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Orderability in the presence of local compactness

Dedicated to Professor Tsugunori Nogura on the occasion of his 60th birthday

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Abstract. We prove that a locally compact paracompact space is suborderable if and only if it has a continuous weak selection. This fits naturally into the pattern of the van Mill and Wattel's characterization [**15**] of compact orderable spaces, and provides a further partial positive answer to a question of theirs. Several applications about the orderability and suborderablity of locally compact spaces are demonstrated. In particular, we show that a locally compact paracompact space has a continuous selection for its Vietoris hyperspace of nonempty closed subsets if and only if it is a topologically well-orderable subspace of some orderable space.

1. Introduction.

For a T_1-space X, let $\mathscr{F}(X)$ be the set of all nonempty closed subsets of X. Usually, we endow $\mathscr{F}(X)$ with the *Vietoris topology* τ_V, and call it the *Vietoris hyperspace* of X. Recall that τ_V is generated by all collections of the form

$$\langle \mathscr{V} \rangle = \Big\{ S \in \mathscr{F}(X) : S \subset \bigcup \mathscr{V} \ \text{ and } \ S \cap V \neq \emptyset, \text{ whenever } V \in \mathscr{V} \Big\},$$

where $\mathscr{V}$ runs over the finite families of open subsets of X.

In the sequel, all spaces are assumed to be at least Hausdorff, while any subset $\mathscr{D} \subset \mathscr{F}(X)$ will carry the relative Vietoris topology τ_V as a subspace of the hyperspace $(\mathscr{F}(X), \tau_V)$. A map $f : \mathscr{D} \to X$ is a *selection* for $\mathscr{D}$ if $f(S) \in S$ for every $S \in \mathscr{D}$. A selection $f : \mathscr{D} \to X$ is *continuous* if it is continuous with respect to the relative Vietoris topology τ_V on $\mathscr{D}$.

Whenever $1 \leq n < \omega$, we let $\mathscr{F}_n(X) = \{S \in \mathscr{F}(X) : |S| \leq n\}$. Note that we may identify each point $x \in X$ with the corresponding singleton $\{x\} \in \mathscr{F}_1(X)$, and, in fact, this gives rise to a homeomorphism between X and the space

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$(\mathscr{F}_1(X), \tau_V)$. The latter means that the Vietoris topology is *admissible*, see [**14**], and is behind the reason that every selection for a family $\mathscr{D} \supset \mathscr{F}_1(X)$ is continuous on the singletons of X.

In the present paper, we are initially interested in continuous selections for $\mathscr{F}_2(X)$, and their impact on the properties of the space X. To this end, let us recall that every selection $f : \mathscr{F}_2(X) \to X$ defines a natural order-like relation $\preceq_f$ on X [**14**] by letting that $x \preceq_f y$ if and only if $f(\{x, y\}) = x$. For convenience, we write that $x \prec_f y$ if $x \preceq_f y$ and $x \neq y$. This relation is very similar to a linear order on X in that it is both total and antisymmetric, but, unfortunately, it may fail to be transitive. In this regard, one of the fundamental questions in the theory of continuous selections for at most 2-point subsets is the following.

Question 1 (van Mill and Wattel, [**15**]). Let X be a space which has a continuous selection for $\mathscr{F}_2(X)$. Does there exist a linear order $\preceq$ on X such that, for each $y \in X$, the sets $\{x \in X : x \preceq y\}$ and $\{x \in X : y \preceq x\}$ are both closed?

A topological space X is *orderable* (or, *linearly orderable*) if the topology of X coincides with the open interval topology $\mathscr{T}_\preceq$ on X generated by a linear ordering $\preceq$ on X. Recall that all "$\preceq$-open" intervals $\{x \in X : x \prec y\}$ and $\{x \in X : y \prec x\}$, $y \in X$, constitute a subbase for $\mathscr{T}_\preceq$. A subset B of a linearly ordered set $(X, \preceq)$ is $(\preceq)$-*convex* if $\{x \in X : y \preceq x \preceq z\} \subset B$ for every $y, z \in B$, with $y \preceq z$. A topological space X is *suborderable* (or, *generalized ordered*) if there exists a linear order $\preceq$ on X such that the corresponding open interval topology $\mathscr{T}_\preceq$ is coarser than the topology $\mathscr{T}$ of X (i.e., $\mathscr{T}_\preceq \subset \mathscr{T}$), and $\mathscr{T}$ has a base of $(\preceq)$-convex sets. It is well-known that a space X is suborderable if and only if it can be embedded into an orderable space. Finally, let us recall that a topological space X is *weakly orderable* [**15**] if there exists a coarser orderable topology $\mathscr{T}_\preceq$ on X with respect to some linear ordering $\preceq$ on X. In all these cases, the order $\preceq$ will be called *compatible*. Also, in this paper, $\mathscr{T}_\preceq$ will always denote the open interval topology on X generated by a linear ordering $\preceq$ on X, while the term "orderable" will be explicitly reserved for orderable topological spaces, if not suggested otherwise.

In this terminology, Question 1 states the hypothesis if a space X is weakly orderable provided it has a continuous selection for $\mathscr{F}_2(X)$. In view of that, a selection $f : \mathscr{F}_2(X) \to X$ is often called a *weak selection* for X. For a detailed discussion on Question 1, we refer the interested reader to [**10**]. Turning to the purpose of this paper, let us explicitly mention that Question 1 was resolved in the affirmative for compact spaces, [**15**, Theorem 1.1].

Theorem 1.1 ([**15**]). *A compact space X is orderable if and only if it has a continuous weak selection.*

In this paper, we prove the following theorem which fits naturally into the pattern of Theorem 1.1, and provides a further partial positive answer to Question 1 (to [**10**, Question 381] as well).

THEOREM 1.2. *A locally compact paracompact space X is suborderable if and only if it has a continuous weak selection.*

Let us explicitly mention that Theorem 1.2 fails if X is supposed to be only collectionwise normal. It was demonstrated in [**2**, Theorem 4.7] (under the Diamond Principle) that there exists a collectionwise normal, locally compact and locally countable space X which has a continuous selection for $\mathscr{F}(X)$ but is not suborderable.

Concerning the proper place of Theorem 1.2, let us make also the following remark. If X is a strongly zero-dimensional locally compact paracompact space, then it has a pairwise disjoint clopen cover consisting of compact subsets of X. Consequently, in this case, the statement of Theorem 1.2 follows by the following simple observation and Theorem 1.1.

PROPOSITION 1.3. *Let X be a space which has a clopen pairwise disjoint cover consisting of suborderable subsets of X. Then, X is itself suborderable.*

PROOF. Let $\mathscr{P}$ be a clopen pairwise disjoint cover of X consisting of suborderable subsets. Take a linear order $\leq$ on $\mathscr{P}$. By hypothesis, each $P \in \mathscr{P}$ is suborderable by a linear ordering $\preceq_P$. Consider the lexicographical order $\preceq$ on X generated by $\leq$ and $\preceq_P$, $P \in \mathscr{P}$. Then, the open interval topology $\mathscr{T}_{\preceq}$ is coarser than the topology of X, and X has a base of $(\preceq)$-convex sets. Hence, it is suborderable. □

The proof of Theorem 1.2 in the general case is based on the same idea, but to reduce the situation to that one we now rely on the technique in [**6**]. In fact, we demonstrate a little bit more general result that every locally compact paracompact space which has a continuous weak selection must be the topological sum of two orderable spaces, Theorem 5.1. These spaces play an interesting role in the paper, they were called *semi-orderable*, and are investigated in Section 4. It should be mentioned that any orderable space is semi-orderable, but the converse is not true, Example 4.1. On the other hand, any semi-orderable space is suborderable, and again the converse is not true, see Example 4.12. That is, semi-orderable spaces are an intermediate class between suborderable and orderable spaces. One of their best properties is that they are invariant with respect to topological sums (Theorem 4.2), hence, in view of Proposition 1.3, their involvement is quite natural. In fact, they will provide the main interface between local properties of orderability of locally compact spaces and the global property of semi-orderability.

Our main tool to study local properties of orderability of locally compact spaces will be the connected components. In this regard, Section 2 contains several facts about such components of spaces which have continuous weak selections. Relying on these properties, in Section 3, we demonstrate that each point of a locally compact space has a special clopen compact-like neighbourhood provided this space has a continuous weak selection (see, Lemmas 3.1 and 3.3). In the presence of such nice neighbourhoods, in Section 5, we complete the proof of Theorem 1.2 relying on the Morita's result [**16**] that every locally compact paracompact space has a clopen partition of Lindelöf spaces.

Several relations between semi-orderable and orderable spaces are obtained in Section 6. For instance, it is demonstrated that every semi-orderable space which is not orderable has an unique partition into a compact orderable space and an "anti-compact" orderable one, see Theorem 6.3. In case of locally compact spaces, the converse is also true (Theorem 6.9), which allows to provide a complete classification of semi-orderable locally compact spaces, see Corollaries 6.12 and 6.13. Relying on this, we get that a locally compact totally disconnected paracompact space is orderable if and only if it has a continuous weak selection, Corollary 6.14.

The last Section 7 of the paper is devoted to the selection problem for $\mathscr{F}(X)$. Relying on the technique developed in the previous sections, we demonstrate that a locally compact paracompact space X is a topologically well-orderable subspace of some orderable space (*topologically well-suborderable*, in our terminology) if and only if it has a continuous selection for $\mathscr{F}(X)$, Theorem 7.3. Combining this result with the characterization of orderable locally compact spaces, we obtain an interesting result that every topologically well-suborderable locally compact paracompact space must be orderable, Corollary 7.6.

2. Selection relations and components.

For a (binary) relation $\mathscr{E} \subset X^2$ and points $x, y \in X$, we usually write $x\mathscr{E}y$ to denote that $\langle x, y\rangle \in \mathscr{E}$. A relation $\mathscr{E} \subset X^2$ is called a *selection relation* [**9**] if $\mathscr{E}$ is total and antisymmetric. Note that a selection relation $\mathscr{E} \subset X^2$ is a linear order on X if and only if it is transitive.

It should be remarked that $\preceq_f$ is a selection relation on X whenever f is a weak selection for X. The converse is also true. Take a selection relation $\mathscr{E} \subset X^2$, and define for $x, y \in X$ that $f_{\mathscr{E}}(\{x, y\}) = x$ if $x\mathscr{E}y$. Then, $f_{\mathscr{E}}$ is a weak selection for X. Thus, there is a one-to-one correspondence between the weak selections for X and the selection relations on X. Motivated by this, we will often write $\preceq_{\mathbf{s}}$ for a selection relation on X. Also, for points $x, y \in X$, we will write $x \prec_{\mathbf{s}} y$ to express that $x \preceq_{\mathbf{s}} y$ and $x \neq y$.

