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An alternative proof of global existence for nonlinear wave equations in an exterior domain

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Abstract. The aim of this article is to present a simplified proof of a global existence result for systems of nonlinear wave equations in an exterior domain. The novelty of our proof is to avoid completely the scaling operator which would make the argument complicated in the mixed problem, by using new weighted pointwise estimates of a tangential derivative to the light cone.

1. Introduction.

Let Ω be an unbounded domain in $\boldsymbol{R}^3$ with compact and smooth boundary $\partial\Omega$. We put $\mathscr{O} := \boldsymbol{R}^3 \setminus \Omega$, which is called an obstacle. $\mathscr{O}$ is supposed to be non-empty. In this paper, we consider the mixed problem for a system of nonlinear wave equations in Ω, with small initial data:

$$\left(\partial_t^2 - c_i^2\Delta_x\right)u_i = F_i(u, \partial u, \nabla_x\, \partial u), \qquad (t,x) \in (0,\infty)\times\Omega, \tag{1.1}$$

$$u(t,x) = 0, \qquad (t,x) \in (0,\infty)\times\partial\Omega, \tag{1.2}$$

$$u(0,x) = \phi(x),\ (\partial_t u)(0,x) = \psi(x), \qquad x \in \Omega, \tag{1.3}$$

for $i = 1, \ldots, N$, where c_i $(1 \le i \le N)$ are given positive constants, and $u = (u_1, \ldots, u_N)$. Here we have set $\partial_0 := \partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$ $(j = 1,2,3)$, $\Delta_x = \sum_{j=1}^3 \partial_j^2$, $\nabla_x\, u = (\partial_1 u, \partial_2 u, \partial_3 u)$ and $\partial u = (\partial_t u, \nabla_x\, u)$. For a while, we assume $\phi,\ \psi \in C_0^\infty(\overline{\Omega}\,; \boldsymbol{R}^N)$, namely they are smooth functions on $\overline{\Omega}$ vanishing outside some ball. In the following, we always suppose that ϕ and ψ are small in some suitable norm. We assume that each nonlinearity F_i is a smooth function vanishing of second order at the origin $(u, \partial u, \nabla_x \partial u) = (0,0,0)$. We suppose that (1.1) is quasi-linear, namely each F_i has the form

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$$F_i(u, \partial u, \nabla_x \partial u) = \sum_{j=1}^{N} c_{ij}^{ka}(u, \partial u)\partial_k \partial_a u_j + \widetilde{F}_i(u, \partial u),$$

where c_{ij}^{ka}'s are smooth functions vanishing of first order at the origin, and $\widetilde{F}_i$'s are smooth functions vanishing of second order at the origin. In the following we always assume that

$$c_{ij}^{ka}(u, \partial u) = c_{ji}^{ka}(u, \partial u) \text{ and } c_{ij}^{k\ell}(u, \partial u) = c_{ij}^{\ell k}(u, \partial u) \tag{1.4}$$

hold for $1 \le i, j \le N$, $1 \le k, \ell \le 3$ and $0 \le a \le 3$, so that the hyperbolicity of the system is assured.

We also suppose that (ϕ, ψ, F) satisfies the compatibility condition to infinite order (see Definition 1.1 below).

Let us recall the known results. In what follows, when we just say the Cauchy problem, we mean the Cauchy problem on $[0, \infty) \times \boldsymbol{R}^3$.

First we consider the single speed case (i.e., $c_1 = c_2 = \cdots = c_N = 1$). If each nonlinearity F_i vanishes of third order at the origin, then it was shown in Shibata – Tsutsumi [**31**] that the mixed problem (1.1)–(1.3) admits a unique global small amplitude solution. If the quadratic terms are present in the nonlinearity, in order to get a global existence result, we need a certain algebraic condition on the quadratic terms (due to the blow-up result for the corresponding Cauchy problem obtained by John [**10**], which also shows the blow-up for the mixed problem in view of the finite speed of propagation). One of such conditions is the null condition introduced by Klainerman [**16**]. Under the null condition, Klainerman [**16**] and Christodoulou [**2**] independently proved global solvability for the Cauchy problem with small initial data by different methods. This result was extended to the mixed problem by Godin [**4**] when the obstacle $\mathscr{O}$ is a ball (assuming the rotational symmetricity of the solution), by Keel – Smith – Sogge [**14**] when it is star-shaped, and by Metcalfe [**23**] when it is non-trapping (for the case of other space dimensions, we refer to [**31**], [**5**]).

Next we consider the multiple speeds case where the propagation speeds c_i $(1 \le i \le N)$ do not necessarily coincide with each other. Metcalfe – Sogge [**26**] and Metcalfe – Nakamura – Sogge [**24**], [**25**] extended the global existence result for the mixed problem with the single speed to the multiple speeds case (see [**17**], [**34**], [**32**], [**33**], [**6**], [**19**], [**11**], and [**13**] for the Cauchy problem in three space dimensions; see also [**7**] for the two space dimensional case). In addition, they treated more general obstacles as we shall describe in Definition 1.2 below.

The aim of this article is to present an alternative approach to these works. Our approach consists of the following two ingredients. One is the usage of weighted L^∞–L^∞ decay estimates for the mixed problem of the linear wave equation given in Theorem 4.2 below, whose counterparts for the Cauchy problem have been widely studied (see Lemmas 3.2, 3.3 and 3.4 below). Equipped with these estimates, we do not need the space–time L^2 estimates which have been adopted in the works [**14**], [**23**], [**24**], [**25**], [**26**]. Moreover, these weighted L^∞–L^∞ estimates directly give us rather detailed decay estimates

$$|u_i(t,x)| \le C(1+t+|x|)^{-1} \log\left(1+\frac{1+c_it+|x|}{1+|c_it-|x|\,|}\right), \tag{1.5}$$

$$|\partial u_i(t,x)| \le C(1+|x|)^{-1}(1+|c_it-|x||)^{-1} \tag{1.6}$$

for $(t,x) \in [0,\infty)\times\overline{\Omega}$, which are refinement of time decay estimates obtained in the previous works for the mixed problems.

The other is making use of stronger decay property of a tangential derivative to the light cone given in Theorem 4.3 below. This idea is recently introduced by the authors [**12**], where the Cauchy problem is studied, and it enables us to deal with the null forms using neither the scaling operator $t\partial_t + x\cdot\nabla_x$ nor Lorentz boost fields $t\partial_j + x_j\partial_t$ $(j=1,2,3)$. In this paper, we will adopt this approach to the mixed problem, and treat the problem without using these vector fields. In contrast, the scaling operator has been used in the previous works, and it makes the argument rather complicated because it does not preserve the Dirichlet boundary condition (1.2) and has the unbounded coefficient near the boundary. Recently Metcalfe – Sogge [**27**] introduced a simplified approach which allows us to use the scaling operator without special care, but their approach is applicable only to star-shaped obstacles, and they assumed that the nonlinearity depends only on derivatives of u.

We will also avoid the argument of a reduction to zero initial data, used in [**14**], [**23**], [**24**], [**25**], [**26**].

In order to state our result precisely, we need some notation, as well as a couple of notions about the initial data, the obstacle and the nonlinearity.

Consider the mixed problem for a single wave equation

$$\bigl(\partial_t^2 - c^2\Delta_x\bigr)v = f, \qquad (t,x)\in(0,T)\times\Omega, \tag{1.7}$$

$$v(t,x) = 0, \qquad (t,x)\in(0,T)\times\partial\Omega, \tag{1.8}$$

$$v(0,x) = v_0(x),\ (\partial_t v)(0,x) = v_1(x), \qquad x\in\Omega \tag{1.9}$$

for a given data $\Xi = (v_0, v_1, f)$, with some propagation speed $c>0$. We sometimes write $\vec{v}_0 = (v_0, v_1)$ in what follows.

Definition 1.1. Let $\vec{v}_0 = (v_0, v_1) \in C^\infty(\overline{\Omega}; \boldsymbol{R}^2)$ and $f \in C^\infty([0,T) \times \overline{\Omega}; \boldsymbol{R})$ with some $T > 0$. We say that (v_0, v_1, f) satisfies *the compatibility condition* to infinite order for (1.7)–(1.9), if $\partial_t^j v(0,x)$, determined formally from (1.7) and (1.9), vanishes on $\partial\Omega$ for any non-negative integer j. More precisely, we say so if $v_j(x) = 0$ for any $x \in \partial\Omega$ and any non-negative integer j, where v_j for $j \geq 2$ is determined successively by

$$v_j(x) \equiv c^2 \Delta_x v_{j-2}(x) + \left(\partial_t^{j-2} f\right)(0,x) \quad \text{for } x \in \overline{\Omega}. \tag{1.10}$$

Similarly, we say that (ϕ, ψ, F) satisfies the *compatibility condition* to infinite order for the mixed problem (1.1)–(1.3) if $(\partial_t^j u)(0,x)$, formally determined by (1.1) and (1.3), vanishes on $\partial\Omega$ for any non-negative integer j (notice that the values $(\partial_t^j u)(0,x)$ are determined by (ϕ, ψ, F) successively as in (1.10); for example we have $\partial_t^2 u_i(0) = c_i^2 \Delta_x \phi_i + F_i(\phi, (\psi, \nabla_x \phi), \nabla_x(\psi, \nabla_x \phi))$, and so on).

Throughout this paper, B_R stands for

$$B_R = \{x \in \boldsymbol{R}^3;\ |x| < R\} \qquad \text{for } R > 0.$$

We remark that we may assume, without loss of generality, that $\mathscr{O} \subset B_1$ by the scaling. Hence we always make this assumption in the following. For $R \geq 1$, we set

$$\Omega_R = \Omega \cap B_R.$$

We denote by $X_c(T)$ the set of all

$$\Xi = (v_0, v_1, f) = (\vec{v}_0, f) \in C_0^\infty(\overline{\Omega}; \boldsymbol{R}^2) \times C_D^\infty([0,\infty) \times \overline{\Omega}; \boldsymbol{R})$$

satisfying the compatibility condition to infinite order for (1.7)–(1.9) with the propagation speed c, where $f \in C_D^\infty([0,\infty) \times \overline{\Omega}; \boldsymbol{R})$ means that $f \in C^\infty([0,\infty) \times \overline{\Omega}; \boldsymbol{R})$ and $f(t,\cdot) \in C_0^\infty(\overline{\Omega})$ for any fixed $t \in [0,\infty)$. In addition, for $a > 1$, $X_{c,a}(T)$ denotes the set of all $\Xi = (v_0, v_1, f) \in X_c(T)$ satisfying

$$v_0(x) = v_1(x) = f(t,x) \equiv 0 \text{ for } |x| \geq a \text{ and } t \in [0,T).$$

We introduce function spaces. For non-negative integers m and s, we define $H^{m,s}(\Omega) = \{\varphi;\ \|\varphi : H^{m,s}(\Omega)\| < \infty\}$, where

$$\left\|\varphi\!:\!H^{m,s}(\Omega)\right\|^2 = \sum_{|\alpha|\le m}\int_{\Omega}\langle x\rangle^{2s}\,|\partial_x^\alpha\varphi(x)|^2dx$$

for $\varphi = \varphi(x)$. Here $\langle x\rangle = \sqrt{1+|x|^2}$ for $x \in \boldsymbol{R}^3$ and $\partial_x^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$ for a multi-index $\alpha = (\alpha_1,\alpha_2,\alpha_3)$. Throughout this paper, we also use the notations $\langle a\rangle = \sqrt{1+|a|^2}$ for $a \in \boldsymbol{R}$, and $\partial^\alpha = \partial_0^{\alpha_0}\partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$ for a multi-index $\alpha = (\alpha_0,\alpha_1,\alpha_2,\alpha_3)$. We set $H^m(\Omega) = H^{m,0}(\Omega)$ and $L^2(\Omega) = H^0(\Omega)$, which are the standard Sobolev and Lebesgue spaces, and we denote their norms of a function φ by $\|\varphi : H^m(\Omega)\|$ and $\|\varphi : L^2(\Omega)\|$, respectively. Besides, $H_0^m(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the $H^m(\Omega)$ norm. We also put $\mathscr{H}^m(\Omega) = H^{m+1}(\Omega)\times H^m(\Omega)$.

Definition 1.2. We say that the obstacle $\mathscr{O}$ is *admissible* if there exist a non-negative integer ℓ and a real constant $\gamma_0 \ge 1$ having the following property: Suppose that $\Xi = (\vec{v}_0, f) \in X_{c,a}(T)$ for some $c > 0$ and $a > 1$. Then for any $b > 1$, any integer $m \ge 1$ and any $\gamma \in (0,\gamma_0]$, there exists a positive constant $C = C(\gamma,a,b,c,m,\Omega)$ such that for $t \in [0,T)$,

$$\sum_{|\alpha|\le m}\langle t\rangle^\gamma\left\|\partial^\alpha v(t,\cdot)\!:\!L^2(\Omega_b)\right\| \le C\left(\left\|\vec{v}_0\!:\!\mathscr{H}^{m+\ell-1}(\Omega)\right\| + \sup_{0\le s\le t}\langle s\rangle^\gamma\sum_{|\alpha|\le m+\ell-1}\left\|\partial^\alpha f(s,\cdot)\!:\!L^2(\Omega)\right\|\right), \tag{1.11}$$

where v is the solution to (1.7)–(1.9) with the propagation speed c.

