and V_i' are $\mathscr{F}_{t_{i-1},t_i}$-measurable. Since $\mathscr{F}_{t_{i-1},t_i}$, $i = 1,2,\dots,n$, are independent, (N_i', V_i'), $i = 1,2,\dots,n$, are independent. And N_i' and V_i' are independent from Lemma 3.5. Hence $N_1', V_1', \dots, N_n', V_n'$ are independent and $\mathscr{R}'$ and $\mathscr{G}'$ are independent. We have completed the proof of the lemma. □

Lemma 3.8. *Let $\Xi_j = \{\Xi_t^j\}$ be a Lévy process on $\mathscr{P}(\boldsymbol{R})$ for $j = 1,2,\dots,m$. If Ξ_k and $\{\Xi_{k+1},\dots,\Xi_m\}$ are independent for $k = 1,2,\dots,m-1$, then $\Xi_1,\dots,\Xi_m$ are independent.*

Proof. Let $W_k = \{\Xi_k,\dots,\Xi_m\}$. It suffices to prove that $\Xi_1,\Xi_2,\dots,\Xi_{k-1}$, W_k are independent for $k = 2,3,\dots,m$. It is obvious if $k = 2$. Suppose that $\Xi_1,\Xi_2,\dots,\Xi_{k-1},W_k$ are independent. Here $W_k = \{\Xi_k, W_{k+1}\}$, and Ξ_k and W_{k+1} are independent. Hence $\Xi_1,\Xi_2,\dots,\Xi_k,W_{k+1}$ are independent. By mathematical induction, it holds in the case where $k = m$. □

Define $\varphi_U(\Xi_t) = V_t(U)$ and $\psi_U(\Xi_t) = N(U;t,\omega)$ for any additive process $\{\Xi_t\}$ on $\mathscr{P}(\boldsymbol{R})$. If $A \cap B = \phi$, then

$$\begin{cases} \varphi_B(\varphi_A) = \varphi_{A+B}, \\ \psi_B(\varphi_A) = \psi_B. \end{cases} \tag{3.11}$$

Lemma 3.9. *If U_i, $i = 1,2,\dots,n$, are pairwise disjoint, then the follow-*

ing $n+1$ *stochastic processes are independent*: $\mathscr{R}_i = \{N(U_i;t,\omega) : t \geq 0\}$, $i = 1,2,\ldots,n$, *and* $\mathscr{V} = \{V_t(\sum_{\nu=1}^{n} U_\nu) : t \geq 0\}$.

Proof. We remark that we can define $N(B;t,\omega)$ in the case of additive processes on $\mathscr{P}(\boldsymbol{R})$. And all of Lemmas 3.1, 3.2, 3.3, 3.5, 3.6, and 3.7 hold for additive processes on $\mathscr{P}(\boldsymbol{R})$. By using these facts, we can prove the lemma as follows.

Put $\chi = \{\Xi_t : t \geq 0\}$ and $\mathscr{V}_i = \{V_t(\sum_{\nu=1}^{i} U_\nu) : t \geq 0\}$, $i = 1,2,\ldots,n$. Then we have $\mathscr{R}_1 = \psi_{U_1}(\chi)$ and $\mathscr{V}_1 = \varphi_{U_1}(\chi)$. By virtue of Lemma 3.6, $\mathscr{V}_1 = \{V_t(U_1)\}$ is an additive process. Denote by $\mathscr{F}_{U_1}$ the σ-algebra generated by $\{V_{\tau_1,\tau_2}(U_1) : 0 \leq \tau_1 < \tau_2 \leq t\}$. From (3.11), we have

$$\mathscr{R}_k = \psi_{U_k}(\chi) = \psi_{U_k}(\varphi_{U_1}(\chi)) = \psi_{U_k}(\mathscr{V}_1),\ k = 2,\ldots,n$$

and

$$\mathscr{V} = \mathscr{V}_n = \varphi_{U_1+\cdots+U_n}(\chi) = \varphi_{U_2+\cdots+U_n}(\varphi_{U_1}(\chi)) = \varphi_{U_2+\cdots+U_n}(\mathscr{V}_1).$$

Recall that $\mathscr{V}_1$ is an additive process. Here Lemmas 3.2 and 3.3 are applied to the additive process $\mathscr{V}_1$. Then it follows that $\mathscr{R}_k, k = 2,3,\ldots,n$, and $\mathscr{V}$ are $\mathscr{F}_{U_1}$-measurable. Therefore, from Lemma 3.7, $\mathscr{R}_1$ and $\{\mathscr{R}_2,\ldots,\mathscr{R}_n,\mathscr{V}\}$ are independent.