For a selection relation $\preceq_{\mathbf{s}}$ on X and $x \in X$, define the $\preceq_{\mathbf{s}}$-*open intervals*:

$$(\leftarrow, x)_{\preceq_{\mathbf{s}}} = \{y \in X : y \prec_{\mathbf{s}} x\} \quad \text{and} \quad (x, \rightarrow)_{\preceq_{\mathbf{s}}} = \{y \in X : x \prec_{\mathbf{s}} y\}.$$

In the same way, define the $\preceq_{\mathbf{s}}$-*closed intervals*:

$$(\leftarrow, x]_{\preceq_{\mathbf{s}}} = \{y \in X : y \preceq_{\mathbf{s}} x\} \quad \text{and} \quad [x, \rightarrow)_{\preceq_{\mathbf{s}}} = \{y \in X : x \preceq_{\mathbf{s}} y\}.$$

Finally, for points $x, y \in X$, define the corresponding composed $\preceq_{\mathbf{s}}$-intervals:

$$\begin{aligned}
(x, y)_{\preceq_{\mathbf{s}}} &= (x, \rightarrow)_{\preceq_{\mathbf{s}}} \cap (\leftarrow, y)_{\preceq_{\mathbf{s}}}, \\
[x, y]_{\preceq_{\mathbf{s}}} &= [x, \rightarrow)_{\preceq_{\mathbf{s}}} \cap (\leftarrow, y]_{\preceq_{\mathbf{s}}}, \\
(x, y]_{\preceq_{\mathbf{s}}} &= (x, \rightarrow)_{\preceq_{\mathbf{s}}} \cap (\leftarrow, y]_{\preceq_{\mathbf{s}}}, \quad \text{and} \\
[x, y)_{\preceq_{\mathbf{s}}} &= [x, \rightarrow)_{\preceq_{\mathbf{s}}} \cap (\leftarrow, y)_{\preceq_{\mathbf{s}}}.
\end{aligned}$$

Since the relation $\preceq_{\mathbf{s}}$ is not necessarily transitive, both intervals $(x, y)_{\preceq_{\mathbf{s}}}$ and $(y, x)_{\preceq_{\mathbf{s}}}$ could be nonempty, similarly for $[x, y]_{\preceq_{\mathbf{s}}}$ and $[y, x]_{\preceq_{\mathbf{s}}}$, etc.

Suppose that X is a topological space. If f is a continuous weak selection for X, then the selection relation $\preceq_f$ is "compatible" with the topology of X. In this case, Michael [**14**] demonstrated that all $\preceq_f$-open intervals $(\leftarrow, x)_{\preceq_f}$ and $(x, \rightarrow)_{\preceq_f}$, $x \in X$, are open in X. Through this paper, we will freely rely on this property of continuous weak selections. In this regard, it should be mentioned that there are weak selections which are not continuous, but all intervals of this type are open (see, [**7**, Example 3.6] and [**9**, Corollary 4.2]). Here is a very simple description of continuity of weak selections in terms of selection relations.

Proposition 2.1. *A weak selection f for a space X is continuous if and only if the selection relation $\preceq_f$ is closed in X^2.*

Proof. Take points $x, y \in X$ and observe that "$x \preceq_f y$" fails if and only if $x \neq y$ and $f(\{x, y\}) = y$, i.e. when $y \prec_f x$. On the other hand, according to [**7**, Theorem 3.1], f is continuous if and only if for any such pair of points x and y, with $y \prec_f x$, there are disjoint open sets V and U such that $y \in V$, $x \in U$ and $t \prec_f s$ for every $t \in V$ and $s \in U$. The last property is clearly equivalent to statement that $U \times V \subset X^2 \setminus \preceq_f$, which completes the proof. □

In view of Proposition 2.1, to emphasize on the orderability, we will often refer to closed selection relations rather than continuous weak selections.

One can easily observe that a linear order $\preceq$ on a space X is closed if and only if all $\preceq$-open intervals $(\leftarrow, x)_{\preceq}$ and $(x, \rightarrow)_{\preceq}$, $x \in X$, are open in X. Consequently, a space X is weakly orderable if and only if it has a closed linear order on it.

From this point of view, the possible difference between weakly orderable spaces and spaces which have continuous weak selections is about the possible lack of transitivity of closed selection relations. It turns out that any closed selection relation on a connected space X is transitive. The following key observations are due to Eilenberg [**3**] and Michael [**14**].

THEOREM 2.2 ([**3**], [**14**]). *Any compatible order on a weakly orderable space is a closed relation, and any closed selection relation on a connected space is a linear order. Moreover, a connected weakly orderable space X has precisely two compatible orders, which are inverse of each other.*

For a set Z and a selection relation $\preceq_{\mathbf{s}}$ on Z, a point $p \in Z$ is $(\preceq_{\mathbf{s}})$-*minimal* (respectively, $(\preceq_{\mathbf{s}})$-*maximal*) if $p \preceq_{\mathbf{s}} z$ (respectively, $z \preceq_{\mathbf{s}} p$) for every $z \in Z$.

THEOREM 2.3 ([**14**]). *If X is connected and f is a continuous selection for some $\mathscr{D} \subset \mathscr{F}(X)$, with $\bigcup\{\mathscr{F}_n(X) : 2 \leq n < \omega\} \subset \mathscr{D}$, then each member of $\mathscr{D}$ has a $(\preceq_f)$-minimal element.*

For a space X and $x \in X$, we will use $\mathscr{C}[x]$ to denote the *component* of the point x, and $\mathscr{C}^*[x]$ — the corresponding *quasi-component*. Recall that

$$\mathscr{C}[x] = \bigcup\{C \subset X : x \in C \text{ and } C \text{ is connected}\}, \quad \text{and}$$
$$\mathscr{C}^*[x] = \bigcap\{C \subset X : x \in C \text{ and } C \text{ is clopen}\}.$$

THEOREM 2.4 ([**8**]). *If X has a closed selection relation, then $\mathscr{C}[x] = \mathscr{C}^*[x]$ for every $x \in X$.*

Here is another important property related to connected subsets of such spaces.

PROPOSITION 2.5 ([**6**]). *If $\preceq_{\mathbf{s}}$ is a closed selection relation on X, Z is a connected subset of X, and $x, y \in Z$, with $x \preceq_{\mathbf{s}} y$, then $[x, y]_{\preceq_{\mathbf{s}}}$ is contained in Z and is connected.*

A point p of a connected space Z is called a cut point if $Z \setminus \{p\}$ is not connected. A point $p \in Z$ is called a *noncut* point if $Z \setminus \{p\}$ is connected. We let $\mathsf{ct}(Z)$ to be the set of all cut points of Z, and, respectively, $\mathsf{nct}(Z)$ that of all noncut points of Z.

PROPOSITION 2.6 ([**6**]). *Let $\preceq_{\mathbf{s}}$ be a closed selection relation on X, Z be a connected subset of X, and let $p \in Z$. Then, $p \in \mathsf{nct}(Z)$ if and only if p is a*

$(\preceq_{\mathbf{s}})$-minimal element of Z or a $(\preceq_{\mathbf{s}})$-maximal one. In particular, $|\,\mathsf{nct}(Z)| \leq 2$ and $\mathsf{ct}(Z)$ is open in X.

We conclude this section with the following observation which was actually established in [**2**, Proposition 1.18] (see, also, [**7**, Proposition 4.3]).

PROPOSITION 2.7. *Let Z be an infinite locally compact connected space with a closed linear order $\preceq$. Then, it is orderable with respect to $\preceq$. Moreover, Z is compact if and only if $|\,\mathsf{nct}(Z)| = 2$.*

PROOF. That Z is orderable with respect to $\preceq$, it follows by [**2**, Proposition 1.18]. To show the second part of this statement, it suffices to show that Z is compact provided $|\,\mathsf{nct}(Z)| = 2$. In this case, by Proposition 2.6, there are points $x, y \in Z$ such that $Z = [x, y]_{\preceq}$. However, by [**7**, Proposition 4.3] (see, also, [**4**], [**11**]), $[x, y]_{\preceq}$ is compact with respect to the corresponding open interval topology $\mathscr{T}_{\preceq}$ on $[x, y]_{\preceq}$. Consequently, Z is also compact. □

3. Orderable compact-like neighbourhoods.

In this section we demonstrate that if X is locally compact and has a closed selection relation, then each point of X has a special compact-like clopen neighbourhood. As the reader may expect, it will turn out that any such neighbourhood is also orderable.

LEMMA 3.1. *Let X be a locally compact space which has a closed selection relation. Then, each $x \in X$ has a clopen neighbourhood Z such that $Z \setminus \mathsf{ct}(\mathscr{C}[x])$ is compact.*

PROOF. For a point $x \in X$, we distinguish the following cases.

(a) If $\mathscr{C}[x] = \{x\}$, then there exists an open set $V \subset X$ such that $x \in V$ and $\overline{V}$ is compact. Set $S = \overline{V} \setminus V$, and note that, by Theorem 2.4, each $y \in S$ has a clopen neighbourhood G_y in X, with $y \in G_y$ and $x \notin G_y$. Since S is compact, $S \subset G = \bigcup\{G_y : y \in F\}$ for some finite set $F \subset S$. In this case, $Z = V \setminus G$ is a compact clopen neighbourhood of x, which is as required because $\mathsf{ct}(\mathscr{C}[x]) = \emptyset$.

(b) If $\mathsf{ct}(\mathscr{C}[x]) = \mathscr{C}[x]$, then, by Proposition 2.6, $\mathscr{C}[x]$ is open in X, hence it is clopen in X. In this case, take $Z = \mathscr{C}[x]$.