We often refer to (1.11) as decay of local energy (or local energy decay).

For $F_i = F_i(u,\partial u,\nabla_x\partial u)$, we denote the quadratic part of F_i by $F_i^{(2)}$. More precisely, writing $\zeta = (\zeta_1,\ldots,\zeta_{17N}) = (u,\partial u,\nabla_x\partial u)$, we define

$$F_i^{(2)}(\zeta) = \sum_{|\alpha|=2}\frac{(\partial_\zeta^\alpha F_i)(0)}{\alpha!}\zeta^\alpha, \tag{1.12}$$

where α is a multi–index with the standard notation.

Definition 1.3. We say that the nonlinearity $F = (F_1,F_2,\ldots,F_N)$ satisfies the *null condition* associated with the propagation speeds $(c_1,c_2,\ldots,c_N)$ if each $F_i^{(2)}$ $(1 \le i \le N)$, given by (1.12), depends only on ∂u and $\nabla_x\partial u$ (namely $F_i^{(2)} = F_i^{(2)}(\partial u,\nabla_x\partial u)$), and satisfies

$$F_i^{(2)}\big((X_a\mu_j),(X_kX_a\nu_j)\big)=0 \tag{1.13}$$

for any μ, $\nu\in\Lambda_i$ and $X=(X_0,X_1,X_2,X_3)\in\boldsymbol{R}^4$ satisfying $X_0^2=c_i^2(X_1^2+X_2^2+X_3^2)$, where

$$\Lambda_i=\big\{(\lambda_1,\lambda_2,\dots,\lambda_N)\in\boldsymbol{R}^N;\lambda_j=0 \text{ if } c_j\neq c_i\big\}.$$

Here the left-hand side of (1.13) means that $X_a\mu_j$ $(a=0,1,2,3;\ j=1,\dots,N)$ and $X_kX_a\nu_j$ $(k=1,2,3;\ a=0,1,2,3;\ j=1,\dots,N)$ are substituted in place of $\partial_a u_j$ and $\partial_k\partial_a u_j$, respectively.

We remark that under the null condition, each $F_i^{(2)}(\partial u,\nabla_x\partial u)$ is expressed as a sum of two groups of terms. The one is a linear combination of $Q_0(u_j,u_k;c_i)$, $Q_{ab}(u_j,u_k)$, where Q_0 and Q_{ab} are the null forms defined by

$$Q_0(\xi,\eta\,;c)=(\partial_t\xi)(\partial_t\eta)-c^2(\nabla_x\,\xi)\cdot(\nabla_x\,\eta), \tag{1.14}$$

$$Q_{ab}(\xi,\eta)=(\partial_a\xi)(\partial_b\eta)-(\partial_b\xi)(\partial_a\eta)\quad(0\le a<b\le 3) \tag{1.15}$$

for a positive constant c, and real valued-functions $\xi=\xi(t,x)$ and $\eta=\eta(t,x)$. The other is a linear combination of such terms $(\partial_a u_j)(\partial_b u_k)$ that at least one of c_i, c_j and c_k is different from the others. More precise expression is given by (5.1) below.

Now we are in a position to state our main result.

Theorem 1.4. *Let* (1.4) *be fulfilled, and* ϕ, $\psi\in C^\infty(\overline{\Omega}\,;\boldsymbol{R}^N)$. *Suppose that* (ϕ,ψ,F) *satisfies the compatibility condition to infinite order for the problem* (1.1)–(1.3), $\mathscr{O}$ *is admissible, and* F *satisfies the null condition associated with* $(c_1,c_2,\dots,c_N)$. *Then there exist a positive constant* ε_0 *and an integer* s *such that the mixed problem* (1.1)–(1.3) *admits a unique solution* $u\in C^\infty([0,\infty)\times\overline{\Omega};\boldsymbol{R}^N)$, *satisfying* (1.5) *and* (1.6), *for any* (ϕ,ψ) *with*

$$\big\|\phi\colon H^{s+2,s}(\Omega)\big\|+\big\|\psi\colon H^{s+1,s}(\Omega)\big\|\le\varepsilon_0.$$

Theorem 1.4 was already presented in [**25**] with a different assumption on the obstacles; they assumed exponential decay of local energy, with possible loss of derivatives as in (1.11), for solutions to the mixed problem of homogeneous wave equations on $[0,\infty)\times\Omega$ (see (B.8) below). The same assumption is made also in [**24**], [**26**]. The known examples satisfying their assumption, given in [**24**], [**25**], [**26**], are non-trapping obstacles, and trapping obstacles which were treated in Ikawa [**8**], [**9**]. All the obstacles satisfying their assumption are also admissible in our sense (see Appendix B below for the proof). Thus our assumption is possibly weaker than theirs. More precisely, exponential decay of local energy is not actually needed in [**24**], [**25**], [**26**], but one needs (1.11) for γ up to some $\gamma_0 > 1$ to apply their method. On the other hand, only (1.11) for $\gamma \le 1$ is required in our method. However we have unfortunately *no* concrete example of admissible obstacle (in our sense) other than those satisfying also their assumption. Hence, at the present time, we may say that there is no essential difference between the practical claims in Theorem 1.4 and [**25**].

Here we emphasize that our main aim in this paper is to obtain a simplified proof of the global existence result in [**25**], and not to weaken the assumption on the obstacles.

This paper is organized as follows. In the next section we collect notation. In Section 3 we give some preliminaries needed later on. Section 4 is devoted to establish pointwise decay estimates. Making use of the estimates from Section 4, we give a proof of Theorem 1.4 in Section 5. The appendices are devoted to discussion on admissible obstacles, as well as the proof of Lemmas 3.1 and 3.5 below.

2. Notation.

Let $c > 0$. For $\Xi = (v_0, v_1, f) \in H_0^1(\Omega) \times L^2(\Omega) \times L^\infty((0,T); L^2(\Omega))$, we denote by $S[\Xi; c](t,x)$ the solution of the mixed problem (1.7)–(1.9). Besides we set $K[\vec{v}_0; c] = S[(\vec{v}_0, 0); c]$ and $L[f; c] = S[(0,0,f); c]$, where $\vec{v}_0 = (v_0, v_1)$, as before.

Similarly, for $(w_0, w_1, g) \in H^1(\boldsymbol{R}^3) \times L^2(\boldsymbol{R}^3) \times L^\infty((0,T); L^2(\boldsymbol{R}^3))$, we denote by $S_0[(w_0, w_1, g); c](t,x)$ the solution of the following Cauchy problem:

$$\left(\partial_t^2 - c^2 \Delta_x\right) w = g, \qquad (t,x) \in (0,T) \times \boldsymbol{R}^3, \tag{2.1}$$

$$w(0,x) = w_0(x), \ (\partial_t w)(0,x) = w_1(x), \qquad x \in \boldsymbol{R}^3. \tag{2.2}$$

Besides we put $K_0[\vec{w}_0; c] = S_0[(\vec{w}_0, 0); c]$ and $L_0[g; c] = S_0[(0,0,g); c]$, where $\vec{w}_0 = (w_0, w_1)$.

Next we introduce vector fields: We denote

$$\partial_0 = \partial_t, \quad \partial_j \ (j = 1,2,3), \quad \Omega_{ij} = x_i \partial_j - x_j \partial_i \ (1 \le i < j \le 3),$$

by Z_j $(j = 0, 1, \ldots, 6)$, respectively. Notice that

$$\left[Z_j, \partial_t^2 - c^2 \Delta_x\right] = 0 \quad (j = 0, 1, \ldots, 6), \tag{2.3}$$

where we put $[A,B] := AB - BA$. Denoting $Z^\alpha = Z_0^{\alpha_0} Z_1^{\alpha_1} \cdots Z_6^{\alpha_6}$ with a multi–index $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_6)$, we set

$$|\varphi(t,x)|_m = \sum_{|\alpha| \le m} |Z^\alpha \varphi(t,x)|, \quad \|\varphi(t)\|_m = \big\| \, |\varphi(t,\cdot)|_m : L^2(\Omega) \big\| \tag{2.4}$$

for a real or $\boldsymbol{R}^N$–valued smooth function $\varphi(t,x)$ and a non-negative integer m.

For ν, $\kappa \in \boldsymbol{R}$, $c \ge 0$ and $c_j > 0$ $(1 \le j \le N)$, we define

$$\Phi_\nu(t,x) = \begin{cases} \langle t + |x| \rangle^\nu & \text{if } \nu < 0, \\ \left\{ \log\left(2 + \dfrac{\langle t + |x| \rangle}{\langle t - |x| \rangle}\right) \right\}^{-1} & \text{if } \nu = 0, \\ \langle t - |x| \rangle^\nu & \text{if } \nu > 0, \end{cases} \tag{2.5}$$

$$W_{\nu,\kappa}(t,x) = \langle t + |x| \rangle^\nu \Big(\min_{0 \le j \le N} \langle c_j t - |x| \rangle \Big)^\kappa, \tag{2.6}$$

$$W^{(c)}_{\nu,\kappa}(t,x) = \langle t + |x| \rangle^\nu \Big(\min_{0 \le j \le N; c_j \ne c} \langle c_j t - |x| \rangle \Big)^\kappa, \tag{2.7}$$

where $c_0 = 0$. We set

$$\nu_*(\rho,\kappa) = \begin{cases} \rho & \text{if } \rho > 0,\ \kappa > 1, \\ \rho + 1 - \kappa & \text{if } \rho > 0,\ 0 \le \kappa < 1. \end{cases} \tag{2.8}$$

We define

$$\|f(t) : N_k(\mathscr{W})\| = \sup_{(s,x) \in [0,t] \times \Omega} \langle x \rangle \, \mathscr{W}(s,x) \, |f(s,x)|_k \tag{2.9}$$

for $t \in [0,T)$, a non-negative integer k and any non-negative function $\mathscr{W}(s,x)$. Similarly we put

$$\|g(t) : M_k(\mathscr{W})\| = \sup_{(s,x) \in [0,t] \times \boldsymbol{R}^3} \langle x \rangle \, \mathscr{W}(s,x) \, |g(s,x)|_k. \tag{2.10}$$

Let $\rho \ge 0$, and k be a non-negative integer. We define

$$\mathscr{A}_{\rho,k}[v_0,v_1] = \sup_{x \in \Omega} \langle x \rangle^\rho \big(|v_0(x)|_k + |\nabla_x v_0(x)|_k + |v_1(x)|_k \big) \tag{2.11}$$

for a smooth function (v_0, v_1) on Ω, while

$$\mathscr{B}_{\rho,k}[w_0, w_1] = \sup_{x\in \boldsymbol{R}^3} \langle x\rangle^{\rho} \big(|w_0(x)|_k + |\nabla_x w_0(x)|_k + |w_1(x)|_k\big) \tag{2.12}$$

for a smooth function (w_0, w_1) on $\boldsymbol{R}^3$.

For $a \geq 1$, let ψ_a be a smooth radially symmetric function on $\boldsymbol{R}^3$ satisfying

$$\psi_a(x) = 0 \ (|x| \leq a), \quad \psi_a(x) = 1 \ (|x| \geq a+1). \tag{2.13}$$

3. Preliminaries.

First we introduce the well-known elliptic estimate, whose proof will be given in Appendix A for the completeness.

Lemma 3.1. *Let $\varphi \in H^m(\Omega) \cap H_0^1(\Omega)$ for some integer $m (\geq 2)$. Then we have*

$$\sum_{|\alpha|=m} \big\|\partial_x^\alpha \varphi : L^2(\Omega)\big\| \leq C\big(\|\Delta_x \varphi : H^{m-2}(\Omega)\| + \|\nabla_x \varphi : L^2(\Omega)\|\big). \tag{3.1}$$

Next we introduce a couple of known estimates for the Cauchy problem. The first one is the decay estimate of solutions to the homogeneous wave equation, due to Asakura [**1**, Proposition 1.1] (observe that the general case can be reduced to the case $k = 0$, thanks to (2.3)). Recall that $\Phi_\nu(t, x)$ is the function defined by (2.5).

Lemma 3.2. *Let $c > 0$. For $\vec{w_0} = (w_0, w_1) \in C_0^\infty(\boldsymbol{R}^3; \boldsymbol{R}^2)$, $\rho > 0$ and a non-negative integer k, there exists a positive constant $C = C(\rho, k, c)$ such that*

$$\langle t + |x|\rangle\, \Phi_{\rho-1}(ct, x) |K_0[\vec{w_0}; c](t, x)|_k \leq C \mathscr{B}_{\rho+1,k}[\vec{w_0}] \tag{3.2}$$

for $(t, x) \in [0, \infty) \times \boldsymbol{R}^3$.

