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A general approach to the Fekete-Szegö problem

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Abstract. In this article, we provide a new method solving the Fekete-Szegö problem for classes of close-to-convex functions defined in terms of subordination. As an example, we apply it to the class of strongly close-to-convex functions.

1. Introduction.

Fekete and Szegö [**5**] proved the striking result that the inequality

$$\left|a_3 - \mu a_2^2\right| \le 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right)$$

holds for any normalized univalent function $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ in the unit disk $\boldsymbol{D} = \{z \in \boldsymbol{C} : |z| < 1\}$ and for $0 \le \mu \le 1$ and that this inequality is sharp for each μ (see also [**3**]). The coefficient functional

$$\Lambda_\mu(f) = a_3 - \mu a_2^2 = \frac{1}{6}\left(f'''(0) - \frac{3\mu}{2}\{f''(0)\}^2\right) \tag{1.1}$$

on normalized analytic functions $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ in the unit disk is important in the sense that this can represent various geometric quantities as well as in the sense that this behaves well with respect to the rotation, namely, $\Lambda_\mu(R_\theta f) = e^{2i\theta}\Lambda_\mu(f)$ for $\theta \in \boldsymbol{R}$. Here $R_\theta f$ denotes the rotation of f by angle θ, more precisely, $R_\theta f(z) = e^{-i\theta} f(e^{i\theta} z)$.

In fact, other than the simplest case when $\Lambda_0(f) = a_3$, we have several important ones. For example, $\Lambda_1(f) = a_3 - a_2^2$ represents $S_f(0)/6$, where S_f denotes the Schwarzian derivative $(f''/f')' - (f''/f')^2/2$ of f. Moreover, the first two non-trivial coefficients of the n-th root transform $\{f(z^n)\}^{1/n} = z + c_{n+1} z^{n+1} + c_{2n+1} z^{2n+1} + \cdots$ of $f(z) = z + a_2 z^2 + \cdots$ are written by $c_{n+1} = a_2/n$ and $c_{2n+1} = (\Lambda_{(n-1)/2n} f)/n = a_3/n - (n-1)a_2^2/2n^2$.

Thus it is quite natural to ask about inequalities for Λ_μ corresponding to subclasses of normalized univalent functions in the unit disk. This is sometimes called the Fekete-

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Szegö problem. Actually, many authors have considered this problem for typical classes of univalent functions (see, for instance, [**1**], [**2**], [**7**], [**9**], [**10**], [**11**], [**12**], [**13**]).

We denote by $\mathscr{A}$ the set of all normalized analytic functions on the unit disk $\boldsymbol{D}$ and denote by $\mathscr{S}$ the subclass of $\mathscr{A}$ consisting of all univalent functions as usual. Let $\mathscr{M}$ be the class of analytic functions φ on the unit disk with $\varphi(0) = 1$ and let $\mathscr{N}$ be the subclass of $\mathscr{M}$ consisting of functions with positive real part. For $\varphi \in \mathscr{M}$, we will denote by $\mathscr{M}(\varphi)$ the subset defined by $\{h \in \mathscr{M} : h \prec \varphi\}$. Here and hereafter, we use the notation $f \prec g$, or, $f(z) \prec g(z)$ in $\boldsymbol{D}$ for analytic functions f and g on $\boldsymbol{D}$ to mean the subordination, namely, that there exists a holomorphic map ω of the unit disk $\boldsymbol{D}$ into itself with $\omega(0) = 0$ such that $f = g \circ \omega$. Note that if g is univalent, $f \prec g$ is equivalent to the condition that $f(0) = g(0)$ and $f(\boldsymbol{D}) \subset g(\boldsymbol{D})$.

Ma and Minda [**12**] gave a complete answer to the Fekete-Szegö problem for the classes $\mathscr{S}^*(\varphi)$ and $\mathscr{K}(\varphi)$ for $\varphi \in \mathscr{M}$ with some mild conditions. (Actually, it is enough to assume that φ is univalent and satisfies $\varphi(\bar{z}) = \overline{\varphi(z)}$ on $\boldsymbol{D}$ and $\varphi'(0) > 0$.) These classes are defined for $\varphi \in \mathscr{M}$ by

$$\mathscr{S}^*(\varphi) = \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \text{ in } \boldsymbol{D} \right\}, \text{ and}$$

$$\mathscr{K}(\varphi) = \left\{ f \in \mathscr{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \text{ in } \boldsymbol{D} \right\}.$$

Note that, with the special choice of $\varphi(z) = (1+z)/(1-z)$ the above classes consist of starlike and convex functions in the standard sense.

In our previous paper [**8**], we gave a partial answer to the problem for the classes

$$\mathscr{C}(\varphi, \psi) = \left\{ f \in \mathscr{A} : \text{there exists a } g \in \mathscr{K}(\varphi) \text{ such that } \frac{f'}{g'} \prec \psi \text{ in } \boldsymbol{D} \right\}.$$

These classes cover close-to-convex functions in some sense as we shall see later. The authors would like to take this opportunity to note that in Lemma 2.2 and in Theorem 3.1 of [**8**] some conditions for φ such as above was dropped, for instance, the conditions that φ is univalent and satisfies $\varphi(\bar{z}) = \overline{\varphi(z)}$ on $\boldsymbol{D}$ and $\varphi'(0) > 0$ should be assumed.

It is worth noting that $R_\theta f \in \mathscr{S}^*(\varphi)$ if and only if $f \in \mathscr{S}^*(\varphi)$ for a fixed real number θ. The same thing can be said for $\mathscr{K}(\varphi)$ and for $\mathscr{C}(\varphi, \psi)$ as well. In particular, these classes are rotation invariant.

The main aim of this paper is to provide a general method to compute the quantity $\sup\{|\Lambda_\mu(f)| : f \in \mathscr{C}(\varphi, \psi)\}$ in terms of φ, ψ and μ. Actually, we will see that this quantity is determined only by the first two non-trivial coefficients of φ and ψ and the parameter μ provided that $\varphi'(0) \neq 0$ and $\psi'(0) \neq 0$. Our method also enables us to find all the extremal functions for the functional $|\Lambda_\mu|$.

Our method developed below can easily be modified to apply to the classes $\mathscr{S}^*(\varphi)$ and $\mathscr{K}(\varphi)$ as well. We, however, will not state it separately because we have satisfactory results due to Ma and Minda [**12**] already.

As a typical example, we treat the class

$$\mathscr{SCC}(\alpha,\beta)=\bigcup_{-\pi\alpha/2<\gamma<\pi\alpha/2}\mathscr{SCC}_\gamma(\alpha,\beta)$$

of strongly close-to-convex functions of order (α,β) for $0<\alpha\le 1$ and $0<\beta\le 1$. Here, $f\in\mathscr{SCC}_\gamma(\alpha,\beta)$ if and only if f belongs to $\mathscr{A}$ and there exists a function $g\in\mathscr{A}$ with $|\arg(1+zg''(z)/g'(z))|<\pi\beta/2$ such that $|\arg(f'(z)/g'(z))-\gamma|<\pi\alpha/2$. Though the class $\mathscr{SCC}(\alpha,\beta)$ seems not to be written in the form $\mathscr{C}(\varphi,\psi)$, each $\mathscr{SCC}_\gamma(\alpha,\beta)$ can be described in such a form, see Section 4. Note that $\mathscr{SCC}(\alpha,1)$ is the class of strongly close-to-convex functions of order α (see [**6**, II, Definition 11.4]), and that $\mathscr{SCC}(1,1)$ is the class of close-to-convex functions in the standard sense. Note also that $\mathscr{SCC}_0(\alpha,\alpha)$ contains the class of strongly starlike functions of order α.

These classes have been considered by many authors, however, no complete answer to the Fekete-Szegö problem has been given in the literature so far. Actually, many authors treated only "normalized" classes contained in $\mathscr{SCC}_0(\alpha,\beta)$ for some α,β. Only the case when $\alpha=\beta=1$ was completely settled by Eenigenburg and Silvia [**4**] (see also [**9**]). For other cases, only partial results are known beyond the normalized classes ([**10**], [**1**]).

In order to exhibit effectiveness of our method, we will give another proof for a recent result of Darus and Thomas [**2**] on Fekete-Szegö inequalities for the class $\mathscr{SCC}_0(\alpha,\beta)$ for all $0<\alpha,\ \beta\le 1$ and, moreover, we will prove that those inequalities hold still true for the class $\mathscr{SCC}(\alpha,\beta)$ when $2/3\le\mu\le 1$.

