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Large time behavior of solutions to Schrödinger equations with a dissipative nonlinearity for arbitrarily large initial data

Dedicated to Professor Kenji Yajima on his sixtieth birthday

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Abstract. We study the asymptotic behavior in time of solutions to the Cauchy problem of nonlinear Schrödinger equations with a long-range dissipative nonlinearity given by $\lambda|u|^{p-1}u$ in one space dimension, where $1 < p \leq 3$ (namely, p is a critical or subcritical exponent) and λ is a complex constant satisfying $\operatorname{Im}\lambda < 0$ and $\big((p-1)/2\sqrt{p}\big)|\operatorname{Re}\lambda| \leq |\operatorname{Im}\lambda|$. We present the time decay estimates and the large-time asymptotics of the solution for arbitrarily large initial data, when "$p = 3$" or "$p < 3$ and p is suitably close to 3".

1. Introduction.

We study the large time behavior of solutions to the initial value problem of the following nonlinear Schrödinger equation in one space dimension:

$$\begin{cases} i\partial_t u + \dfrac{1}{2}\partial_x^2 u = \lambda \mathscr{N}(u), & t > 0,\ x \in \boldsymbol{R}, \\ u(0,x) = u_0(x), & x \in \boldsymbol{R}, \end{cases} \tag{1.1}$$

where u_0 is a complex-valued (given) initial data, u is a complex-valued unknown function, $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$, and $\mathscr{N}(u)$ is the gauge invariant power type nonlinearity discribed as

$$\mathscr{N}(u) = |u|^{p-1}u$$

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with $1 < p \leq 3$. As for the complex constant λ, we denote $\lambda_1 = \operatorname{Re}\lambda$ and $\lambda_2 = \operatorname{Im}\lambda$, and we assume the large dissipative condition, i.e.,

$$\lambda_2 < 0 \tag{1.2}$$

and

$$\frac{p-1}{2\sqrt{p}}|\lambda_1| \leq |\lambda_2|. \tag{1.3}$$

From the physical point of view, (1.1) with $p = 3$ is said to be a governing equation of the light traveling through optical fibers, in which $|u(t,x)|$ describes the amplitude of electric field, t denotes the position along the fiber and x stands for the temporal parameter expressing a form of pulse. As for the nonlinear coefficient, λ_1 denotes the magnitude of the nonlinear Kerr effect and λ_2 implies the magnitude of dissipation due to nonlinear Ohm's law (see e.g. [**1**] for detail). Therefore, $\lambda\mathscr{N}(u)$ causes a loss of energy, and we easily expect the decay of $u(t)$ for large t. One of our aims in this paper is the justification of the decaying property of $u(t)$.

There are a lot of mathematical works concerning the large-time asymptotic profiles of the solution to (1.1) for various kinds of nonlinearities. Most of these works deals with a real λ, but the ideas are still applicable to a complex λ (for small initial data). For instance, if $p > 3$, $\lambda \in \boldsymbol{C}$ and the size of $u_0(x)$ is sufficiently small, it is well-known that the solution $u(t)$ behaves like a free solution $U(t)\phi$ for large t (see e.g. [**3**] and references therein), where $U(t) = \exp((it/2)\partial_x^2)$ denotes the solution operator of the free Schrödinger equation and ϕ is called the scattering state which is determined in terms of the initial data. The strategy for this free asymptotic profile is largely relies on the rapid decay of nonlinearity. In other words, the integrability of $\mathscr{N}(u(t))/u(t) = |u(t)|^{p-1}$ around $t = \infty$ allows the nonlinearity to be regarded as negligible in the long-time dynamics, and it occurs if and only if $p > 3$ since $\int_1^\infty |u(t)|^{p-1}\,dt \sim \int_1^\infty t^{-(p-1)/2}\,dt < \infty$ by expecting that $u(t)$ decays like a free solution. On the other hand, in the case $p \leq 3$, the situation changes. In this case, we can not expect the free asymptotic profile of the solution, but some modification is required. In the case that the space dimension $n = 1$, $p = 3$ and λ is real, Ozawa [**12**] constructed modified wave operators to the equation (1.1) for small scattering states, and Hayashi and Naumkin [**5**] proved the time decay and the large-time asymptotics of $u(t)$ for small initial data. According to their results, if $\lambda \in \boldsymbol{R}$, then the small solution $u(t)$ asymptotically tends to a modified free solution like $\mathscr{F}^{-1}\exp(i\lambda|\phi(\xi)|^2\log t)\mathscr{F}U(t)\phi$ as $t \to \infty$ and the L^∞-norm of $u(t)$ decays similarly to the free solution, where $\mathscr{F}$ denotes the Fourier transform. The nonlinear Schrödinger equations have been so far treated in non-dissipative structures of nonlinearities. In [**13**], the second author of this paper has studied the dissipative critical nonlinear case, i.e., $p = 3$ and $\lambda_2 < 0$ (that is, the condition (1.2)), by imposing smallness assumption on the initial data, in which the negativity of λ_2 visibly affects the decay rate of $\|u(t)\|_{L^\infty}$ and, actually, it decays like $t^{-1/2}(\log t)^{-1/2}$ (This tells us that $u(t)$ decays more rapidly in comparison with the free solution). To derive these decaying properties in dissipative or non-dissipative structure, they wrote $u(t,x)$ as $u(t,x) = (it)^{-1/2}\exp(ix^2/2t)\mathscr{F}v(t,x/t) + (\text{error term})$ where $v(t) = U(-t)u(t)$ and estimated $\mathscr{F}v(t)$ by applying certain gauge transform. The estimate of the error terms were established in terms of the operator J, where $J = x + it\partial_x = U(t)xU(-t)$. This idea is also applied to the case of cubic derivative nonlinearity with dissipative structure due to Hayashi-Naumkin-Sunagawa [**7**]. Then our next concern is to observe the subcritical case, i.e., $p < 3$. When $\lambda_2 = 0$, Hayashi-Kaikina-Naumkin [**4**] treated this case for small data solution, and proved that $u(t)$ asymptotically behaves like a modified free solution. Their idea requires the non-zero condition, strong decay and analiticity of the initial data to overcome the difficulty of non-smooth nonlinearity and derivative loss in their estimates. If $\lambda_2 < 0$ (which is the condition (1.2)), Hayashi and Naumkin [**6**] also studied "the final value problem" with the generalized linear dispersion like $(-\partial_x^2)^{\rho/2}$ $(\rho \geq 1)$ included, where they proved the existence and uniqueness of the solution which asymptotically tends to a given modified free solution. On the other hand, when the linear dispersion is of the second order, we considered "the initial value problem" for small initial data and showed the time decay estimate of $u(t)$ as well as the asymptotic formula for large time [**9**]. Note that, in [**9**], the smallness assumption is required to minimize the growth order of $\|Ju(t)\|_{L^2}$. We also note that the analogy in [**9**] is not applied when $(-\partial_x^2)^{\rho/2}$ with $\rho \neq 2$ is included in the equation, since our argument largely depends on the concrete structure of $\exp((it/2)\partial_x^2)$. As far as the authors know, there seems to be no result established for the asymptotic behavior of large data solutions to (1.1), and in this paper, we are aimed at the time decay and asymptotic profile of $u(t)$ without any smallness assumptions on u_0 for the long-range case (i.e. $p \leq 3$). The key to remove the smallness of u_0 is to use the dissipative structure described as (1.2) and (1.3) which helps us obtain the time-global estimate of $\|\partial_x u(t)\|_{L^2} + \|Ju(t)\|_{L^2}$ and regard any error terms in our argument as negligible. Our first result is concerning the global existence and time decay rate of the solution.

THEOREM 1.1. *Assume that* $1 < p \leq 3$ *and* λ *satisfies the conditions* (1.2) *and* (1.3). *Let* $u_0 \in H^{1,0} \cap H^{0,1}$. *Then there exists a unique solution* u *to the initial value problem* (1.1) *satisfying*

$$u \in C([0,\infty); H^{1,0}), \quad Ju \in C([0,\infty); L^2), \tag{1.4}$$

$$\|u(t)\|_{H^{1,0}} + \|Ju(t)\|_{L^2} \le \|u_0\|_{H^{1,0}\cap H^{0,1}} \tag{1.5}$$

for any $t \ge 0$. Furthermore, if $(5+\sqrt{33})/4 < p \le 3$, then there exists a constant $C_0 > 0$ such that

$$\|u(t)\|_{L^\infty} \le C_0\{(2+t)\log(2+t)\}^{-1/2}, \quad \text{when} \quad p = 3, \tag{1.6}$$

$$\|u(t)\|_{L^\infty} \le C_0(1+t)^{-1/(p-1)}, \quad \text{when} \quad \tfrac{5+\sqrt{33}}{4} < p < 3 \tag{1.7}$$

for any $t \ge 0$.

Remark 1.1. We want to emphasize that Theorem 1.1 is valid without any smallness conditions on the initial data u_0.