The second one is the decay estimate for the inhomogeneous wave equation.

Lemma 3.3. *Let $c > 0$, $\rho > 0$, $\kappa \geq 0$ with $\kappa \neq 1$, and k be a non-negative integer. Then there exists a positive constant $C = C(\rho, \kappa, k, c)$ such that*

$$\langle t + |x|\rangle\, \Phi_{\rho-1}(ct, x) |L_0[g; c](t, x)|_k \leq C \big\|g(t) : M_k(W_{\nu_*(\rho,\kappa),\kappa})\big\|, \tag{3.3}$$

for $(t,x)\in[0,T)\times \boldsymbol{R}^3$, *where* $\nu_*(\rho,\kappa)$ *is given by* (2.8).

Proof. The desired estimate for $k=0$ was shown in Theorem 3.4 of Kubota – Yokoyama [**19**] (see also Lemmas 3.2 and 8.1 in Katayama – Yokoyama [**13**], and Lemma 3.2 in the authors [**12**]).

Let $|\alpha|\le k$. Then it follows from (2.3) that

$$Z^\alpha L_0[g;c]=L_0[Z^\alpha g;c]+K_0[(\phi_\alpha,\psi_\alpha);c], \tag{3.4}$$

where we put $\phi_\alpha(x)=(Z^\alpha L_0[g;c])(0,x)$, $\psi_\alpha(x)=(\partial_t Z^\alpha L_0[g;c])(0,x)$. The first term on the right-hand side of (3.4) can be easily estimated by (3.3) for $k=0$. On the other hand, as for the second term, from the equation (2.1) we get

$$\phi_\alpha(x)=\sum_{|\beta|\le|\alpha|-2} C_\beta(Z^\beta g)(0,x), \quad \psi_\alpha(x)=\sum_{|\beta|\le|\alpha|-1} C'_\beta(Z^\beta g)(0,x)$$

with suitable constants C_β and C'_β (cf. (1.10)). Therefore, by virtue of Lemma 3.2, we obtain

$$\langle t+|x|\rangle\, \Phi_{\rho-1}(ct,x)|K_0[\phi_\alpha,\psi_\alpha;c](t,x)|\le C\sup_{y\in\boldsymbol{R}^3}\langle y\rangle^{\rho+1}\,|g(0,y)|_{k-1}.$$

Since we have $\nu_*(\rho,\kappa)+\kappa\ge\rho+1$, it follows that

$$\begin{aligned}\sup_{y\in\boldsymbol{R}^3}\langle y\rangle^{\rho+1}\,|g(0,y)|_{k-1} &\le \sup_{y\in\boldsymbol{R}^3}\langle y\rangle^{\rho+2}\,|g(0,y)|_{k}\\ &\le C\big\|g(t):M_k(W_{\nu_*(\rho,\kappa),\kappa})\big\|. \end{aligned}\tag{3.5}$$

This completes the proof. □

The third one is the decay estimate for derivatives of solutions to the inhomogeneous wave equation.

Lemma 3.4. *Let* $c>0$, *and* k *be a non-negative integer.*

If $\rho>1$ *and* $\kappa>1$, *or alternatively if* $0<\rho\le 1$ *and* $0<\kappa<\rho$, *then there exists a positive constant* $C=C(c,\rho,\kappa,k)$ *such that*

$$\langle x\rangle\,\langle ct-|x|\rangle^\rho|\partial L_0[g;c](t,x)|_k\le C\big\|g(t):M_{k+1}(W_{\nu_*(\rho,\kappa),\kappa})\big\| \tag{3.6}$$

for $(t,x)\in[0,T)\times\boldsymbol{R}^3$.

On the other hand, if $\rho > 0$ and $\kappa > 1$, then we have

$$\langle x\rangle \langle ct-|x|\rangle^{\rho}|\partial L_0[g;c](t,x)|_k \le C\big\|g(t):M_{k+1}(W^{(c)}_{\rho,\kappa})\big\| \tag{3.7}$$

for $(t,x)\in[0,T)\times \boldsymbol{R}^3$.

Proof. Let $0\le a\le 3$. In view of Lemma 3.2 in [**19**], Lemma 8.2 and the proof of Lemma 3.2 in [**13**], we find that

$$\langle x\rangle \langle ct-|x|\rangle^{\rho}\,|L_0[\partial_a g;c](t,x)| \le C\big\|g(t):M_1(W_{\nu_*(\rho,\kappa),\kappa})\big\| \tag{3.8}$$

for $\rho>1$ and $\kappa>1$, or for $0<\rho\le 1$ and $0<\kappa<\rho$, as well as

$$\langle x\rangle \langle ct-|x|\rangle^{\rho}\,|L_0[\partial_a g;c](t,x)| \le C\big\|g(t):M_1(W^{(c)}_{\rho,\kappa})\big\|, \tag{3.9}$$

for $\rho>0$ and $\kappa>1$ (see also [**12**]).

Since $\partial_a L_0[g;c] = L_0[\partial_a g;c]+\delta_{a0}K_0[(0,g(0,\cdot));c]$ for $0\le a\le 3$ with the Kronecker delta δ_{ab}, (3.6) and (3.7) follow from (3.4), (3.8), (3.9), and Lemma 3.2, with the help of (3.5) and its variant obtained by replacing $W_{\nu_*(\rho,\kappa),\kappa}$ by $W^{(c)}_{\rho,\kappa}$. This completes the proof. □

In order to associate decay estimates with the energy estimate, we use the following variant of the Sobolev type inequality, whose counterpart for the Cauchy problem is due to Klainerman [**15**]:

Lemma 3.5. *Let $\varphi\in C^2_0(\overline{\Omega})$. Then we have*

$$\sup_{x\in\Omega}\langle x\rangle\,|\varphi(x)| \le C\sum_{|\alpha|\le 2}\big\|\widetilde{Z}^{\alpha}\varphi:L^2(\Omega)\big\|, \tag{3.10}$$

where $\widetilde{Z}=\{\partial_1,\partial_2,\partial_3,\Omega_{12},\Omega_{23},\Omega_{13}\}$.

The proof of Lemma 3.5 will be given in Appendix C.

Finally, we recall the estimates of the null forms from [**12**].

Lemma 3.6. *Let c be a positive constant and $u=(u_1,\dots,u_N)$. Suppose that Q is one of the null forms defined by* (1.14) *and* (1.15)*. Then, for a non-negative integer k, there exists a positive constant $C=C(c,k)$ such that*

$$|Q(u_j,u_k)|_k \le C\Big\{|\partial u|_{[k/2]}\sum_{|\alpha|\le k}|D_{+,c}Z^\alpha u| + |\partial u|_k \sum_{|\alpha|\le [k/2]}|D_{+,c}Z^\alpha u| + \frac{1}{r}\big(|\partial u|_{[k/2]}|u|_{k+1} + |u|_{[k/2]+1}|\partial u|_k\big)\Big\},$$

where we put $D_{+,c} = \partial_t + c\,\partial_r$ *with* $r = |x|$ *and* $\partial_r = (x/r)\cdot\nabla_x$.

4. Basic estimates.

The aim of this section is to establish pointwise decay estimates for the mixed problem, which are deduced from corresponding estimates for the Cauchy problem in combination with the local energy decay (1.11). To prove such estimates we use the following lemma. Remember that we have assumed $\mathscr{O} \subset B_1$.

Lemma 4.1. *Let $\mathscr{O}$ be admissible, and ℓ and γ_0 be the constants in Definition 1.2. Let $b > 1$, $c > 0$, $\rho > 0$, and $\kappa \ge 0$ with $\kappa \ne 1$, while m is a non-negative integer.*

(i) *Suppose that χ is a smooth radially symmetric function on $\boldsymbol{R}^3$ satisfying* $\operatorname{supp}\chi \subset B_b$. *If $\rho \le \gamma_0$, and $\Xi = (\vec{v_0}, f) \in X_{c,a}(T)$ for some $a(> 1)$, then there exists a positive constant $C = C(\rho,a,b,c,m,\Omega)$ such that*

$$\langle t\rangle^\rho |\chi S[\Xi;c](t,x)|_m \le C\mathscr{A}_{\rho+1,m+\ell+1}[\vec{v}_0] + C\sum_{|\beta|\le m+\ell+1}\ \sup_{(s,x)\in[0,t]\times\Omega_a}\langle s\rangle^\rho|\partial^\beta f(s,x)| \tag{4.1}$$

for $(t,x) \in [0,T)\times\overline{\Omega}$.

(ii) *Let $\vec{w}$ and g are smooth functions on $\boldsymbol{R}^3$ and on $[0,T)\times\boldsymbol{R}^3$, respectively.*

If $\operatorname{supp}\vec{w}_0 \cup \operatorname{supp} g(t,\cdot) \subset \overline{B_a\setminus B_1}$ *for any $t\in[0,T)$ with some $a > 1$, then there exists a positive constant $C = C(\rho,a,c,m)$ such that*

$$\langle t+|x|\rangle\,\Phi_{\rho-1}(ct,x)|S_0[(\vec{w}_0,g);c](t,x)|_m \le C\mathscr{A}_{\rho+1,m}[\vec{w}_0] + C\sum_{|\beta|\le m}\ \sup_{(s,x)\in[0,t]\times\Omega_a}\langle s\rangle^\rho|\partial^\beta g(s,x)| \tag{4.2}$$

and

$$\langle x\rangle \langle ct-|x|\rangle^{\rho}|\partial S_0[(\vec{w}_0,g);c](t,x)|_m$$
$$\le C\mathscr{A}_{\rho+2,m+1}[\vec{w}_0]+C\sum_{|\beta|\le m+1}\sup_{(s,x)\in[0,t]\times\Omega_a}\langle s\rangle^{\rho}|\partial^{\beta}g(s,x)| \tag{4.3}$$

for $(t,x)\in[0,T)\times\overline{\Omega}$.

On the other hand, if $\vec{w_0}(x)=g(t,x)=0$ *for any* $x\in B_1$ *and any* $t\in[0,T)$, *then there exists a positive constant* $C=C(\rho,a,b,c,m)$ *such that*

$$\langle t\rangle^{\rho}\sum_{|\beta|\le m}|\partial^{\beta}S_0[(\vec{w_0},g);c](t,x)|$$
$$\le C\mathscr{A}_{\rho+1,m}[\vec{w}_0]+C\sum_{|\beta|\le m}\big\|\partial^{\beta}g(t)\colon N_0(W_{\nu_*(\rho,\kappa),\kappa})\big\| \tag{4.4}$$

for $(t,x)\in[0,T)\times\overline{\Omega_b}$.

Proof. First we note that we have

$$|h(t,x)|_m\le C\sum_{|\beta|\le m}|\partial^{\beta}h(t,x)| \tag{4.5}$$

for any smooth function h on $[0,T)\times\overline{\Omega}$ (or on $[0,T)\times\boldsymbol{R}^3$) with $\operatorname{supp}h(t,\cdot)\subset B_R$ for some $R(>1)$.

Let $\Xi\in X_{c,a}(T)$, and $\rho\le\gamma_0$. For $(t,x)\in[0,T)\times\overline{\Omega}$, by (4.5), the Sobolev inequality and (1.11), we obtain

$$\langle t\rangle^{\rho}|\chi S[\Xi;c](t,x)|_m$$
$$\le C\langle t\rangle^{\rho}\sum_{|\beta|\le m+2}\big\|\partial^{\beta}S[\Xi;c](t)\colon L^2(\Omega_b)\big\|$$
$$\le C\big\|\vec{v}_0\colon\mathscr{H}^{m+\ell+1}(\Omega)\big\|+C\sup_{s\in[0,t]}\langle s\rangle^{\rho}\sum_{|\beta|\le m+\ell+1}\big\|\partial^{\beta}f(s)\colon L^2(\Omega)\big\|,$$

which yields (4.1), since $\operatorname{supp}f(t,\cdot)\subset\overline{\Omega_a}$ implies $\big\|\partial^{\beta}f(s)\colon L^2(\Omega)\big\|\le C\big\|\partial^{\beta}f(s)\colon L^{\infty}(\Omega_a)\big\|$.