2. General approach to the Fekete-Szegö problem.

Let $\varphi(z)=1+A_1z+A_2z^2+\cdots$, $\psi(z)=1+B_1z+B_2z^2+\cdots$ be analytic in the unit disk with $A_1B_1\neq 0$. For an arbitrary complex number μ, consider the functional Λ_μ on $\mathscr{A}$ given by (1.1). Since the class $\mathscr{C}(\varphi,\psi)$ is rotation invariant, the range set $\Delta(\varphi,\psi,\mu)=\{\Lambda_\mu(f):f\in\mathscr{C}(\varphi,\psi)\}$ of Λ_μ is rotation invariant, too. Therefore, the outer boundary of the set is a circle centered at the origin. (We will describe this set in a detailed way in Proposition 2.5 below.) Then the radius, denoted by $\rho(\varphi,\psi,\mu)$, of the circle is nothing but the quantity which we want to compute. Namely,

$$\rho(\varphi,\psi,\mu)=\sup\big\{|\Lambda_\mu(f)|:f\in\mathscr{C}(\varphi,\psi)\big\} \tag{2.1}$$

$$=\sup\big\{\operatorname{Re}\Lambda_\mu(f):f\in\mathscr{C}(\varphi,\psi)\big\}. \tag{2.2}$$

First of all, we remark that the quantity $\rho(\varphi,\psi,\mu)$ has an obvious convexity property in μ.

Lemma 2.1. *Let s and t be non-negative numbers with $s+t=1$. Then, the inequality*

$$\rho(\varphi,\psi,s\mu_0+t\mu_1)\le s\rho(\varphi,\psi,\mu_0)+t\rho(\varphi,\psi,\mu_1)$$

holds for $\mu_0,\mu_1\in\boldsymbol{C}$.

Proof. By definition,

$$\Lambda_{s\mu_0+t\mu_1}(f) = s\Lambda_{\mu_0}(f) + t\Lambda_{\mu_1}(f)$$

holds for each $f \in \mathscr{C}(\varphi, \psi)$. Therefore, the triangle inequality

$$|\Lambda_{s\mu_0+t\mu_1}(f)| \le s|\Lambda_{\mu_0}(f)| + t|\Lambda_{\mu_1}(f)|$$

yields the required inequality. □

For each function $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ in $\mathscr{C}(\varphi, \psi)$, by definition, we can take a function $g(z) = z + b_2 z^2 + b_3 z^3 + \cdots$ from $\mathscr{K}(\varphi)$ such that $h := f'/g' \prec \psi$. If we write $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$, by a simple calculation, we have the relation

$$\begin{aligned}\Lambda_\mu(f) = a_3 - \mu a_2^2 &= \left(b_3 - \mu b_2^2\right) + \frac{1}{3}\left(c_2 - \frac{3\mu}{4}c_1^2\right) + \left(\frac{2}{3} - \mu\right)b_2 c_1 \\ &= \Pi_\mu(b_2, b_3, c_1, c_2),\end{aligned}$$

where Π_μ is a polynomial given by the above formula. Note also that this quantity can be written in the form $\Lambda_\mu(g) + \frac{1}{3}\Lambda_{3\mu/4}(zh) + (\frac{2}{3} - \mu)b_2 c_1$. By the triangle inequality, we could get an estimate for $|\Lambda_\mu(f)|$ by using the knowledge of $|\Lambda_\mu|$ for $\mathscr{K}(\varphi)$, $|\Lambda_{3\mu/4}|$ for $zh(z)$ where $h \in \mathscr{M}(\psi)$ and $|b_2 c_1|$. This was the basic idea in our previous paper [**8**]. Unfortunately, this estimate is not always sharp. Therefore, in order to get a sharp result, we may not divide the terms in this way generally.

To obtain information about $\rho(\varphi, \psi, \mu)$, we now consider the coefficient regions

$$\begin{aligned}U_\varphi &= \left\{(u_2, u_3) \in \boldsymbol{C}^2 : \exists g \in \mathcal{K}(\varphi) \text{ s.t. } g(z) = z + u_2 z^2 + u_3 z^3 + O(z^4)\right\}, \text{ and} \\ W_\psi &= \left\{(w_1, w_2) \in \boldsymbol{C}^2 : \exists h \in \mathscr{M}(\psi) \text{ s.t. } h(z) = 1 + w_1 z + w_2 z^2 + O(z^3)\right\}.\end{aligned}$$

Then, by definition, $a \in \Delta(\varphi, \psi, \mu)$ if and only if $a = \Pi_\mu(u_2, u_3, w_1, w_2)$ for some $(u_2, u_3) \in U_\varphi$ and $(w_1, w_2) \in W_\psi$. In other words,

$$\Delta(\varphi, \psi, \mu) = \Pi_\mu(U_\varphi \times W_\psi).$$

To describe these sets, we introduce the universal set V defined by

$$V = \left\{(v_1, v_2) \in \boldsymbol{C}^2 : \exists \omega : \boldsymbol{D} \to \boldsymbol{D} \text{ holomorphic and satisfying } \omega(z) = v_1 z + v_2 z^2 + O(z^3)\right\}.$$

Then the following result can immediately be obtained by applying the Schwarz-Pick lemma to the function $\omega(z)/z$ (see, for instance, [**14**, p. 108]).

Lemma 2.2.

$$V = \left\{(v_1, v_2) \in \boldsymbol{C}^2 : |v_1|^2 + |v_2| \le 1\right\}.$$

Furthermore, $(v_1, v_2) \in \partial V = \{|v_1|^2 + |v_2| = 1\}$ *for an analytic function* $\omega(z) = v_1 z + v_2 z^2 + \cdots$ *on the unit disk with* $|\omega| < 1$ *if and only if either* $\omega(z) = v_1 z$ *(in case of* $|v_1| = 1$*) or* $\omega(z) = az(z + \bar{a}v_1)/(1 + a\overline{v_1}z)$ *(in case of* $|v_1| < 1$*), where* $a = v_2/(1 - |v_1|^2) \in \partial \boldsymbol{D}$*.*

In particular, V is a compact connected set and its interior is a Reinhardt domain in $\boldsymbol{C}^2$.

Let $g \in \mathscr{K}(\varphi)$. We now take a holomorphich map $\omega : \boldsymbol{D} \to \boldsymbol{D}$ with $\omega(0) = 0$ so that $1 + zg''/g' = \varphi \circ \omega$. Writing $\omega(z) = v_1 z + v_2 z^2 + \cdots$, we have the relation $(u_2, u_3) = F_\varphi(v_1, v_2)$, where $F_\varphi : \boldsymbol{C}^2 \to \boldsymbol{C}^2$ is the map determined by

$$F_\varphi(v_1, v_2) = \big(A_1 v_1/2, \big(A_1 v_2 + \big(A_2 + A_1^2\big)v_1^2\big)/6\big).$$

Since F_φ is an analytic automorphism, namely, a biholomorphic map of $\boldsymbol{C}^2$, we have the following.

Lemma 2.3. *$U_\varphi = F_\varphi(V)$. In particular, U_φ is a compact connected set and $\partial U_\varphi = F_\varphi(\partial V)$.*

Next, suppose that $h(z) = 1 + w_1 z + w_2 z^2 + \cdots = \psi(\omega(z))$ for some analytic function $\omega(z) = v_1 z + v_2 z^2 + \cdots$ on $\boldsymbol{D}$ with $|\omega| < 1$. Then, we have the relation $(w_1, w_2) = G_\psi(v_1, v_2)$, where

$$G_\psi(v_1, v_2) = \big(B_1 v_1, B_2 v_1^2 + B_1 v_2\big).$$

Since $G_\psi : \boldsymbol{C}^2 \to \boldsymbol{C}^2$ is also biholomorphic, we have

Lemma 2.4. *$W_\psi = G_\psi(V)$. In particular, W_ψ is a compact connected set and $\partial W_\psi = G_\psi(\partial V)$.*

Let (u_1, u_2) and (v_1, v_2) be points in ∂V. Then, by Lemma 2.2, functions ω_1, ω_2 are uniquely determined by the conditions $|\omega_j| < 1$, $\omega_1(z) = u_1 z + u_2 z^2 + \cdots$ and $\omega_2(z) = v_1 z + v_2 z^2 + \cdots$. Therefore, a function $f \in \mathscr{C}(\varphi, \psi)$ is determined by $f'/g' = \psi \circ \omega_2$, where $1 + zg''/g' = \varphi \circ \omega_1$. We denote by $E[u_1, u_2, v_1, v_2]$ or, more specifically, by $E_{\varphi,\psi}[u_1, u_2, v_1, v_2]$ the function f determined above.