Remark 1.2. The nonlinear effect is visible in the time decay estimates (1.6) and (1.7). In fact, the solution decays faster than the non-trivial free solution. (Recall that the L^∞-norm of the non-trivial free solution decays like $t^{-1/2}$ in one space dimension.) We also remark that, in (1.7), $u(t)$ decays like an ODE-solution satisfying $i\partial_t u = \lambda\mathscr{N}(u)$. This implies that, in subcritical case, the nonlinear dissipation is dominant over the linear dispersion for large time.

Remark 1.3. The additional assumption $p > (5+\sqrt{33})/4 = 2.686\cdots$ in Theorem 1.1 comes from the estimate of the error term (see (3.7) below).

We next present the result on the large time asymptotics of the global solution as in Theorem 1.1. To obtain Theorem 1.2, we need to make the lower bound of p enlarged.

Theorem 1.2. *Suppose that the assumptions in Theorem* 1.1 *are satisfied, and let u be the global solution as in Theorem* 1.1. *In addition, assume that $(21+\sqrt{177})/12 < p \le 3$. Then the followings hold*:

(I) *Let*

$$\Phi(t,\cdot) = \int_1^t \tau^{-(p-1)/2}|\mathscr{F}U(-\tau)u(\tau)|^{p-1}\,d\tau. \tag{1.8}$$

Then, there exists a unique $\phi \in L^2 \cap L^\infty$ such that

$$\left\|e^{i\lambda\Phi(t)}\mathscr{F}U(-t)u(t) - \phi\right\|_{L^2\cap L^\infty} = O(t^{-\beta}),$$

as $t \to \infty$ for some $\beta > 3-p$.

(II) *There exists a unique real valued function* $\eta \in L^\infty$ *such that*

$$e^{(p-1)|\lambda_2|\Phi(t,x)} = 1 + (p-1)|\lambda_2||\phi(x)|^{p-1}\int_1^t \tau^{-(p-1)/2}\,d\tau + \eta(x) + O(t^{-(\beta-(3-p)/2)})$$

in L^∞ *as* $t \to \infty$, *where* β *is the same as in* (I). *Furthermore there exists a constant* $\widetilde{T} \geq 1$ *such that*

$$1 + (p-1)|\lambda_2||\phi(x)|^{p-1}\int_1^t \tau^{-(p-1)/2}\,d\tau + \eta(x) \geq \frac{1}{2}$$

for almost every $x \in \boldsymbol{R}$ *and for any* $t \geq \widetilde{T}$.

(III) *For* $t \geq \widetilde{T}$, *let*

$$\begin{aligned} A(t,x) &= \frac{1}{(p-1)|\lambda_2|}\log\left(1 + (p-1)|\lambda_2||\phi(x)|^{p-1}\int_1^t \tau^{-(p-1)/2}\,d\tau + \eta(x)\right) \\ &= \begin{cases} \dfrac{1}{2|\lambda_2|}\log(1 + 2|\lambda_2||\phi(x)|^2\log t + \eta(x)), & \text{when } p = 3, \\ \dfrac{1}{(p-1)|\lambda_2|}\log\left(1 + \dfrac{2(p-1)|\lambda_2|}{3-p}|\phi(x)|^{p-1}(t^{\frac{3-p}{2}} - 1) + \eta(x)\right), & \\ \qquad\qquad \text{when } \frac{21+\sqrt{177}}{12} < p < 3. \end{cases} \end{aligned} \tag{1.9}$$

Then the followings hold:

$$\begin{aligned} u(t,x) &= (it)^{-1/2}e^{i|x|^2/2t}e^{-i\lambda A(t,x/t)}\phi(x/t) \\ &\quad + \begin{cases} o((t\log t)^{-1/2}), & \text{when } p = 3, \\ o(t^{-1/(p-1)}), & \text{when } \frac{21+\sqrt{177}}{12} < p < 3 \end{cases} \end{aligned} \tag{1.10}$$

in L^∞, *as* $t \to \infty$. *Furthermore*

$$\begin{aligned} &\left\|u(t) - U(t)\mathscr{F}^{-1}(e^{-i\lambda A(t,\cdot)}\phi)\right\|_{L^2} \\ &= \begin{cases} o((\log t)^{-1/2}), & \text{when } p = 3, \\ o(t^{-(1/(p-1)-1/2)}), & \text{when } \frac{21+\sqrt{177}}{12} < p < 3 \end{cases} \end{aligned} \tag{1.11}$$

as $t \to \infty$, and

$$\lim_{t\to\infty} \|u(t)\|_{L^2} = 0. \tag{1.12}$$

REMARK 1.4. The additional assumption $p > (21 + \sqrt{177})/12 = 2.858\cdots$ in Theorem 1.2 comes from the inequality (4.1) below.

REMARK 1.5. According to the asymptotic formulas (1.10) and (1.11), we see that the solution u is not asymptotically free. By the definition (1.9) of A, we can write the modification factor $e^{-i\lambda A(t,x)}$ explicitly:

$$e^{-i\lambda A(t,x)} = \frac{\exp\left\{-\frac{i\lambda_1}{2|\lambda_2|}\log(1+2|\lambda_2||\phi(x)|^2\log t+\eta(x))\right\}}{(1+2|\lambda_2||\phi(x)|^2\log t+\eta(x))^{1/2}}$$

when $p = 3$, and

$$e^{-i\lambda A(t,x)} = \frac{\exp\left\{-\frac{i\lambda_1}{(p-1)|\lambda_2|}\log\left(1+\frac{2(p-1)|\lambda_2|}{3-p}|\phi(x)|^{p-1}(t^{(3-p)/2}-1)+\eta(x)\right)\right\}}{\left(1+\frac{2(p-1)|\lambda_2|}{3-p}|\phi(x)|^{p-1}(t^{(3-p)/2}-1)+\eta(x)\right)^{1/(p-1)}}$$

when $(21+\sqrt{177})/12 < p < 3$.

This paper is organized as follows. In Section 2, we collect several lemmas required to bound $\|u(t)\|_{H^{1,0}}$ and $\|Ju(t)\|_{L^2}$ by $\|u_0\|_{H^{1,0}\cap H^{0,1}}$. In Sections 3, the global existence of the solution and the key proposition to obtain the decay estimate of $u(t)$ are proved (see Proposition 3.2). In Proposition 3.2, we derive $\|\mathscr{F}U(-t)u(t)\|_{L^\infty} \le K(\log t)^{-1/2}$ if $p = 3$, and $\le Kt^{1/2-1/(p-1)}$ otherwise for large t. Note that the positive constant K is independent of the size of initial data, and this contributes to removing the smallness assumption on u_0. (In [**9**], the dependence of $\|u_0\|_{H^{1,0}\cap H^{0,1}}$ remains in K, and so we require to impose the smallness of u_0.) In Section 4, we prove Theorem 1.2 by combining the estimates given in the former sections.

We close this section by presenting several notations. For $\psi \in \mathscr{S}'(\boldsymbol{R})$, the Fourier transform of ψ is denoted by $\mathscr{F}\psi$ or $\hat{\psi}$. For $\psi \in L^1(\boldsymbol{R})$, $\mathscr{F}\psi$ is represented as

$$\mathscr{F}\psi(\xi) = (2\pi)^{-1/2}\int_{\boldsymbol{R}} \psi(x)e^{-ix\xi}\,dx.$$

For $m, s \ge 0$, the weighted Sobolev space is defined by

$$H^{m,s} = \{\psi \in L^2(\boldsymbol{R});\ \|\psi\|_{H^{m,s}} < \infty\},$$

where

$$\|\psi\|_{H^{m,s}} = \big\|(1+|x|^2)^{s/2}(1-\partial_x^2)^{m/2}\psi\big\|_{L^2}.$$

$\|\psi\|_{L^{q_1}\cap L^{q_2}}$ and $\|\psi\|_{H^{m_1,s_1}\cap H^{m_2,s_2}}$ denote $\|\psi\|_{L^{q_1}} + \|\psi\|_{L^{q_2}}$ and $\|\psi\|_{H^{m_1,s_1}}$ $+\|\psi\|_{H^{m_2,s_2}}$, respectively. Throughout this paper, we use the following operators:

$$U(t) = e^{it\partial_x^2/2} = \mathscr{F}^{-1}e^{-it|\xi|^2/2}\mathscr{F}, \quad J = x + it\partial_x.$$

We often use the factorization : $U(t) = M(t)D(t)\mathscr{F}M(t)$, where $M(t)$ and $D(t)$ are the following operators:

$$(M(t)\psi)(x) = e^{i|x|^2/2t}\psi(x), \quad (D(t)\psi)(x) = (it)^{-1/2}\psi(x/t).$$

2. Preliminaries.

In this section, we collect several lemmas for the proof of our main theorems.

Lemma 2.1. *For $z_1, z_2 \in \boldsymbol{C}$ and $q > 1$, the following inequality holds*:

$$\begin{aligned}&\big|\operatorname{Im}\{(|z_1|^{q-1}z_1 - |z_2|^{q-1}z_2)(\overline{z_1 - z_2})\}\big|\\ &\quad \le \frac{q-1}{2\sqrt{q}}\operatorname{Re}\big\{(|z_1|^{q-1}z_1 - |z_2|^{q-1}z_2)(\overline{z_1 - z_2})\big\}.\end{aligned}$$

Lemma 2.1 is obtained by Liskevich and Perelmuter [**10**, Lemma 2.2]. According to Okazawa and Yokota [**11**], this lemma is also applicable to the theory of the complex Ginzburg-Landau equation, where they showed the global existence and smoothing property of the solution.