Furthermore, from (3.11), we have $\mathscr{R}_2 = \psi_{U_2}(\chi) = \psi_{U_2}(\varphi_{U_1}(\chi)) = \psi_{U_2}(\mathscr{V}_1)$ and $\mathscr{V}_2 = \varphi_{U_1+U_2}(\chi) = \varphi_{U_2}(\varphi_{U_1}(\chi)) = \varphi_{U_2}(\mathscr{V}_1)$. And we have

$$\mathscr{R}_k = \psi_{U_k}(\chi) = \psi_{U_k}(\varphi_{U_1+U_2}(\chi)) = \psi_{U_k}(\mathscr{V}_2),\ k = 3,4,\ldots,n$$

and

$$\mathscr{V} = \mathscr{V}_n = \varphi_{U_1+U_2+\cdots+U_n}(\chi) = \varphi_{U_3+\cdots+U_n}(\varphi_{U_1+U_2}(\chi)) = \varphi_{U_3+\cdots+U_n}(\mathscr{V}_2).$$

Here $\mathscr{V}_2$ is an additive process from Lemma 3.6. Hence Lemmas 3.2 and 3.3 are applied to the additive process $\mathscr{V}_2$. Then $\mathscr{R}_2$ and $\{\mathscr{R}_3,\ldots,\mathscr{R}_n,\mathscr{V}\}$ are independent from Lemma 3.7. Repeating the same way as above, we can obtain that $\mathscr{R}_k$ and $\{\mathscr{R}_{k+1},\ldots,\mathscr{R}_n,\mathscr{V}\}$ are independent for $k = 3,\ldots,n-1$, and that $\mathscr{R}_n$ and $\mathscr{V}$ are independent. Therefore, by virtue of Lemma 3.8, $\mathscr{R}_1,\ldots,\mathscr{R}_n,\mathscr{V}$ are independent. □

Lemma 3.10. *For* $U \in \mathscr{B}(A^{(\epsilon)})$, $\{S_t(U)\}$ *and* $\{V_t(U)\}$ *are independent.*

Proof. Let $\{\nu_j\}$ be a dense set in U. Then, for any $\delta > 0$, we have $U = \bigcup_{j=1}^{\infty} B_j$, where $B_j = \{\mu \in \mathscr{P}(\boldsymbol{R}) : d(\nu_j, \mu) < \delta\} \cap U$. Here we set $B_j^c = U \backslash B_j$ for $j = 1, 2, \ldots$. Then we have $U = \bigcup_{j=1}^{\infty} B_j = \sum_{j=1}^{\infty} C_j$, where $C_1 = B_1$ and $C_j = B_1^c \cap B_2^c \cap \cdots \cap B_{j-1}^c \cap B_j$ for $j \geq 2$. We remark that $d(\nu_j, \mu) < \delta$ for any $\mu \in C_j$. Furthermore, we define $\nu_j^{N(C_j;t,\omega)} = \nu_j^{n*}$, if $n = N(C_j; t, \omega)$. Here ν_j^{n*} is the n-hold convolution of ν_j. When $n = 0$, ν_j^{n*} is understood to be δ_0.

Let $n = N(U; t, \omega)$. For each ω, $S_t(U)$ is represented as

$$\mu_{1,1} * \mu_{1,2} * \cdots * \mu_{1,m(1)} * \cdots * \mu_{l,1} * \cdots * \mu_{l,m(l)},$$

where $\mu_{i,j}$ belongs to some $C_{\sigma(i)}$ for all $1 \leq j \leq m(i)$. Here $m(1) > 0, \ldots, m(l) > 0$, $m(1)+m(2)+\cdots+m(l) = n$, and $m(1) = N(C_{\sigma(1)}; t, \omega), \ldots, m(l) = N(C_{\sigma(l)}; t, \omega)$. Then $N(C_i; t, \omega) = 0$ if $i \notin \{\sigma(1), \sigma(2), \ldots, \sigma(l)\}$. Hence we have

$$\begin{aligned} d\Big(S_t(U), \mathop{*}_{i=1}^{\infty} \nu_i^{N(C_i;t,\omega)}\Big) &= d\Big(S_t(U), \mathop{*}_{i=1}^{l} \nu_{\sigma(i)}^{N(C_{\sigma(i)};t,\omega)}\Big) \\ &\leq \sum_{i=1}^{l} \sum_{j=1}^{m(i)} d(\mu_{i,j}, \nu_{\sigma(i)}) \leq \delta n. \end{aligned}$$

Therefore $*_{i=1}^{\infty} \nu_i^{N(C_i;t,\omega)} \to S_t(U)$ as $\delta \to 0$ a.s.