By Lemmas 2.3 and 2.4, we have $\Delta(\varphi, \psi, \mu) = \Pi_\mu \circ (F_\varphi \times G_\psi)(V \times V)$. In particular, the quantity $\rho = \rho(\varphi, \psi, \mu)$ depends only on A_1, A_2, B_1, B_2 and μ. Therefore, we can express ρ also as a function H in these variables, namely, we can write

$$\rho(\varphi, \psi, \mu) = H(A_1, A_2, B_1, B_2, \mu), \tag{2.3}$$

where $\varphi(z) = 1 + A_1 z + A_2 z^2 + \cdots$ and $\psi(z) = 1 + B_1 z + B_2 z^2 + \cdots$. Also, we can now show the following.

Proposition 2.5. *Let $\varphi, \psi \in \mathscr{M}$ and suppose that $\varphi'(0) \neq 0$ and $\psi'(0) \neq 0$. Then the range set $\Delta(\varphi, \psi, \mu)$ of the functional Λ_μ on $\mathscr{C}(\varphi, \psi)$ is a closed disk centered at the origin.*

Proof. The relation $\Pi_\mu(0,0,0,0) = 0$ implies that $0 \in \Delta = \Delta(\varphi,\psi,\mu)$. Since $\Delta = \Pi_\mu(U_\varphi \times W_\psi)$ is closed and connected, there is a point $z \in \Delta$ with $|z| = r$ for each $r \in [0, \rho(\varphi,\psi,\mu)]$. As we have seen, the set Δ is rotation invariant, and therefore, Δ must be a closed disk $\{|z| \le \rho(\varphi,\psi,\mu)\}$. $\square$

A direct calculation shows

$$\begin{aligned}\Pi_\mu\big(F_\varphi(u_1,u_2), G_\psi(v_1,v_2)\big) &= \left(\left(\frac{1}{6} - \frac{\mu}{4}\right)A_1^2 + \frac{A_2}{6}\right)u_1^2 + \frac{A_1}{6}u_2 \\ &\quad + \left(\frac{B_2}{3} - \frac{B_1^2\mu}{4}\right)v_1^2 + \frac{B_1}{3}v_2 + \left(\frac{1}{3} - \frac{\mu}{2}\right)A_1B_1u_1v_1 \\ &= Au_2 + Bv_2 + Ku_1^2 + 2Mu_1v_1 + Lv_1^2.\end{aligned}$$

Here, for later convenience as well, we have set

$$\begin{aligned} A &= \frac{A_1}{6}, \\ B &= \frac{B_1}{3}, \\ K &= \frac{A_2}{6} + \frac{1}{4}\left(\frac{2}{3} - \mu\right)A_1^2, \\ L &= \frac{B_2}{3} - \frac{\mu}{4}B_1^2, \\ M &= \frac{1}{4}\left(\frac{2}{3} - \mu\right)A_1B_1. \end{aligned} \tag{2.4}$$

Based on the above facts, we can reduce our problem to an algebraic one.

Lemma 2.6. *The quantity $\rho(\varphi,\psi,\mu)$ is given by the function Ω defined by*

$$\Omega(A,B,K,L,M) = \max_{\substack{|u_1|^2+|u_2|\le 1\\ |v_1|^2+|v_2|\le 1}} \left|Au_2 + Bv_2 + Ku_1^2 + Lv_1^2 + 2Mu_1v_1\right| \tag{2.5}$$

$$= \max_{\substack{|u_1|^2+|u_2|\le 1\\ |v_1|^2+|v_2|\le 1}} \mathrm{Re}\left[Au_2 + Bv_2 + Ku_1^2 + Lv_1^2 + 2Mu_1v_1\right], \tag{2.6}$$

where A, B, K, L, M are related to φ and ψ by (2.4). *Furthermore, one can replace the range in the above maxima by $|u_1|^2 + |u_2| = 1$ and $|v_1|^2 + |v_2| = 1$.*

Actually, expressions (2.5) and (2.6) follow from (2.1) and (2.2), respectively, together with the above lemmas. The last assertion is an immediate consequence of the maximum principle. (This also follows from the fact that $\partial V \times \partial V$ is the Sirov boundary of the domain $\mathrm{Int}\, V \times \mathrm{Int}\, V$.)

The function H given by (2.3) is now described by the relation

$$H(A_1, A_2, B_1, B_2, \mu) = \Omega\left[\frac{A_1}{6}, \frac{B_1}{3}, \frac{A_2}{6} + \frac{1}{4}\left(\frac{2}{3} - \mu\right)A_1^2, \frac{B_2}{3} - \frac{\mu}{4}B_1^2, \frac{1}{4}\left(\frac{2}{3} - \mu\right)A_1B_1\right].$$

In the rest of the section, we give several simplifications of the expression of Ω and, equivalently, of H. The expressions (2.5) and (2.6) have their own advantage. We begin with examination of the first expression. Note that each variable of $(v_1, v_2) \in V$ can have arbitrary argument independently in view of the shape of V (Lemma 2.2). This means that, for $(u_1, u_2), (v_1, v_2) \in V$, in the chain of trivial inequalities

$$\begin{aligned}
&\left|Au_2 + Bv_2 + Ku_1^2 + 2Mu_1v_1 + Lv_1^2\right| \\
&\quad \le |A||u_2| + |B||v_2| + \left|Ku_1^2 + 2Mu_1v_1 + Lv_1^2\right| && (2.7)\\
&\quad \le |A|(1 - |u_1|^2) + |B|(1 - |v_1|^2) + \left|Ku_1^2 + 2Mu_1v_1 + Lv_1^2\right| && (2.8)
\end{aligned}$$

all equalities can hold at once. Therefore, we can deduce the following expression of Ω from (2.5):

$$\Omega(A, B, K, L, M) = \max_{|u|\le 1, |v|\le 1}\left[|A|(1 - |u|^2) + |B|(1 - |v|^2) + \left|Ku^2 + 2Muv + Lv^2\right|\right].$$

Now we introduce the auxiliary functions P and Q on $[0, +\infty) \times [0, +\infty)$ defined by

$$\begin{aligned}
P(s, t) &= Q(s, t) - |A|s^2 - |B|t^2, \\
Q(s, t) &= \max_{|u|=s, |v|=t}\left|Ku^2 + 2Muv + Lv^2\right| \\
&= \max_{\theta, \tau \in \boldsymbol{R}}\left|Ks^2e^{2i\theta} + 2Mste^{i(\theta+\tau)} + Lt^2e^{2i\tau}\right| \\
&= \max_{\theta \in \boldsymbol{R}}\left|Ks^2 + 2Mste^{i\theta} + Lt^2e^{2i\theta}\right|.
\end{aligned}$$

Then, we have the simple-looking expression

$$\Omega = |A| + |B| + \max_{0 \le s, t \le 1} P(s, t). \tag{2.9}$$

By definition, these functions are homogeneous of degree 2, precisely, $P(rs, rt) = r^2P(s, t)$ and $Q(rs, rt) = r^2Q(s, t)$ hold for each $r \ge 0$. Therefore, setting $F(r) = P(rs, rt)$, we observe that $F(r)$ is non-decreasing if $F(1) \ge 0$ and non-increasing if $F(1) \le 0$. In particular, we can see the relation

$$\max_{0 \le s, t \le 1} P(s, t) = \max\{\Phi_1, \Phi_2, 0\},$$

where Φ_1 and Φ_2 are the functions in A, B, K, L, M defined by

$$\Phi_1 = \max_{0 \le s \le 1} P(s, 1) \quad \text{and} \quad \Phi_2 = \max_{0 \le t \le 1} P(1, t). \tag{2.10}$$

Thus, we have reached the following result.

THEOREM 2.7. *The quantity* $\Omega = \Omega(A, B, K, L, M)$ *can be represented by*

$$\Omega = |A| + |B| + \max\{\Phi_1, \Phi_2, 0\},$$

where $\Phi_j = \Phi_j(A, B, K, L, M)$ *is defined by* (2.10) *for* $j = 1, 2$.