(c) Suppose finally that $\mathsf{ct}(\mathscr{C}[x]) \neq \emptyset \neq \mathsf{nct}(\mathscr{C}[x])$. Take a closed selection relation $\preceq_{\mathbf{s}}$ on X, and $p \in \mathsf{nct}(\mathscr{C}[x])$. By Proposition 2.6, p is the $(\preceq_{\mathbf{s}})$-minimal or the $(\preceq_{\mathbf{s}})$-maximal element of $\mathscr{C}[x]$. If p is the $(\preceq_{\mathbf{s}})$-minimal element of $\mathscr{C}[x]$, then consider the subset $X_p = (\leftarrow, p]_{\preceq_{\mathbf{s}}} \subset X$ which is clopen in $X \setminus \mathsf{ct}(\mathscr{C}[x])$, and is itself locally compact being closed in X. Also, the component of p in X_p is $\{p\}$. Hence, by (a), p has a clopen compact neighbourhood Z_p in X_p, and in

$X \setminus \mathsf{ct}(\mathscr{C}[x])$ as well. In the same way, p has a clopen compact neighbourhood Z_p in $[p, \to)_{\preceq_{\mathbf{s}}}$ if p is the $(\preceq_{\mathbf{s}})$-maximal element of $\mathscr{C}[x]$. Then, $Z = \bigcup \big\{ Z_p \cup \mathscr{C}[x] : p \in \mathsf{nct}(\mathscr{C}[x]) \big\}$ is as required. Indeed, each $Z_p \cup \mathscr{C}[x]$, $p \in \mathsf{nct}(\mathscr{C}[x])$, is closed in X as an union of two closed sets, hence, by Proposition 2.6, the same is true for Z. To see that Z is also open, take $p \in \mathsf{nct}(\mathscr{C}[x])$ and $y \in \mathsf{ct}(\mathscr{C}[x])$, and assume, for instance, that $p \prec_{\mathbf{s}} y$. Since $Z_p \subset (\leftarrow, p]_{\preceq_{\mathbf{s}}} \subset (\leftarrow, y)_{\preceq_{\mathbf{s}}}$, there exists an open in $(\leftarrow, y)_{\preceq_{\mathbf{s}}}$ subset $V \subset (\leftarrow, y)_{\preceq_{\mathbf{s}}}$, with $Z_p = V \cap (\leftarrow, p]_{\preceq_{\mathbf{s}}}$. However, $(\leftarrow, y)_{\preceq_{\mathbf{s}}}$ is open in X, therefore so is V. This, in fact, completes the verification because $Z_p \cup \mathsf{ct}(\mathscr{C}[x]) = V \cup \mathsf{ct}(\mathscr{C}[x])$ while, by Proposition 2.6, $\mathsf{ct}(\mathscr{C}[x])$ is open in X. This also completes the proof because $Z = \bigcup \big\{ Z_p \cup \mathsf{ct}(\mathscr{C}[x]) : p \in \mathsf{nct}(\mathscr{C}[x]) \big\}$. □

It should be mentioned that even clopen subsets of orderable spaces may fail to be orderable. In this regard, the following simple observation will be found useful.

PROPOSITION 3.2. *Let X be a space, and let $\preceq$ be a linear order on X such that, for some points $x, y \in X$, with $x \preceq y$, the subsets $(\leftarrow, y]_{\preceq}$ and $[x, \to)_{\preceq}$ are both closed in X and orderable spaces with respect to $\preceq$. Then, X is also an orderable space with respect to $\preceq$.*

PROOF. Let $\mathscr{T}$ be the topology on X, and let $\mathscr{T}_{\preceq}$ be the open interval topology on X generated by $\preceq$. Then, the identity map $\mathrm{id}_X : (X, \mathscr{T}) \to (X, \mathscr{T}_{\preceq})$ is continuous because the sets $(\leftarrow, y]_{\preceq}$ and $[x, \to)_{\preceq}$ compose a cover of X and are both $\mathscr{T}$-closed and $\mathscr{T}_{\preceq}$-closed. By the same reason, id_X is also closed. Hence, it is a homeomorphism. □

LEMMA 3.3. *Let Z be a locally compact space which has a closed selection relation and a point $z \in Z$ such that $\big|\mathsf{nct}(\mathscr{C}[z])\big| = 1$, $\mathsf{ct}(\mathscr{C}[z]) \neq \emptyset$ and $Z \setminus \mathsf{ct}(\mathscr{C}[z])$ is compact. Then, Z is orderable by a linear order $\preceq$ such that $\mathscr{C}[z]$ has a first element with respect to $\preceq$ and $x \prec y$ for every $x \in Z \setminus \mathscr{C}[z]$ and $y \in \mathscr{C}[z]$.*

PROOF. Take p to be the point of the singleton $\mathsf{nct}(\mathscr{C}[z])$, and, using Theorem 2.2 and Proposition 2.6, take a closed selection relation $\preceq_{\mathbf{s}}$ on Z such that p is the $(\preceq_{\mathbf{s}})$-minimal element of $\mathscr{C}[z]$. According to Proposition 2.6 once again, $\mathscr{C}[z]$ has no $(\preceq_{\mathbf{s}})$-maximal element because $\big|\mathsf{nct}(\mathscr{C}[z])\big| = 1$ and $\mathsf{ct}(\mathscr{C}[z]) \neq \emptyset$. Hence, there exists a point $y \in \mathscr{C}[z]$ such that $p \preceq_{\mathbf{s}} z \prec_{\mathbf{s}} y$. Set $Y = \big(Z \setminus \mathsf{ct}(\mathscr{C}[z])\big) \cup [p, y]_{\preceq_{\mathbf{s}}}$, and observe that Y is compact. Indeed, Y is locally compact being closed in Z (by Proposition 2.6), and the component of z in Y is $[p, y]_{\preceq_{\mathbf{s}}}$ (by Proposition 2.5). Hence, by Proposition 2.7, $[p, y]_{\preceq_{\mathbf{s}}}$ is compact because, by Proposition 2.6, $\big|\mathsf{nct}([p, y]_{\preceq_{\mathbf{s}}})\big| = 2$. This implies that Y is itself compact because, by hypothesis, $Z \setminus \mathsf{ct}(\mathscr{C}[z])$ is compact. Thus, by Theorem 1.1, Y is orderable. Then, reversing a compatible order on Y if necessary, take a compatible order $\preceq_1$ on Y, with $p \prec_1 y$.

We are going to show that Y has another compatible order $\preceq_0$ such that y is the $(\preceq_0)$-maximal element of Y. Namely, if y is the $(\preceq_1)$-maximal element of Y, then take $\preceq_0$ to be $\preceq_1$. Otherwise, observe that $(p, y]_{\preceq_{\mathbf{s}}} = \mathsf{ct}(\mathscr{C}[z]) \cap Y$ is a neighbourhood of y in Y (by Proposition 2.6), hence there are points $s, x \in Y$, with $y \in (s, x)_{\preceq_1} \subset (p, y]_{\preceq_{\mathbf{s}}}$. However, by Theorem 2.2, both orders $\preceq_1$ and $\preceq_{\mathbf{s}}$ coincide on $[p, y]_{\preceq_{\mathbf{s}}}$ because it is connected (by Proposition 2.5) and $p \prec_1 y$. Hence, this now implies that

$$(p, x)_{\preceq_1} = (p, s]_{\preceq_1} \cup (s, x)_{\preceq_1} = (p, y]_{\preceq_{\mathbf{s}}} = (p, y]_{\preceq_1}.$$

That is, $(y, x)_{\preceq_1} = \emptyset$ and, therefore, $(\leftarrow, x)_{\preceq_1} = (\leftarrow, y]_{\preceq_1}$ and $(y, \rightarrow)_{\preceq_1} = [x, \rightarrow)_{\preceq_1}$ are disjoint clopen subsets of Y. Then, define $\preceq_0$ by preserving the order $\preceq_1$ on the sets $(\leftarrow, y]_{\preceq_1}$ and $[x, \rightarrow)_{\preceq_1}$, and making all points of $[x, \rightarrow)_{\preceq_1}$ to be $\preceq_0$-less than the points of $(\leftarrow, y]_{\preceq_1}$. Since Y is compact, it is orderable by $\preceq_0$. Since y is the $(\preceq_1)$-maximal element of $(\leftarrow, y]_{\preceq_1}$, it is the $(\preceq_0)$-maximal element of Y.

We finalize the proof as follows. Define a linear order $\preceq$ on Z by letting for points $s, t \in Z$ that $s \preceq t$ if $s, t \in Y$ and $s \preceq_0 t$, or $s, t \in \mathscr{C}[z]$ and $s \preceq_{\mathbf{s}} t$, or $s \in Y$ and $t \notin Y$. Observe that $\preceq$ is well-defined because $s, t \in Y \cap \mathscr{C}[z]$ implies that $s \preceq_0 t$ if and only if $s \preceq_{\mathbf{s}} t$. From one hand, by construction, Y is orderable with respect to $\preceq$ because $\preceq$ coincides with $\preceq_0$ on Y. From another hand, by Proposition 2.7, $\mathscr{C}[z]$ is also orderable with respect to $\preceq$ because, on this set, $\preceq$ and $\preceq_{\mathbf{s}}$ coincide. Finally, take in mind that Y and $\mathscr{C}[z]$ are closed in Z, $Y = (\leftarrow, y]_{\preceq}$ and $\mathscr{C}[z] = [p, \rightarrow)_{\preceq}$. Hence, Proposition 3.2 completes the proof. $\square$

4. Semi-orderable spaces.

A family $\mathscr{P}$ of subsets of a set X is usually called a *partition* of X if it is a pairwise disjoint cover of X. If X is a topological space, we say that $\mathscr{P}$ is a *clopen* partition of X if it consists of clopen (equivalently, open) subsets of X. Note that, if $\mathscr{P}$ is a clopen partition of X, then X is, in fact, the topological sum $\biguplus \mathscr{P}$ of the elements of the partition.

In the present section, we are interested in topological sums of orderable topological spaces. Turning to this, let us mention that such sums may fail to be orderable. Here is a very simple example.

Example 4.1. Let $Z = \{0\} \cup (1, 2) \subset \boldsymbol{R}$. Then, Z is the sum of two orderable spaces, but is itself not orderable.

Motivated by this, we shall say that a topological space X is *semi-orderable* if it has a clopen partition into two orderable spaces, or, equivalently, if it is the topological sum of two orderable spaces. Clearly, every orderable space is semi-orderable, and, by Proposition 1.3, every semi-orderable space is suborderable. However, no one of these implications is invertible. Example 4.1 demonstrates that a semi-orderable space may fail to be orderable. As we will see, the suborderable spaces are not necessarily semi-orderable, Example 4.12.

The purpose of this section is to prove that a topological sum of orderable spaces is, in fact, always a semi-orderable space.