Applying Lemma 2.1, we derive the following two fundamental inequalities which lead us to the important estimates of $\|u(t)\|_{H^{1,0}}$ and $\|Ju(t)\|_{L^2}$.

Lemma 2.2. *Let $p > 1$, and let $\lambda = \lambda_1 + i\lambda_2 \in \boldsymbol{C}$ satisfy the conditions* (1.2) *and* (1.3). *Then for $z_1, z_2 \in \boldsymbol{C}$, the following inequality holds*:

$$\operatorname{Im}\big\{\lambda(|z_1|^{p-1}z_1 - |z_2|^{p-1}z_2)(\overline{z_1 - z_2})\big\} \le 0.$$

Proof. It follows from the conditions (1.2)–(1.3) and Lemma 2.1 that

$$\begin{aligned}
&\operatorname{Im}\left\{\lambda(|z_1|^{p-1}z_1-|z_2|^{p-1}z_2)(\overline{z_1-z_2})\right\}\\
&=\lambda_1\operatorname{Im}\left\{(|z_1|^{p-1}z_1-|z_2|^{p-1}z_2)(\overline{z_1-z_2})\right\}\\
&\quad+\lambda_2\operatorname{Re}\left\{(|z_1|^{p-1}z_1-|z_2|^{p-1}z_2)(\overline{z_1-z_2})\right\}\\
&=\lambda_1\operatorname{Im}\left\{(|z_1|^{p-1}z_1-|z_2|^{p-1}z_2)(\overline{z_1-z_2})\right\}\\
&\quad-|\lambda_2|\operatorname{Re}\left\{(|z_1|^{p-1}z_1-|z_2|^{p-1}z_2)(\overline{z_1-z_2})\right\}\\
&\le|\lambda_1|\big|\operatorname{Im}\{(|z_1|^{p-1}z_1-|z_2|^{p-1}z_2)(\overline{z_1-z_2})\}\big|\\
&\quad-|\lambda_2|\frac{2\sqrt{p}}{p-1}\big|\operatorname{Im}\{(|z_1|^{p-1}z_1-|z_2|^{p-1}z_2)(\overline{z_1-z_2})\}\big|\\
&=\left(|\lambda_1|-\frac{2\sqrt{p}}{p-1}|\lambda_2|\right)\big|\operatorname{Im}\{(|z_1|^{p-1}z_1-|z_2|^{p-1}z_2)(\overline{z_1-z_2})\}\big|\\
&\le 0.
\end{aligned}$$

Therefore this lemma is proved. □

Lemma 2.3. *Let $\lambda=\lambda_1+i\lambda_2\in\boldsymbol{C}$ satisfy the conditions* (1.2) *and* (1.3). *Let $w=w(t,x)$ be a complex valued function satisfying $w\in C([0,\infty);H^{1,0})$ and $Jw\in C([0,\infty);L^2)$. Then for all $t\ge 0$, the inequalities*

$$\operatorname{Im}\left\{\lambda\partial_x\mathscr{N}(w)(t,x)\cdot\overline{\partial_x w(t,x)}\right\}\le 0, \tag{2.1}$$

$$\operatorname{Im}\left\{\lambda(J\mathscr{N}(w))(t,x)\cdot\overline{(Jw)(t,x)}\right\}\le 0 \tag{2.2}$$

hold for almost every $x\in\boldsymbol{R}$.

Proof. Let $w\in C([0,\infty);H^{1,0})$ and $Jw\in C([0,\infty);L^2)$. We fix $t_0\ge 0$ arbitrarily. Since $w(t_0,\cdot)\in H^{1,0}$, $w(t_0,x)$ is absolutely continuous with respect to x on any bounded interval of $\boldsymbol{R}$. Therefore $w(t_0,x)$ is classically differentiable for almost every $x\in\boldsymbol{R}$. We fix a (classically) differentiable point $x_0\in\boldsymbol{R}$ of the function $w(t_0,\cdot)$ arbitrarily.

First we prove the inequality (2.1) for almost every $x\in\boldsymbol{R}$ and for all $t\ge 0$. By the conditions (1.2)–(1.3) and Lemma 2.2, we see that for $h\in\boldsymbol{R}\setminus\{0\}$,

$$\operatorname{Im}\left\{\lambda\frac{\mathscr{N}(w)(t_0,x_0+h)-\mathscr{N}(w)(t_0,x_0)}{h}\,\frac{\overline{w(t_0,x_0+h)-w(t_0,x_0)}}{h}\right\}\le 0, \tag{2.3}$$

because of $h^2>0$. Since $w(t_0,\cdot)$ is differentiable at x_0, letting $h\to 0$ in the inequality (2.3), we have

$$\operatorname{Im}\left\{\lambda\partial_x\mathscr{N}(w)(t_0,x_0)\cdot\overline{\partial_x w(t_0,x_0)}\right\}\le 0.$$

Therefore for all $t\ge 0$, we obtain the inequality (2.1) for almost every $x\in\boldsymbol{R}$.

Next we prove the inequality (2.2) for almost every $x\in\boldsymbol{R}$ and for all $t\ge 0$. Let $M(t,x)=e^{ix^2/2t}$. We note that

$$J=M(t,x)(it\partial_x)M(t,x)^{-1}. \tag{2.4}$$

Applying the above calculation to $M^{-1}w$ instead of w, we see that

$$\operatorname{Im}\left\{\lambda\partial_x\mathscr{N}(M^{-1}w)(t_0,x_0)\cdot\overline{\partial_x(M^{-1}w)(t_0,x_0)}\right\}\le 0.$$

By the gauge invariance $M^{-1}\mathscr{N}(w)=\mathscr{N}(M^{-1}w)$, we have

$$\operatorname{Im}\left\{\lambda\partial_x(M^{-1}\mathscr{N}(w))(t_0,x_0)\cdot\overline{\partial_x(M^{-1}w)(t_0,x_0)}\right\}\le 0.$$

Since $t_0\ge 0$ is arbitrary, x_0 is an arbitrary classically differentiable point of $w(t_0,\cdot)$ and $M^{-1}(t_0,\cdot)w(t_0,\cdot)$, and the functions $w(t_0,x)$ and $M^{-1}(t_0,x)w(t_0,x)$ is classically differentiable for almost every $x\in\boldsymbol{R}$, it follows that for $t\ge 0$, the inequality

$$\operatorname{Im}\left\{\lambda\partial_x(M^{-1}\mathscr{N}(w))(t,x)\cdot\overline{\partial_x(M^{-1}w)(t,x)}\right\}\le 0$$

holds for almost every $x\in\boldsymbol{R}$. This implies that for any $t\ge 0$, the estimate

$$\operatorname{Im}\left\{\lambda(M(it\partial_x)M^{-1}\mathscr{N}(w))(t,x)\cdot\overline{(M(it\partial_x)M^{-1}w)(t,x)}\right\}\le 0 \tag{2.5}$$

holds for almost every $x\in\boldsymbol{R}$. By the identity (2.4) and the inequality (2.5), for any $t\ge 0$, we obtain the inequality (2.2) for almost every $x\in\boldsymbol{R}$. Therefore this lemma is proved. □

3. Proof of Theorem 1.1.

Throughout this section, we assume that, the space dimension is one, $1<p\le 3$ and $\lambda\in\boldsymbol{C}$ satisfies the conditions (1.2) and (1.3). (For the proof of the time decay estimates, we will assume the additional condition $(5+\sqrt{33})/4<p\le 3$.)

For $T>0$, let

$$X_T = \{ w \in C([0,T]; H^{1,0});\ Jw \in C([0,T]; L^2),\ \|w\|_{X_T} < \infty \},$$

where

$$\|w\|_{X_T} = \sup_{0 \le t \le T} \|w(t)\|_{H^{1,0}} + \sup_{0 \le t \le T} \|Jw(t)\|_{L^2}.$$

The following proposition is concerned with the global existence and uniqueness, that is, the first half of Theorem 1.1.

Proposition 3.1. *Let $u_0 \in H^{1,0} \cap H^{0,1}$. Then there exists a unique global solution u to the initial value problem* (1.1) *satisfying* (1.4)–(1.5).

Proof. For initial data $u_0 \in H^{1,0} \cap H^{0,1}$, the local existence and uniqueness in X_T of a solution to the Cauchy problem (1.1) easily follow from the embedding $H^{1,0}(\boldsymbol{R}) \hookrightarrow L^\infty(\boldsymbol{R})$ (See, e.g., Cazenave [**2**], Ginibre [**3**] or Kato [**8**]). Therefore we derive a priori estimates of a solution $u \in X_T$ to (1.1).