Unfortunately, it is not always easy to calculate $Q(s,t)$ and thus $P(s,t)$. If K, L and M are all real, however, we can do that as we shall see in the next section.

Now we make a brief discussion on extremal points. Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ be an extremal function in $\mathscr{C}(\varphi, \psi)$ for the functional $|\Lambda_\mu|$. Note here that the rotation $R_\theta f$ is an extremal one because $\Lambda_\mu(R_\theta f) = e^{2i\theta}\Lambda_\mu(f)$. Thus, we may regard $R_\theta f$, $\theta \in \boldsymbol{R}$, as a trivial one-parameter family of extremal functions.

By definition, there is a function $g \in \mathscr{K}(\varphi)$ such that $f'/g' \prec \psi$. We take holomorphic maps ω_1, $\omega_2 : \boldsymbol{D} \to \boldsymbol{D}$ with $\omega_1(0) = \omega_2(0) = 0$ so that $1 + zg''/g' = \varphi \circ \omega_1$ and that $f'/g' = \psi \circ \omega_2$. Let (u_1, u_2, v_1, v_2) be a corresponding point in $V \times V$, namely, $\omega_1(z) = u_1z + u_2z^2 + \cdots$ and $\omega_2(z) = v_1z + v_2z^2 + \cdots$. For this point, equalities must hold in (2.7) and (2.8) simultaneously. In particular, $(u_1, u_2, v_1, v_2) \in \partial V \times \partial V$. Now, by Lemma 2.2, the forms of ω_1, ω_2 are exactly determined by u_1, u_2, v_1, v_2. In this way, the extremal function f can be expressed in terms of $\varphi, \psi, u_1, u_2, v_1, v_2$ and will be denoted by $E_{\varphi,\psi}[u_1, u_2, v_1, v_2]$.

We further analyze several possible cases and explain how to determine u_1, u_2, v_1, v_2. Recall that we have set $s = |u_1|$ and $t = |v_1|$.

CASE 1: $\max\{\Phi_1, \Phi_2\} > 0$. Assume, for instance, $\Phi_1 \geq \Phi_2$. In this case, we have $\Omega = |A| + |B| + \Phi_1$. Then there exists an $s_0 \in [0, 1]$ such that $\Phi_1 = P(s_0, 1)$. As we shall see by example in the next section, such s_0 is not unique in general, however, $\Phi_1 > P(s, t)$ holds for any $0 \leq s \leq 1$ and for any $0 \leq t < 1$ by the homogeneity of P. Therefore, in this case, $|v_1| = 1$ and $v_2 = 0$ hold. Choose u_1 and v_1 with $|u_1| = s_0, |v_1| = 1$ so that $Q(s_0, 1) = |Ku_1^2 + 2Mu_1v_1 + Lv_1^2|$. Then choose u_2 with $|u_2| = 1 - s_0^2$ so that equality holds in (2.7), in other words, Au_2 and $Ku_1^2 + 2Mu_1v_1 + Lv_1^2$ have the same argument. The case when $\Phi_2 \geq \Phi_1$ can be treated smilarly.

CASE 2: $\max\{\Phi_1, \Phi_2\} = 0$. This is a degenerate case. Assume, for instance, $\Phi_1 = 0$. Then, there exists an $s_0 \in [0, 1]$ such that $P(s_0, 1) = 0$. Let (u_1, u_2, v_1, v_2) be the point in $\partial V \times \partial V$ determined in the same way as in Case 1.

In the present case, $P(s_0t, t) = 0$ holds for any $t \in [0, 1]$ by the homogeneity of $P(s, t)$. We choose a complex number b with $|b| = 1$ so that Bb and $Ku_1^2 + 2Mu_1v_1 + Lv_1^2$ have the same argument. For simplicity, we consider only the generic case when $s_0 < 1$. Choose a non-negative number $k = k(t)$ so that $|tu_1|^2 + |ku_2| = 1$, namely, $k = (1 - s_0^2t^2)/(1 - s_0^2)$. We then have a non-trivial one-parameter family of extremal functions $f_t = E_{\varphi,\psi}[tu_1, (1 - s_0^2t^2)u_2/(1 - s_0^2), tv_1, (1 - t^2)b]$ $(0 \leq t \leq 1)$.

CASE 3: $\max\{\Phi_1, \Phi_2\} < 0$. This is one of the simplest cases. The maximum in (2.5) is attained only in the case when $u_1 = v_1 = 0$. Therefore, by Lemma 2.2, ω_1 and ω_2 above must take the forms $\omega_1(z) = az^2$ and $\omega_2(z) = bz^2$, where a and b are unimodular constants chosen so that Aa and Bb have the same argument.

We end this section with examination of expression (2.6). Actually, in some cases, this method is more appropriate. As above, we easily have the expression

$$\Omega(A,B,K,L,M) = |A| + |B| + \max_{|u|\le 1, |v|\le 1} \operatorname{Re}\left[Ku^2 + 2Muv + Lv^2 - |A||u|^2 - |B||v|^2\right]. \tag{2.11}$$

Introducing real coordinates $u = x_1 + ix_2$ and $v = x_3 + ix_4$, we obtain the expression of Ω in terms of a real quadratic form in x_1, x_2, x_3, x_4.

Theorem 2.8. *The quantity Ω can be described in the form*

$$\Omega(A,B,K,L,M) = |A| + |B| + \max_{\substack{x_1^2+x_2^2\le 1\\ x_3^2+x_4^2\le 1}} \sum_{j,k=1}^{4} c_{jk}x_jx_k,$$

where c_{jk} are the entries of the real symmetric matrix

$$C = \begin{pmatrix} \operatorname{Re}K - |A| & -\operatorname{Im}K & \operatorname{Re}M & -\operatorname{Im}M \\ -\operatorname{Im}K & -\operatorname{Re}K - |A| & \operatorname{Im}M & -\operatorname{Re}M \\ \operatorname{Re}M & \operatorname{Im}M & \operatorname{Re}L - |B| & -\operatorname{Im}L \\ -\operatorname{Im}M & -\operatorname{Re}M & -\operatorname{Im}L & -\operatorname{Re}L - |B| \end{pmatrix}.$$

It seems quite difficult to treat the above matrix, however, when some of entries vanish, the situation becomes easier to analyze. For example, if the matrix C is non-positive, we know that $\Omega = |A| + |B|$. Sometimes it is easier to check this condition than to calculate Φ_j directly in the above if truly $\Omega = |A| + |B|$ holds.

3. The case when K, L, M are all real.

When K, L and M are all real, we can actually calculate the quantity $\Omega(A,B,K,L,M)$. For convenience, we define $\operatorname{Sign}[x]$ for $x \in \boldsymbol{R}$ by

$$\operatorname{Sign}[x] = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ \pm 1 & \text{if } x = 0. \end{cases}$$

For example, the assertion $K = \operatorname{Sign}[x]L$ means that $K = L$ or $K = -L$ when $x = 0$. We are now ready to state our main result in this section.

Theorem 3.1. *Let $A, B \in \boldsymbol{C}$ and $K, L, M \in \boldsymbol{R}$. If $KL \ge 0$, the quantity $\Omega(A,B,K,L,M)$ in* (2.5) *is given by*

$$\Omega(A,B,K,L,M)=\begin{cases} |A|+|B| & \textit{if}\ \ |A|+|B|\ge|K|+|L|\ \textit{and}\ D\ge 0\\[2mm] |A|+|L|-\dfrac{M^2}{|K|-|A|} & \textit{if}\ \ |A|>|M|+|K|\ \textit{and}\ D<0\\[2mm] |B|+|K|-\dfrac{M^2}{|L|-|B|} & \textit{if}\ \ |B|>|M|+|L|\ \textit{and}\ D<0\\[2mm] |K|+2|M|+|L| & \textit{otherwise,}\end{cases}$$

where $D=(|K|-|A|)(|L|-|B|)-M^2$. *If* $KL<0$, *the quantity* $\Omega(A,B,K,L,M)$ *is given by*