Let ξ and η be functions on $\Lambda(\subset\boldsymbol{R}\times\boldsymbol{R}^3)$. We write $\xi(t,x)\sim\eta(t,x)$ for $(t,x)\in\Lambda$, if there exists a positive constant C such that

$$C^{-1}\xi(t,x)\le\eta(t,x)\le C\xi(t,x) \text{ for any } (t,x)\in\Lambda.$$

Observing that we have $W_{\rho,\kappa}(t,x) \le W^{(c)}_{\rho,\kappa}(t,x) \le C\langle t+|x|\rangle^\rho \langle |x|\rangle^\kappa$ for $(t,x) \in [0,\infty)\times \boldsymbol{R}^3$, we obtain

$$\begin{aligned}\langle t\rangle^\rho &\sim \langle x\rangle\, W_{\rho,\kappa}(t,x) \sim \langle x\rangle\, W^{(c)}_{\rho,\kappa}(t,x)\\ &\sim \langle t+|x|\rangle \Phi_{\rho-1}(ct,x) \sim \langle x\rangle\, \langle ct-|x|\rangle^\rho \end{aligned} \tag{4.6}$$

for $(t,x) \in [0,\infty)\times B_R$, where $R>0$, $\rho\ge 0$, $c>0$, and $\kappa \ge 0$.

By (3.2) and (3.3) with $\kappa > 1$, we find that the left-hand side on (4.2) is estimated by $C\mathscr{B}_{\rho+1,m}[\vec{w}_0] + C\big\|g(t): M_m(W_{\rho,\kappa})\big\|$, and we obtain (4.2) in view of (4.6), since $\operatorname{supp}\vec{w}_0 \cup \operatorname{supp} g(t,\cdot) \subset \overline{B_a\setminus B_1} \subset \overline{\Omega_a}$. Similarly, if we use (3.7) instead of (3.3), then we get (4.3).

On the other hand, replacing Z^α by ∂^α in the proof of (3.3), and using (4.6), we find

$$\langle t\rangle^\rho \sum_{|\beta|\le m} |\partial^\beta S_0[(\vec{w}_0,g);c](t,x)| \le C\mathscr{B}_{\rho+1,m}[\vec{w}_0] + C\sum_{|\beta|\le m} \big\|\partial^\beta g(t): M_0(W_{\nu_*(\rho,\kappa),\kappa})\big\|$$

for $(t,x)\in[0,T)\times\overline{\Omega_b}$, which leads to (4.4), because of the assumption on $\vec{w}_0$ and g. This completes the proof. □

Theorem 4.2. *Let $\mathscr{O}$ be admissible, ℓ and γ_0 be the constants in Definition 1.2, and $c>0$. Suppose that $\Xi = (\vec{v}_0, f) \in X_c(T)$ and $f = f_1 + f_2$.*
(i) *Let $\rho \in (0,\gamma_0]$, $\kappa_i \ge 0$ and $\kappa_i \ne 1$ $(i=1,2)$. Then there exists a constant $C = C(\rho,\kappa_1,\kappa_2,c) > 0$ such that*

$$\begin{aligned}&\langle t+|x|\rangle \Phi_{\rho-1}(ct,x) |S[\Xi;c](t,x)|_k\\ &\quad\le C\mathscr{A}_{\rho+1,k+\ell+3}[\vec{v}_0] + C\sum_{|\beta|\le \ell+3}\sum_{i=1}^{2} \big\|\partial^\beta f_i(t): N_k(W_{\nu_*(\rho,\kappa_i),\kappa_i})\big\|\end{aligned} \tag{4.7}$$

for $(t,x) \in [0,T)\times\overline{\Omega}$.
(ii) *Let $\kappa_2 > 1$. If $\gamma_0 > 1$, $\rho \in (1,\gamma_0)$ and $\kappa_1 > 1$, or alternatively if $0<\rho\le 1$ and $0<\kappa_1<\rho$, then we have*

$$\begin{aligned}&\langle x\rangle\,\langle ct-|x|\rangle^\rho |\partial S[\Xi;c](t,x)|_k\\ &\quad\le C\mathscr{A}_{\rho+2,k+\ell+4}[\vec{v}_0] + C\big\|f_1(t): N_{k+\ell+4}(W_{\nu_*(\rho,\kappa_1),\kappa_1})\big\|\\ &\qquad + C\big\|f_2(t): N_{k+\ell+4}(W^{(c)}_{\rho,\kappa_2})\big\|\end{aligned} \tag{4.8}$$

for $(t,x) \in [0,T) \times \overline{\Omega}$.

Proof. First we remark that, under the same assumption on $(\rho, \kappa_1, \kappa_2)$ for (4.7) (resp. (4.8)), $\sum_{|\beta|\le m} \sup_{(s,x)\in[0,t]\times\Omega_3} \langle s\rangle^\rho\, |\partial^\beta f(s,x)|$ with $m = k+\ell+3$ (resp. $m = k+\ell+4$) is bounded by the right-hand side of (4.7) (resp. (4.8)), because we have (4.6) and $\nu_*(\rho,\kappa_i) \ge \rho$ $(i=1,2)$. Hence we only have to prove (4.7) and (4.8) with these terms added on their right-hand sides.

Here we recall the following representation formula based on the cut–off method developed by Shibata [**29**], and also by Shibata – Tsutsumi [**31**] where L^p–L^q time decay estimates for the mixed problem were obtained (see also [**18**]):

$$S[\Xi;c](t,x) = \psi_1(x) S_0[\psi_2\Xi;c](t,x) + \sum_{i=1}^{4} S_i[\Xi](t,x) \tag{4.9}$$

for $(t,x) \in [0,T) \times \overline{\Omega}$, where ψ_a is defined by (2.13) and we have set

$$S_1[\Xi](t,x) = (1-\psi_2(x)) L\big[\,[\psi_1, -c^2\Delta_x] S_0[\psi_2\Xi;c];c\big](t,x), \tag{4.10}$$

$$S_2[\Xi](t,x) = -L_0\big[\,[\psi_2, -c^2\Delta_x] L\big[\,[\psi_1, -c^2\Delta_x] S_0[\psi_2\Xi;c];c\big];c\big](t,x), \tag{4.11}$$

$$S_3[\Xi](t,x) = (1-\psi_3(x)) S[(1-\psi_2)\Xi;c](t,x), \tag{4.12}$$

$$S_4[\Xi](t,x) = -L_0\big[\,[\psi_3, -c^2\Delta_x] S[(1-\psi_2)\Xi;c];c\big](t,x). \tag{4.13}$$

Writing $\zeta_0 = S_0[\psi_2\Xi;c]$, we get

$$\begin{aligned} &\langle x\rangle \langle ct-|x|\rangle^\rho \left|\partial_a\big(\psi_1\zeta_0(t,x)\big)\right|_k \\ &\quad\le \langle x\rangle \langle ct-|x|\rangle^\rho \big(|\psi_1(x)(\partial_a\zeta_0)(t,x)|_k + |(\partial_a\psi_1)(x)\zeta_0(t,x)|_k\big) \\ &\quad\le C\langle x\rangle \langle ct-|x|\rangle^\rho |\partial_a\zeta_0(t,x)|_k + C|\partial_a\psi_1(x)|\,\langle t\rangle^\rho \sum_{|\beta|\le k} |\partial^\beta\zeta_0(t,x)|, \end{aligned}$$

where the last inequality is obtained by (4.5) and (4.6), because we have $\operatorname{supp}\partial_a\psi_1 \subset \overline{B_2}$. Now, it follows from Lemmas 3.2, 3.3, and 3.4, together with (4.4), that $\psi_1 S_0[\psi_2\Xi;c]$ has the desired bound, since we can write $\Xi = (\vec{v}_0, 0) + \sum_{i=1}^2 (0,0,f_i)$.

We assume $0 < \rho \le \gamma_0$ and $\kappa_i \ge 0$ with $\kappa_i \ne 1$ in the following. It is easy to check that

$$[\psi_a, -\Delta_x] h(t,x) = h(t,x)\Delta_x\psi_a(x) + 2\nabla_x h(t,x)\cdot\nabla_x\psi_a(x)$$

for $(t,x) \in [0,T) \times \overline{\Omega}$, $a \geq 1$ and any smooth function h. Note that this identity implies $(0,0,[\psi_a, -c^2\Delta_x]h) \in X_{c,a+1}(T)$ for any smooth function h and $a \geq 1$, because $\operatorname{supp} \nabla_x \psi_a \cup \operatorname{supp} \Delta_x \psi_a \subset \overline{B_{a+1} \setminus B_a}$. Therefore, by (4.1) and (4.4), we obtain

$$\begin{aligned}
&\langle t\rangle^\rho |\partial^\alpha S_1[\Xi](t,x)|_k \\
&\quad \leq C \sum_{|\beta|\leq k+\ell+2+|\alpha|} \sup_{(s,x)\in[0,t]\times\Omega_2} \langle s\rangle^\rho |\partial^\beta S_0[\psi_2\Xi](s,x)| \\
&\quad \leq C\mathscr{A}_{\rho+1,k+\ell+2+|\alpha|}[\vec{v_0}] + C \sum_{|\beta|\leq k+\ell+2+|\alpha|} \sum_{i=1}^{2} \|\partial^\beta f_i(t) : N_0(W_{\nu_*(\rho,\kappa_i),\kappa_i})\|
\end{aligned} \tag{4.14}$$

for $(t,x) \in [0,T) \times \overline{\Omega}$ and $|\alpha| \leq 1$. Similarly, since we have $(1-\psi_2)\Xi \in X_{c,3}(T)$ for any $\Xi \in X_c(T)$, (4.1) leads to

$$\begin{aligned}
&\langle t\rangle^\rho |\partial^\alpha S_3[\Xi](t,x)|_k \\
&\quad \leq C\mathscr{A}_{\rho+1,k+\ell+1+|\alpha|}[\vec{v_0}] + C \sum_{|\beta|\leq k+\ell+1+|\alpha|} \sup_{(s,x)\in[0,t]\times\Omega_3} \langle s\rangle^\rho |\partial^\beta f(s,x)|
\end{aligned} \tag{4.15}$$

for $(t,x) \in [0,T) \times \overline{\Omega}$ and $|\alpha| \leq 1$. Since $\operatorname{supp} S_i[\Xi](t,x) \subset \overline{B_4}$ for $i = 1,3$, (4.14) and (4.15), together with (4.6), imply the desired estimates for $S_1[\Xi]$ and $S_3[\Xi]$ (note that we have $W_{\nu_*(\rho,\kappa_2),\kappa_2} \leq W^{(c)}_{\rho,\kappa_2}$ on $[0,\infty) \times \boldsymbol{R}^3$ for $\kappa_2 > 1$).

Set $g_j[\Xi] = (\partial_t^2 - c^2\Delta_x)S_j[\Xi]$ for $j = 2,4$. Observing that g_2 and g_4 have the almost same structures as S_1 and S_3, respectively, by (4.1) and (4.4) we obtain

$$\begin{aligned}
&\sum_{|\beta|\leq m} \sup_{(s,x)\in[0,t]\times\Omega_3} \langle s\rangle^\rho |\partial^\beta g_2[\Xi](s,x)| \\
&\quad \leq C\mathscr{A}_{\rho+1,m+\ell+3}[\vec{v_0}] + C \sum_{|\beta|\leq m+\ell+3} \sum_{i=1}^{2} \|\partial^\beta f_i(t) : N_0(W_{\nu_*(\rho,\kappa_i),\kappa_i})\|,
\end{aligned} \tag{4.16}$$

$$\begin{aligned}
&\sum_{|\beta|\leq m} \sup_{(s,x)\in[0,t]\times\Omega_4} \langle s\rangle^\rho |\partial^\beta g_4[\Xi](s,x)| \\
&\quad \leq C\mathscr{A}_{\rho+1,m+\ell+2}[\vec{v_0}] + C \sum_{|\beta|\leq m+\ell+2} \sup_{(s,x)\in[0,t]\times\Omega_3} \langle s\rangle^\rho |\partial^\beta f(s,x)|
\end{aligned} \tag{4.17}$$

for any $m \geq 0$. Thus, since g_2 and g_4 are supported on $\overline{B_4 \setminus B_2}$, (4.2) and (4.3) with $\vec{w}_0 = (0,0)$ imply the desired estimates for $S_2[\Xi]$ and $S_4[\Xi]$. This completes the proof. $\square$

In order to handle the null forms, we also need the following estimate of a tangential derivative to the light cone $ct = |x|$ which is denoted by $D_{+,c} = \partial_t + c\partial_r$.

THEOREM 4.3. *Let the assumptions in Theorem* 4.2 *be fulfilled, and let* $1 \le \rho \le \min\{2, \gamma_0\}$, $\kappa_i \ge 0$ *and* $\kappa_i \ne 1$ $(i = 1, 2)$. *Then there exists a constant* $C = C(\rho, \kappa_1, \kappa_2, c) > 0$ *such that*

$$\frac{\langle x\rangle \langle t+|x|\rangle \langle ct-|x|\rangle^{\rho-1}}{\log(2+t+|x|)} \sum_{|\alpha|\le k} |D_{+,c} Z^\alpha S[\Xi; c](t,x)|$$
$$\le C\mathscr{A}_{\rho+1,k+\ell+5}[\vec{v}_0] + C\sum_{i=1}^{2} \left\| f_i(t) : N_{k+\ell+5}(W_{\nu_*(\rho,\kappa_i),\kappa_i}) \right\| \tag{4.18}$$

for $(t,x) \in [0,T) \times \overline{\Omega}$.