For $u_0 \in H^{1,0} \cap H^{0,1}$, let $u \in X_T$ be a solution to the Cauchy problem (1.1). By the equation (1.1), the relation $J(i\partial_t + (1/2)\partial_x^2) = (i\partial_t + (1/2)\partial_x^2)J$, the conditions (1.2)–(1.3) and Lemma 2.3, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 &= \lambda_2 \|u(t)\|_{L^{p+1}}^{p+1} \le 0, \\
\frac{1}{2}\frac{d}{dt}\|\partial_x u(t)\|_{L^2}^2 &= \operatorname{Im}\{\lambda \langle (\partial_x \mathscr{N}(u))(t), \partial_x u(t) \rangle\} \le 0, \\
\frac{1}{2}\frac{d}{dt}\|Ju(t)\|_{L^2}^2 &= \operatorname{Im}\{\lambda \langle (J\mathscr{N}(u))(t), Ju(t) \rangle\} \le 0
\end{aligned}$$

for any $t \in [0,T]$. Therefore

$$\begin{aligned}
\|u(t)\|_{L^2} &\le \|u_0\|_{L^2}, \\
\|\partial_x u(t)\|_{L^2} &\le \|\partial_x u_0\|_{L^2}, \\
\|Ju(t)\|_{L^2} &\le \|xu_0\|_{L^2}
\end{aligned}$$

for any $t \in [0,T]$, and hence

$$\|u(t)\|_{H^{1,0}} + \|Ju(t)\|_{L^2} \le \|u_0\|_{H^{1,0} \cap H^{0,1}}$$

for any $t \in [0,T]$. By this estimate, we can extend the solution to the interval $[0,\infty)$, and the estimate (1.5) holds. Therefore this proposition is proved. □

It remains to show the time decay estimates (1.6)–(1.7). Let $u_0 \in H^{1,0} \cap H^{0,1}$ and let u be the global solution to the Cauchy problem (1.1) obtained in Proposition 3.1. To proceed in our argument, let $v(t) = U(-t)u(t)$. Note that $U(t)$ is factorized like $U(t) = MD\mathscr{F}M$. Then, according to the gauge invariance property of $\mathscr{N}(u)$, we see that $v(t)$ satisfies

$$i\partial_t(\mathscr{F}v) = \lambda t^{-(p-1)/2}\mathscr{N}(\mathscr{F}v) + R(t), \tag{3.1}$$

where $R(t)$ denotes the rapidly decaying error term written as

$$R(t) = R_1(t) + R_2(t)$$

with

$$R_1(t) = \lambda t^{-(p-1)/2}\mathscr{F}(M^{-1} - 1)\mathscr{F}^{-1}\mathscr{N}(\mathscr{F}Mv),$$
$$R_2(t) = \lambda t^{-(p-1)/2}\big(\mathscr{N}(\mathscr{F}Mv) - \mathscr{N}(\mathscr{F}v)\big).$$

The error $R(t)$ is estimated as in the following lemma.

Lemma 3.1. *Let μ satisfy $0 < \mu < 1/4$. Then, there exists some positive constant C_μ such that*

$$\|R(t)\|_{L^2\cap L^\infty} \le C_\mu t^{-(p-1)/2-\mu}\|u_0\|^p_{H^{1,0}\cap H^{0,1}}$$

for all $t \ge 1$.

To prove Lemma 3.1, we first show that $\|R(t)\|_{L^2\cap L^\infty} \le C_\mu t^{-(p-1)/2-\mu}\|u\|^p_{X_T}$ which similarly follows from the proof of Lemma 2.2 in [**9**] with $n = 1$, $s = 1$, $\sigma = 0$. By using Theorem 1.1 (1.5), $\|u\|_{X_T} \le \|u_0\|_{H^{1,0}\cap H^{0,1}}$ and then Lemma 3.1 is obtained.

The following proposition plays an important role for deriving the desired L^∞-decay estimate of u.

Proposition 3.2. *Assume that $(5+\sqrt{33})/4 < p \le 3$. Let*

$$K = \begin{cases} \sqrt{\dfrac{2}{4|\lambda_2| - \varepsilon}}, & \textit{when} \;\; p = 3, \\ \left(\dfrac{3-p}{2(p-1)|\lambda_2| - \varepsilon}\right)^{\frac{1}{p-1}}, & \textit{when} \;\; \frac{5+\sqrt{33}}{4} < p < 3, \end{cases} \tag{3.2}$$

where ε *is a constant such that* $0 < \varepsilon < 2(p-1)|\lambda_2|$. *Then there exists some constant* $T_0 = T_0(\|u_0\|_{H^{1,0}\cap H^{0,1}}, \varepsilon) \geq 1$ *such that*

$$(\log t)^{1/2}\|\mathscr{F}v(t)\|_{L^\infty} \leq K, \quad \textit{when} \;\; p = 3, \tag{3.3}$$

$$t^{1/(p-1)-1/2}\|\mathscr{F}v(t)\|_{L^\infty} \leq K, \quad \textit{when} \;\; \tfrac{5+\sqrt{33}}{4} < p < 3 \tag{3.4}$$

for any $t > T_0$.

To prove Proposition 3.2, we require the following lemma.

Lemma 3.2. *Suppose that the assumptions of Proposition* 3.2 *are satisfied. Let*

$$A_\varepsilon = \begin{cases} \{t \in [2,\infty);\; K < (\log t)^{1/2}\|\mathscr{F}v(t)\|_{L^\infty}\}, & \textit{when} \;\; p = 3, \\ \{t \in [2,\infty);\; K < t^{1/(p-1)-1/2}\|\mathscr{F}v(t)\|_{L^\infty}\}, & \textit{when} \;\; \frac{5+\sqrt{33}}{4} < p < 3, \end{cases}$$

where K *is the constant defined by* (3.2). *Then,* A_ε *is a bounded subset of* $\boldsymbol{R}$.

Proof. We describe only the proof of this lemma for the subcritical case $(5+\sqrt{33})/4 < p < 3$. For the critical case $p = 3$, this lemma is proved in the same way. Let $(5+\sqrt{33})/4 < p < 3$. We prove it by the contradiction argument. Assume that A_ε is unbounded. Then for any $n \in \boldsymbol{N}$, there exist some $t_n \in [n,\infty)$ and $\xi_n \in \boldsymbol{R}$ such that

$$K < t_n^{1/(p-1)-1/2}|\mathscr{F}v(t_n,\xi_n)| = t_n^{(3-p)/2(p-1)}|\mathscr{F}v(t_n,\xi_n)|.$$

(It is possible to take $\{t_n\}$ as a monotone increasing sequence if needed.) Let fix $t_0 \gg 1$. We first show that for any $n > t_0$,

$$K < t^{(3-p)/2(p-1)}|\mathscr{F}v(t,\xi_n)| \tag{3.5}$$

holds for $t \in [t_0, t_n]$. If not so, there exist some $n > t_0$ and $t_{*,n} \in [t_0, t_n)$ such that

$$K = t_{*,n}^{(3-p)/2(p-1)}|\mathscr{F}v(t_{*,n},\xi_n)| \quad \text{and} \quad K < t^{(3-p)/2(p-1)}|\mathscr{F}v(t,\xi_n)| \tag{3.6}$$

for $t \in (t_{*,n}, t_n)$, since for each $\xi \in \boldsymbol{R}$, $\mathscr{F}v(t,\xi)$ is continuous in t, which follows from $v \in C([0,\infty); H^{0,1})$ due to Theorem 1.1 and the embedding $H^{1,0}(\boldsymbol{R}) \hookrightarrow L^\infty(\boldsymbol{R})$. We note that

$$0 < p\left(\frac{1}{p-1} - \frac{1}{2}\right) < \frac{1}{4}, \tag{3.7}$$

since $(5+\sqrt{33})/4 < p < 3$. Hence we can choose a constant b such that

$$\frac{p(3-p)}{2(p-1)} = p\left(\frac{1}{p-1} - \frac{1}{2}\right) < b < \frac{1}{4}. \tag{3.8}$$