Let $0 \leq t_0 < t_2 < \cdots < t_n$. Then we have

$$\lim_{\delta \to 0} d\Big(\mathop{*}_{i=1}^{\infty} \nu_i^{N(C_i;t_j,\omega)}, S_{t_j}(U)\Big) = 0$$

for $j = 1, 2, \ldots, n$ a.s. From Lemma 3.9, $\{*_{i=1}^{k} \nu_i^{N(C_i;t_j,\omega)} : j = 1, 2, \ldots, n\}$ and $\{V_t(\sum_{i=1}^{k} C_i) : t \geq 0\}$ are independent. In the same way as in the proof of Lemma 2.4, we can show that $\{*_{i=1}^{\infty} \nu_i^{N(C_i;t_j,\omega)} : j = 1, 2, \ldots, n\}$ and $\{V_t(U) : t \geq 0\}$ are independent and, furthermore, that $S_{t_1}(U), \ldots, S_{t_n}(U)$ and $\{V_t(U) : t \geq 0\}$ are independent. We have completed the proof. $\square$

Lemma 3.11. *For $U \in \mathscr{B}(A^{(\epsilon)})$, $\{N(U; t, \omega) : t \geq 0\}$ is a Poisson process.*

Proof. Let $T > 0$. From Lemma 3.1 the number of discontinuous points of $N(U; t, \omega)$ is finite in $[0, T]$ almost surely. Furthermore, $N(U; t, \omega)$ is an increasing right continuous step function with jumps of height 1 almost surely.

Set $N_{s,t}^U = N(U; t, \omega) - N(U; s, \omega)$. Let $0 \leq t_0 < t_1 < \cdots < t_n \leq T$. From Lemma 3.2, N_{t_{i-1},t_i}^U, $i = 1, 2, \ldots, n$, are $\mathscr{F}_{t_{i-1},t_i}$-measurable, $i = 1, 2, \ldots, n$, respectively. And, from the definition of Lévy processes on $\mathscr{P}(\boldsymbol{R})$, it follows that

$\mathscr{F}_{t_{i-1},t_i}$, $i = 1, 2, \ldots, n$, are independent. Hence, $N^U_{t_{i-1},t_i}$, $i = 1, 2, \ldots, n$, are independent.

And we have

$$P(N(U;t-,\omega) \neq N(U;t,\omega)) \leq P(\Xi^\omega_{t-} \neq \Xi^\omega_t) = 0,$$

so that $N(U;t,\omega)$ is stochastically continuous.

In order to show $N(U;t,\omega) - N(U;s,\omega) \overset{(d)}{=} N(U;t-s,\omega)$, it suffices to prove

$$P(N(U;t,\omega) - N(U;s,\omega) \geq k) = P(N(U;t-s,\omega) \geq k) \tag{3.12}$$

for $k = 1, 2, \ldots$. Take $\mathscr{O}$ in the proof of Lemma 3.2. The finite dimensional distributions of $\{\Xi_t\}$ are identical with those of $\{\Xi_{s,s+t} : t \geq 0\}$. As mentioned in the proof of Lemma 3.2, we take $U = U_{\epsilon_1}(\widetilde{\nu}_1) \cap \cdots \cap U_{\epsilon_m}(\widetilde{\nu}_m)$ and $U_n = U_{\epsilon_1+n^{-1}}(\widetilde{\nu}_1) \cap \cdots \cap U_{\epsilon_m+n^{-1}}(\widetilde{\nu}_m)$. Then, from (3.4), we have

$$\begin{aligned} P\big(N^U_{s,t}(\omega) \geq 1\big) &= P\Bigg(\bigcap_{n=1}^{\infty} \bigcup_{\substack{s<r<r'\leq t \\ r'-r<n^{-1} \\ r,r' \in \boldsymbol{Q}\cup\{t\}}} \big\{\omega \in \Omega_0 : \Xi^\omega_{r,r'} \in U_n\big\}\Bigg) \\ &= P\Bigg(\bigcap_{n=1}^{\infty} \bigcup_{\substack{0<r<r'\leq t-s \\ r'-r<n^{-1} \\ r,r' \in \boldsymbol{Q}\cup\{t-s\}}} \big\{\omega \in \Omega_0 : \Xi^\omega_{r,r'} \in U_n\big\}\Bigg) = P(N^U_{0,t-s} \geq 1). \end{aligned} \tag{3.13}$$

If (3.12) holds for $k-1$, then we have

$$\begin{aligned} P\big(N^U_{s,t}(\omega) \geq k\big) &= \sum_{\substack{s<r<t \\ r\in\boldsymbol{Q}}} P\big(N^U_{s,r} \geq k-1\big) P\big(N^U_{r,t} \geq 1\big) \\ &= \sum_{\substack{0<r<t-s \\ r\in\boldsymbol{Q}}} P\big(N^U_{0,r} \geq k-1\big) P\big(N^U_{r,t-s} \geq 1\big) = P\big(N^U_{0,t-s} \geq k\big). \end{aligned}$$