THEOREM 4.2. *A space X is semi-orderable if and only if it has a clopen partition consisting of orderable spaces.*

To prepare for the proof of Theorem 4.2, we will deal with a few propositions about orderability of topological sums of orderable spaces. In what follows, for a linearly ordered set $(Z,\preceq)$, we let

$$\xi_{\min}(Z,\preceq) = \big|\{p \in Z : p \text{ is } (\preceq)\text{-minimal}\}\big|$$

and, respectively,

$$\xi_{\max}(Z,\preceq) = \big|\{p \in Z : p \text{ is } (\preceq)\text{-maximal}\}\big|.$$

Further, to any linearly ordered set $(Z,\preceq)$ we associate the number $\xi_{\text{end}}(Z,\preceq)$, suggesting the *end point characteristic number*, which is defined by

$$\xi_{\text{end}}(Z,\preceq) = \xi_{\min}(Z,\preceq) + \xi_{\max}(Z,\preceq).$$

Finally, whenever Z is an orderable space, we let

$$\xi_{\text{end}}(Z) = \min\big\{\, \xi_{\text{end}}(Z,\preceq) : \preceq \text{ is a compatible order on } Z\big\}.$$

Note that, for a singleton Z, we have that $\xi_{\text{end}}(Z) = 2$, but $|Z| = 1$.

Most of our arguments in the proof of Theorem 4.2 will be based on the following two observations. Their proofs are easy, and are left to the reader.

PROPOSITION 4.3. *Let X be an orderable space by a linear order $\preceq$. For a $(\preceq)$-convex subset $Z \subset X$, the following holds*:

(a) *Z is open provided $\xi_{\text{end}}(Z,\preceq) = 0$.*
(b) *Z is closed provided $\xi_{\text{end}}(Z,\preceq) = 2$.*

PROPOSITION 4.4. *Let X be a space, $\preceq$ be a linear ordering on X, and let $\mathscr{P}$ be a clopen partition of X such each $P \in \mathscr{P}$ is orderable by $\preceq$. Then, X*

is orderable by $\preceq$ if and only if $\mathscr{P}$ is clopen in the corresponding open interval topology $\mathscr{T}_\preceq$ on X.

Finally, here is also a standard construction of a linear order on X generated by linear orders on the elements of a partition of X. Suppose that $\mathscr{P}$ is a partition of X and $\preceq_P$ is a linear ordering on P for every $P \in \mathscr{P}$. Whenever $\leq$ is a linear order on $\mathscr{P}$, we will consider the *lexicographical order* $\preceq$ on X generated by $\leq$. For points $s, t \in X$, it is defined by $s \preceq t$ if $s, t \in P \in \mathscr{P}$ and $s \preceq_P t$, or $s \in S \in \mathscr{P}$, $t \in T \in \mathscr{P}$ and $S < T$.

PROPOSITION 4.5. *Let X be a space, and let $\mathscr{P}$ be a clopen partition of X such that each $P \in \mathscr{P}$ is an orderable spaces, with $\xi_{\mathrm{end}}(P) = 0$. Then, X is also an orderable space, with $\xi_{\mathrm{end}}(X) = 0$.*

PROOF. Take a linear order $\leq$ on $\mathscr{P}$. By hypothesis, each $P \in \mathscr{P}$ is orderable by a linear order $\preceq_P$ such that $\xi_{\mathrm{end}}(P, \preceq_P) = 0$. Let $\preceq$ be the lexicographical order on X generated by $\leq$, and let $\mathscr{T}_\preceq$ be the corresponding open interval topology on X. Then, $\xi_{\mathrm{end}}(X, \preceq) = 0$, while, by Proposition 4.3, each $P \in \mathscr{P}$ is $\mathscr{T}_\preceq$-open because $\xi_{\mathrm{end}}(P, \preceq) = \xi_{\mathrm{end}}(P, \preceq_P) = 0$. Hence, by Proposition 4.4, $X = \biguplus \mathscr{P}$ is orderable by $\preceq$. □

PROPOSITION 4.6. *Let Z be a space, and let $\{Z_0, Z_1\}$ be a clopen partition of Z such that each Z_i, $i < 2$, is a nonempty orderable space by a linear order $\preceq_i$, with $\xi_{\mathrm{end}}(Z_i, \preceq_i) \leq 1$. Then, Z is an orderable space by a linear order $\preceq$ such that*

$$\xi_{\mathrm{end}}(Z, \preceq) = \xi_{\mathrm{end}}(Z_0, \preceq_0) + \xi_{\mathrm{end}}(Z_1, \preceq_1) \pmod 2.$$

PROOF. If $\xi_{\mathrm{end}}(Z_0, \preceq_0) = 1 = \xi_{\mathrm{end}}(Z_1, \preceq_1)$, reversing one of the orders if necessary, we are in the situation that Z_0 has a $(\preceq_0)$-maximal element and Z_1 has a $(\preceq_1)$-minimal one. Take $\preceq$ to be the lexicographical order on Z generated by the relation "$Z_0 < Z_1$". Then, $\xi_{\mathrm{end}}(X, \preceq) = 0$, while each Z_i, $i < 2$, is orderable by $\preceq$, and is closed in the corresponding open interval topology $\mathscr{T}_\preceq$ on Z. Hence, by Proposition 4.4, Z is also orderable by $\preceq$. If $\xi_{\mathrm{end}}(Z_0, \preceq_0) \neq \xi_{\mathrm{end}}(Z_1, \preceq_1)$, we may assume that Z_0 has no $(\preceq_0)$-maximal element and Z_1 has no $(\preceq_1)$-minimal element. Consider again $\preceq$ to be the lexicographical order on Z generated by "$Z_0 < Z_1$". Just like before, each Z_i, $i < 2$, is orderable by $\preceq$, but now $\xi_{\mathrm{end}}(X, \preceq) = 1$ because only one of these sets has some end point. Finally, observe that each Z_i, $i < 2$, is open in the corresponding open interval topology $\mathscr{T}_\preceq$ on Z. Hence, by Proposition 4.4, Z is again orderable by $\preceq$. Since the third case is covered by Proposition 4.5, the proof completes. □

PROPOSITION 4.7. *Let X be a space, and let $\mathscr{P}$ be a clopen partition of X such that each $P \in \mathscr{P}$ is orderable by a linear order $\preceq_P$, with $\xi_{\mathrm{end}}(P, \preceq_P) = 1$. Then, X is an orderable space, with $\xi_{\mathrm{end}}(X) \leq 1$.*

PROOF. If $\mathscr{P}$ is a finite partition of X, then the statement follows by Proposition 4.6. If $\mathscr{P}$ is infinite, then there exists a bijective map $h : \mathscr{P} \times 2 \to \mathscr{P}$. Whenever $P \in \mathscr{P}$, let $Q_P = h(P, 0) \cup h(P, 1)$. Note that $\mathscr{Q} = \{Q_P : P \in \mathscr{P}\}$ is also a clopen partition of X such that, by Proposition 4.6, $\xi_{\mathrm{end}}(Q_P) = 0$. Hence, Proposition 4.5 completes the proof. □

PROPOSITION 4.8. *Let X be a space, and let $\mathscr{P}$ be a clopen partition of X such that each $P \in \mathscr{P}$ is orderable by a linear order $\preceq_P$, with $\xi_{\mathrm{end}}(P, \preceq_P) = 2$. Then, X is also an orderable space.*

PROOF. If $\mathscr{P}$ is finite, take $\preceq$ to be the lexicographical order on X generated by a linear ordering $\leq$ on $\mathscr{P}$. Then, by Propositions 4.3 and 4.4, X is orderable by $\preceq$. If $\mathscr{P}$ is infinite, then there exists a bijective map $h : \mathscr{P} \times \omega \to \mathscr{P}$, where ω is the first infinite ordinal. Whenever $P \in \mathscr{P}$, consider the countable infinite family $\mathscr{Q}_P = \{h(P, n) : n < \omega\}$, and let $Q_P = \bigcup \mathscr{Q}_P$. Take $\preceq_{(P,\omega)}$ to be the lexicographical order on Q_P generated by the linear order $\leq$ on $\mathscr{Q}_P$ as that of ω, i.e. $h(P, n) \leq h(P, m)$ if and only if $n \leq m$. For convenience, let $\mathscr{T}_{\preceq_{(P,\omega)}}$ be the corresponding open interval topology on Q_P, and let $h(P, n) = [\ell_n, r_n]_{\preceq_{(P,\omega)}}$, $n < \omega$, which is possible by hypothesis. Then, each $h(P, n)$, $n < \omega$, is $\mathscr{T}_{\preceq_{(P,\omega)}}$-clopen because $h(P, 0) = (\leftarrow, \ell_1)_{\preceq_{(P,\omega)}}$ and $h(P, n+1) = (r_n, \ell_{n+2})_{\preceq_{(P,\omega)}}$, $n < \omega$. Hence, by Proposition 4.4, $Q_P = \biguplus \mathscr{Q}_P$ is orderable by $\preceq_{(P,\omega)}$. Since $\xi_{\mathrm{end}}(Q_P, \preceq_{(P,\omega)}) = 1$, $P \in \mathscr{P}$, the clopen partition $\mathscr{Q} = \{Q_P : P \in \mathscr{P}\}$ of X is as in Proposition 4.7, which completes the proof. □

PROOF OF THEOREM 4.2. Let $\mathscr{P}$ be a clopen partition of X consisting of orderable spaces. Whenever $i < 3$, let

$$\mathscr{P}_i = \big\{P \in \mathscr{P} : \xi_{\mathrm{end}}(P) = i\big\}.$$

According to Propositions 4.5, 4.7 and 4.8, each topological sum $P_i = \biguplus \mathscr{P}_i$, $i < 3$, is an orderable space, with $\xi_{\mathrm{end}}(P_i) \leq i$. Since $\xi_{\mathrm{end}}(P_i) \leq 1$, $i < 2$, it now follows by Proposition 4.6 that $P_0 \cup P_1$ is also an orderable space. Hence, $\{P_0 \cup P_1, P_2\}$ is a partition of X into two clopen orderable spaces, which completes the proof. □

According to Theorem 4.2, we have the following immediate consequence.

COROLLARY 4.9. *Let X be a space which has a clopen partition consisting of semi-orderable spaces. Then, X is itself semi-orderable.*

Suppose that X is a semi-orderable space. By definition, X is the topological sum of two orderable spaces X_0 and X_1. Take a compatible order $\preceq_i$ on X_i, $i < 2$, and consider the lexicographical order $\preceq$ on X generated by "$X_0 < X_1$". Then, $\preceq$ is a compatible order on X as a suborderable space, but the subsets X_i, $i < 2$, are now both orderable by $\preceq$ and $(\preceq)$-convex. In the sequel, such a compatible order on X will be called *canonical*.