$$\Omega(A,B,K,L,M)=|A|+|B|+\max\{0,R\},$$

where R *is defined by the table*

	(a)	(b)	(c)
(1)	0	0	$\lvert L\rvert-\lvert B\rvert+\dfrac{M^2}{\lvert A\rvert+\lvert K\rvert}$
(2)	0	(no possibility)	$-\lvert L\rvert-\lvert B\rvert+\dfrac{M^2}{\lvert A\rvert-\lvert K\rvert}$
(3)	$\lvert K\rvert-\lvert A\rvert+\dfrac{M^2}{\lvert B\rvert+\lvert L\rvert}$	$-\lvert K\rvert-\lvert A\rvert+\dfrac{M^2}{\lvert B\rvert-\lvert L\rvert}$	$S-\lvert A\rvert-\lvert B\rvert$

according to the combination of the cases

(1) $|A|\ge\max\left\{|K|\sqrt{1-\frac{M^2}{KL}},|M|-|K|\right\}$,

(2) $|K|+|M|\le|A|<|K|\sqrt{1-\frac{M^2}{KL}}$,

(3) *otherwise,*

and the cases

(a) $|B|\ge\max\left\{|L|\sqrt{1-\frac{M^2}{KL}},|M|-|L|\right\}$,

(b) $|L|+|M|\le|B|<|L|\sqrt{1-\frac{M^2}{KL}}$,

(c) *otherwise.*

Here, S *is defined by*

$$S=\begin{cases} -|K|+2|M|+|L| & \textit{if}\ \ |MK|-|ML|\le-2|KL|\\[2mm] |K-L|\sqrt{1-\dfrac{M^2}{KL}} & \textit{if}\ \ -2|KL|\le|MK|-|ML|\le 2|KL|\\[2mm] |K|+2|M|-|L| & \textit{if}\ \ 2|KL|\le|MK|-|ML|.\end{cases}$$

In the rest of the section, we prove the above theorem. The information on extremal functions can be obtained from Proposition 3.4 when $KL \geq 0$ and Proposition 3.7 when $KL < 0$. We start with the investigation of the preparatory quantity

$$q(a,b,c,\theta) = \left|a + 2be^{i\theta} + ce^{2i\theta}\right|$$

for $a,b,c,\theta \in \boldsymbol{R}$. First we need the following lemma.

LEMMA 3.2. *When $ac \geq 0$, one has*

$$q(a,b,c,\theta) \leq |a| + 2|b| + |c|,$$

where equality holds when $e^{i\theta} = \mathrm{Sign}(b(a+c))$.

Note that a and c have the same signature if $ac \geq 0$. In the case when $ac < 0$ the situation becomes a bit complicated.

LEMMA 3.3. *When $ac < 0$, letting $\xi = -b(a+c)/2ac$, one has*

$$q(a,b,c,\theta) \leq \begin{cases} |-a+2b-c| = \mathrm{Sign}[b](-a+2b-c) & \text{if } \xi \leq -1 \\ \sqrt{1 - \dfrac{b^2}{ac}}\,|a-c| & \text{if } -1 \leq \xi \leq 1 \\ |a+2b+c| = \mathrm{Sign}[b](a+2b+c) & \text{if } 1 \leq \xi, \end{cases}$$

where equality holds when $\cos\theta = -1$, $\cos\theta = \xi$ and $\cos\theta = 1$, respectively.

PROOF. We set $x = \cos\theta$. Then

$$\begin{aligned} q(a,b,c,\theta)^2 &= 4acx^2 + 4b(a+c)x + a^2 + 4b^2 + c^2 - 2ac \\ &= 4ac(x-\xi)^2 + \left(\frac{1-b^2}{ac}\right)(a-c)^2. \end{aligned}$$

By analyzing the behaviour of this quadratic polynomial in x, we obtain the desired result. □

We now return to our problem. Noting the relation $Q(s,t) = \max_{\theta \in \boldsymbol{R}} q(Ks^2, Mst, Lt^2, \theta)$, we can explicitly calculate $Q(s,t)$ by Lemmas 3.2 and 3.3.

First, we treat the easier case when $KL \geq 0$. In this case, $P(s,t) = (|K| - |A|)s^2 + 2|M|st + (|L| - |B|)t^2$. We denote by D the descriminant of this quadratic form, precisely, $D = (|K| - |A|)(|L| - |B|) - M^2$. Then the next statement is easily verified.

PROPOSITION 3.4. *Assume that $KL \geq 0$. Then*

$$\Omega(A,B,K,L,M)-|A|-|B|=\max_{0\le s,t\le 1}P(s,t)$$

$$=\begin{cases} P(0,0)=0 & \textit{if } |K|+|L|\le |A|+|B| \textit{ and } D\ge 0\\ P\left(\dfrac{|M|}{|A|-|K|},1\right)=|L|-|B|-\dfrac{M^2}{|K|-|A|} & \textit{if } |M|<|A|-|K| \textit{ and } D<0\\ P\left(1,\dfrac{|M|}{|B|-|L|}\right)=|K|-|A|-\dfrac{M^2}{|L|-|B|} & \textit{if } |M|<|B|-|L| \textit{ and } D<0\\ P(1,1)=|K|+2|M|+|L|-|A|-|B| & \textit{otherwise.}\end{cases}$$

In each case, the maximum point of $P(s,t)$ is unique in $(s,t)\in[0,1]\times[0,1]$ except for the following cases:

(a) *When $D=0$ and $|K|+|L|<|A|+|B|$, one has $P(s,t)=0$ if and only if $s(|K|-|A|)+t|M|=0$ and $s|M|+t(|L|-|B|)=0$.*
(b) *When $M=0, |K|=|A|$ and $|L|=|B|$, one has $P(s,t)=0$ for all s,t.*

Next, we consider the case when $KL<0$ and $M\ne 0$. In what follows, the terms “increasing” and “decreasing” will be used in the strict sense.

Let r_0 and r_1 be the positive numbers determined by the relations

$$\left|\frac{M}{2KL}\right|\left(|K|r_0-\frac{|L|}{r_0}\right)=-1 \quad\text{and}\quad \left|\frac{M}{2KL}\right|\left(|K|r_1-\frac{|L|}{r_1}\right)=1.$$

Since the function $|K|r-|L|/r$ is increasing in $r>0$, the inequality $r_0<r_1$ holds. We set $I_1=[0,r_0]$, $I_2=[r_0,r_1]$, $I_3=[r_1,+\infty]$. Then we can show the following.

Lemma 3.5. *Let $K,L,M\in\boldsymbol{R}$ with $KL<0$ and $M\ne 0$. Then*

$$Q(s,t)=\begin{cases} -|K|s^2+2|M|st+|L|t^2 & \textit{if } s/t\in I_1=[0,r_0]\\ \sqrt{1-\dfrac{M^2}{KL}}\,|Ks^2-Lt^2| & \textit{if } s/t\in I_2=[r_0,r_1]\\ |K|s^2+2|M|st-|L|t^2 & \textit{if } s/t\in I_3=[r_1,\infty].\end{cases}$$

Proof. Let $\varepsilon=\mathrm{Sign}[KM]$. We now apply Lemma 3.3 to the case when $a=Ks^2$, $b=Mst$, $c=Lt^2$. Then we have

$$\xi=\frac{-b(a+c)}{2ac}=-\frac{M}{2KL}\left(K\frac{s}{t}+L\frac{t}{s}\right)=\varepsilon\left|\frac{M}{2KL}\right|\left(|K|\frac{s}{t}-|L|\frac{t}{s}\right).$$

When $\varepsilon=+1$, the condition $\xi\le -1$ holds precisely if $s/t\le r_0$, i.e., $s/t\in I_1$. In this case, by Lemma 3.3, we have

$$Q(s,t)=\mathrm{Sign}[M](-Ks^2+2Mst-Lt^2)=-|K|s^2+2|M|st+|L|t^2.$$

When $\varepsilon = -1$, the condition $\xi \geq 1$ holds precisely if $s/t \leq r_0$, i.e., $s/t \in I_1$. In this case, again by Lemma 3.3, we have

$$Q(s,t) = \mathrm{Sign}[M](Ks^2 + 2Mst + Lt^2) = -|K|s^2 + 2|M|st + |L|t^2.$$

Therefore, we show the assertion in the case when $s/t \in I_1$. The other two cases can be dealt with similarly. $\square$

We need later the following properties of r_0 and r_1.

Lemma 3.6. *Let $K, L, M \in \boldsymbol{R}$ with $KL < 0$ and $M \neq 0$ and consider the following conditions*:

(a) $r_0 \leq |M|/(|A| + |K|)$,
(b) $r_1 \leq |M|/(|A| - |K|)$, *and*
(c) $|A| \leq |K|\sqrt{1 - \frac{M^2}{KL}}$.