Set $\mathscr{A} = \{U \subseteq A^{(\epsilon)} : N^U_{s,t} \overset{(d)}{=} N^U_{t-s}\}$. In the same way as in the proof of Lemma 3.2, we can show that $A^{(\epsilon)} \in \mathscr{A}$. And, let $U_1, U_2, \cdots \in \mathscr{A}$ and $U_m \cap U_n = \phi$ for $m \neq n$. Then, from Lemma 3.9, $E\exp[iz\sum_{j=1}^\infty N^{U_j}_{s,t}] = \Pi_{j=1}^\infty E\exp[izN^{U_j}_{t-s}] = E\exp[iz\sum_{j=1}^\infty N^{U_j}_{t-s}]$, so we have $\bigcup_{j=1}^\infty U_j \in \mathscr{A}$. Furthermore, let $U \in \mathscr{A}$. By virtue of Lemma 3.9, $N^U_{s,t}$ and $N^{A^{(\epsilon)}\setminus U}_{s,t}$ are independent, and N^U_{t-s} and $N^{A^{(\epsilon)}\setminus U}_{t-s}$

are independent. Hence we have

$$E\exp\left[izN_{s,t}^{A^{(\epsilon)}\setminus U}\right]=\frac{E\exp\left[izN_{s,t}^{A^{(\epsilon)}}\right]}{E\exp\left[izN_{s,t}^{U}\right]}=\frac{E\exp\left[izN_{t-s}^{A^{(\epsilon)}}\right]}{E\exp\left[izN_{t-s}^{U}\right]}=E\exp\left[izN_{t-s}^{A^{(\epsilon)}\setminus U}\right],$$

so that we have $A^{(\epsilon)}\setminus U\in\mathscr{A}$. There $\mathscr{A}$ is λ-system. By using the π-λ theorem in [**1**, p. 42], (3.12) holds for $U\in\mathscr{B}(A^{(\epsilon)})$. □

Lemma 3.12. *The additive process* $\{V_t(A^{(\epsilon)})\}$ *satisfies the condition* (D.4.5). *Hence* $\{V_t(A^{(\epsilon)})\}$ *is a Lévy process on* $\mathscr{P}(\boldsymbol{R})$ *with the condition* (D.4).

Proof. Let $0<\epsilon<\epsilon'$. As V_{ni}, $i=1,2,\dots,n$, are $\mathscr{F}_{t_{n(i-1)},t_{ni}}$-measurable from Lemma 3.2, they are independent. Hence $\Phi_{V_{s,t}(\bar{A}^{(\epsilon')})}(\boldsymbol{z})=\lim_{n\to\infty}\Pi_{i=1}^{n}\Phi_{V_{ni}}(\boldsymbol{z})$. Here $\bar{A}^{(\epsilon')}$ is the closure of $A^{(\epsilon')}$. Let $U\in\mathscr{B}(\mathscr{P}(\boldsymbol{R}))$ with $\delta_0\notin U$. Put $u_{ni}=t_{ni}-s$. In the same way as (3.13) in the proof of Lemma 3.11, we have

$$\begin{aligned}
&P(V_{ni}\in U)\\
&\quad=P\Big(\Xi_{n,i}\in U,N_{n,i}^{\bar{A}^{(\epsilon')}}=0\Big)=P(\Xi_{n,i}\in U)-P\Big(\Xi_{n,i}\in U,N_{n,i}^{\bar{A}^{(\epsilon')}}\geq 1\Big)\\
&\quad=P(\Xi_{n,i}\in U)\\
&\qquad-P\Bigg(\bigcap_{k=k_0}^{\infty}\bigcup_{\substack{t_{n(i-1)}<r<r'\leq t_{ni}\\ r'-r<k^{-1}\\ r,r'\in\boldsymbol{Q}\cup\{t_{ni}\}}}\big\{\omega\in\Omega_0:\Xi_{t_{n(i-1)},t_{ni}}\in U,\Xi_{r,r'}^{\omega}\in\bar{A}^{(\epsilon'-k^{-1})}\big\}\Bigg)\\
&\quad=P\big(\Xi_{u_{n(i-1)},u_{ni}}\in U\big)\\
&\qquad-P\Bigg(\bigcap_{k=k_0}^{\infty}\bigcup_{\substack{u_{n(i-1)}<r<r'\leq u_{ni}\\ r'-r<k^{-1}\\ r,r'\in\boldsymbol{Q}\cup\{u_{ni}\}}}\big\{\omega\in\Omega_0:\Xi_{u_{n(i-1)},u_{ni}}\in U,\Xi_{r,r'}^{\omega}\in\bar{A}^{(\epsilon'-k^{-1})}\big\}\Bigg)\\
&\quad=P(V_{u_{n(i-1)},u_{ni}}\in U).
\end{aligned}\tag{3.14}$$