By multiplying $|\mathscr{F}v(t,\xi_n)|^{-(p+1)}\overline{\mathscr{F}v(t,\xi_n)}$ on both hand sides of (3.1) and taking the imaginary part, Lemma 3.1 with $\mu = b$ gives

$$\begin{aligned} -\frac{1}{p-1}\partial_t|\mathscr{F}v(t,\xi_n)|^{-(p-1)} &= \lambda_2 t^{-(p-1)/2} + \mathrm{Im}\frac{\overline{\mathscr{F}v(t,\xi_n)}R(t)}{|\mathscr{F}v(t,\xi_n)|^{p+1}} \\ &\le \lambda_2 t^{-(p-1)/2} + C\|u_0\|^p_{H^{1,0}\cap H^{0,1}} K^{-p} t^{-(p-1)/2-\alpha} \end{aligned} \tag{3.9}$$

for $t \in (t_{*,n}, t_n)$, where $\alpha = b - p(1/(p-1) - 1/2)$, which is positive according to (3.8). (We here note that, in (3.9), the mollification of $\mathscr{F}v(t,\xi)$ with respective to t is required for the rigorous argument. However, we proceed in formal way to avoid the complexity of the proof.) Then, (3.9) gives

$$\begin{aligned} |\mathscr{F}v(t,\xi_n)|^{-(p-1)} &\ge |\mathscr{F}v(t_{*,n},\xi_n)|^{-(p-1)} + \frac{2|\lambda_2|(p-1)}{3-p}\big(t^{(3-p)/2} - t_{*,n}^{(3-p)/2}\big) \\ &\quad - \frac{2C(p-1)}{3-p-2\alpha}\|u_0\|^p_{H^{1,0}\cap H^{0,1}} K^{-p}\big(t^{(3-p)/2-\alpha} - t_{*,n}^{(3-p)/2-\alpha}\big) \\ &= K^{-(p-1)} t_{*,n}^{(3-p)/2} + \frac{2|\lambda_2|(p-1)}{3-p}\big(t^{(3-p)/2} - t_{*,n}^{(3-p)/2}\big) \\ &\quad - \frac{2C(p-1)}{3-p-2\alpha}\|u_0\|^p_{H^{1,0}\cap H^{0,1}} K^{-p}\big(t^{(3-p)/2-\alpha} - t_{*,n}^{(3-p)/2-\alpha}\big). \end{aligned}$$

This implies that

$$\begin{aligned} &\left|t^{(3-p)/2(p-1)}\mathscr{F}v(t,\xi_n)\right|^{-(p-1)} \\ &\quad\ge K^{-(p-1)}\left(\frac{t_{*,n}}{t}\right)^{(3-p)/2} + \frac{2|\lambda_2|(p-1)}{3-p}\left(1 - \left(\frac{t_{*,n}}{t}\right)^{(3-p)/2}\right) \\ &\qquad - \frac{2C(p-1)}{3-p-2\alpha}\|u_0\|^p_{H^{1,0}\cap H^{0,1}} K^{-p}\big(t^{-\alpha} - t_{*,n}^{(3-p)/2-\alpha} t^{-(3-p)/2}\big) \\ &\quad\equiv f(t). \end{aligned}$$

We see that $f(t)$ is monotone increasing around $t = t_{*,n}$. Indeed,

$$\begin{aligned} f'(t_{*,n}) &= t_{*,n}^{-1}\left(-\frac{3-p}{2K^{p-1}} + |\lambda_2|(p-1) - C(p-1)\|u_0\|_{H^{1,0}\cap H^{0,1}}^p K^{-p} t_{*,n}^{-\alpha} \right) \\ &\geq t_{*,n}^{-1}\big(\varepsilon - C(p-1)\|u_0\|_{H^{1,0}\cap H^{0,1}}^p K^{-p} t_0^{-\alpha}\big) \\ &> 0 \end{aligned}$$

if t_0 is so large that $C(p-1)\|u_0\|_{H^{1,0}\cap H^{0,1}}^p K^{-p} t_0^{-\alpha} < \varepsilon/2$. Thus, if t is slightly larger than $t_{*,n}$, then $t^{(3-p)/2(p-1)}|\mathscr{F}v(t,\xi_n)|^{-(p-1)} > K^{-(p-1)}$. This contradicts to (3.6), and hence (3.5) is valid.

The relations (3.1) and (3.5) yield

$$\begin{aligned} |\mathscr{F}v(t_n,\xi_n)|^{-(p-1)} \geq{}& |\mathscr{F}v(t_0,\xi_n)|^{-(p-1)} + \frac{2|\lambda_2|(p-1)}{3-p}\big(t_n^{(3-p)/2} - t_0^{(3-p)/2}\big) \\ & - \frac{2C(p-1)}{3-p-2\alpha}\|u_0\|_{H^{1,0}\cap H^{0,1}}^p K^{-p}\big(t_n^{(3-p)/2-\alpha} - t_0^{(3-p)/2-\alpha}\big). \end{aligned}$$

Since $K^{-(p-1)} t_n^{(3-p)/2} > |\mathscr{F}v(t_n,\xi_n)|^{-(p-1)}$, we have

$$\begin{aligned} K^{-(p-1)} >{}& t_n^{-(3-p)/2}|\mathscr{F}v(t_0,\xi_n)|^{-(p-1)} + \frac{2|\lambda_2|(p-1)}{3-p}\left(1 - \left(\frac{t_0}{t_n}\right)^{(3-p)/2}\right) \\ & - \frac{2C(p-1)}{3-p-2\alpha}\|u_0\|_{H^{1,0}\cap H^{0,1}}^p K^{-p}\big(t_n^{-\alpha} - t_0^{(3-p)/2-\alpha} t_n^{-(3-p)/2}\big) \end{aligned} \tag{3.10}$$

Note that the first term on the right hand side of (3.10) is estimated as

$$0 \leq t_n^{-(3-p)/2}|\mathscr{F}v(t_0,\xi_n)|^{-(p-1)} \leq t_n^{-(3-p)/2} t_0^{(3-p)/2} K^{-(p-1)},$$

and so $\lim_{n\to\infty} t_n^{-(3-p)/2}|\mathscr{F}v(t_0,\xi_n)|^{-(p-1)} = 0$. Then, taking $n \to \infty$ in (3.10), we have

$$K^{-(p-1)} \geq \frac{2|\lambda_2|(p-1)}{3-p}.$$

This inequality is a contradiction. Hence, A_ε is a bounded subset of $[0,\infty)$. □

We turn to the proof of Proposition 3.2.

Proof of Proposition 3.2. By Lemma 3.2, we can choose T_0 as $\sup A_\varepsilon$ if $A_\varepsilon \neq \emptyset$ and as 2 if $A_\varepsilon = \emptyset$ so that $\|\mathscr{F}v(t,\cdot)\|_{L^\infty} \leq Kt^{-(1/(p-1)-1/2)}$ for $t > T_0$. Thus the estimate (3.4) (in the subcritical case) is proved. In the same way, the estimate (3.3) (in the critical case) follows. □

Proposition 3.3. *Let $(5+\sqrt{33})/4 < p \leq 3$. Then there exists a constant $C > 0$ such that for any $t \geq 0$,*

$$\|u(t)\|_{L^\infty} \leq (K + C\|u_0\|_{H^{1,0}\cap H^{0,1}})\{(2+t)\log(2+t)\}^{-1/2} \tag{3.11}$$

when $p = 3$, and

$$\|u(t)\|_{L^\infty} \leq (K + C\|u_0\|_{H^{1,0}\cap H^{0,1}})(1+t)^{-1/(p-1)} \tag{3.12}$$

when $(5+\sqrt{33})/4 < p < 3$, where K is the positive constant defined by (3.2).

Proof. We prove only the estimate (3.12) (in the subcritical case). The estimate (3.11) (in the critical case) is proved in the same way, and the proof of (3.11) is easier than that of the estimate (3.12). Let $(5+\sqrt{33})/4 < p < 3$, and let $T_0 \geq 1$ be the constant appearing in Proposition 3.2. Note that

$$\begin{aligned} u(t) &= U(t)v(t) \\ &= MD\mathscr{F}v(t) + MD\mathscr{F}(M-1)v(t). \end{aligned}$$

Then, by taking a constant b satisfying (3.8) and applying $H^{0,1-2b}(\boldsymbol{R}) \hookrightarrow L^1(\boldsymbol{R})$, Proposition 3.2 gives

$$\begin{aligned} \|u(t)\|_{L^\infty} &\leq Kt^{-1/2}\cdot t^{-(1/(p-1)-1/2)} + Ct^{-1/2}\|(M-1)v(t)\|_{L^1} \\ &\leq Kt^{-1/(p-1)} + Ct^{-1/2-b}\||x|^{2b}v(t)\|_{L^1} \\ &\leq Kt^{-1/(p-1)} + Ct^{-1/2-b}\||x|^{2b}v(t)\|_{H^{0,1-2b}} \\ &\leq Kt^{-1/(p-1)} + Ct^{-1/2-b}\|v(t)\|_{H^{0,1}} \\ &\leq Kt^{-1/(p-1)} + Ct^{-1/2-b}(\|u(t)\|_{L^2} + \|Ju(t)\|_{L^2}) \\ &\leq Kt^{-1/(p-1)} + Ct^{-1/2-b}\|u_0\|_{H^{1,0}\cap H^{0,1}} \end{aligned} \tag{3.13}$$

for $t > T_0$. In (3.13), we see that $-1/2-b < -1/(p-1)$, since $b > p\{1/(p-1) - 1/2\} > 1/(p-1) - 1/2$ by (3.8). Therefore

$$\|u(t)\|_{L^\infty} \le (K + C\|u_0\|_{H^{1,0}\cap H^{0,1}})t^{-1/(p-1)} \tag{3.14}$$

for $t > T_0$. On the other hand, by the estimate (1.5), we have

$$\|u(t)\|_{L^\infty} \le C\|u(t)\|_{H^{1,0}} \le C\|u_0\|_{H^{1,0}\cap H^{0,1}} \tag{3.15}$$

for any $t \ge 0$. The inequalities (3.14) and (3.15) imply the estimate (3.12). □

Now we prove Theorem 1.1.