PROOF. When $|x| \le 1$, (4.18) follows from (4.7) immediately. While, if $|x| > 1$, then we can proceed as in the proof of Theorem 1.2 in [**12**], because $\mathscr{O} \subset B_1$. Here we only give an outline of the proof. Setting $U_\alpha(t,r,\omega) = rZ^\alpha S[\Xi; c](t, r\omega)$ for $r > 1$, $\omega \in S^2$ and $|\alpha| \le k$, we have

$$D_{-,c} D_{+,c} U_\alpha(t,r,\omega) = rZ^\alpha f(t, r\omega) + \frac{c^2}{r} \sum_{1\le i<j\le 3} \Omega_{ij}^2 Z^\alpha S[\Xi; c](t, r\omega), \tag{4.19}$$

where $D_{-,c} = \partial_t - c\partial_r$. Let $t_0 > 0$, $r_0 > 1$ and $\omega_0 \in S^2$. Then we have

$$|rZ^\alpha f(t, r\omega)| \le C \sum_{i=1}^{2} W^{-1}_{\nu_*(\rho,\kappa_i),\kappa_i}(t,x) \left\| f_i(t_0) : N_k(W_{\nu_*(\rho,\kappa_i),\kappa_i}) \right\|$$

for $t \le t_0$. Applying (4.7) to estimate the second term on the right-hand side of (4.19), we find that $|D_{-,c} D_{+,c} U(t,r,\omega)|$ is bounded from above by the right-hand side of (4.18) (with $t = t_0$) multiplied by

$$\sum_{i=1}^{2} W^{-1}_{\nu_*(\rho,\kappa_i),\kappa_i}(t,x) + \langle x\rangle^{-1} \langle t+|x|\rangle^{-1} \Phi^{-1}_{\rho-1}(ct,x)$$

for $t \le t_0$. Integrating the obtained inequality along the ray

$$\{(t, (r_0 + c(t_0 - t))\omega_0); 0 \le t \le t_0\}$$

(note that this ray lies in Ω), we obtain

$$\begin{aligned} &\frac{\langle t_0 + r_0\rangle^\rho}{\log(2 + t_0 + r_0)} |D_{+,c} U_\alpha(t_0, r_0, \omega_0)| \\ &\quad \le C \mathscr{A}_{m+\rho+1,\ell+5}[\vec{v}_0] + C \sum_{i=1}^{2} \big\| f_i(t_0) : N_{m+\ell+5}(W_{\nu_*(\rho,\kappa_i),\kappa_i}) \big\|. \end{aligned} \tag{4.20}$$

Since $r D_{+,c} Z^\alpha S[\Xi; c](t, r\omega) = D_{+,c} U_\alpha(t, r, \omega) - c Z^\alpha S[\Xi; c](t, r\omega)$, (4.20) and (4.7) imply (4.18) for $|x| \ge 1$. This completes the proof. □

5. Proof of Theorem 1.4.

In this section we prove Theorem 1.4. We assume $\mathscr{O} \subset B_1$ as before. Let all the assumptions of Theorem 1.4 be fulfilled.

Though there is no essential difficulty in treating the quasi-linear case[1], we concentrate on the semilinear case to keep our exposition simple. Hence we assume $F = F(u, \partial u)$ in what follows. We also suppose that $(\phi, \psi) \in C_0^\infty(\overline{\Omega}; \boldsymbol{R}^N \times \boldsymbol{R}^N)$ in the following. Observing that the argument below is independent of the size of the support of (ϕ, ψ), one can immediately obtain the result for the general data by the standard approximation argument.

From the null condition associated with $(c_1, c_2, \ldots, c_N)$, we see that the quadratic part $F_i^{(2)}$ of F_i can be written as

$$F_i^{(2)}(\partial u) = F_i^{\mathrm{null}}(\partial u) + R_{I,i}(\partial u) + R_{II,i}(\partial u), \tag{5.1}$$

where

$$F_i^{\mathrm{null}}(\partial u) = \sum_{\substack{1 \le j,k \le N \\ c_j = c_k = c_i}} \bigg(A_i^{jk} Q_0(u_j, u_k; c_i) + \sum_{0 \le a < b \le 3} B_i^{jk,ab} Q_{ab}(u_j, u_k) \bigg),$$

[1]In fact, to treat the quasi-linear case, we have only to replace the energy inequality for the wave equation in Subsections 5.1, 5.2 and 5.4 below with that for systems of perturbed wave equations which is also standard (remember that the symmetry condition (1.4) is assumed). Such replacement is not needed for pointwise decay estimates, because loss of derivatives is allowed there.

$$R_{I,i}(\partial u) = \sum_{\substack{1\le j,k\le N\\ c_j\neq c_k}} \sum_{0\le a,b\le 3} C_i^{jk,ab}(\partial_a u_j)(\partial_b u_k),$$

$$R_{II,i}(\partial u) = \sum_{\substack{1\le j,k\le N\\ c_j=c_k\neq c_i}} \sum_{0\le a,b\le 3} D_i^{jk,ab}(\partial_a u_j)(\partial_b u_k)$$

with suitable constants A_i^{jk}, $B_i^{jk,ab}$, $C_i^{jk,ab}$ and $D_i^{jk,ab}$. We put

$$H_i(u,\partial u) = F_i(u,\partial u) - F_i^{(2)}(\partial u)$$

for $i = 1, 2, \ldots, N$, so that $H_i(u,\partial u) = O(|u|^3 + |\partial u|^3)$ near $(u,\partial u) = (0,0)$.

Let $u = (u_1, u_2, \ldots, u_N)$ be a smooth solution to (1.1)–(1.3) on $[0,T)\times\overline{\Omega}$. We set

$$\begin{aligned} e_{k,i}[u_i](t,x) = {} & \langle t+|x|\rangle\, \Phi_0(c_i t,x)|u_i(t,x)|_{k+1} + \langle x\rangle\, \langle c_i t - |x|\rangle\, |\partial u_i(t,x)|_k \\ & + \frac{\langle x\rangle\,\langle t+|x|\rangle}{\log(2+t+|x|)} \sum_{|\alpha|\le k-1} \left| D_{+,c_i} Z^\alpha u_i(t,x)\right| \end{aligned}$$

for $1 \le i \le N$. We also set $e_k[u](t,x) = \sum_{i=1}^N e_{k,i}[u_i](t,x)$.

We fix $k \ge 6\ell + 28$, and suppose that

$$\left\|\phi : H^{2k+1,2k-1}(\Omega)\right\| + \left\|\psi : H^{2k,2k-1}(\Omega)\right\| \le \varepsilon. \tag{5.2}$$

Note that, by the Sobolev inequality, we have

$$\sum_{|\alpha|\le 2k-1} \left| \langle x\rangle^{2k-1}\, \partial_x^\alpha \phi(x)\right| + \sum_{|\alpha|\le 2k-2} \left| \langle x\rangle^{2k-1}\, \partial_x^\alpha \psi(x)\right| \le C\varepsilon$$

for any $x \in \overline{\Omega}$. Especially we have $e_k[u](0) \le C\varepsilon$.

Since the local existence for the mixed problem (1.1)–(1.3) has been shown by [**31**], what we need for the proof of Theorem 1.4 is a suitable *a priori* estimate. Assume that

$$\sup_{0\le t<T} \left\| e_k[u](t) : L^\infty(\Omega)\right\| \le M\varepsilon \tag{5.3}$$

holds for some large $M(>1)$ and small $\varepsilon(>0)$, satisfying $M\varepsilon \le 1$. We will prove that (5.3) implies

$$\sup_{0\le t<T} \left\| e_k[u](t) : L^\infty(\Omega) \right\| \le C\varepsilon + CM^2\varepsilon^2, \tag{5.4}$$

where C is a constant independent of M, ε and T. From (5.4) we find that (5.3) with M replaced by $M/2$ is true for $M \ge 4C$ and $\varepsilon \le 1/(4CM)$. Then, for small ε, the standard continuity argument implies that $e_k[u](t)$ stays bounded as long as the solution u exists (observe that $\left\| e_k[u](t) : L^\infty(\Omega) \right\|$ is continuous with respect to t, because u is smooth and $\operatorname{supp} u(t,\cdot) \subset B_{t+R}$ for $t \in [0,T)$ with some $R > 0$). Theorem 1.4 follows immediately from this *a priori* bound.

To this end, the following energy estimate is crucial:

$$\|\partial u(t)\|_{2k-\ell-7} \le CM\varepsilon(1+t)^{C_*M\varepsilon+\rho_*} \quad \text{for } t \in [0,T), \tag{5.5}$$

where C, C_* and ρ_* are positive constants independent of M, ε and T. Moreover ρ_* can be chosen arbitrarily small. Once we find (5.5), we can proceed as in the case of the corresponding Cauchy problem (though we need careful evaluation of the possible nonlinearity u^3, because of loss of derivatives in (4.7), which is not present in (3.3)). While, unlike the case of the Cauchy problem, it is not so simple to get (5.5), because boundary terms coming from the integration–by–parts argument may cause some loss of derivatives. For this reason, we estimate the space–time gradient and generalized derivatives separately and improve the estimate of the latter by using some decay estimate.

In the following, we set $r = |x|$. We define

$$w_-(t,r) = \min_{0\le j\le N} \langle c_j t - r\rangle, \quad w_-^{(c)}(t,r) = \min_{0\le j\le N; c_j\ne c} \langle c_j t - r\rangle$$

for $c \ge 0$, with $c_0 = 0$. Note that, for $0 \le j,\ k \le N$, $c_j \ne c_k$ implies

$$\langle c_j t - r\rangle^{-1} \langle c_k t - r\rangle^{-1} \le C \langle t + r\rangle^{-1} \min\{\langle c_j t - r\rangle, \langle c_k t - r\rangle\}^{-1}.$$

Notice also that, for any $\mu > 0$ and $c > 0$, we have

$$\Phi_0(ct,x)^{-1} \le C \langle t + r\rangle^{\mu} \langle ct - r\rangle^{-\mu},$$

where C is a positive constant depending only on μ and c.

In the arguments below, we always suppose that M is large enough, while ε is small enough to satisfy $M\varepsilon \ll 1$.

Here we also remark that if (ϕ, ψ, F) satisfies the compatibility condition for (1.1)–(1.3), then $(\phi_i, \psi_i, f_i) \in X_{c_i}(T)$ for $1 \le i \le N$, where $f_i(t,x) = F_i(u(t,x), \partial u(t,x), \nabla_x \partial u(t,x))$.

5.1. Estimates of the energy.

In this subsection, we will prove

$$\sum_{|\alpha| \le 2k} \left\| \partial^\alpha \partial u(t) : L^2(\Omega) \right\| \le CM\varepsilon(1+t)^{C_0 M\varepsilon}, \tag{5.6}$$

where C_0 is a universal constant which is independent of M, ε and T.