Here we took k_0 with $\epsilon'-k_0^{-1}>0$. If $U\in\mathscr{B}(\mathscr{P}(\boldsymbol{R}))$ with $\delta_0\in U$, then $P(V_{ni}\in U)=P(V_{u_{n(i-1)},u_{ni}}\in U)$ in the same way as above. Hence

$$\Phi_{V_{s,t}(\bar{A}^{(\epsilon')})}(\boldsymbol{z})=\lim_{n\to\infty}\prod_{i=1}^{n}\Phi_{V_{ni}}(\boldsymbol{z})=\lim_{n\to\infty}\prod_{i=1}^{n}\Phi_{V_{u_{n(i-1)},u_{ni}}}(\boldsymbol{z})=\Phi_{V_{t-s}(\bar{A}^{(\epsilon')})}(\boldsymbol{z}).$$

We have $\Phi_{V_{s,t}(A^{(\epsilon)})}(\boldsymbol{z}) = \Phi_{V_{t-s}(A^{(\epsilon)})}(\boldsymbol{z})$ as $\epsilon' \downarrow \epsilon$, so that the condition (D.4.5) is satisfied. By virtue of Lemma 3.6, $\{V_t(A^{(\epsilon)})\}$ is a Lévy process with the condition (D.4). $\square$

LEMMA 3.13. *The process $\{S_t(A^{(\epsilon)})\}$ is a Lévy process on $\mathscr{P}(\boldsymbol{R})$ with the condition* (D.4). *The characteristic function of $\{S_t(A^{(\epsilon)})\}$ is represented as*

$$\Phi_{S_t(A^{(\epsilon)})}(\boldsymbol{z}) = \exp\left[t\int_{A^{(\epsilon)}} \langle \mu^{\otimes \boldsymbol{N}}, e^{i\langle \boldsymbol{z},\boldsymbol{x}\rangle} - 1\rangle m(d\mu)\right] \tag{3.15}$$

for $\boldsymbol{z} \in \ell_0(\boldsymbol{N})$.

PROOF. As $\{\Xi_t\}$ is stochastically continuous, we have

$$P(S_{t-}(A^{(\epsilon)}) \neq S_t(A^{(\epsilon)})) = P(S_{t-,t}(A^{(\epsilon)}) \neq \delta_0) \leq P(\Xi_{t-} \neq \Xi_t) = 0.$$

Hence $S_t(A^{(\epsilon)})$ is stochastically continuous.

The conditions (D.4.1), (D.4.2), and (D.4.3) are satisfied from the way of making $S_{s,t}(A^{(\epsilon)})$. Let $0 = t_0 < t_1 < \cdots < t_n$. From Lemma 3.3, $S_{t_{j-1},t_j}(A^{(\epsilon)})$ is $\mathscr{F}_{t_{j-1},t_j}$-measurable for j with $1 \leq j \leq n$. Hence $\{S_{t_{j-1},t_j}(A^{(\epsilon)})\}_{1\leq j\leq n}$ are independent. In the same way as in the proof of Lemma 3.12, the condition (D.4.5) is satisfied. Hence $\{S_t(A^{(\epsilon)})\}$ is a Lévy process with the condition (D.4).

The process $S_t(A^{(\epsilon)})$ is represented as $S_t(A^{(\epsilon)}) = \int_{A^{(\epsilon)}} \xi * N(d\xi; t, \omega)$. Hence, in the same way as in the proof of Lemma 2.5, Lemma 3.1 in [**9**] is applied to $S_t(A^{(\epsilon)})$ and we have (3.15). $\square$

LEMMA 3.14. *Let $\{V_t\}$ be a Lévy process on $\mathscr{P}(\boldsymbol{R})$ with the condition* (D.4). *If sample functions of $\{V_t\}$ are continuous a.s., then the Lévy measure of V_t is zero.*