Proof of Theorem 1.1. Suppose that the assumptions of Theorem 1.1 are satisfied. Then the unique existence of a solution u to the Cauchy problem (1.1) satisfying (1.4) and (1.5) is proved in Proposition 3.1. Furthermore under the additional assumption $(5+\sqrt{33})/4 < p \le 3$, the time decay estimates (1.6) and (1.7) are shown in Proposition 3.3. These complete the proof of Theorem 1.1. □

4. Proof of Theorem 1.2.

Throughout this section, we assume that, the space dimension is one, $(21+\sqrt{177})/12 < p \le 3$ and $\lambda \in \boldsymbol{C}$ satisfies the conditions (1.2) and (1.3).

Let $u_0 \in H^{1,0} \cap H^{0,1}$ and u be the global solution to the initial value problem (1.1) obtained in Theorem 1.1. Also, let $v = U(-t)u$ as in the previous section. Recall that Φ is the function defined by (1.8). We write

$$\Phi(t,\cdot) = \int_1^t \tau^{-(p-1)/2} |\mathscr{F} v(\tau)|^{p-1}\, d\tau$$

by using v. Then, the equation (3.1) is deformed into

$$\partial_t (e^{i\lambda\Phi(t)} \mathscr{F} v(t)) = -ie^{i\lambda\Phi(t)} R(t).$$

First we investigate the large-time behavior of $e^{i\lambda\Phi(t)}\mathscr{F} v(t)$.

Proposition 4.1. *If* $(21+\sqrt{177})/12 < p \le 3$, *then there exist a constant* $T_1 \ge 1$ *and a unique* $\phi \in L^2 \cap L^\infty$ *such that*

$$\left\| e^{i\lambda\Phi(t)} \mathscr{F} v(t) - \phi \right\|_{L^2\cap L^\infty} \le Ct^{-\beta} \qquad \text{for any } t > T_1,$$

for some $\beta > 3 - p$. *In particular,* $\lim_{t\to\infty} e^{i\lambda\Phi(t)}\mathscr{F} v(t) = \phi$ *in* $L^2 \cap L^\infty$.

PROOF. We note that

$$\frac{(3p-2)(3-p)}{2(p-1)} < \frac{1}{4}, \tag{4.1}$$

since $(21+\sqrt{177})/12 < p \leq 3$. Let d be a constant satisfying

$$\frac{(3p-2)(3-p)}{2(p-1)} < d < \frac{1}{4}. \tag{4.2}$$

First we show this proposition in the subcritical case $(5+\sqrt{33})/4 < p < 3$. By the definition (3.2) of K, K is represented by

$$K = \left(\frac{3-p}{2(p-1)|\lambda_2|} + \frac{\delta_\varepsilon}{|\lambda_2|} \right)^{1/(p-1)}, \tag{4.3}$$

where

$$\delta_\varepsilon = |\lambda_2| \left(\frac{3-p}{2(p-1)|\lambda_2| - \varepsilon} - \frac{3-p}{2(p-1)|\lambda_2|} \right). \tag{4.4}$$

Note that $\lim_{\varepsilon \downarrow 0} \delta_\varepsilon = 0$. According to (4.2), we can take $\varepsilon > 0$ sufficiently small so that

$$0 < \delta_\varepsilon < d - \frac{(3p-2)(3-p)}{2(p-1)}. \tag{4.5}$$

Let $t > T_0$, where T_0 is the positive constant appearing in Proposition 3.2. By Proposition 3.2, we have

$$\begin{aligned} \|\Phi(t)\|_{L^\infty} &\leq \int_1^t \tau^{-(p-1)/2} \|\mathscr{F} v(\tau)\|_{L^\infty}^{p-1} \, d\tau \\ &\leq K^{p-1} \int_1^t \tau^{-1} \, d\tau \\ &= K^{p-1} \log t, \end{aligned}$$

and so Lemma 3.1 with $\mu = d$ gives

$$\begin{aligned}
&\left\|e^{i\lambda\Phi(t)}\mathscr{F}v(t)-e^{i\lambda\Phi(t')}\mathscr{F}v(t')\right\|_{L^2\cap L^\infty}\\
&\quad\le\int_{t'}^{t}\left\|e^{|\lambda_2|\Phi(\tau)}R(\tau)\right\|_{L^2\cap L^\infty}d\tau\\
&\quad\le C\int_{t'}^{t}e^{|\lambda_2|\|\Phi(\tau)\|_{L^\infty}}\|R(\tau)\|_{L^2\cap L^\infty}\,d\tau\\
&\quad\le C\|u_0\|^p_{H^{1,0}\cap H^{0,1}}\int_{t'}^{t}e^{|\lambda_2|K^{p-1}\log\tau}\tau^{-(p-1)/2-d}\,d\tau\\
&\quad= C\|u_0\|^p_{H^{1,0}\cap H^{0,1}}\int_{t'}^{t}\tau^{-1-\beta}\,d\tau
\end{aligned}$$

for $T_0<t'<t$, where

$$\beta=-1+\frac{(p-1)}{2}+d-|\lambda_2|K^{p-1}=d-\frac{p(3-p)}{2(p-1)}-\delta_\varepsilon. \tag{4.6}$$

Here we note that $\beta>3-p$, since the equality (4.6) and the inequality (4.5) imply

$$\begin{aligned}
\beta-(3-p)&=d-\frac{p(3-p)}{2(p-1)}-\delta_\varepsilon-(3-p)\\
&>d-\frac{(3p-2)(3-p)}{2(p-1)}-\delta_\varepsilon>0.
\end{aligned}$$

Therefore

$$\left\|e^{i\lambda\Phi(t)}\mathscr{F}v(t)-e^{i\lambda\Phi(t')}\mathscr{F}v(t')\right\|_{L^2\cap L^\infty}\le C\|u_0\|^p_{H^{1,0}\cap H^{0,1}}t'^{-\beta}$$

for $T_0<t'<t$. This implies that there exists a unique $\phi\in L^2\cap L^\infty$ such that

$$\left\|e^{i\lambda\Phi(t)}\mathscr{F}v(t)-\phi\right\|_{L^2\cap L^\infty}\le C\|u_0\|^p_{H^{1,0}\cap H^{0,1}}t^{-\beta}$$

for $t>T_0$.

Next we consider the critical case $p=3$. Let $t>T_0$. By Proposition 3.2, we have

$$\begin{aligned}
\|\Phi(t)\|_{L^\infty}&\le\int_1^t\tau^{-1}\|\mathscr{F}v(\tau)\|^2_{L^\infty}\,d\tau\le K^2\int_1^t\tau^{-1}(\log\tau)^{-1}\,d\tau\\
&=K^2\log(\log t).
\end{aligned}$$

Lemma 3.1 with $\mu = d$ gives

$$\begin{aligned}
&\left\| e^{i\lambda\Phi(t)}\mathscr{F}v(t) - e^{i\lambda\Phi(t')}\mathscr{F}v(t')\right\|_{L^2\cap L^\infty} \\
&\le \int_{t'}^{t} \left\| e^{|\lambda_2|\Phi(\tau)}R(\tau)\right\|_{L^2\cap L^\infty} d\tau \\
&\le C\int_{t'}^{t} e^{|\lambda_2|\|\Phi(\tau)\|_{L^\infty}} \|R(\tau)\|_{L^2\cap L^\infty}\, d\tau \\
&\le C\|u_0\|^3_{H^{1,0}\cap H^{0,1}} \int_{t'}^{t} e^{|\lambda_2|K^2\log(\log\tau)}\tau^{-1-d}\, d\tau \\
&= C\|u_0\|^3_{H^{1,0}\cap H^{0,1}} \int_{t'}^{t} (\log\tau)^{|\lambda_2|K^2}\tau^{-1-d}\, d\tau \\
&\le C\|u_0\|^3_{H^{1,0}\cap H^{0,1}} t'^{-\beta}
\end{aligned}$$

for $T_0 < t' < t$, where β is a constant such that $0 < \beta < d$. This implies that there exists a unique $\phi \in L^2 \cap L^\infty$ such that

$$\left\| e^{i\lambda\Phi(t)}\mathscr{F}v(t) - \phi\right\|_{L^2\cap L^\infty} \le C\|u_0\|^3_{H^{1,0}\cap H^{0,1}} t^{-\beta}$$

for $t > T_0$. This completes the proof of Proposition 4.1. □

Let us next observe the asymptotic behavior of $\Phi(t)$. Noting that

$$\begin{aligned}
\partial_t\Phi(t) &= t^{-(p-1)/2}|\mathscr{F}v(t)|^{p-1} \\
&= t^{-(p-1)/2}e^{(p-1)\lambda_2\Phi(t)}\left|e^{i\lambda\Phi(t)}\mathscr{F}v(t)\right|^{p-1} \\
&= t^{-(p-1)/2}e^{-(p-1)|\lambda_2|\Phi(t)}\left|e^{i\lambda\Phi(t)}\mathscr{F}v(t)\right|^{p-1},
\end{aligned}$$

we see that

$$\partial_t e^{(p-1)|\lambda_2|\Phi(t)} = (p-1)|\lambda_2|t^{-(p-1)/2}\left|e^{i\lambda\Phi(t)}\mathscr{F}v(t)\right|^{p-1}.$$