Then (a) *and* (c) *are equivalent. Furthermore, when $|A| > |K|$,* (b) *and* (c) *are equivalent. Also,*

$$r_0 \leq 1 \;\Leftrightarrow\; -2|KL| \leq |MK| - |ML| \quad \textit{and} \quad r_1 \leq 1 \;\Leftrightarrow\; 2|KL| \leq |MK| - |ML|.$$

Proof. We observe

$$\begin{aligned}
r_0 &\leq \frac{|M|}{|A| + |K|} \\
&\Leftrightarrow \; -1 \leq \left|\frac{M}{2KL}\right|\left(|K|\frac{|M|}{|A| + |K|} - |L|\frac{|A| + |K|}{|M|}\right) \\
&\Leftrightarrow \; -2|KL| \leq \frac{|K|M^2}{|A| + |K|} - |L|(|A| + |K|) \\
&\Leftrightarrow \; |L|(|A| - |K|) \leq \frac{|K|M^2}{|A| + |K|} \\
&\Leftrightarrow \; A^2 - K^2 \leq \frac{|K|M^2}{|L|} = -\frac{KM^2}{L} \\
&\Leftrightarrow \; A^2 \leq K^2\left(1 - \frac{M^2}{KL}\right).
\end{aligned}$$

Next, $r_0 \leq 1$ if and only if $-1 \leq \left|\frac{M}{2KL}\right|\bigl(|K| - |L|\bigr)$, which is equivalent to $-2|KL| \leq |MK| - |ML|$. The other cases can be treated in the similar way. $\square$

Now we are ready to show the following result.

Proposition 3.7. *Let $A, B \in \boldsymbol{C}$ and $K, L, M \in \boldsymbol{R}$ with $AB \neq 0$ and $KL < 0$. Then,*

$$\Phi_1 = \begin{cases} P\Big(\dfrac{|M|}{|A|+|K|}, 1\Big) = \dfrac{M^2}{|A|+|K|} + |L| - |B| & \\ \qquad\qquad \textit{if } \ |A| \ge \max\Big\{|K|\sqrt{1-\dfrac{M^2}{KL}}, |M|-|K|\Big\} & \\ P\Big(\dfrac{|M|}{|A|-|K|}, 1\Big) = \dfrac{M^2}{|A|-|K|} - |L| - |B| & \\ \qquad\qquad \textit{if } \ |K|+|M| \le |A| < |K|\sqrt{1-\dfrac{M^2}{KL}} & \\ P(1,1) \qquad\qquad \textit{otherwise.} & \end{cases}$$

In each case, the maximum point of $P(s,1)$ is unique in $s \in [0,1]$ except for the following two cases:

(a) *When $M = 0$ and $|A| = |K|$, the function $P(s,1)$ is constant in $0 \le s \le 1$.*
(b) *When $M \ne 0$, $|A| = |K|\sqrt{1-\frac{M^2}{KL}}$ and $-2|KL| < |MK| - |ML|$, the maximum of $P(s,1)$ in $s \in [0,1]$ is attained at each point in the interval $I_2 \cap [0,1]$ which has positive length.*

Proof. We first consider the case when $KM > 0$. When $s \in I_1 \cap [0,1]$, by Lemma 3.5, we have $P(s,1) = (-|K|s^2 + 2|M|s + |L|) - |A|s^2 - |B| = -(|A|+|K|)s^2 + 2|M|s + |L| - |B|$. The axis of symmetry of the graph of this quadratic polynomial in s is $s = |M|/(|A|+|K|)$. By Lemma 3.6, we see that $|M|/(|A|+|K|) \in I_1 \cap [0,1]$ if and only if $|A| \ge \max\{|K|\sqrt{1-\frac{M^2}{KL}}, |M|-|K|\}$. Hence, we can see the following.

Claim 1.

1. *If $|A| \ge \max\{|K|\sqrt{1-\frac{M^2}{KL}}, |M|-|K|\}$, then $s = M/(|A|+|K|)$ is the unique maximum point of $P(s,1)$ in $I_1 \cap [0,1]$, thus $P(s,1) \le P(M/(|A|+|K|),1)$ there.*
2. *Otherwise, $P(s,1)$ is increasing in $I_1 \cap [0,1]$.*

When $s \in I_2 \cap [0,1]$, we have $P(s,1) = \sqrt{1-\frac{M^2}{KL}}(|K|s^2 + |L|) - |A|s^2 - |B|$. Then it is easy to see the following

Claim 2.

1. *If $|K|\sqrt{1-\frac{M^2}{KL}} > |A|$, the function $P(s,1)$ is increasing in $I_2 \cap [0,1]$.*
2. *If $|K|\sqrt{1-\frac{M^2}{KL}} < |A|$, the function $P(s,1)$ is decreasing in $I_2 \cap [0,1]$.*
3. *If $|K|\sqrt{1-\frac{M^2}{KL}} = |A|$, the function $P(s,1)$ is constant in $I_2 \cap [0,1]$.*

Here, we note that the interval $I_2 \cap [0,1]$ has positive length if and only if $r_0 < 1$. In view of Lemma 3.6, the last condition is equivalent to the inequality $-2|KL| < |MK| - |ML|$.

When $s \in I_3 \cap [0,1]$, we have $P(s,1) = (|K|s^2 + 2|M|s - |L|) - |A|s^2 - |B| = -(|A|-|K|)s^2 + 2|M|s - |L| - |B|$. If $|A| = |K|$, then $P(s,1)$ is increasing in s. We now assume that $|A| \ne |K|$. Then the axis of symmetry is $s = |M|/(|A|-|K|)$ and, by Lemma 3.6, this value is contained in $I_3 \cap [0,1]$ if and only if $|A| - |K| > 0, |M| \le |A| - |K|$ and if $|A| \geq |K|\sqrt{1-\frac{M^2}{KL}}(> |K|)$. Thus, we obtain

Claim 3.

1. *If $|A| \leq |K| + |M|$, the function $P(s, 1)$ is increasing in $I_3 \cap [0, 1]$.*
2. *If $|A| \geq |K|\sqrt{1-\frac{M^2}{KL}}$, then the function $P(s, 1)$ is decreasing in $I_3 \cap [0, 1]$.*
3. *If $|K|+|M| < |A| < |K|\sqrt{1-\frac{M^2}{KL}}$, then $s = M/(|A|-|K|)$ is the unique maximum point of $P(s, 1)$ in $I_3 \cap [0, 1]$, thus $P(s, 1) \leq P(M/(|A| - |K|), 1)$ there.*

Summarizing the above three claims, we obtain the statement in our proposition including the discussion on the uniqueness of maximum points.

When $M = 0$, we have $P(s, 1) = (K - |A|)s^2$+const., so the desired conclusion can be directly deduced. □

By interchanging the roles of K, L and A, B, respectively, we can deduce the corresponding result for Φ_2 from the above proposition. Concretely, we have

$$\Phi_2 = \begin{cases} P\left(1, \dfrac{|M|}{|B|+|L|}\right) = \dfrac{M^2}{|B|+|L|} + |K| - |A| \\ \qquad\qquad\qquad\qquad \text{if } |B| \geq \max\left\{|L|\sqrt{1-\dfrac{M^2}{KL}}, |M| - |L|\right\} \\ P\left(1, \dfrac{|M|}{|B|-|L|}\right) = \dfrac{M^2}{|B|-|L|} - |K| - |A| \\ \qquad\qquad\qquad\qquad \text{if } |L| + |M| \leq |B| < |L|\sqrt{1-\dfrac{M^2}{KL}} \\ P(1, 1) \qquad\qquad\qquad \text{otherwise.} \end{cases}$$

Proof of Theorem 3.1. The first part is an immediate consequence of Proposition 3.4. We show now the second part of the theorem. Assume that $KL < 0$. By Theorem 2.8, we see that the quantity $\tilde{R} = \max\{\Phi_1, \Phi_2\}$ satisfies $\Omega = |A|+|B|+\max\{0, \tilde{R}\}$. Therefore, it is enough to see that $\max\{0, \tilde{R}\} = \max\{0, R\}$, where R is the quantity given in the theorem. Below, for instance, by case (1a), we mean the case when conditions (1) and (a) in the theorem both hold.