PROOF. Suppose that the characteristic function of V_t is represented as

$$\begin{aligned}
&\Phi_{V_t}(\boldsymbol{z}) \\
&= \exp\Biggl[t\Biggl(\sum_{j\in\boldsymbol{N}}\Biggl(-2^{-1}\alpha z_j^2 + i\gamma z_j + \int_{\boldsymbol{R}\setminus\{0\}}(e^{iz_jx} - 1 - iz_jxI_{[|x|\leq 1]}(x))\rho(dx)\Biggr) \\
&\qquad - \frac{\beta}{2}\Biggl(\sum_{j\in\boldsymbol{N}} z_j\Biggr)^2 \\
&\qquad + \int_{\mathscr{P}_*(\boldsymbol{R})}\Biggl\langle \mu^{\otimes\boldsymbol{N}}, e^{i\langle\boldsymbol{z},\boldsymbol{x}\rangle} - 1 - i\Biggl(\sum_{j\in\boldsymbol{N}} z_jx_jI_{[|x_j|\leq 1]}\Biggr)\Biggr\rangle m(d\mu)\Biggr)\Biggr], \qquad (3.16)
\end{aligned}$$

where the Lévy measure m is not zero. Let $\{V_t\}$ be defined on a probability space $(\Omega, \mathscr{F}, P)$, and let the conditions (D.4.1), (D.4.2) and (D.4.3) be satisfied on $\Omega_0 \in \mathscr{F}$ with $P(\Omega_0) = 1$. And, from Theorem 2.3, there also exists a Lévy process $\{W_t\}$ on a probability space $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \widetilde{P})$ such that the characteristic function is represented as (3.16). Then, let the conditions (D.4.1), (D.4.2) and (D.4.3) be satisfied on $\widetilde{\Omega}_0 \in \widetilde{\mathscr{F}}$ with $\widetilde{P}(\widetilde{\Omega}_0) = 1$. Let $T > 0$. From Remark 2.2, the probability that W_t has a discontinuous point on $[0, T]$ is positive.

Now let $\boldsymbol{D}$ be the totality of $\mathscr{P}(\boldsymbol{R})$-valued càdlàg paths on $[0, T]$. Then we define mappings $\psi : \Omega \to \boldsymbol{D}$ and $\varphi : \widetilde{\Omega} \to \boldsymbol{D}$ by

$$\psi(\omega)(t) = \begin{cases} V_t^\omega & \text{for } \omega \in \Omega_0, \\ \delta_0 & \text{for } \omega \notin \Omega_0, \end{cases} \qquad \varphi(\omega)(t) = \begin{cases} W_t^\omega & \text{for } \omega \in \widetilde{\Omega}_0, \\ \delta_0 & \text{for } \omega \notin \widetilde{\Omega}_0. \end{cases}$$

By the equality in law of $\{V_t\}$ and $\{W_t\}$, we have $P(\psi^{-1}(G)) = \widetilde{P}(\varphi^{-1}(G))$ if G is a cylinder set in $\boldsymbol{D}$. Let $\boldsymbol{C}$ be the totality of $\mathscr{P}(\boldsymbol{R})$-valued continuous paths on $[0, T]$. From right continuity, we have $\boldsymbol{C} = \bigcap_{m=1}^\infty \bigcup_{l=1}^\infty \bigcap_{\substack{r,s \in \boldsymbol{Q}_+ \cup \{T\} \\ |r-s| < l^{-1}}} \{\Xi'_t \in \boldsymbol{D} : d(\Xi'_r, \Xi'_s) < m^{-1}\}$. Here $\boldsymbol{Q}_+$ is the collection of nonnegative rational numbers. Hence $\boldsymbol{C} \in \mathscr{B}(\boldsymbol{D})$ and

$$1 = P(\psi^{-1}(\boldsymbol{C})) = \widetilde{P}(\varphi^{-1}(\boldsymbol{C})).$$

This implies that W_t is continuous on $[0, T]$ a.s. It is a contradiction. Hence $m = 0$. $\square$

Lemma 3.15. *Let $m(U) = E[N(U; 1, \omega)]$ for $U \in \mathscr{B}(A^{(\epsilon)})$. Then*

$$\int_{\mathscr{P}_*(\mathscr{R})} \langle \mu, x^2 \wedge 1 \rangle m(d\mu) < \infty.$$

Proof. Let $\widetilde{m}$ be a Lévy measure of Ξ_1. The characteristic function of Ξ_1 is represented as

$$\Phi_{\Xi_1}(\boldsymbol{z}) = \exp\left[-2^{-1}(\alpha + \beta) z_1^2 + i\gamma z_1 + \int_{\boldsymbol{R} \setminus \{0\}} \left(e^{iz_1 x} - 1 - iz_1 x I_{[|x| \le 1]}(x) \right) \Pi(dx) \right]$$

for $\boldsymbol{z} = (z_1, 0, 0, \ldots)$, where $\Pi(dx) = \rho(dx) + \int_{\mathscr{P}_*(\boldsymbol{R})} \mu(dx) \widetilde{m}(d\mu)$.