Integrating the above equation from 1 to t, we have

$$e^{(p-1)|\lambda_2|\Phi(t)} = 1 + (p-1)|\lambda_2|\int_1^t \tau^{-(p-1)/2}\left|e^{i\lambda\Phi(\tau)}\mathscr{F}v(\tau)\right|^{p-1} d\tau.$$

Therefore we see that

$$e^{(p-1)|\lambda_2|\Phi(t)}$$
$$= E(t) + (p-1)|\lambda_2| \int_1^t \tau^{-(p-1)/2} \big(|e^{i\lambda\Phi(\tau)} \mathscr{F} v(\tau)|^{p-1} - |\phi|^{p-1} \big)\, d\tau, \qquad (4.7)$$

where

$$E(t,x) = 1 + (p-1)|\lambda_2||\phi(x)|^{p-1} \int_1^t \tau^{-(p-1)/2}\, d\tau$$
$$= \begin{cases} 1 + 2|\lambda_2||\phi(x)|^2 \log t, & \text{when } \; p = 3, \\ 1 + \dfrac{2(p-1)|\lambda_2|}{3-p} |\phi(x)|^{p-1} (t^{(3-p)/2} - 1), & \text{when } \; \frac{21+\sqrt{177}}{12} < p < 3. \end{cases}$$

By (4.7), we can derive the asymptotic formula of $e^{(p-1)|\lambda_2|\Phi(t)}$.

Proposition 4.2. *Assume that* $(21+\sqrt{177})/12 < p \le 3$. *Then there exist a constant* $T_2 \ge 1$ *and a unique real valued function* $\eta \in L^\infty$ *such that*

$$\big\| e^{(p-1)|\lambda_2|\Phi(t)} - E(t) - \eta \big\|_{L^\infty} \le C t^{-(\beta-(3-p)/2)} \qquad (4.8)$$

for any $t \ge T_2$, *where* β *is the constant satisfying* $\beta > 3-p$, *which appears in Proposition* 4.1. *Furthermore, there exists a constant* $\widetilde{T} \ge 1$ *such that* $\widetilde{T} \ge T_2$ *and*

$$E(t,x) + \eta(x) \ge \frac{1}{2} \qquad (4.9)$$

for almost every $x \in \boldsymbol{R}$ *and for any* $t \ge \widetilde{T}$.

Proof. We show the estimate (4.8). First we consider the subcritical case $(21+\sqrt{177})/12 < p < 3$. We show that the function $e^{(p-1)|\lambda_2|\Phi(t)} - E(t)$ has a limit in L^∞ as $t \to \infty$. It follows from the identity (4.7) that

$$\big(e^{(p-1)|\lambda_2|\Phi(t)} - E(t)\big) - \big(e^{(p-1)|\lambda_2|\Phi(t')} - E(t')\big)$$
$$= (p-1)|\lambda_2| \int_{t'}^t \tau^{-(p-1)/2} \big(|e^{i\lambda\Phi(\tau)} \mathscr{F} v(\tau)|^{p-1} - |\phi|^{p-1} \big)\, d\tau. \qquad (4.10)$$

By Proposition 4.1 and the fact that $1 < p-1 \le 2$, we see that if $T_2 \ge T_1$ and T_2 is sufficiently large, then

$$\begin{aligned}
&\big\||e^{i\lambda\Phi(\tau)}\mathscr{F}v(\tau)|^{p-1}-|\phi|^{p-1}\big\|_{L^\infty}\\
&\le C\max\left\{\|e^{i\lambda\Phi(\tau)}\mathscr{F}v(\tau)\|_{L^\infty}^{p-2},\ \|\phi\|_{L^\infty}^{p-2}\right\}\big\||e^{i\lambda\Phi(\tau)}\mathscr{F}v(\tau)|-|\phi|\big\|_{L^\infty}\\
&\le C\big\|e^{i\lambda\Phi(\tau)}\mathscr{F}v(\tau)-\phi\big\|_{L^\infty}\\
&\le C\tau^{-\beta} \qquad (4.11)
\end{aligned}$$

for $\tau \ge T_2$, where T_1 is a positive constant appearing in Proposition 4.1. Therefore by the identity (4.10) and the estimate (4.11), we obtain

$$\begin{aligned}
&\big\|(e^{(p-1)|\lambda_2|\Phi(t)}-E(t))-(e^{(p-1)|\lambda_2|\Phi(t')}-E(t'))\big\|_{L^\infty}\\
&\le (p-1)|\lambda_2|\int_t^{t'}\tau^{-(p-1)/2}\big\||e^{i\lambda\Phi(\tau)}\mathscr{F}v(\tau)|^{p-1}-|\phi|^{p-1}\big\|_{L^\infty}\,d\tau\\
&\le C\int_t^{t'}\tau^{-\beta-(p-1)/2}\,d\tau\\
&\le Ct^{-(\beta-(3-p)/2)} \qquad (4.12)
\end{aligned}$$

for $T_2 \le t < t'$. Therefore there exists a unique function $\eta \in L^\infty$ such that

$$\big\|e^{(p-1)|\lambda_2|\Phi(t)}-E(t)-\eta\big\|_{L^\infty}\le Ct^{-(\beta-(3-p)/2)}$$

for $t \ge T_2$. Hence the estimate (4.8) is proved in the subcritical case $(21+\sqrt{177})/12 < p < 3$. In the same way as above, when $p = 3$ (that is, the critical case), we can show the estimate (4.8). (Recall that $\beta > 0$ when $p = 3$.)

We note that $e^{(p-1)|\lambda_2|\Phi(t)} \ge 1$ if $t \ge 1$. Therefore by the estimate (4.8), we see that there exists a sufficiently large $\widetilde{T} \ge T_2$ such that

$$E(t,x)+\eta(x)\ge\frac{1}{2}$$

for almost every $x \in \boldsymbol{R}$ and for any $t \ge \widetilde{T}$. Hence the estimate (4.9) is proved. □

For $t \ge \widetilde{T}$, let A be the function defined by (1.9). Then

$$A(t,x)=\frac{1}{(p-1)|\lambda_2|}\log(E(t,x)+\eta(x)),$$

and then

$$\begin{aligned}
e^{(p-1)|\lambda_2|A(t,x)} &= E(t,x)+\eta(x)\\
&= 1+(p-1)|\lambda_2||\phi(x)|^{p-1}\int_1^t \tau^{-(p-1)/2}\,d\tau+\eta(x),\\
e^{|\lambda_2|A(t,x)} &= (E(t,x)+\eta(x))^{1/(p-1)}\\
&= \left(1+(p-1)|\lambda_2||\phi(x)|^{p-1}\int_1^t \tau^{-(p-1)/2}\,d\tau+\eta(x)\right)^{1/(p-1)}.
\end{aligned} \tag{4.13}$$

Then we have the asymptotic profile of the modification factor $e^{-i\lambda\Phi(t)}$ as given below.

Lemma 4.1. *Assume that* $(21+\sqrt{177})/12 < p \le 3$. *Then there exists a constant* $T_3 > 1$ *such that the following estimate holds for* $t \ge T_3$:

$$\left\|(e^{-i\lambda\Phi(t)}-e^{-i\lambda A(t)})\phi\right\|_{L^2\cap L^\infty} \le Ct^{-(\frac{\beta}{p-1}-\frac{3-p}{2(p-1)})},$$

where β *is the constant satisfying* $\beta > 3-p$, *which appears in Proposition* 4.1.

Proof. Let $t \ge \widetilde{T}$, where $\widetilde{T} \ge 1$ is the constant appearing in Proposition 4.2. We write

$$\begin{aligned}
\left(e^{-i\lambda\Phi(t)}-e^{-i\lambda A(t)}\right)\phi &= e^{-i\lambda_1\Phi(t)}e^{-|\lambda_2|\Phi(t)}\left(e^{|\lambda_2|\Phi(t)}-e^{|\lambda_2|A(t)}\right)e^{-|\lambda_2|A(t)}\phi\\
&\quad+\left(e^{-i\lambda_1\Phi(t)}-e^{-i\lambda_1 A(t)}\right)e^{-|\lambda_2|A(t)}\phi\\
&\equiv P_1(t)+P_2(t).
\end{aligned}$$

We here remark $\|e^{-|\lambda_2|\Phi(t)}\|_{L^\infty} \le 1$ and $\|e^{-|\lambda_2|A(t)}\|_{L^\infty} \le 2^{1/(p-1)}$ by the inequality (4.9), and it follows from Proposition 4.2 that

$$\begin{aligned}
\left\|e^{|\lambda_2|\Phi(t)}-e^{|\lambda_2|A(t)}\right\|_{L^\infty} &\le \left\|\left(e^{(p-1)|\lambda_2|\Phi(t)}\right)^{1/(p-1)}-\left(e^{(p-1)|\lambda_2|A(t)}\right)^{1/(p-1)}\right\|_{L^\infty}\\
&\le C\left\|e^{(p-1)|\lambda_2|\Phi(t)}-e^{(p-1)|\lambda_2|A(t)}\right\|_{L^\infty}^{1/(p-1)}\\
&\le Ct^{-(\frac{\beta}{p-1}-\frac{3-p}{2(p-1)})}.
\end{aligned}$$