Case (1a). We show that $\tilde{R} \leq 0$, and hence, $\max\{0, \tilde{R}\} = \max\{0, R\}$. By condition (a), we have $|B| \geq |L|\sqrt{1 - M^2/KL}$. By Proposition 3.7,

$$\begin{aligned} \Phi_1 &= \frac{M^2}{|A|+|K|} + |L| - |B| \leq \frac{M^2}{|A|+|K|} + |L| - |L|\sqrt{1-\frac{M^2}{KL}} \\ &= \frac{M^2}{|A|+|K|} - \frac{M^2}{|K|\big(1+\sqrt{1-M^2/KL}\big)} = \frac{M^2\big(|K|\sqrt{1-M^2/KL} - |A|\big)}{(|A|+|K|)|K|\big(1+\sqrt{1-M^2/KL}\big)}. \end{aligned}$$

Now condition (1) implies the inequality $\Phi_1 \leq 0$. In the same way, one can show that $\Phi_2 \leq 0$.

Cases (2a) and (1b). First consider case (2a). Note that $(|A|-|K|)(|L|+|B|) \geq M^2$ holds. Then, by Proposition 3.7, we see that

$$\Phi_1 = \frac{M^2}{|A|-|K|} - |L| - |B| = \frac{M^2 - (|A|-|K|)(|L|+|B|)}{|A|-|K|} \leq 0$$

and

$$\Phi_2 = \frac{M^2}{|B|+|L|} + |K| - |A| = \frac{M^2 - (|A|-|K|)(|L|+|B|)}{|B|+|L|} \leq 0.$$

Therefore, we conclude that $\tilde{R} \leq 0$. Case (1b) can be treated similarly.

Case (2b). This case never occurs. To show this, suppose that conditions (2) and (b) both hold. Set $a = |A|, b = |B|, k = |K|, l = |L|$ and $m = |M|$. Then $k + m \leq a < k\sqrt{1 + m^2/kl}$ and $l + m \leq b < l\sqrt{1 + m^2/kl}$. In particular, $m \neq 0$. By squaring $k+m < k\sqrt{1+m^2/kl}$, we have $2km+m^2 < km^2/l$, which is equivalent to $2kl < (k-l)m$. By using the other inequality, we also have $2kl < (l-k)m$. This is a contradiction.

The other cases. By Proposition 3.7, we obtain the required expression for R up to the relation $S = Q(1,1)$ in case (3c). When $M = 0$, this relation is verified in a straightforward way. We now suppose that $M \neq 0$. In view of Lemma 3.5, the quantity $Q(1,1)$ can be computed according to the interval I_j to which 1 belongs. For instance, $1 \in I_1 = (0, r_0]$ if and only if $1 \leq r_0$, which is equivalent to the condition $|MK| - |ML| \leq -2|KL|$ by Lemma 3.6. In this way, the relation $S = Q(1,1)$ is deduced. □

4. Strongly close-to-convex functions.

In Section 1, we introduced the class $\mathscr{SCC}(\alpha, \beta)$ of strongly close-to-convex functions of order (α, β). This class itself is out of our category of the classes $\mathscr{C}(\varphi, \psi)$, however, each subclass $\mathscr{SCC}_\gamma(\alpha, \beta)$ can be described in this form. Therefore, our method is applicable.

Indeed, we define the univalent function $\varphi_{\alpha,\gamma}$ on $\boldsymbol{D}$ for $0 < \alpha \leq 1$ and $-\pi\alpha/2 < \gamma < \pi\alpha/2$ by

$$\begin{aligned}\varphi_{\alpha,\gamma}(z) &= \left(\frac{1 + e^{i\gamma/\alpha}z}{1 - e^{-i\gamma/\alpha}z}\right)^\alpha \\ &= 1 + \left(2\alpha\cos\frac{\gamma}{\alpha}\right)z + 2\alpha\cos\frac{\gamma}{\alpha}\left(\alpha\cos\frac{\gamma}{\alpha} - i\sin\frac{\gamma}{\alpha}\right)z^2 + \cdots\end{aligned}$$

and set $\varphi_\alpha = \varphi_{\alpha,0}$ for simplicity. Then the function $\varphi_{\alpha,\gamma}$ maps the unit disk conformally onto the sector $\{z \in \boldsymbol{C}^* : |\arg z - \gamma| < \pi\alpha/2\}$. Note also that $\varphi_{\alpha,\gamma}(0) = 1$ and $\varphi'_{\alpha,\gamma}(0) > 0$. Therefore, by definition, we now see $\mathscr{SCC}_\gamma(\alpha, \beta) = \mathscr{C}(\varphi_\beta, \varphi_{\alpha,\gamma})$.

In the following, we concentrate on the case when $\varphi = \varphi_\beta$ and $\psi = \varphi_{\alpha,\gamma}$. According to (2.4), set

$$\begin{aligned}
A &= \frac{\beta}{3}, \\
B_\gamma &= \frac{2\alpha}{3}\cos\frac{\gamma}{\alpha}, \\
K &= \beta^2(1-\mu), \\
L_\gamma &= \frac{2\alpha}{3}\cos\frac{\gamma}{\alpha}\left(\left(1-\frac{3\mu}{2}\right)\alpha\cos\frac{\gamma}{\alpha} - i\sin\frac{\gamma}{\alpha}\right), \\
M_\gamma &= \left(\frac{2}{3}-\mu\right)\alpha\beta\cos\frac{\gamma}{\alpha}.
\end{aligned} \tag{4.1}$$

Especially, if $\gamma = 0$, we have

$$A = \frac{\beta}{3},\ B_0 = \frac{2\alpha}{3},\ K = \beta^2(1-\mu),\ L_0 = \alpha^2\left(\frac{2}{3}-\mu\right),\ M_0 = \alpha\beta\left(\frac{2}{3}-\mu\right).$$

Therefore, Theorem 3.1 can be used to deduce the following result due to Darus and Thomas [**2**].

THEOREM 4.1. *Let $0 < \alpha, \beta \le 1$ and $\mu \in \boldsymbol{R}$. Then, every function $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ in the class $\mathscr{SCC}_0(\alpha,\beta)$ satisfies*

$$|a_3 - \mu a_2^2| \le \begin{cases} (\alpha+\beta)^2\left(\frac{2}{3}-\mu\right) + \frac{\beta^2}{3} & \text{if } \mu \le \mu_1 \\ \frac{1}{3\alpha}\left[2\alpha^2 + \alpha\beta^2 - 2\beta^2 + \frac{4\beta^2}{2-2\alpha+3\alpha\mu}\right] & \text{if } \mu_1 \le \mu \le \mu_2 \\ \frac{2\alpha+\beta}{3} & \text{if } \mu_2 \le \mu \le \mu_2' \\ \frac{1}{3\alpha}\left[2\alpha^2 - \alpha\beta^2 - 2\beta^2 + \frac{4\beta^2}{2+2\alpha-3\alpha\mu}\right] & \text{if } \mu_2' \le \mu \le \mu_1' \\ (\alpha+\beta)^2\left(\mu-\frac{2}{3}\right) - \frac{\beta^2}{3} & \text{if } \mu_1' \le \mu, \end{cases}$$

where $\mu_1 = \frac{2}{3}\left(1-\frac{1}{\alpha+\beta}\right)$, $\mu_2 = \frac{2}{3}\left(1-\frac{1-\beta}{\alpha+2\beta-\alpha\beta}\right)$, $\mu_2' = \frac{2}{3}\left(1+\frac{1+\beta}{\alpha+2\beta+\alpha\beta}\right)$, $\mu_1' = \frac{2}{3}\left(1+\frac{1}{\alpha+\beta}\right)$. Furthermore, the above inequality is sharp in the sense that for each μ there is a function in $\mathscr{SCC}_0(\alpha,\beta)$ for which equality holds.

Note that the inequalities $\mu_1 < \mu_2 \le \frac{2}{3} < 1 \le \mu_2' \le \mu_1'$ hold always, and that $\mu_2 = \frac{2}{3}$ and $\mu_2' = \mu_1'$ if $\beta = 1$.

PROOF. For simplicity, we write $B = B_0$, $L = L_0$ and $M = M_0$ in the proof. Set $D = (|K| - A)(|L| - B) - M^2$ and set $\rho = \rho(\varphi_\beta, \varphi_\alpha, \mu)$. First noting that $KL \ge 0$ if and only if either $\mu \le 2/3$ or $\mu \ge 1$, we divide the proof into three cases according to the value of μ.