Now we have $\Xi_t = S_t(A^{(\epsilon)}) * V_t(A^{(\epsilon)})$. The processes $\{S_t(A^{(\epsilon)})\}$ and $\{V_t(A^{(\epsilon)})\}$ are Lévy processes, so that $S_1(A^{(\epsilon)})$ and $V_1(A^{(\epsilon)})$ are inifinitely divisible. Let m_1

and m_2 be Lévy measures of $S_1(A^{(\epsilon)})$ and $V_1(A^{(\epsilon)})$, respectively. From Lemma 3.13 it follows that $m_1 = m|_{A^{(\epsilon)}}$. From the uniqueness of Lévy-Khintchin representation for infinitely divisible distributions on $\boldsymbol{R}$, the measure Π uniquely exists. Hence we have

$$\begin{aligned}\infty &> \int_{\boldsymbol{R}\setminus\{0\}} (x^2 \wedge 1)\Pi(dx) \\ &\geq \int_{\mathscr{P}_*(\boldsymbol{R})} \langle \mu, x^2 \wedge 1\rangle m_1(d\mu) + \int_{\mathscr{P}_*(\boldsymbol{R})} \langle \mu, x^2 \wedge 1\rangle m_2(d\mu) \\ &\geq \int_{A^{(\epsilon)}} \langle \mu, x^2 \wedge 1\rangle m(d\mu).\end{aligned}$$

As $\epsilon \to 0$, we have $\int_{\mathscr{P}_*(\boldsymbol{R})} \langle \mu, x^2 \wedge 1\rangle m(d\mu) < \infty$. We have completed the proof. $\square$

Now we prove Theorem 1.2.

Proof of Theorem 1.2.

(i) By virtue of Lemma 3.11, $N(U;t,\omega)$ has a Poisson distribution with mean $tm(U)$. From Lemma 3.9, if $U_1, U_2, \dots, U_n$ are pairwise disjoint, then $N(U_1;t,\omega), \dots, N(U_n;t,\omega)$ are independent. It is obvious that $N(U;t,\omega)$ is a finite measure in $U \in A^{(\epsilon)}$ for every $\epsilon > 0$. From Lemma 3.15, we have

$$\int_{\mathscr{P}_*(\boldsymbol{R})} \langle \mu, x^2 \wedge 1\rangle m(d\mu) < \infty.$$

(ii) Let $t > s \geq 0$. The process $S_{s,t}(A^{(\epsilon)})$ is represented as $\int_{A^{(\epsilon)}} \xi * (N(d\xi;t,\omega) - N(d\xi;s,\omega))$. Here we define

$$S_{s,t}^{(\epsilon)} = \theta_{a_\epsilon(t-s)} \cdot S_{s,t}(A^{(\epsilon)}) = \theta_{a_\epsilon(t-s)} \cdot \int_{A^{(\epsilon)}} \xi * (N(d\xi;t,\omega) - N(d\xi;s,\omega)).$$

In particular, we define $S_t^{(\epsilon)} = S_{0,t}^{(\epsilon)}$. And we define

$$S_{s,t}^{(\epsilon,\epsilon_0)} = \theta_{a_{\epsilon,\epsilon_0}(t-s)} \cdot \int_{A^{(\epsilon,\epsilon_0)}} \xi * (N(d\xi;t,\omega) - N(d\xi;s,\omega)).$$

In particular, we define $S_t^{(\epsilon,\epsilon_0)} = \theta_{a_{\epsilon,\epsilon_0}t} \cdot \int_{A^{(\epsilon,\epsilon_0)}} \xi * N(d\xi;t,\omega)$. We have already defined $a_\epsilon = \int_{A^{(\epsilon)}} \langle \mu, xI_{[|x|\leq 1]}\rangle m(d\mu)$ and $a_{\epsilon,\epsilon_0} =$

$\int_{A^{(\epsilon,\epsilon_0)}}\langle\mu, xI_{[|x|\le 1]}\rangle m(d\mu)$ in Section 2. In the same way as in the proof of Theorem 2.1, there exists a Lévy process $\{S_t\}$ on $\mathscr{P}(\boldsymbol{R})$ such that, almost surely,

$$\lim_{\epsilon\to 0}\sup_{0\le t\le T} d(S_t^{(\epsilon)}, S_t) = 0 \quad (\forall T > 0). \tag{3.17}$$

By virtue of Lemma 3.13, the characteristic function of S_t is represented as (1.3).