For $0 \le m \le 2k$, we define $z_m(t) = \sum_{p=0}^{2k-m} \left\| \partial_t^p \partial u(t) : H^m(\Omega) \right\|$. To prove (5.6), it suffices to prove

$$z_m(t) \le CM\varepsilon(1+t)^{C_0 M\varepsilon} \quad \text{for } 0 \le m \le 2k. \tag{5.7}$$

First we evaluate $z_0(t)$. For $0 \le p \le 2k$, from (5.3) we get

$$\left| \partial_t^p F^{(2)}(\partial u)(t,x) \right| \le CM\varepsilon \langle t \rangle^{-1} \sum_{q=0}^{2k} \left| \partial_t^q \partial u(t,x) \right|,$$

and

$$\begin{aligned} \left| \partial_t^p H(u, \partial u)(t,x) \right| &\le C|u(t,x)|^3 + C \sum_{q=0}^{k} \sum_{|\alpha| \le 1} \left| \partial_t^q \partial^\alpha u(t,x) \right|^2 \sum_{q=0}^{2k} \left| \partial_t^q \partial u(t,x) \right| \\ &\le CM^3\varepsilon^3 \langle t+r \rangle^{-3+3\mu} w_-(t,r)^{-3\mu} \\ &\quad + CM^2\varepsilon^2 \langle t+r \rangle^{-2+2\mu} w_-(t,r)^{-2\mu} \sum_{q=0}^{2k} \left| \partial_t^q \partial u(t,x) \right| \end{aligned}$$

with small $\mu > 0$. Since we have

$$\left\| \langle t + |\cdot| \rangle^{-3+3\mu} \langle c_j t - |\cdot| \rangle^{-3\mu} : L^2(\boldsymbol{R}^3) \right\| \le C_\mu \langle t \rangle^{-3/2}$$

for $\mu > 0$ and $0 \le j \le N$, we get

$$\left\| \partial_t^p F(u, \partial u)(t) : L^2(\Omega) \right\| \le C_0 M\varepsilon(1+t)^{-1} z_0(t) + CM^3\varepsilon^3(1+t)^{-3/2}$$

for $0 \le p \le 2k$. Noting that the boundary condition (1.2) implies $\partial_t^p u(t,x) = 0$ for $(t,x) \in [0,T) \times \partial\Omega$ and $0 \le p \le 2k+1$, we see from the energy inequality for the wave equation that

$$\frac{dz_0}{dt}(t) \le C_0 M\varepsilon(1+t)^{-1} z_0(t) + CM^3\varepsilon^3(1+t)^{-3/2},$$

which yields

$$z_0(t) \le (z_0(0) + CM^3\varepsilon^3)(1+t)^{C_0 M\varepsilon} \le CM\varepsilon(1+t)^{C_0 M\varepsilon}. \tag{5.8}$$

Next suppose $m \ge 1$. Then, from the definition of z_m, we have

$$\begin{aligned} z_m(t) \le C \sum_{p=0}^{2k-m} \Bigg(\big\|\partial_t^p \partial u(t) : L^2(\Omega)\big\| + \sum_{1\le|\alpha|\le m} \big\|\partial_t^p \partial_x^\alpha \partial_t u(t) : L^2(\Omega)\big\| \\ + \sum_{1\le|\alpha|\le m} \big\|\partial_t^p \partial_x^\alpha \nabla_x u(t) : L^2(\Omega)\big\| \Bigg) \\ \le C\Bigg(z_0(t) + z_{m-1}(t) + \sum_{p=0}^{2k-m} \sum_{2\le|\alpha|\le m+1} \big\|\partial_t^p \partial_x^\alpha u(t) : L^2(\Omega)\big\| \Bigg), \end{aligned}$$

where we have used

$$\sum_{1\le|\alpha|\le m} \big\|\partial_t^p \partial_x^\alpha \partial_t u(t) : L^2(\Omega)\big\| \le C \sum_{|\alpha'|\le m-1} \big\|\partial_t^{p+1} \partial_x^{\alpha'} \nabla_x u(t) : L^2(\Omega)\big\|.$$

For $2 \le |\alpha| \le m+1$, (3.1) yields

$$\big\|\partial_t^p \partial_x^\alpha u(t) : L^2(\Omega)\big\| \le C\big(\|\Delta_x \partial_t^p u(t) : H^{m-1}(\Omega)\| + \|\nabla_x\, \partial_t^p u(t) : L^2(\Omega)\|\big).$$

For $0 \le p \le 2k-m$, we see that the second term on the right-hand side in the above is bounded by $z_0(t)$. While, using (1.1), the first term is estimated by

$$C\big(\|\partial_t^{p+2} u(t) : H^{m-1}(\Omega)\| + \|\partial_t^p F(u,\partial u)(t) : H^{m-1}(\Omega)\|\big),$$

whose first term is bounded by $z_{m-1}(t)$ for $0 \le p \le 2k-m$. On the other hand, we have

$$\left\|\partial_t^p F(u,\partial u)(t):H^{m-1}(\Omega)\right\| \le CM\varepsilon(1+t)^{-1}z_{m-1}(t)+CM^3\varepsilon^3(1+t)^{-3/2}$$

for $0\le p\le 2k-m$, as before. In conclusion, we get[2]

$$z_m(t)\le C\bigl(z_{m-1}(t)+z_0(t)+M^3\varepsilon^3(1+t)^{-3/2}\bigr) \tag{5.9}$$

for $m\ge 1$. Using (5.8), we obtain (5.6) by the inductive argument in $m(\ge 1)$.

5.2. Estimates of the generalized energy, part 1.

In this subsection we evaluate the generalized derivatives $\partial Z^\alpha u$ in $L^2(\Omega)$ for $|\alpha|\le 2k-1$. It follows from (2.3) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\int_\Omega \bigl(|\partial_t Z^\alpha u_i|^2+|\nabla_x Z^\alpha u_i|^2\bigr)\,dx\\ &=\int_\Omega Z^\alpha F_i(u,\partial u)\,\partial_t Z^\alpha u_i\,dx + c_i^2\int_{\partial\Omega}(\nu\cdot\nabla_x Z^\alpha u_i)\,(\partial_t Z^\alpha u_i)\,dS,\end{aligned} \tag{5.10}$$

where $\nu=\nu(x)$ is the unit outer normal vector at $x\in\partial\Omega$, and dS is the surface measure on $\partial\Omega$.

Let α and β be multi–indices with $|\alpha|+|\beta|\le 2k-1$. Since $|\partial^{\beta'}Z^{\alpha'}u|\le C_{\alpha',\beta'}\bigl(|\partial u|_{|\alpha'|+|\beta'|}+|u|_{|\alpha'|}\bigr)$ for any multi–indices α' and β', from (5.3) we get

$$\begin{aligned}\bigl|\partial^\beta Z^\alpha F(u,\partial u)(t,x)\bigr| \le\, & CM\varepsilon\,\langle t+r\rangle^{-1}\,w_-(t,r)^{-1}|\partial u(t,x)|_{|\alpha|+|\beta|}\\ &+CM^2\varepsilon^2\,\langle t+r\rangle^{-2+2\mu}\,w_-(t,r)^{-2\mu}|u(t,x)|_{|\alpha|}\end{aligned} \tag{5.11}$$

for arbitrarily fixed $\mu>0$.

Fix small $\mu_0>0$. Observing that $|Z\eta|\le C\,\langle r\rangle\,|\partial\eta|$ for any function η, we get $|u|_{|\alpha|}\le C\bigl(|u|+\langle r\rangle\,|\partial u|_{|\alpha|-1}\bigr)$ for $|\alpha|\ge 1$. Therefore, from (5.11) with $|\beta|=0$ we obtain

$$\begin{aligned}&\bigl\|Z^\alpha F(u,\partial u)(t):L^2(\Omega)\bigr\|\\ &\quad\le CM\varepsilon(1+t)^{-1}\|\partial u(t)\|_{|\alpha|}+CM^2\varepsilon^2(1+t)^{-1+2\mu_0}\|\partial u(t)\|_{|\alpha|-1}\\ &\qquad+CM^3\varepsilon^3(1+t)^{-3/2}\end{aligned} \tag{5.12}$$

[2]We note that, when we consider the quasi-linear case, (5.9) is replaced by

$$z_m(t)\le CM\varepsilon z_m(t)+C\bigl(z_{m-1}(t)+z_0(t)+M^3\varepsilon^3(1+t)^{-3/2}\bigr),$$

but we can easily recover (5.9) from this inequality, because ε is small.

for $|\alpha| \leq 2k-1$.

While, $\partial\Omega \subset B_1$ implies $|\partial Z^\alpha u(t,x)| \leq C\sum_{|\beta|\leq|\alpha|} |\partial^\beta \partial u(t,x)|$ for $(t,x) \in [0,T) \times \partial\Omega$. Hence, by the trace theorem, we see that the second term on the right-hand side of (5.10) is evaluated by

$$C \sum_{|\beta|\leq|\alpha|+1} \left\|\partial^\beta \partial u(t) : L^2(\Omega_2)\right\|^2,$$

which is bounded from above by $CM^2\varepsilon^2(1+t)^{2C_0M\varepsilon}$ in view of (5.6).

Now, from (5.10), (5.12) and Young's inequality, there exist positive constants C_1 and C such that

$$\frac{d}{dt}\|\partial u(t)\|_m^2 \leq C_1 M\varepsilon(1+t)^{-1}\|\partial u(t)\|_m^2 + CM^3\varepsilon^3(1+t)^{-1+4\mu_0}\|\partial u(t)\|_{m-1}^2 + CM^2\varepsilon^2(1+t)^{2C_0M\varepsilon}$$

for $m \leq 2k-1$, from which we inductively obtain

$$\|\partial u(t)\|_m^2 \leq CM^2\varepsilon^2(1+t)^{2C_0M\varepsilon+4\mu_0(m-1)+1} \tag{5.13}$$

for $m \leq 2k-1$, provided that ε is small enough to satisfy $C_1M\varepsilon \leq 1$. Setting $\gamma = 4(k-1)\mu_0$, we obtain

$$\|\partial u(t)\|_{2k-1} \leq CM\varepsilon(1+t)^{C_0M\varepsilon+\gamma+(1/2)}. \tag{5.14}$$

5.3. Pointwise estimates, part 1.

By (3.10) and (5.14) we have

$$\langle x\rangle\, |\partial u(t,x)|_{2k-3} \leq C\|\partial u(t)\|_{2k-1} \leq CM\varepsilon(1+t)^{C_0M\varepsilon+\gamma+(1/2)}. \tag{5.15}$$

Let α and β be multi–indices with $|\alpha|+|\beta| \leq 2k-3$. We put

$$U_{m,\lambda}(t) = \sup_{(s,x)\in[0,t]\times\Omega} \sum_{i=1}^N \langle s+|x|\rangle^{1-\lambda}\, \Phi_0(c_is,x)|u_i(s,x)|_m \tag{5.16}$$

for $\lambda \geq 0$. Then (5.11) yields

$$\left|\partial^\beta Z^\alpha F(u,\partial u)(t,x)\right| \le CM\varepsilon \langle t+r\rangle^{-1} w_-(t,r)^{-1}|\partial u(t,x)|_{|\alpha|+|\beta|}$$
$$+ CM^2\varepsilon^2 \langle t+r\rangle^{\lambda-3+3\mu} w_-(t,r)^{-3\mu} U_{|\alpha|,\lambda}(t). \qquad (5.17)$$

Let χ be a non-negative $C^\infty(\boldsymbol{R})$–function satisfying $\chi(\tau)=1$ for $\tau\le 0$, and $\chi(\tau)=0$ for $\tau\ge 1$. We define

$$\chi_{c,t_0,x_0}(t,x) = \chi\big((ct+\langle x\rangle)-(ct_0+\langle x_0\rangle)\big) \qquad (5.18)$$

for $c>0$ and $(t_0,x_0)\in[0,T)\times\Omega$. Observe that if $t\in[0,t_0]$ and $ct+|x|\le ct_0+|x_0|$, then $\chi_{c,t_0,x_0}(t,x)=1$. We also have $Z^\alpha\chi_{c,t_0,x_0}(t,x)\le C_{m,c}$ for $(t,x)\in[0,\infty)\times\boldsymbol{R}^3$ and $|\alpha|=m$, where $C_{m,c}$ is a constant depending only on m and c. Then, taking the domain of dependence for (t_0,x_0) into account, we get

$$L[g;c](t_0,x_0) = L[\chi_{c,t_0,x_0} g;c](t_0,x_0). \qquad (5.19)$$

We also have

$$\langle t+|x|\rangle \le C\,\langle t_0+|x_0|\rangle \qquad (5.20)$$

for any $(t,x)\in\operatorname{supp}\chi_{c,t_0,x_0}$ with $t\ge 0$, and any $(t_0,x_0)\in[0,T)\times\Omega$, where C is a constant depending only on c.

Now we set $\lambda=\big(C_0M\varepsilon+\gamma+(1/2)\big)+\gamma$. Using (5.15) and (5.17) for $|\alpha|\le 2k-\ell-6$, $|\beta|\le\ell+3$ and $\mu=(1-\gamma)/3$, we find

$$\sum_{|\beta|\le\ell+3}\left\|\partial^\beta(\chi_{c_i,t_0,x_0}F_i(u,\partial u))(t_0) : N_{2k-\ell-6}(W_{1+\gamma,1-\gamma})\right\|$$
$$\le CM^2\varepsilon^2(1+U_{2k-\ell-6,\lambda}(t_0))\,\langle t_0+|x_0|\rangle^\lambda .$$

In view of (5.19), by using (4.7) with $(\rho,\kappa_1)=(1,1-\gamma)$ and $(f_1,f_2)=(\chi_{c_i,t_0,x_0}F_i,0)$, we obtain

$$U_{2k-\ell-6,\lambda}(t)\le C\varepsilon + CM^2\varepsilon^2(1+U_{2k-\ell-6,\lambda}(t)),$$

which leads to

$$\sum_{i=1}^N \langle t+|x|\rangle^{(1/2)-C_0M\varepsilon-2\gamma}\,\Phi_0(c_it,x)|u_i(t,x)|_{2k-\ell-6}\le CM\varepsilon \qquad (5.21)$$

for $(t,x) \in [0,T) \times \overline{\Omega}$, since we may assume $CM^2\varepsilon^2 \le 1/2$.