Then it follows from the above estimates that

$$\begin{aligned}\|P_1(t)\|_{L^2\cap L^\infty} &\le C\big\|e^{-|\lambda_2|\Phi(t)}\big\|_{L^\infty}\big\|e^{|\lambda_2|\Phi(t)}-e^{|\lambda_2|A(t)}\big\|_{L^\infty}\big\|e^{-|\lambda_2|A(t)}\phi\big\|_{L^2\cap L^\infty}\\ &\le Ct^{-(\frac{\beta}{p-1}-\frac{3-p}{2(p-1)})}\end{aligned}$$

and

$$\begin{aligned}\|P_2(t)\|_{L^2\cap L^\infty} &\le C\big\|e^{-i\lambda_1\Phi(t)}-e^{-i\lambda_1 A(t)}\big\|_{L^\infty}\big\|e^{-|\lambda_2|A(t)}\phi\big\|_{L^2\cap L^\infty}\\ &\le C\|\Phi(t)-A(t)\|_{L^\infty}\\ &\le C\big\|e^{|\lambda_2|\Phi(t)}-e^{|\lambda_2|A(t)}\big\|_{L^\infty}\\ &\le Ct^{-(\frac{\beta}{p-1}-\frac{3-p}{2(p-1)})}.\end{aligned}$$

Therefore Lemma 4.1 is proved. □

Now we prove Theorem 1.2.

Proof of Theorem 1.2. We have already proved Part (I) and Part (II) of Theorem 1.2 in Proposition 4.1 and Proposition 4.2. It remains to prove Part (III) of Theorem 1.2.

Suppose that the assumptions in Theorem 1.2 are satisfied. First we show the asymptotic formula (1.10). Since

$$\begin{aligned}u(t)-MDe^{-i\lambda A}\phi &= U(t)v(t)-MDe^{-i\lambda A(t)}\phi\\ &= MD\mathscr{F}(M-1)v(t)+MDe^{-i\lambda\Phi(t)}(e^{i\lambda\Phi(t)}\mathscr{F}v(t)-\phi)\\ &\quad+MD(e^{-i\lambda\Phi(t)}-e^{-i\lambda A(t)})\phi,\end{aligned}$$

we have

$$\begin{aligned}&\big\|u(t)-MDe^{-i\lambda A}\phi\big\|_{L^\infty}\\ &\le \|MD\mathscr{F}(M-1)v(t)\|_{L^\infty}+\big\|MDe^{-i\lambda\Phi(t)}(e^{i\lambda\Phi(t)}\mathscr{F}v(t)-\phi)\big\|_{L^\infty}\\ &\quad+\big\|MD(e^{-i\lambda\Phi(t)}-e^{-i\lambda A(t)})\phi\big\|_{L^\infty}\\ &\le t^{-1/2}\|\mathscr{F}(M-1)v(t)\|_{L^\infty}+t^{-1/2}\big\|e^{i\lambda\Phi(t)}\mathscr{F}v(t)-\phi\big\|_{L^\infty}\\ &\quad+t^{-1/2}\big\|(e^{-i\lambda\Phi(t)}-e^{-i\lambda A(t)})\phi\big\|_{L^\infty}\\ &\equiv I_1(t)+I_2(t)+I_3(t) \qquad (4.14)\end{aligned}$$

Here we have noted that $\|e^{-i\lambda\Phi(t)}\|_{L^\infty} = \|e^{-|\lambda_2|\Phi(t)}\|_{L^\infty} \leq 1$. Lemma 4.1 yields

$$I_3(t) \leq Ct^{-\frac{1}{2}-(\frac{\beta}{p-1}-\frac{3-p}{2(p-1)})} = Ct^{-(p+\beta-2)/(p-1)}.$$

Hence we see that

$$I_3(t) = \begin{cases} o((t\log t)^{-1/2}), & \text{when } p=3, \\ o(t^{-1/(p-1)}), & \text{when } \frac{21+\sqrt{177}}{12} < p < 3, \end{cases} \tag{4.15}$$

since $(p+\beta-2)/(p-1) > 1/(p-1)$ which follows from the fact $\beta > 3-p$. Proposition 4.1 gives

$$I_2(t) \leq Ct^{-1/2-\beta}.$$

Note that $\beta > 3-p > (3-p)/2(p-1) = 1/(p-1)-1/2$, since $2(p-1) > 1$. Hence

$$I_2(t) = \begin{cases} o((t\log t)^{-1/2}), & \text{when } p=3, \\ o(t^{-1/(p-1)}), & \text{when } \frac{21+\sqrt{177}}{12} < p < 3. \end{cases} \tag{4.16}$$

As we proved in the estimate (3.13), we obtain

$$I_1(t) \leq Ct^{-1/2-b},$$

where $b > 0$ is the constant satisfying (3.8). By the inequality (3.8), we see that $b > p\{1/(p-1)-1/2\} > 1/(p-1)-1/2$, and hence

$$I_1(t) = \begin{cases} o((t\log t)^{-1/2}), & \text{when } p=3, \\ o(t^{-1/(p-1)}), & \text{when } \frac{21+\sqrt{177}}{12} < p < 3. \end{cases} \tag{4.17}$$

By the estimates (4.14)–(4.17), we have the asymptotic formula (1.10).

Next we prove the asymptotic formula (1.11). The following holds:

$$\begin{aligned} &\left\|u(t) - U(t)\mathscr{F}^{-1}(e^{-i\lambda A(t)}\phi)\right\|_{L^2} \\ &\quad = \left\|\mathscr{F}U(-t)u(t) - e^{-i\lambda A(t)}\phi\right\|_{L^2} \\ &\quad = \left\|\mathscr{F}v(t) - e^{-i\lambda A(t)}\phi\right\|_{L^2} \end{aligned}$$

$$\begin{aligned}
&\leq \left\| \mathscr{F} v(t) - e^{-i\lambda\Phi(t)}\phi \right\|_{L^2} + \left\| (e^{-i\lambda\Phi(t)} - e^{-i\lambda A(t)})\phi \right\|_{L^2} \\
&\leq \left\| e^{i\lambda\Phi(t)} \mathscr{F} v(t) - \phi \right\|_{L^2} + \left\| (e^{-i\lambda\Phi(t)} - e^{-i\lambda A(t)})\phi \right\|_{L^2} \\
&\equiv I_4(t) + I_5(t). \qquad (4.18)
\end{aligned}$$

In the same way as in the proof of the estimate (4.16) together with Proposition 4.1, we obtain

$$I_4(t) = \begin{cases} o((\log t)^{-1/2}), & \text{when } \; p = 3, \\ o(t^{-(1/(p-1)-1/2)}), & \text{when } \; \frac{21+\sqrt{177}}{12} < p < 3. \end{cases} \qquad (4.19)$$

By Lemma 4.1, we can show

$$I_5(t) = \begin{cases} o((\log t)^{-1/2}), & \text{when } \; p = 3, \\ o(t^{-(1/(p-1)-1/2)}), & \text{when } \; \frac{21+\sqrt{177}}{12} < p < 3. \end{cases} \qquad (4.20)$$

in the same way as in the proof of the estimate (4.15). The estimates (4.18)–(4.20) imply the asymptotic formula (1.11).

Finally we prove (1.12). The unitarity of $U(t)$ and $\mathscr{F}$ gives

$$\left\| U(t)\mathscr{F}^{-1}(e^{-i\lambda A(t)}\phi) \right\|_{L^2} = \left\| e^{-i\lambda A(t)}\phi \right\|_{L^2} = \left\| e^{-|\lambda_2| A(t)}\phi \right\|_{L^2}.$$

By the identity (4.13), we have

$$\lim_{t\to\infty} e^{-|\lambda_2| A(t,x)}\phi(x) = 0$$

for almost every $x \in \boldsymbol{R}$. By the estimate (4.9), note that for $x \in \boldsymbol{R}$ and $t > \widetilde{T}$,

$$\begin{aligned}
\left| e^{-|\lambda_2| A(t,x)}\phi(x) \right| &= (E(t,x) + \eta(x))^{-1/(p-1)} |\phi(x)| \\
&\leq C|\phi(x)|.
\end{aligned}$$

Then $C|\phi(x)|$ is regarded as a dominating function of $|e^{-|\lambda_2| A(t,x)}\phi(x)|$, and we have

$$\lim_{t\to\infty} \left\| e^{-|\lambda_2| A(t)}\phi \right\|_{L^2} = 0$$

by Lebesgue's dominated convergence theorem. Therefore

$$\lim_{t\to\infty} \left\| U(t)\mathscr{F}^{-1}(e^{-i\lambda A(t)}\phi) \right\|_{L^2} = 0. \tag{4.21}$$

By (4.21) and the asymptotic formula (1.11), we have (1.12). This completes the proof of Theorem 1.2. □

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