(iii) Furthermore, we define

$$V_{s,t}^{(\epsilon)} = \theta_{-a_\epsilon(t-s)}\cdot V_{s,t}(A^{(\epsilon)})$$

for $t > s \ge 0$. In particular, we define $V_t^{(\epsilon)} = V_{0,t}^{(\epsilon)}$. If $\epsilon_0 > \epsilon > 0$, we have $V_{s,t}(A^{(\epsilon_0)}) = S_{s,t}(A^{(\epsilon,\epsilon_0)}) * V_{s,t}(A^{(\epsilon)})$. Then it follows that

$$d\big(\theta_{-a_{\epsilon_0}t}\cdot V_t(A^{(\epsilon_0)}), \theta_{-a_\epsilon t}\cdot V_t(A^{(\epsilon)})\big) \le d\big(\theta_{a_{\epsilon,\epsilon_0}t}\cdot S_t(A^{(\epsilon,\epsilon_0)}), \delta_0\big).$$

Here we use d_T of (2.3). For $V_t^{(\epsilon)}$, Lemma 6.3 in [**9**] becomes as follows:

$$P\Big(\sup_{0<\epsilon<\epsilon_0} d_T(V^{(\epsilon)}, V^{(\epsilon_0)}) \ge 3\eta\Big) \le 3\sup_{0<\epsilon<\epsilon_0} P\big(d_T(S^{(\epsilon,\epsilon_0)}, \delta_0) \ge \eta\big).$$

Furthermore, Lemma 6.5 in [**9**] holds for $S_{s,t}^{(\epsilon,\epsilon_0)}$. As $\{V_t^{(\epsilon)}\}$ is a Lévy process on $\mathscr{P}(\boldsymbol{R})$ with the condition (D.4) from Lemma 3.12, in the same way as in the proof of Theorem 2.1, we can show that there exists a Lévy process $\{V_t\}$ on $\mathscr{P}(\boldsymbol{R})$ with the condition (D.4) such that, almost surely,

$$\lim_{\epsilon\to 0}\sup_{0\le t\le T} d\big(V_t^{(\epsilon)}, V_t\big) = 0 \quad (\forall T > 0). \tag{3.18}$$

Then there is $\Omega_0' \in \mathscr{F}$ with $P(\Omega_0') = 1$ such that (3.17) and (3.18) hold for every $\omega \in \Omega_0'$. Set $\Omega_1 = \Omega_0 \cap \Omega_0'$. If Ξ_t has a jump with jump size μ_0, we can take $\epsilon > 0$ satisfying $\langle\mu_0, x^2\wedge 1\rangle > \epsilon$. For the ϵ and any $\omega \in \Omega_1$, $S_t^{(\epsilon)}$ and Ξ_t have the same jump with jump size μ_0 simultaneously. Hence, as $\epsilon \to 0$, it follows that S_t^ω and Ξ_t^ω have the same jumps at the same times for any $\omega \in \Omega_1$. Consequently, V_t^ω is continuous in t for $\omega \in \Omega_1$. By virtue of Lemma 3.14, the characteristic function of V_t is represented as (1.5). And, since $\Xi_t = S_t^{(\epsilon)} * V_t^{(\epsilon)}$, we obtain that $\Xi_t^\omega = S_t^\omega * V_t^\omega$ for $\omega \in \Omega_1$.

(iv) From Lemma 3.10, $\{S_t^{(\epsilon)}\}$ and $\{V_t^{(\epsilon)}\}$ are independent. Hence, in the same way as in the proof of Lemma 2.4, we can show that $\{S_t\}$ and $\{V_t\}$ are independent. $\square$

References

[1] P. Billingsley, Probability and Measure, 3rd ed., Wiley, New York, 1995.

[2] E. B. Dynkin, Criteria of continuity and absence of discontinuity of the second kind for trajectories of a Markov process, Izv. Akad. Nauk SSSR Ser. Mat., **16** (1952), 563–572.

[3] K. Itô, On stochastic processes, I (Infinitely divisible laws of probability), Japan J. Math., **18** (1942), 261–301.

[4] K. Itô, Stochastic processes, Lectures given at Aarhus University, Reprint of the 1969 original, Edited by Ole E. Barndorff-Nielsen and Ken-iti Sato, Springer-Verlag, 2004.

[5] J. R. Kinney, Continuity properties of sample functions of Markov processes, Trans. Amer. Math. Soc., **74** (1953), 280–302.

[6] P. Lévy, Sur les intégrales dont les éléments sont des variables aléatoires indépendantes, Ann. Scuola Norm. Sup. Pisa (2), **3** (1934), 337–366; **4** (1934), 217–218.

[7] P. Lévy, Théorie de l'Addition des Variables Aléatoires, Gauthier-Villars, Paris, 1937.

[8] K. Sato, Lévy Processes and Infinitely Divisible Distributions, Cambridge University Press, Cambridge, 1999.

[9] T. Shiga and H. Tanaka, Infinitely divisible random probability distributions with an application to a random motion in a random environment, Electronic J. Prob., **11** (2006), 1144–1183.

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