5.4. Estimates of the generalized energy, part 2.

Since $\Phi_0(c_i t, x)$ is bounded for $(t,x) \in [0,\infty) \times \partial\Omega$, from (5.21) we get

$$\begin{aligned}\left\|\partial Z^\alpha u(t) : L^2(\partial\Omega)\right\| &\le C\left\||u(t)|_{2k-\ell-6} : L^\infty(\partial\Omega)\right\| \\ &\le CM\varepsilon \langle t\rangle^{-(1/2)+C_0M\varepsilon+2\gamma}, \end{aligned} \tag{5.22}$$

for $|\alpha| \le 2k-\ell-7$. Now (5.10), (5.12) and (5.22) yield

$$\begin{aligned}\frac{d}{dt}\|\partial u(t)\|_m^2 \le{}& C_2 M\varepsilon(1+t)^{-1}\|\partial u(t)\|_m^2 + CM^3\varepsilon^3(1+t)^{-1+4\mu_0}\|\partial u(t)\|_{m-1}^2 \\ &+ CM^2\varepsilon^2(1+t)^{-1+2C_0M\varepsilon+4\gamma}\end{aligned}$$

for $m \le 2k-\ell-7$ with some positive constant C_2, which inductively leads to

$$\|\partial u(t)\|_m^2 \le CM^2\varepsilon^2(1+t)^{(2C_0+C_2)M\varepsilon+4\gamma+4(m-1)\mu_0}$$

for $m \le 2k-\ell-7$. Finally we obtain (5.5) if we take $C_* = C_0 + C_2/2$ and $\rho_* = 3\gamma\,(=12(k-1)\mu_0)$ for example.

5.5. Pointwise estimates, part 2.

(3.10) and (5.5) imply

$$\langle x\rangle\, |\partial u(t,x)|_{2k-\ell-9} \le CM\varepsilon(1+t)^\delta \tag{5.23}$$

for $0 < \varepsilon < \rho_*/(C_*M)$, where we have set $\delta = 2\rho_*$. Note that we can take ρ_* arbitrarily small, hence we may assume that δ is small enough in the following.

Using (5.23) and (5.17) with $|\alpha| \le 2k-2\ell-12$, $|\beta| \le \ell+3$, $\lambda = 2\delta$, and $\mu = (1-\delta)/3$, we find

$$\begin{aligned}&\sum_{|\beta|\le \ell+3}\left\|\partial^\beta(\chi_{c_i,t_0,x_0}F_i(u,\partial u))(t_0) : N_{2k-2\ell-12}(W_{1+\delta,1-\delta})\right\| \\ &\qquad \le CM^2\varepsilon^2(1+U_{2k-2\ell-12,2\delta}(t_0))\,\langle t_0+|x_0|\rangle^{2\delta}.\end{aligned}$$

Similarly to (5.21), this estimate ends up with

$$\langle t+|x|\rangle^{1-2\delta}\,\Phi_0(c_i t,x)|u_i(t,x)|_{2k-2\ell-12} \le CM\varepsilon \tag{5.24}$$

for $1 \le i \le N$ and $(t,x) \in [0,T) \times \overline{\Omega}$.

From (5.17) (with $\mu = (1-\delta)/3$), (5.23) and (5.24), we get

$$\left\| \chi_{c_i,t_0,x_0} F_i(u,\partial u)(t_0) : N_{2k-2\ell-12}(W_{1+\delta,1-\delta}) \right\| \le CM^2\varepsilon^2 \langle t_0 + |x_0| \rangle^{2\delta}.$$

From (4.8) and (4.18) with $\rho = 1$, $\kappa_1 = 1-\delta$ and $(f_1, f_2) = (\chi_{c_i,t_0,x_0} F_i, 0)$, we thus obtain

$$\langle r \rangle \langle t+r \rangle^{-2\delta} \langle c_i t - r \rangle |\partial u_i(t,x)|_{2k-3\ell-16} \le CM\varepsilon, \tag{5.25}$$

$$\langle r \rangle \langle t+r \rangle^{1-3\delta} \sum_{|\alpha| \le 2k-3\ell-17} \left| D_{+,c_i} Z^\alpha u_i(t,x) \right| \le CM\varepsilon \tag{5.26}$$

for $1 \le i \le N$ and $(t,x) \in [0,T) \times \overline{\Omega}$, where we have used the fact that $\log(2+t+r) \le C \langle t+r \rangle^\delta$.

5.6. Pointwise estimates, part 3.

From now on, we take advantage of detailed structure of our nonlinearity, and we shall show

$$\langle r \rangle \langle c_i t - r \rangle^{1-2\delta} |\partial u_i(t,x)|_{2k-4\ell-21} \le CM\varepsilon. \tag{5.27}$$

Note that r is equivalent to $\langle t+r \rangle$, when $r \ge 1$ and $|c_i t - r| < c_i t/2$. By Lemma 3.6, with the help of (5.3), (5.24), (5.25), and (5.26), we obtain

$$\left| F_i^{\rm null}(\partial u)(t,x) \right|_{2k-3\ell-17} \le CM^2\varepsilon^2 \langle t+r \rangle^{-3+3\delta} \langle c_i t - r \rangle^{-1} \tag{5.28}$$

for (t,x) satisfying $r \ge 1$ and $|c_i t - r| < c_i t/2$.

On the other hand, $\langle c_i t - r \rangle$ is equivalent to $\langle t+r \rangle$, when $r < 1$ or $|c_i t - r| \ge (c_i t/2)$. Hence, observing that $F_i^{\rm null}$ is quadratic with respect to ∂u, from (5.3) and (5.25) we get

$$\left| F_i^{\rm null}(\partial u)(t,x) \right|_{2k-3\ell-17} \le CM^2\varepsilon^2 \langle t+r \rangle^{-2+2\delta} \langle r \rangle^{-2} \tag{5.29}$$

for (t,x) satisfying $r < 1$ or $|c_i t - r| \ge (c_i t/2)$.

Now we find

$$\left\| F_i^{\rm null}(\partial u)(t) : N_{2k-3\ell-17}(W_{2-3\delta,1}) \right\| \le CM^2\varepsilon^2. \tag{5.30}$$

While, (5.3) and (5.25) yield

$$|R_{I,i}(\partial u)(t,x)|_{2k-3\ell-17} \le CM^2\varepsilon^2 \langle r\rangle^{-2} \langle t+r\rangle^{2\delta} \sum_{c_j\ne c_k} \langle c_j t-r\rangle^{-1} \langle c_k t-r\rangle^{-1}$$

$$\le CM^2\varepsilon^2 \langle r\rangle^{-1} \langle t+r\rangle^{-2+2\delta} w_-(t,r)^{-1} \tag{5.31}$$

for $(t,x)\in[0,T)\times\overline{\Omega}$, and hence we obtain

$$\left\| R_{I,i}(\partial u)(t) : N_{2k-3\ell-17}(W_{2-2\delta,1}) \right\| \le CM^2\varepsilon^2. \tag{5.32}$$

Similarly, we have

$$|R_{II,i}(\partial u)(t,x)|_{2k-3\ell-17} \le CM^2\varepsilon^2 \langle r\rangle^{-1} \langle t+r\rangle^{-1+2\delta} w_-^{(c_i)}(t,r)^{-2}, \tag{5.33}$$

which yields

$$\left\| R_{II,i}(\partial u)(t) : N_{2k-3\ell-17}(W^{(c_i)}_{1-2\delta,2}) \right\| \le CM^2\varepsilon^2. \tag{5.34}$$

From (5.3), (5.24) and (5.25) we have

$$|H_i(u,\partial u)(t,x)|_{2k-3\ell-17} \le CM^3\varepsilon^3 \langle t+r\rangle^{-3+3\mu+2\delta} w_-(t,r)^{-3\mu} \tag{5.35}$$

for arbitrarily fixed $\mu>0$, which implies

$$\left\| H_i(u,\partial u)(t) : N_{2k-3\ell-17}(W_{1+\delta,1-3\delta}) \right\| \le CM^2\varepsilon^2, \tag{5.36}$$

if we choose $\mu=(1-3\delta)/3$.

Finally, applying (4.8) with $\rho=1-2\delta$, $\kappa_1=1-3\delta(<\rho)$ (so that $\nu_*(\rho,\kappa_1)=1+\delta$), $\kappa_2=2$, $f_1=F_i^{\rm null}(\partial u)+R_{I,i}(\partial u)+H_i(u,\partial u)$ and $f_2=R_{II,i}(\partial u)$, we find (5.27), since we may assume $1+\delta<2-3\delta$.

5.7. Pointwise estimates, the final part.

By (5.3) and (5.27), we obtain

$$|R_{II,i}(\partial u)(t,x)|_{2k-4\ell-21} \le CM^2\varepsilon^2 \langle r\rangle^{-1} \langle t+r\rangle^{-1} w_-^{(c_i)}(t,r)^{-2+2\delta}, \tag{5.37}$$

which leads to

$$\left\| R_{II,i}(\partial u)(t) : N_{2k-4\ell-21}(W^{(c_i)}_{1,2-2\delta}) \right\| \le CM^2\varepsilon^2. \tag{5.38}$$

By (5.3) and (5.27), we also obtain

$$\begin{aligned} \sum_{|\beta|\le \ell+3} & |\partial^\beta H_i(u,\partial u)(t,x)|_{2k-5\ell-24} \\ & \le CM^3\varepsilon^3 \langle r\rangle^{-1} \langle t+r\rangle^{-2+2\mu} w_-(t,r)^{-1+2\delta-2\mu} \\ & \quad + CM^2\varepsilon^2 \langle t+r\rangle^{-3+3\mu} w_-(t,r)^{-3\mu} U_{2k-5\ell-24,0}(t) \end{aligned} \tag{5.39}$$

for fixed $\mu > 0$, where $U_{m,\lambda}$ is given by (5.16). Choosing $\mu = (1-\delta)/3$, we have

$$\begin{aligned} \sum_{|\beta|\le \ell+3} & \left\| \partial^\beta H_i(u,\partial u)(t) : N_{2k-5\ell-24}(W_{1+\delta,1-\delta}) \right\| \\ & \le CM^2\varepsilon^2 (M\varepsilon + U_{2k-5\ell-24,0}(t)). \end{aligned} \tag{5.40}$$

In view of (5.30), (5.32), (5.38), and (5.40), the application of (4.7) for $\rho = 1$, $\kappa_1 = 1-\delta (<1)$ (so that $\nu_*(\rho,\kappa_1) = 1+\delta$), and $\kappa_2 = 2-2\delta (>1)$, with the same choice of f_1 and f_2 as before, leads to

$$\langle t+r\rangle \Phi_0(c_i t, x) |u_i(t,x)|_{2k-5\ell-24} \le C\varepsilon + CM^2\varepsilon^2 (1 + U_{2k-5\ell-24,0}(t)) \tag{5.41}$$

(observe that we have $W_{1,\kappa_2} \le W^{(c_i)}_{1,\kappa_2}$ for $\kappa_2 > 1$). Now (5.41) yields

$$\langle t+r\rangle \Phi_0(c_i t, x) |u_i(t,x)|_{2k-5\ell-24} \le C\varepsilon + CM^2\varepsilon^2, \tag{5.42}$$

provided that ε is sufficiently small. From (5.40) and (5.42), we obtain

$$\left\| H_i(u,\partial u)(t) : N_{2k-5\ell-24}(W_{1+\delta,1-\delta}) \right\| \le CM^3\varepsilon^3.$$

Now (4.8) and (4.18) with $(\rho,\kappa_1,\kappa_2) = (1, 1-\delta, 2-2\delta)$ and (f_1, f_2) as before imply

$$\langle r\rangle \langle c_i t - r\rangle |\partial u_i(t,x)|_{2k-6\ell-28} \le C\varepsilon + CM^2\varepsilon^2, \tag{5.43}$$

$$\frac{\langle r\rangle \langle t+r\rangle}{\log(2+t+r)} \sum_{|\alpha|\le 2k-6\ell-29} \left| D_{+,c_i} Z^\alpha u_i(t,x) \right| \le C\varepsilon + CM^2\varepsilon^2. \tag{5.44}$$

Finally, since $2k-6\ell-28\ge k$, from (5.42), (5.43) and (5.44), we obtain (5.4). This completes the proof. □

Appendix A: Proof of Lemma 3.1

Suppose $m\ge 2$ and $\varphi\in H^m(\Omega)\cap H_0^1(\Omega)$. Let χ be a $C_0^\infty(\boldsymbol{R}^3)$ function such that $\chi\equiv 1$ in a neighborhood of $\mathscr{O}$. Let $\operatorname{supp}\chi\subset B_R$ for some $R>1$. We set $\varphi_1=\chi\varphi$ and $\varphi_2=(1-\chi)\varphi$, so that $\varphi=\varphi_1+\varphi_2$.