Corollary 4.10. *Let X be a semi-orderable space and let $Y \subset X$ be an open subset. Then, Y is also semi-orderable.*

Proof. By definition, X is the topological sum of two orderable spaces X_0 and X_1. Take a compatible canonical order $\preceq$ on X generated by X_0 and X_1. Then, X_0 and X_1 are both orderable by $\preceq$ and $(\preceq)$-convex, and for every $y \in Y$ there is exactly one $i(y) < 2$, with $y \in X_{i(y)}$. Finally, let $P[y]$ be the $(\preceq)$-convex component of y in $X_{i(y)}$ with respect to Y, i.e.

$$P[y] = \bigcup \big\{ [s,t]_\preceq : s,t \in X_{i(y)},\ s \preceq y \preceq t \text{ and } [s,t]_\preceq \subset Y \big\}.$$

Then, $P[y] \subset Y$ is an open $(\preceq)$-convex subset of $X_{i(y)}$ because $X_{i(y)}$ and Y are open in X and $X_{i(y)}$ is orderable by $\preceq$. In fact, this implies that $P[y]$ is also orderable by $\preceq$. Thus, we get a clopen partition $\mathscr{P} = \{P[y] : y \in Y\}$ of Y consisting of orderable spaces. Hence, by Theorem 4.2, Y is also semi-orderable. □

In conclusion, let us remark that the semi-orderable spaces have several common properties with orderable ones. Below, we mention only one of them that helps to provide an example of suborderable spaces which are not semi-orderable.

Proposition 4.11. *A semi-orderable space is metrizable if and only if it has a G_δ-diagonal.*

Proof. By definition, X has a clopen finite partition $\mathscr{P}$ consisting of orderable spaces. Then, X has a G_δ-diagonal if and only if each $P \in \mathscr{P}$ has a G_δ-diagonal. However, according to a result of Lutzer [**13**], an orderable space is metrizable if and only if it has a G_δ-diagonal. Hence, X has a G_δ-diagonal if and only if each $P \in \mathscr{P}$ is metrizable, which completes the proof. □

It is well-known that the Sorgenfrey line is a suborderable space which has a G_δ-diagonal but its Cartesian square is not normal [**17**] (see, [**4**, Example 2.3.12]). Consequently, the Sorgenfrey line is not semi-orderable. Readers who are more familiar with the Michael line one can use it as another example of a suborderable space which fails to be semi-orderable, see [**4**, Michael's Example 5.1.32]. Thus, we have the following example.

EXAMPLE 4.12. The Sorgenfrey line and the Michael line are suborderable spaces which are not semi-orderable.

5. Weak orderability and semi-orderability.

In this section, we finalize the proof of Theorem 1.2. In fact, this theorem is a consequence of the following more general result.

THEOREM 5.1. *For a locally compact paracompact space X, the following are equivalent*:
(a) *X has a continuous weak selection.*
(b) *X is semi-orderable.*
(c) *X is suborderable.*
(d) *X is weakly orderable.*

To prepare for the proof of Theorem 5.1, we need to derive some consequences from the technique developed in the previous sections. To this end, for a space X, we consider the family $\mathscr{O}(X)$ consisting of all clopen subsets $U \subset X$ such that U is connected or there is a point $x \in U$ such that $\mathsf{nct}(\mathscr{C}[x]) \neq \emptyset$ and $U \setminus \mathsf{ct}(\mathscr{C}[x])$ is compact.

The family $\mathscr{O}(X)$ was naturally suggested by Lemmas 3.1 and 3.3, and we have the following immediate consequences for its members.

COROLLARY 5.2. *Let X be a locally compact space which has a closed selection relation. Then, each member of $\mathscr{O}(X)$ is orderable.*

PROOF. Let $U \in \mathscr{O}(X)$. If U is connected, then $U = \mathscr{C}[x]$ for some (every) point $x \in U$. Hence, it is orderable by Theorem 2.2 and Proposition 2.7. If U is not connected, then there exists a point $x \in U$ such that $\mathsf{nct}(\mathscr{C}[x]) \neq \emptyset$ and $U \setminus \mathsf{ct}(\mathscr{C}[x])$ is compact. If $\mathscr{C}[x]$ is compact, then so is U, and it is orderable by Theorem 1.1. If $\mathscr{C}[x]$ is not compact, then, by Proposition 2.7, $\left|\mathsf{nct}(\mathscr{C}[x])\right| = 1$, and the statement follows by Lemma 3.3. □

COROLLARY 5.3. *Let X be a locally compact space which has a closed selection relation. Then, each $x \in X$ is contained in some member of $\mathscr{O}(X)$.*

PROOF. Take a point $x \in X$. If $\mathsf{nct}(\mathscr{C}[x]) = \emptyset$, then $\mathscr{C}[x] = \mathsf{ct}(\mathscr{C}[x])$, hence, by Proposition 2.6, $\mathscr{C}[x]$ is a clopen neighbourhood of x. If $\mathsf{nct}(\mathscr{C}[x]) \neq \emptyset$, then, by Lemma 3.1, x has a clopen neighbourhood U such that $U \setminus \mathsf{ct}(\mathscr{C}[x])$ is compact. □

According to Example 4.1, the family $\mathscr{O}(X)$ is not closed with respect to finite unions of its members. However, it is closed with respect to finite intersections, which will be crucial for our proof.

PROPOSITION 5.4. *Let X be a space, $W \subset X$ be clopen, and $U \in \mathscr{O}(X)$. Then, $W \cap U \in \mathscr{O}(X)$.*

PROOF. If U is connected, then either $W \cap U = \emptyset$ or $W \cap U = U$. Consequently, $W \cap U \in \mathscr{O}(X)$ as well. If U is not connected, then it has a point x such that $\mathsf{nct}(\mathscr{C}[x]) \neq \emptyset$ and $U \setminus \mathsf{ct}(\mathscr{C}[x])$ is compact. If $x \in W \cap U$, then the same is true for $W \cap U$ because $\mathscr{C}[x] \subset W \cap U$. If $x \notin W \cap U$, then $W \cap U \subset U \setminus \mathsf{ct}(\mathscr{C}[x])$ and, therefore, it will be compact. In this case, take a point $z \in W \cap U$, and observe that, by Proposition 2.7, $\mathsf{nct}(\mathscr{C}[z]) \neq \emptyset$ and $(W \cap U) \setminus \mathsf{ct}(\mathscr{C}[z])$ is compact because $W \cap U$ is itself compact. Thus, again, $W \cap U \in \mathscr{O}(X)$. □

We finalize the preparation for the proof of Theorem 1.2 with the following result about locally compact Lindelöf spaces.

PROPOSITION 5.5. *Let X be a locally compact Lindelöf space which has a closed selection relation. Then, X has a clopen partition $\mathscr{P} \subset \mathscr{O}(X)$.*

PROOF. By Corollary 5.3, $\mathscr{O}(X)$ is a clopen cover of X. Since X is Lindelöf, there exists a countable family $\{U_n : n < \omega\} \subset \mathscr{O}(X)$, with $X = \bigcup\{U_n : n < \omega\}$. Standard arguments now provide that $\{U_n : n < \omega\}$ has a pairwise disjoint refinement $\{P_n : n < \omega\} \subset \mathscr{O}(X)$ which is still a cover of X. Namely, set $P_0 = U_0$, and, whenever $n < \omega$, define

$$P_{n+1} = U_{n+1} \setminus \bigcup\{U_k : k \leq n\} = U_{n+1} \cap \Big(X \setminus \bigcup\{U_k : k \leq n\}\Big).$$

According to Proposition 5.4, each P_n, $n < \omega$, is a member of $\mathscr{O}(X)$, while, by construction, $\{P_n : n < \omega\}$ is a pairwise disjoint cover of X. □

PROOF OF THEOREM 5.1. To show that (a) ⇒ (b), suppose that X is as in (a). According to a result of Morita [**16**] (see, [**4**, Theorem 5.1.27]), X has a clopen partition $\mathscr{P}$ consisting of Lindelöf spaces. By Proposition 5.5, each $P \in \mathscr{P}$ has a clopen partition $\mathscr{Q}_P \subset \mathscr{O}(P) \subset \mathscr{O}(X)$. Hence, $\mathscr{Q} = \bigcup\{\mathscr{Q}_P : P \in \mathscr{P}\} \subset \mathscr{O}(X)$ is a clopen partition of X. By Corollary 5.2, each $Q \in \mathscr{Q}$ is orderable. Then, by Theorem 4.2, X is semi-orderable. Since (b) ⇒ (c) ⇒ (d) are obvious, while (d) ⇒ (a) follows by Theorem 2.2, the proof completes. □

The following is an immediate consequence of Theorem 5.1 (compare with Corollary 4.10).

Corollary 5.6. *Let X be a semi-orderable locally compact paracompact space. Then, any closed subset of X is also semi-orderable.*

6. Orderability and semi-orderability.

In the present section, we are interested in the difference between semi-orderable and orderable spaces. Turning to this, let us observe the following property of compact orderable spaces which is known in a little bit different terms.

Proposition 6.1. *An orderable space X is compact if and only if there is a compatible order $\preceq$ on X such that $\xi_{\text{end}}(Z, \preceq) = 2$ for every nonempty clopen subset $Z \subset X$.*

Proof. Any compact orderable space has this property with respect to any compatible order on it. To see the converse, suppose that $\preceq$ is a compatible order on X as in the statement of the proposition. Since X has a $(\preceq)$-maximal element, by a result of [**11**] (see, also, [**4**, 3.12.3]), it suffices to show that every nonempty closed subset of X has a $(\preceq)$-minimal element. Suppose if possible that this fails for some nonempty closed subset $F \subset X$. Consider the set $E = \bigcup \big\{(x, \rightarrow)_{\preceq} : x \in F\big\}$ which is open in X and is $(\preceq)$-convex. Since F has no $(\preceq)$-minimal element, the same is true for E. Hence, by hypothesis, E is not closed in X and there exists a point $y \in \overline{E} \setminus E$. Take a $(\preceq)$-convex neighbourhood U of y. Then, $U \cap (x, \rightarrow)_{\preceq} \neq \emptyset$ for some $x \in F$, and, therefore, $x \in U \cap F$ because U is $(\preceq)$-convex and $y \prec z$ for every $z \in E$. This implies that $y \in F$ because F is closed. However, $y \preceq z$ for every $z \in F$, which is a contradiction. $\square$

Motivated by Proposition 6.1, we shall say that an orderable space X is *anti-compact orderable* if, whenever $\preceq$ is a compatible order on X, we have that $\xi_{\text{end}}(Z, \preceq) = 0$ for every clopen subset $Z \subset X$. According to Theorem 2.2 and Propositions 2.6 and 4.5, we have the following immediate example of anti-compact orderable spaces.

Corollary 6.2. *An orderable connected space X is anti-compact orderable if and only if $\mathsf{nct}(X) = \emptyset$. Moreover, any topological sum of connected anti-compact orderable spaces is also an anti-compact orderable space.*

For the proper understanding of the term "anti-compact", let us recall that a pair (D, E) of subsets of a linearly ordered set $(X, \preceq)$ is called a $(\preceq)$-*cut* if $X = D \cup E$, $D \neq \emptyset \neq E$, and $x \prec y$ for every $x \in D$ and $y \in E$. In this case, D is called the *lower section* of the cut, and E — the *upper section.* A cut (D, E) is called a *jump* if the lower section D has a $(\preceq)$-maximal element and the upper section E has a $(\preceq)$-minimal one. A $(\preceq)$-cut (D, E) is called a *gap* if the lower section D has no $(\preceq)$-maximal element and upper section E has no $(\preceq)$-minimal element. A linearly ordered set $(X, \preceq)$ is called *densely ordered* if no $(\preceq)$-cut of X is a jump. Typical examples of densely ordered sets are the real numbers $\boldsymbol{R}$, the rational numbers $\boldsymbol{Q}$, also the irrational ones $\boldsymbol{P}$, considered with respect to the usual order on them.

It follows immediately by the definitions that if X is an anti-compact orderable space and $\preceq$ is a compatible order on it, then $\xi_{\mathrm{end}}(X, \preceq) = 0$ and $(X, \preceq)$ is a densely ordered set. In this regard, there are orderable spaces which are densely ordered sets with respect to some compatible order on them, but they fail to be anti-compact orderable. Such examples are the set of the rational numbers $\boldsymbol{Q}$, also the irrational numbers $\boldsymbol{P}$. For instance, let $\boldsymbol{Q}_0 = \{q \in \boldsymbol{Q} : q \leq 0\}$ and $\boldsymbol{Q}_1 = \{q \in \boldsymbol{Q} : q \geq 0\}$, and let $X = \boldsymbol{Q}_0 \oplus \boldsymbol{Q}_1$ be the disjoint sum of these sets equipped with the sum topology. Then, X is homeomorphic to $\boldsymbol{Q}$, but it is not a densely ordered set when $\boldsymbol{Q}_0$ and $\boldsymbol{Q}_1$ are ordered by the usual order, and all points of $\boldsymbol{Q}_0$ are less than the points of $\boldsymbol{Q}_1$.

We are now ready to state the main result of this section.

THEOREM 6.3. *Let X be a semi-orderable space which is not orderable. Then, X is the topological sum of a nonempty compact orderable space and a nonempty anti-compact orderable one.*

To prepare for the proof of Theorem 6.3, we are going to examine "order"-cuts in orderable spaces. In what follows, if X is a suborderable space with respect to a linear order $\preceq$, and (D, E) is a $(\preceq)$-cut of X, then we shall say that (D, E) is *clopen* if both sets D and E are clopen (equivalently, open or closed) in X.

LEMMA 6.4. *A suborderable space X is orderable with respect to a compatible linear order $\preceq$ on it if and only if each clopen $(\preceq)$-cut of X is either a gap or a jump.*

PROOF. Suppose that X is suborderable by a linear order $\preceq$ such that each clopen $(\preceq)$-cut of X is either a gap or a jump. For convenience, let $\mathscr{T}$ be the topology on X, and let $\mathscr{T}_{\preceq}$ be the open interval one. To show that $\mathscr{T} = \mathscr{T}_{\preceq}$, take a point $x \in X$ such that $[x, \rightarrow)_{\preceq} \in \mathscr{T}$. Then, $[x, \rightarrow)_{\preceq}$ is clopen in X because $\mathscr{T}_{\preceq} \subset \mathscr{T}$. If $X = [x, \rightarrow)_{\preceq}$, then $[x, \rightarrow)_{\preceq} \in \mathscr{T}_{\preceq}$. If $X \neq [x, \rightarrow)_{\preceq}$, then consider the clopen $(\preceq)$-cut (D, E) of X, where $D = (\leftarrow, x)_{\preceq}$ and $E = [x, \rightarrow)_{\preceq}$. Since x is the $(\preceq)$-minimal element of E, the $(\preceq)$-cut (D, E) must be a jump, consequently D has a $(\preceq)$-maximal element y. In this case, $[x, \rightarrow)_{\preceq} = (y, \rightarrow)_{\preceq} \in \mathscr{T}_{\preceq}$. In the same way, $(\leftarrow, x]_{\preceq} \in \mathscr{T}_{\preceq}$ provided $(\leftarrow, x]_{\preceq} \in \mathscr{T}$. Hence, $\mathscr{T} = \mathscr{T}_{\preceq}$.

Suppose now that X is an orderable space by a linear order $\preceq$, and take a clopen $(\preceq)$-cut (D, E) of X. Further, suppose that E has a $(\preceq)$-minimal element

$x \in E$. Since E is open and $x \in [x, \to)_{\preceq} = E$, there exists a $y \in X$, with $x \in (y, \to)_{\preceq} \subset E$. Hence, $(y, \to)_{\preceq} = [x, \to)_{\preceq}$ which implies that $y \notin E$ and $(y, x)_{\preceq} = \emptyset$. That is, $y \in D$ and it is the $(\preceq)$-maximal element of D. Thus, (D, E) is a jump. In the same way, (D, E) is a jump if D has a $(\preceq)$-maximal element, which completes the proof. □

Lemma 6.5. *Let Y be an orderable space with respect to a linear order $\leq$ such that $\xi_{\mathrm{end}}(Y, \leq) = 0$ or $\xi_{\mathrm{end}}(Y, \leq) = 2$, and let $Z \subset Y$ be a clopen $(\leq)$-convex subset of Y, with $\xi_{\mathrm{end}}(Z, \leq) = 1$. Then, Y is also orderable by a linear order $\preceq$ such that $\xi_{\mathrm{end}}(Y, \preceq) + \xi_{\mathrm{end}}(Y, \leq) = 2$.*

Proof. Suppose, that Z has a $(\leq)$-minimal element z, the another case is symmetric. If $\xi_{\mathrm{end}}(Y, \leq) = 0$, then $D = (\leftarrow, z)_{\leq} \neq \emptyset$, while $E = [z, \to)_{\leq}$ is clopen in Y because Z is open in Y and $[z, \to)_{\leq} = Z \cup (z, \to)_{\leq}$. Hence, (D, E) is a clopen $(\leq)$-cut of Y, and, by Lemma 6.4, it must be a jump. Hence, $D = (\leftarrow, y]_{\leq}$ for some $y \in D$. In this case, define a linear order $\preceq$ on Y by preserving the $\leq$-order on the sets D and E, and making all points of E to be $\preceq$-less than the points of D. Thus, in fact, the clopen $(\leq)$-cut (D, E) is transformed into a clopen $(\preceq)$-cut (E, D) of Y which is now a gap, and $\xi_{\mathrm{end}}(Y, \preceq) = 2$. The sets D and E are clearly open (hence, clopen as well) in the open interval topology $\mathscr{T}_{\preceq}$ on Y, so, by Proposition 4.4, Y is orderable by $\preceq$. If $\xi_{\mathrm{end}}(Y, \leq) = 2$, then $D = (\leftarrow, z]_{\leq} \cup Z = (\leftarrow, z)_{\leq} \cup Z$ is clopen in Y and has no $(\leq)$-maximal element. Then, $E = Y \setminus D \neq \emptyset$ and it is also clopen. Since Z is $(\leq)$-convex, we also have that $x < y$ for every $x \in D$ and $y \in E$. Hence, (D, E) is a clopen $(\leq)$-cut of Y which, by Lemma 6.4, must be a gap. Then, just like before, define a linear order $\preceq$ on Y by preserving the $\leq$-order on the sets D and E, and making all points of E to be $\preceq$-less than the points of D. Thus, Y is orderable by $\preceq$, but now $\xi_{\mathrm{end}}(Y, \preceq) = 0$. □

We finalize the preparation for the proof of Theorem 6.3 with the following observation.

Lemma 6.6. *Let X be a semi-orderable space by a canonical ordering $\leq$, and let Z be a clopen $(\leq)$-convex subset of X, which is orderable by $\leq$ and $\xi_{\mathrm{end}}(Z, \leq) = 1$. Then, X is orderable.*

Proof. By definition, X is the topological sum of two $(\leq)$-convex subsets X_0 and X_1 which are orderable by $\leq$, say we have $x_0 < x_1$ for every $x_i \in X_i$, $i < 2$. If one of these sets is empty, then X is itself orderable. If $X_0 \neq \emptyset \neq X_1$, then (X_0, X_1) is a clopen $(\leq)$-cut of X. If this $(\leq)$-cut is a gap or a jump, then X_0 and X_1 are also clopen in the open interval topology $\mathscr{T}_{\leq}$ on X, hence, by Proposition 4.4, X should be orderable. Suppose finally that (X_0, X_1) is neither a gap nor a jump. The following cases are possible. If $\xi_{\mathrm{end}}(X_i) \leq 1$, $i < 2$, then, by Proposition 4.6, X is orderable. Thus, it remains, for instance, that $\xi_{\mathrm{end}}(X_1, \leq) = 2$. If $\xi_{\mathrm{end}}(X_0, \leq) = 1$, then X_0 must have a $(\leq)$-minimal element because (X_0, X_1) is neither a gap nor a jump. In this case, define a linear order $\preceq$ on X by preserving the $\leq$-order on the sets X_0 and X_1, and making all points of X_1 to be $\preceq$-less than the points of X_0. This makes X orderable by $\preceq$ because now the sets X_0 and X_1 are clopen in the open interval topology $\mathscr{T}_{\preceq}$ on X. Suppose finally that $\xi_{\mathrm{end}}(X_0, \leq) = 0$. In this case, $Z \cap X_0 = \emptyset$ or $Z \cap X_1 = \emptyset$. Indeed, if $D = Z \cap X_0 \neq \emptyset$ and $E = Z \cap X_1 \neq \emptyset$, then (D, E) must be a $(\leq)$-cut in Z which is neither a gap nor a jump. However, Z is orderable by $\leq$, hence, by Lemma 6.4, this is impossible. If $Z \subset X_0$, then, by Lemma 6.5, X_0 is orderable by a linear order $\leq_0$, with $\xi_{\mathrm{end}}(X_0, \leq_0) = 2$. So, by Proposition 4.8, X is orderable as well. If $Z \subset X_1$, then, by Lemma 6.5, X_1 is orderable by a linear order $\leq_1$, with $\xi_{\mathrm{end}}(X_1, \leq_1) = 0$, and, by Proposition 4.5, X is orderable again. □