First we estimate φ_1. The following elliptic estimate is well-known (see Chapter 9 in [**3**] for instance):

$$\|w:H^{k+2}(\Omega_R)\|\le C(\|\Delta_x w:H^k(\Omega_R)\|+\|w:L^2(\Omega_R)\|) \tag{A.1}$$

holds for $w\in H^{k+2}(\Omega_R)\cap H_0^1(\Omega_R)$ with a non-negative integer k. It is also well-known that we have

$$\|w:L^2(\Omega_R)\|\le CR^2\|\nabla_x w:L^2(\Omega)\| \tag{A.2}$$

for $w\in H_0^1(\Omega)$ and $R>1$ (see [**20**] for the proof).

Since $\varphi\in H_0^1(\Omega)$ and $\operatorname{supp}\chi\subset B_R$, we have $\varphi_1\in H_0^1(\Omega_R)$. Therefore, the application of (A.1) in combination with (A.2) gives

$$\|\varphi_1:H^m(\Omega)\|\le C\big(\|\Delta_x\varphi:H^{m-2}(\Omega)\|+\|\nabla_x\varphi:L^2(\Omega)\|\big). \tag{A.3}$$

Now our task is to show

$$\sum_{|\alpha|=m}\|\partial_x^\alpha\varphi_2:L^2(\Omega)\|\le C\big(\|\Delta_x\varphi:H^{m-2}(\Omega)\|+\|\nabla_x\varphi:L^2(\Omega)\|\big), \tag{A.4}$$

because it implies (3.1) in view of (A.3).

Since $\|\partial^\alpha w:L^2(\boldsymbol{R}^3)\|\le C\|\Delta_x w:L^2(\boldsymbol{R}^3)\|$ for $|\alpha|=2$ and $w\in H^2(\boldsymbol{R}^3)$, the left-hand side of (A.4) with $m=2$ is estimated by

$$C\|\Delta_x\varphi_2:L^2(\Omega)\|\le C\big(\|\Delta_x\varphi:L^2(\Omega)\|+\|\nabla_x\varphi:L^2(\Omega)\|+\|\varphi:L^2(\Omega_R)\|\big).$$

Hence, using (A.2), we obtain (A.4) for $m=2$.

For $k\ge 3$, similar argument to the above gives

$$\sum_{|\alpha|=k} \left\|\partial_x^\alpha \varphi_2 : L^2(\Omega)\right\| \le C\left(\left\|\Delta_x \phi : H^{k-2}(\Omega)\right\| + \left\|\nabla_x \varphi : H^{k-2}(\Omega)\right\|\right), \tag{A.5}$$

and the second term on the right-hand side of (A.5) is bounded by $C(\|\Delta_x\varphi : H^{k-3}(\Omega)\| + \|\nabla_x\varphi : L^2(\Omega)\|)$, if we know (3.1) for $m = k-1$. Hence we inductively obtain (A.4) (and consequently (3.1)) for $m \ge 2$. □

Appendix B: Admissible Obstacles

First we assume that $\mathscr{O}$ is non-trapping, and we shall show that it is admissible in our sense. For a, $b > 1$, it is known that there exist positive constants C and σ depending on a, b and Ω such that

$$\sum_{|\alpha|\le 1} \left\|\partial^\alpha K[\phi_0, \phi_1; c](t, \cdot) : L^2(\Omega_b)\right\| \le Ce^{-\sigma t}\left\|\vec{\phi}_0 : \mathscr{H}^0(\Omega)\right\| \tag{B.1}$$

for any $\vec{\phi}_0 = (\phi_0, \phi_1) \in H_0^1(\Omega) \times L^2(\Omega)$ satisfying $\phi_0(x) = \phi_1(x) \equiv 0$ for $|x| \ge a$ (see for instance Melrose [**22**], Shibata – Tsutsumi [**30**]).

Now let $(\vec{v}_0, f) = (v_0, v_1, f) \in X_{c,a}(T)$ with some $a > 1$. Then, by Duhamel's principle, it follows that

$$\begin{aligned}
&\partial_t^j S[(\vec{v}_0, f); c](t, x) \\
&\quad = K[(v_j, v_{j+1}); c](t, x) + \int_0^t K[(0, (\partial_t^j f)(s)); c](t-s, x)ds
\end{aligned} \tag{B.2}$$

for any non-negative integer j and any $(t, x) \in [0, T) \times \Omega$, where v_j are given by (1.10). Apparently we have $(\partial_t^j f)(s, \cdot) \in L^2(\Omega)$ for $0 \le s \le t$. Thanks to the compatibility condition, we also find $v_j \in H_0^1(\Omega)$ for any $j \ge 0$. Therefore, by (B.1), for $|\alpha| \le 1$ we have

$$\begin{aligned}
\left\|\partial^\alpha K[\vec{v}_j; c](t) : L^2(\Omega_b)\right\| &\le Ce^{-\sigma t}\left\|\vec{v}_j : \mathscr{H}^0(\Omega)\right\| \\
&\le Ce^{-\sigma t}\left(\left\|\vec{v}_0 : \mathscr{H}^j(\Omega)\right\| + \sum_{|\alpha|\le j-1} \left\|(\partial^\alpha f)(0, \cdot) : L^2(\Omega)\right\|\right)
\end{aligned} \tag{B.3}$$

and

$$\begin{aligned}\int_0^t &\big\|\partial^\alpha K[(0,(\partial_t^j f)(s));c](t-s) : L^2(\Omega_b)\big\| ds \\ &\le C\int_0^t e^{-\sigma(t-s)}\big\|(\partial_t^j f)(s) : L^2(\Omega)\big\| ds \\ &\le C(1+t)^{-\gamma}\sup_{0\le s\le t}(1+s)^\gamma\big\|(\partial_t^j f)(s) : L^2(\Omega)\big\| \end{aligned} \tag{B.4}$$

for any $\gamma>0$, where we have put $\vec{v}_j=(v_j,v_{j+1})$. Therefore for $|\alpha|\le 1$ and any non-negative integer j, we have

$$\begin{aligned}&\big\|\partial^\alpha\partial_t^j S[(\vec{v}_0,f);c](t) : L^2(\Omega_b)\big\| \\ &\le C(1+t)^{-\gamma}\Big(\big\|\vec{v}_0 : \mathscr{H}^j(\Omega)\big\| + \sum_{|\alpha|\le j}\sup_{0\le s\le t}(1+s)^\gamma\big\|\partial^\alpha f(s) : L^2(\Omega)\big\|\Big). \end{aligned} \tag{B.5}$$

In order to evaluate $\partial^\alpha v$ for $|\alpha|\le m$, we have only to combine (B.5) with a variant of (3.1):

$$\|\varphi : H^m(\Omega_b)\| \le C(\big\|\Delta_x\varphi : H^{m-2}(\Omega_{b'})\big\| + \big\|\varphi : H^1(\Omega_{b'})\big\|, \tag{B.6}$$

where $1<b<b'$ and $\varphi\in H^m(\Omega)\cap H_0^1(\Omega)$ with $m\ge 2$. In this way, we obtain (1.11) for any $\gamma>0$ with $\ell=0$. Hence we see that the non-trapping obstacle $\mathscr{O}$ is admissible.

Now let the obstacle $\mathscr{O}$ satisfy one of the assumptions from Ikawa [**8**], [**9**]. The assumption in [**8**] is:

(I–1) $\mathscr{O}$ is a union of disjoint compact sets $\mathscr{O}_1$ and $\mathscr{O}_2$ whose Gaussian curvatures are strictly positive at every point of their boundaries.

We do not describe the precise assumption in [**9**], to which we refer as (I–2). For example, it is fulfilled when

(I–2') $\mathscr{O}$ is a union of any numbers of disjoint balls $\mathscr{O}_i$ of the same radius, the distance between arbitrarily chosen two balls $\mathscr{O}_j$ and $\mathscr{O}_k$ is sufficiently large, and the convex hull of $\mathscr{O}_j$ and $\mathscr{O}_k$ has no intersection with any other balls.

Note that these obstacles are trapping.

Under (I–1) or (I–2), it was proved that

$$\sum_{|\alpha|\le 1}\big\|\partial^\alpha K[\phi_0,\phi_1;c](t,\cdot) : L^2(\Omega_b)\big\| \le Ce^{-\sigma t}\big\|\vec{\phi}_0 : \mathscr{H}^\ell(\Omega)\big\| \tag{B.7}$$

holds for any $(\phi_0, \phi_1, 0) \in X_{c,a}(T)$, where $\vec{\phi}_0 = (\phi_0, \phi_1)$. Here $\ell = 5$ for (I–1), and $\ell = 2$ for (I–2) (see [**8**], [**9**]). But these numbers are not important, because we may assume that ℓ is as small as we wish.

In fact, suppose that we have (B.7) for some $\ell = \ell_0 > 1$. For a while, we identify a function on Ω_a with its natural extension on Ω obtained by setting its value being 0 on $\Omega \setminus \Omega_a$. Since $\vec{\phi}_0 \in \left(C_0^\infty(\Omega_a)\right)^2$ implies $(\vec{\phi}_0, 0) \in X_{c,a}(T)$, we have (B.7) for such $\vec{\phi}_0$. Then the standard approximation argument shows that (B.7) is valid for $\vec{\phi}_0 \in H_0^{\ell_0+1}(\Omega_a) \times H_0^{\ell_0}(\Omega_a)$. Let $0 < m < 1/2$. By taking interpolation between (B.7) with $\ell = \ell_0$ and the standard energy inequality (which, in combination with (A.2), gives (B.7) with $\ell = 0$ and $\sigma = 0$ for $\vec{\phi}_0 \in H_0^1(\Omega_a) \times L^2(\Omega_a)$), we find that (B.7) with $\ell = m$ and σ replaced by $\sigma_m \equiv (m\sigma)/\ell_0$ is valid for $\vec{\phi}_0 \in H_0^{1+m}(\Omega_a) \times H_0^m(\Omega_a)$. Since we have $H_0^m(\Omega_a) = H^m(\Omega_a)$ and $H_0^{1+m}(\Omega_a) = H^{1+m}(\Omega_a) \cap H_0^1(\Omega_a)$ for $0 < m < 1/2$ (see Lions–Magenes [**21**, Chapter 1, Theorems 11.1 and 11.5] for example), finally it follows that there exists a positive constant σ such that

$$\sum_{|\alpha| \le 1} \left\| \partial^\alpha K[\phi_0, \phi_1; c](t, \cdot) : L^2(\Omega_b) \right\| \le C e^{-\sigma t} \left\| \vec{\phi}_0 : \mathscr{H}^1(\Omega) \right\| \tag{B.8}$$

for any $\vec{\phi}_0 \in \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times H^1(\Omega)$ with $\vec{\phi}_0 \equiv 0$ for $|x| \ge a$ (note that we have $\vec{\phi}_0|_{\Omega_a} \in (H^{1+m}(\Omega_a) \cap H_0^1(\Omega_a)) \times H^m(\Omega_a)$ for such $\vec{\phi}_0$). This is the exact assumption for the obstacles in [**26**] (and its successors [**24**], [**25**]).

For $(\vec{v}_0, f) = (v_0, v_1, f) \in X_{c,a}(T)$, we have $v_j \in H^2(\Omega) \cap H_0^1(\Omega)$ for any $j \ge 0$, and $(\partial_t^j f)(s, \cdot) \in H^1(\Omega)$ for any $s \in [0, T)$ and any $j \ge 0$. The support condition is also satisfied. Now, following similar lines to (B.2)–(B.6), we see that (B.8) implies (1.11) for any $\gamma > 0$ with $\ell = 1$. Hence obstacles satisfying (B.8) are admissible. Especially, trapping obstacles satisfying (I–1) or (I–2) are admissible.

Appendix C: Proof of Lemma 3.5.

It is well-known that for $w \in C_0^2(\boldsymbol{R}^3)$ we have

$$\sup_{x \in \boldsymbol{R}^3} |x||w(x)| \le C \sum_{|\alpha| \le 2} \left\| \widetilde{Z}^\alpha w : L^2(\boldsymbol{R}^3) \right\|$$

(for the proof, see e.g. [**15**]). Rewriting φ as $\varphi = \psi_1 \varphi + (1 - \psi_1)\varphi$ with ψ_1 in (2.13), we see that the left-hand side on (3.10) is evaluated by

$$
\begin{aligned}
&C \sup_{x\in \boldsymbol{R}^3} |x||\psi_1(x)\varphi(x)| + C \sup_{x\in\Omega} |(1-\psi_1(x))\varphi(x)| \\
&\quad \le C \sum_{|\alpha|\le 2} \big\|\widetilde{Z}^\alpha(\psi_1\varphi) : L^2(\boldsymbol{R}^3)\big\| + C \sum_{|\alpha|\le 2} \big\|\partial_x^\alpha((1-\psi_1)\varphi) : L^2(\Omega_2)\big\| \\
&\quad \le C \sum_{|\alpha|\le 2} \big\|\widetilde{Z}^\alpha \varphi : L^2(\Omega)\big\|, \qquad \text{(C.1)}
\end{aligned}
$$

where we have used the standard Sobolev inequality to estimate the second term on the left-hand side. □

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