Proof of Theorem 6.3. Suppose that X is as in that theorem. By definition, X is the topological sum of two orderable spaces K and L. Since X is not orderable, both sets K and L must be nonempty. Take a compatible canonical order $\preceq$ on X so that K and L are orderable by $\preceq$ and $s \prec t$ for every $s \in K$ and $t \in L$. Then, (K, L) is a clopen $(\preceq)$-cut of X which, by Proposition 4.4, is not clopen in the corresponding open interval topology $\mathscr{T}_{\preceq}$. We now have that, for instance, K has a $(\preceq)$-maximal element and L has no $(\preceq)$-minimal one. According once again on the hypothesis that X is not orderable, we get, by Lemma 6.6, that K must have a $(\preceq)$-minimal element and L has no $(\preceq)$-maximal element. First, we are going to show that K is compact. Take a nonempty closed subset $F \subset K$, and suppose if possible that F has no $(\preceq)$-minimal element. We proceed just like in the proof of Proposition 6.1. Namely, consider the set $E = \bigcup \big\{(x, \to)_{\preceq} \cap K : x \in F\big\}$ which is open in K (hence, in X) and is $(\preceq)$-convex. Since F has no $(\preceq)$-minimal element, the same is true for E and we now get that $\xi_{\mathrm{end}}(E, \preceq) = 1$. According to Lemma 6.6, E cannot be closed. Hence, there exists a point $y \in \overline{E} \setminus E \subset K$. Take a $(\preceq)$-convex neighbourhood $U \subset K$ of y. Then, $U \cap (x, \to)_{\preceq} \neq \emptyset$ for some $x \in F$, and, therefore, $x \in U \cap F$ because $y \prec z$ for every $z \in E$. That is, $y \in F$ because F is closed, which is impossible because $y \preceq z$ for every $z \in F$. A contradiction, which, together with Proposition 6.1, implies that K is compact.

To finish the proof, let us show that $(L, \preceq)$ is a densely ordered set. Take a $(\preceq)$-cut (D, E) of L. If (D, E) is a jump, then both sets D and E must be closed in L, hence clopen in X. However, $\xi_{\mathrm{end}}(L, \preceq) = 0$ which implies that, in this case, $\xi_{\mathrm{end}}(D, \preceq) = 1 = \xi_{\mathrm{end}}(E, \preceq)$. According to Lemma 6.6, this is impossible. Thus, L is densely ordered by $\preceq$. Finally, observe that any compatible order on L as an orderable space generates a compatible canonical order on X as a semi-orderable space. Hence, L has the same property with respect to any compatible order on it, consequently it is anti-compact orderable. □

It is an interesting question if the topological sum of a nonempty compact orderable space and a nonempty anti-compact orderable one gives rise always to a non-orderable space. In this regard, let us observe that the partition in Theorem 6.3 is not as arbitrary as one might look at first. The following is an immediate consequence of the definition of anti-compact orderable spaces.

COROLLARY 6.7. *Let X be a space which is the topological sum of a compact orderable space K and an anti-compact orderable space L. Then, K contains any clopen compact subset of X, i.e.*

$$K = \bigcup\{A \subset X : A \text{ is clopen and compact}\}.$$

Motivated by this, for a semi-orderable space X, we define an invariant $\mathsf{soc}(X)$, suggesting the *semi-orderable character* of X, as the cardinality of the minimal clopen partition of X into orderable spaces. Thus, X is orderable if and only if $\mathsf{soc}(X) = 1$, and X is not orderable if and only if $\mathsf{soc}(X) = 2$.

COROLLARY 6.8. *Let X be a semi-orderable space, and let $\mathscr{P}$ and $\mathscr{Q}$ be clopen partitions of X into orderable spaces such that $|\mathscr{P}| = \mathsf{soc}(X) = |\mathscr{Q}|$. Then, $\mathscr{P} = \mathscr{Q}$.*

PROOF. If $\mathsf{soc}(X) = 1$, this is obvious. If $\mathsf{soc}(X) = 2$, then it follows by Theorem 6.3 and Corollary 6.7. □

Going back to our question it should be mentioned that, in fact, the author was unable to verify if an anti-compact orderable space L which is a clopen subset of an orderable space X will be orderable with respect to any compatible order on X. In the presence of the local compactness, the situation is different, and we have the following theorem.

THEOREM 6.9. *A locally compact semi-orderable space X is not orderable if and only if it is the topological sum of a nonempty compact orderable space and a nonempty anti-compact orderable one.*

As the reader may expect, to achieve this result it seems crucial to provide a topological description of the possible anti-compact orderable component of a semi-orderable space. This is what we will do in the preparation for the proof of Theorem 6.9.

PROPOSITION 6.10. *Let X be a locally compact space which is the topological* *sum of a compact orderable space K and an anti-compact orderable space L. Then, X doesn't contain a point x such that $\left|\mathsf{nct}(\mathscr{C}[x])\right| = 1$ and $\mathsf{ct}(\mathscr{C}[x]) \neq \emptyset$.*

Proof. Suppose that this fails, i.e. that X has a point x such that $\left|\mathsf{nct}(\mathscr{C}[x])\right| = 1$ and $\mathsf{ct}(\mathscr{C}[x]) \neq \emptyset$. Then, $\mathscr{C}[x] \subset K$ or $\mathscr{C}[x] \subset L$ because both K and L are clopen. Since $\mathscr{C}[x]$ is not compact (by Proposition 2.7), we have that $\mathscr{C}[x] \subset L$. Take a compatible order $\preceq$ on L. Then, by Proposition 2.6, $\mathscr{C}[x]$ has, for instance, a $(\preceq)$-minimal element y which is the point of the singleton $\mathsf{nct}(\mathscr{C}[x])$. On the other hand, by Lemma 3.1, there exists a clopen neighbourhood Z of x in L such that $Z \setminus \mathsf{ct}(\mathscr{C}[x])$ is compact. By assumption, $Z \setminus \mathsf{ct}(\mathscr{C}[x]) \neq \emptyset$, hence it has a $(\preceq)$-minimal element z_0 and a $(\preceq)$-maximal one z_1. Since $y \in Z \setminus \mathsf{ct}(\mathscr{C}[x])$, we now have that $z_0 \preceq y \preceq z_1$. If y is not the $(\preceq)$-minimal element of Z, we get that $z_0 \prec y$ which, by Proposition 2.5, implies that $z_0 \prec t$ for every $t \in \mathscr{C}[x]$. Thus, Z must have a $(\preceq)$-minimal element which contradicts the property of L. □

Proposition 6.11. *Let X be a locally compact space which is the topological sum of a compact orderable space K and an anti-compact orderable space L. Then,*

$$K = \left\{x \in X : \mathsf{nct}(\mathscr{C}[x]) \neq \emptyset\right\} \text{ and } L = \left\{x \in X : \mathsf{nct}(\mathscr{C}[x]) = \emptyset\right\}.$$

Proof. Take a point $x \in X$. Then, $\mathscr{C}[x] \subset K$ or $\mathscr{C}[x] \subset L$ because both K and L are clopen. If $\mathsf{nct}(\mathscr{C}[x]) \neq \emptyset$, then, by Proposition 6.10, either $\mathscr{C}[x]$ is a singleton or $\left|\mathsf{nct}(\mathscr{C}[x])\right| = 2$. In this case, according to Proposition 2.7, $\mathscr{C}[x]$ must be compact. Hence, by Lemma 3.1, $\mathscr{C}[x]$ has a clopen compact neighbourhood Z and, by Corollary 6.7, $\mathscr{C}[x] \subset Z \subset K$. If $\mathsf{nct}(\mathscr{C}[x]) = \emptyset$, then $\mathscr{C}[x]$ is a clopen subset of X (by Proposition 2.6) and is not compact (by Proposition 2.7). Hence, it is not a subset of K, consequently $\mathscr{C}[x] \subset L$. □

Proof of Theorem 6.9. The one direction follows by Theorem 6.3. To show the converse, suppose that X is orderable by a linear order $\preceq$ and is the topological sum of a nonempty compact orderable space K and a nonempty anti-compact orderable one L. Let

$$\mathscr{P} = \left\{\mathscr{C}[x] : x \in X \text{ and } \mathsf{nct}(\mathscr{C}[x]) = \emptyset\right\}.$$

According to Propositions 2.6 and 6.11, $\mathscr{P}$ is a clopen partition of L, while, by Proposition 2.7, each $P \in \mathscr{P}$ is orderable by $\preceq$. Hence, by Proposition 4.4, L is also orderable by $\preceq$. In this case, one of the sets K or L must be not $(\preceq)$-convex. Namely, if both K and L are $(\preceq)$-convex, then either (K, L) or (L, K) must be a clopen $(\preceq)$-cut of X. However, by Lemma 6.4, this is impossible because $\xi_{\mathrm{end}}(K, \preceq) = 2$ and $\xi_{\mathrm{end}}(L, \preceq) = 0$. Thus one of the sets K and L is not $(\preceq)$-

convex, and there are points $t \in L$ and $z \in K$ such that $t \prec z$. Let Z be the maximal $(\preceq)$-convex set which contains z and is contained in K, i.e.

$$Z = \bigcup \big\{[x,y]_{\preceq} : x,y \in X,\ x \preceq z \preceq y \text{ and } [x,y]_{\preceq} \subset K\big\}.$$

Then, Z has a $(\preceq)$-minimal element y because it is a nonempty compact subset of X. In fact, Z is clopen in X because K is clopen in X and is orderable by $\preceq$. Then, consider the set $E = [y,\to)_{\preceq} = Z \cup (y,\to)_{\preceq}$ which is also clopen in X and $D = X \setminus E \neq \emptyset$ because $t \notin E$. Thus, (D,E) is a clopen $(\preceq)$-cut in X which, by Lemma 6.4, must be a jump, so $D = (\leftarrow, x]_{\preceq}$ for some point $x \in D$. Since $x \in \mathscr{C}[x] \subset L$ and $\mathscr{C}[x]$ is orderable by $\preceq$, by Proposition 2.6, $x \in \mathsf{nct}(\mathscr{C}[x])$. According to Proposition 6.11, this is impossible which completes the proof. □

We finalize this section with several applications about the orderability of locally compact spaces.

Corollary 6.12. *Let X be a locally compact noncompact semi-orderable space which does not contain a point $x \in X$, with $\big|\mathsf{nct}(\mathscr{C}[x])\big| = 1$ and $\mathsf{ct}(\mathscr{C}[x]) \neq \emptyset$. Then X is not orderable if and only if the union of all compact components of X is nonempty and compact.*

Proof. Consider the set $L = \{x \in X : \mathsf{nct}(\mathscr{C}[x]) = \emptyset\}$. Then, L is an open subset of X, and, by Proposition 2.7 and Corollary 6.2, L is an anti-compact orderable space. Let $K = X \setminus L$. Then, K is closed, while $x \in K$ if and only if $\mathscr{C}[x]$ is compact. Hence, by Lemma 3.1, K is also open in X. Thus, we get a clopen partition of X into a space K an anti-compact orderable one L. Observe that $L \neq \emptyset$ provided K is nonempty and compact because X is not compact. Hence, by Proposition 6.11 and Theorem 6.9, X is not orderable if and only if K is nonempty and compact. □

Corollary 6.13. *A semi-orderable locally compact noncompact space X is orderable if and only if one of the following holds*:

(a) *X contains a point x such that $\big|\mathsf{nct}(\mathscr{C}[x])\big| = 1$ and $\mathsf{ct}(\mathscr{C}[x]) \neq \emptyset$.*
(b) *The union of all compact components of X is either empty or not compact.*

Proof. Suppose that (a) fails. Then, by Corollary 6.12, (b) fails if and only if X is not orderable. Finally, observe that if (a) holds, then, by Proposition 6.10, X must be orderable. This completes the proof. □

Corollary 6.14. *Every semi-orderable locally compact totally disconnected space X is orderable. In particular, a locally compact totally disconnected paracompact space X is orderable if and only if it has a continuous weak selection.*

PROOF. If X is compact, then it is orderable. Suppose that X is not compact. Since X is totally disconnected, by Theorem 2.4, $\mathsf{ct}(\mathscr{C}[x]) = \emptyset$ for every $x \in X$. Hence, X is the union of all compact components of X and, by Corollary 6.12, it must be orderable. Now, Theorem 5.1 completes the second part of the statement. □

7. Topological well-suborderability and semi-orderability.

This last section of the paper goes further in the orderability-like properties of X but now implicated by continuous selections for $\mathscr{F}(X)$. To this end, let Y be an orderable space, and let $\preceq$ be a compatible order on Y. Recall that a subset $X \subset Y$ is called a *topologically well-ordered subspace* [**5**] if any nonempty relatively closed subset of X has a ($\preceq$)-minimal element. In this case, to avoid any source of ambiguity, let us agree to say that X is *topologically well-suborderable.* Thus, to reserve the term *topologically well-orderable* for an orderable space Y such that, with respect to a compatible order $\preceq$ on Y, each nonempty closed subset of Y has a ($\preceq$)-minimal element. It should be mentioned that there are orderable spaces which are neither topologically well-orderable nor topologically well-suborderable. For instance, the real line $\boldsymbol{R}$ is an example in this direction, see [**5**]. Here is another simple example of a space which is orderable, topologically well-suborderable, but is not topologically well-orderable.

EXAMPLE 7.1. Let $X = [0,1) \cup [2,3) \subset \boldsymbol{R}$. Then, X is a locally compact space which is orderable and topologically well-suborderable, but it fails to be topologically well-orderable.

PROOF. The space X is topologically well-suborderable with respect to the usual order on $\boldsymbol{R}$. It is also easy to see that X is orderable, use, for instance, Proposition 4.6. To see that X is not topologically well-orderable, take a compatible order $\preceq$ on X so that X is orderable with respect to it. Then, we have that $\xi_{\mathrm{end}}(X,\preceq) = 0$ or $\xi_{\mathrm{end}}(X,\preceq) = 2$. If $\xi_{\mathrm{end}}(X,\preceq) = 0$, then X has no ($\preceq$)-minimal element. Hence, it remains $\xi_{\mathrm{end}}(X,\preceq) = 2$. In this case, X has a partition consisting of nonempty clopen subsets $X_0, X_1 \subset X$ such that X_0 has no ($\preceq$)-maximal element and X_1 has no ($\preceq$)-minimal one. Hence, it fails to be topologically well-orderable. □

Turning to selections for $\mathscr{F}(X)$ and the role of topologically well-orderable spaces, let us explicitly recall the following partial case of [**14**, Lemma 7.5.1].

PROPOSITION 7.2 ([**14**]). *If X is a topologically well-suborderable space, then $\mathscr{F}(X)$ has a continuous selection.*

We are now ready to state also the main purpose of this section. Namely, in this section we will prove the following theorem.

Theorem 7.3. *For a locally compact paracompact space X, the following are equivalent*:

(a) *X is topologically well-suborderable.*
(b) *$\mathscr{F}(X)$ has a continuous selection.*
(c) *X has a continuous weak selection and $\mathsf{nct}(\mathscr{C}[x]) \neq \emptyset$ for every $x \in X$.*

The proof of Theorem 7.3 consists of a reduction to the same statement but now for the members of the family $\mathscr{O}(X)$ which was defined in Section 5. To prepare for this, we proceed with several statements about the members of this family.

Proposition 7.4. *For a locally compact connected space Z, the following are equivalent*:

(a) *Z is topologically well-orderable.*
(b) *$\mathscr{F}(Z)$ has a continuous selection.*
(c) *Z has a closed selection relation and $\mathsf{nct}(Z) \neq \emptyset$.*

Proof. The implication (a) $\Rightarrow$ (b) follows by Proposition 7.2. As for (b) $\Rightarrow$ (c), it follows by Theorem 2.3 and Proposition 2.6. To show finally that (c) $\Rightarrow$ (a), take a point $p \in \mathsf{nct}(Z)$. By Theorem 2.2 and Proposition 2.6, Z has a closed linear order $\preceq$ such that p is the ($\preceq$)-minimal element of Z. According to Proposition 2.7, Z is orderable with respect to $\preceq$. Take a nonempty closed subset $F \subset Z$, and a point $y \in F$. Consider the space $Y = [p, y]_\preceq$ which is locally compact, and, by Proposition 2.5, it is also connected. Hence, by Proposition 2.7, it is compact. Therefore, $F \cap [p, y]_\preceq$ has a ($\preceq$)-minimal element which is, in fact, the ($\preceq$)-minimal element of F because p is the first ($\preceq$)-element of Z. Hence, Z is topologically well-orderable. □

Lemma 7.5. *Let X be a locally compact space. Then, for a member $Z \in \mathscr{O}(X)$, with $Z \neq \emptyset$, the following are equivalent*:

(a) *Z is topologically well-orderable.*
(b) *$\mathscr{F}(Z)$ has a continuous selection.*
(c) *Z has a closed selection relation and $\mathsf{nct}(\mathscr{C}[z]) \neq \emptyset$ for every $z \in Z$.*

Proof. The implication (a) $\Rightarrow$ (b) follows by Proposition 7.2. The implications (b) $\Rightarrow$ (c) $\Rightarrow$ (a) follow by Theorem 1.1 if Z is compact, and by Proposition 7.4 if Z is connected. Suppose that Z is neither compact nor connected. In this case, by definition, there exists a point $z \in Z$ such that $\mathsf{nct}(\mathscr{C}[z]) \neq \emptyset$ and

$Z \setminus \mathsf{ct}(\mathscr{C}[z])$ is compact. Since $\mathscr{C}[z]$ cannot be compact, by Proposition 2.7, we have that $\left|\mathsf{nct}(\mathscr{C}[z])\right| = 1$. Thus, to finish the proof, it only remains to show that in this case Z is topologically well-orderable provided it has a closed selection relation. To this end, observe that $\mathsf{ct}(\mathscr{C}[z]) \neq \emptyset$ because Z is not compact while $Z \setminus \mathsf{ct}(\mathscr{C}[z]) \neq \emptyset$ is compact. Hence, by Lemma 3.3, Z is orderable by an order $\preceq$ such that $\mathscr{C}[z]$ has a $(\preceq)$-minimal element p and $x \prec y$ for every $x \in Z \setminus \mathscr{C}[z]$ and $y \in \mathscr{C}[z]$. Take an $F \in \mathscr{F}(Z)$. If $F \subset \mathscr{C}[z]$, then, just like in the proof of Proposition 7.4, take a point $y \in F$ and observe that $[p, y]_{\preceq}$ is compact. Hence, F has a $(\preceq)$-minimal element in $[p, y]_{\preceq}$ which is the $(\preceq)$-minimal element of F because $F \subset [p, \rightarrow)_{\preceq}$. If $F \cap (Z \setminus \mathsf{ct}(\mathscr{C}[z])) \neq \emptyset$, then $F \cap (Z \setminus \mathsf{ct}(\mathscr{C}[z]))$ has a $(\preceq)$-minimal element $x(F)$ in $Z \setminus \mathsf{ct}(\mathscr{C}[z])$. Since $x(F) \preceq y$ for every $y \in \mathscr{C}[z]$, $x(F)$ is the $(\preceq)$-minimal element of F. This completes the proof. □

Proof of Theorem 7.3. The implication (a) $\Rightarrow$ (b) is Proposition 7.2, while the implication (b) $\Rightarrow$ (c) follows by Corollary 5.3 and Lemma 7.5. Suppose finally that X is as in (c). We follow the proof of Theorem 5.1. Just like in that proof, X has a clopen partition $\mathscr{Q} \subset \mathscr{O}(X)$. By Lemma 7.5, each member of $\mathscr{O}(X)$ is topologically well-orderable, hence so is each $Q \in \mathscr{Q}$. Then, by [**1**, Proposition 1.4], $X = \biguplus \mathscr{Q}$ is topologically well-suborderable. □

According to Example 7.1, the topological well-suborderability in Theorem 7.3 cannot be replaced by the topological well-orderability. However, we have the following immediate consequence.

Corollary 7.6. *Let X be a topologically well-suborderable locally compact paracompact space. Then, X is orderable.*

Proof. By Theorem 7.3, $\mathsf{nct}(\mathscr{C}[x]) \neq \emptyset$ for every $x \in X$. Hence, by Proposition 6.11, X is not the topological sum of a compact orderable space and a nonempty anti-compact orderable one. Therefore, by Theorem 6.9, X must be orderable. □

We conclude this paper with the following interesting consequence.

Corollary 7.7. *Let X be a locally compact paracompact space which has a continuous selection for $\mathscr{F}(X)$. Then, X is orderable.*

Note that the converse of Corollary 7.7 is not true. As it was already mentioned, the real line $\boldsymbol{R}$ is not topologically well-suborderable, and there is no continuous selection for $\mathscr{F}(\boldsymbol{R})$, [**5**].

Addendum (August, 2007). At the time when this manuscript was in process to be accepted for publication, Michael Hrušák and Iván Martínez-Ruiz announced that they answered Question 1 in the negative by constructing a separable, first countable locally compact space which admits a continuous weak selection but is not weakly orderable. Their manuscript [**12**] is in preparation.

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