Case 1: $\mu \leq 2/3$. In this case, $K \geq 0$, $L \geq 0$ and $M \geq 0$. A simple computation shows that $D < 0$ if and only if $\mu < \mu_2$. For a while, we consider this case. Also, $|M| + |K| < |A|$ is equivalent to the condition $\mu > \nu_1 := (2 - (1-\beta)/(\alpha+\beta))/3$. However, since $\nu_1 - \mu_2 = \alpha(1-\beta^2)/(\alpha+\beta)(\alpha+2\beta-\alpha\beta) \geq 0$, this case never occurs. On the other hand, $|M| + |L| < |B|$ is equivalent to the condition $\mu > \mu_1$. In this case, $|M|/(|B| - |L|) = \beta(2-3\mu)/(2-\alpha(2-3\mu)) \in (0,1)$, and Theorem 3.1 yields $\rho = K + B - M^2/(L-B) = (2\alpha^2 + \alpha\beta^2 - 2\beta^2 + 4\beta^2/(2-\alpha(2-3\mu)))/3\alpha$.

The other case, namely, when $\mu \leq \mu_1$, by Theorem 3.1, we see that $\rho = K + 2M + L = (\alpha+\beta)^2(3/2 - \mu) + \beta^2/3$.

Next, we consider the case $D \geq 0$, in other words, $\mu_2 \leq \mu \leq 2/3$. Then, $A + B \geq |K| + |L|$ if and only if $\mu \geq \nu_2 := (2 - (2\alpha + \beta(1-\beta))/(\alpha^2+\beta^2))/3$. Since $\nu_2 < \mu_2$ is true, the above condition holds. Hence, by Theorem 3.1, we obtain $\rho = A + B = (2\alpha+\beta)/3$.

Case 2: $\mu \geq 1$. Then, $K \leq 0, L \leq 0$ and $M \leq 0$. Note first that $D < 0$ if and only if $\mu > \mu_2'$. Then, $|M| + |K| < |A|$ is equivalent to the condition $\mu < (2 + (1+\beta)/(\alpha+\beta))/3 (< \mu_2')$. Therefore, this case never happens. On the other hand, $|M| + |L| < |B|$ is equivalent to the condition $\mu < \mu_1'$ and in this case $-M/(B+L) = \beta(3\mu - 2)/(2 - \alpha(3\mu-2)) \in (0,1)$. Now Theorem 3.1 implies $\rho = B - K + M^2/(B+L) = (2\alpha - \alpha\beta^2 - 2\beta^2 + 4\beta^2/(2-\alpha(3\mu-2)))/3\alpha$ when $\mu_2' < \mu < \mu_1'$. When $\mu \geq \mu_1'$, we have $\rho = -K - 2M - L = (\alpha+\beta)^2(\mu - 3/2) - \beta^2/3$.

Next, consider the case when $1 \leq \mu \leq \mu_2'$. Then, $|K| + |L| \leq |A| + |B|$ if and only if $\mu \leq (2 + (2\alpha + \beta + \beta^2)/(\alpha^2+\beta^2))/3 (> \mu_2')$. Therefore, by Theorem 3.1, we have $\rho = A + B = (2\alpha+\beta)/3$.

Case 3: $2/3 < \mu < 1$. Though the present case will be covered by Theorem 4.2, we give a proof as an application of Theorem 3.1. In this case, $K > 0, L < 0$ and $M < 0$. Since $|K|\sqrt{1 - M^2/KL} = \beta^2\sqrt{(1-\mu)/3} \leq \beta^2/3 \leq A$ and $|M| - |K| = \beta[(\alpha+\beta)\mu - 2\alpha/3 - \beta] \leq \alpha\beta/3 \leq A$, condition (1) is fulfilled. On the other hand, since

$$|B| - |L| - |M| = \alpha\left[\frac{2}{3} - (\alpha+\beta)\left(\mu - \frac{2}{3}\right)\right] \geq 0,$$

one of the conditions (a) and (b) is satisfied. Therefore, by Theorem 3.1, we obtain $\rho = A + B = (2\alpha+\beta)/3$. □

As an application of our results, we finally treat the case when K, L and M are not necessarily real numbers.

Theorem 4.2. *Let $2/3 \leq \mu \leq 1$ and $\alpha, \beta \in (0,1]$. For each function $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ in $\mathscr{SCC}(\alpha,\beta)$, then the inequality*

$$|a_3 - \mu a_2^2| \leq \frac{2\alpha+\beta}{3}$$

holds. Furthermore, each inequality is sharp in the sense that there is a function $f \in$

$\mathscr{SCC}_0(\alpha,\beta)$ for which equality holds.

This theorem means that the inequality in Theorem 4.1 is still valid for the full class $\mathscr{SCC}(\alpha,\beta)$ in the case when $2/3 \le \mu \le 1$. It is naturally expected that the same thing can be said to an arbitrary $\mu \in \boldsymbol{R}$. Indeed, the case when $\alpha = \beta = 1$ was confirmed by Eenigenburg and Silvia [**4**].

Proof. We use the notation given by (4.1) and set $\rho_\gamma(\mu) = \rho(\varphi_\beta, \varphi_{\alpha,\gamma}, \mu)$.

Case 1: $\mu = 2/3$. In this case, $K = \beta^2/3 > 0$ and $M_\gamma = 0$. Therefore,

$$Q(s,t) = \max_{\theta\in\boldsymbol{R}} \left|Ks^2 + L_\gamma t^2 e^{2i\theta}\right| = Ks^2 + |L_\gamma| t^2,$$

and thus, $P(s,t) = (K-A)s^2 + (|L_\gamma| - B_\gamma)t^2$. Since $K - A = -\beta(1-\beta)/3 \le 0$ and $|L_\gamma| - B_\gamma = -(2\alpha/3)\cos(\gamma/\alpha)(1-\sin(\gamma/\alpha)) \le 0$, we see that $P(s,t) \le P(0,0) = 0$. By (2.9), we obtain

$$\begin{aligned}\rho_\gamma\left(\frac{2}{3}\right) &= A + B_\gamma = \frac{2\alpha\cos(\gamma/\alpha) + \beta}{3} \\ &\le \frac{2\alpha + \beta}{3} = \rho_0\left(\frac{2}{3}\right).\end{aligned}$$

Case 2: $\mu = 1$. In this case, $K = 0$ and $M_\gamma < 0$. Therefore,

$$Q(s,t) = \max_{\theta\in\boldsymbol{R}} \left|2M_\gamma st e^{i\theta} + L_\gamma t^2 e^{2i\theta}\right| = -2M_\gamma st + |L_\gamma| t^2$$

and thus, $P(s,t) = -As^2 - 2M_\gamma st + (|L_\gamma| - B_\gamma)t^2$. We write $x = \cos(\gamma/\alpha)$ and see that $|L_\gamma| = (\alpha x/3)\sqrt{4-(4-\alpha^2)x^2} < 2\alpha x/3 = B_\gamma$. We now investigate the descriminant

$$D = A(B_\gamma - |L_\gamma|) - M_\gamma^2 = \frac{\alpha\beta x}{9}\left(2 - \sqrt{4-(4-\alpha^2)x^2} - \alpha\beta x\right).$$

Since

$$\begin{aligned}D \ge 0 &\Leftrightarrow\ (2-\alpha\beta x)^2 \ge 4 - (4-\alpha^2)x^2 \\ &\Leftrightarrow\ x \ge x_0 := \frac{\alpha\beta}{1 - \alpha^2(1-\beta^2)/4}.\end{aligned}$$

Note that $0 < x_0 \le 1$. When $x \ge x_0$, we obtain $P(s,t) \le P(0,0) = 0$. When $0 < x < x_0$, since the inequality $|M_\gamma| = \alpha\beta x/3 < \beta/3 = A - |K|$ holds, Proposition 3.4 implies that $P(s,t) \le P(|M_\gamma|/A, 1) = P(\alpha x, 1) = |L_\gamma| - B_\gamma + M_\gamma^2/A$. Summarizing the above, we obtain

$$\rho_\gamma(1) = \begin{cases} \dfrac{2\alpha x + \beta}{3} & \text{if } x \ge x_0 \\[2ex] \dfrac{2\alpha x\sqrt{1-(1-\alpha^2/4)x^2} + \alpha^2\beta x^2 + \beta}{3} & \text{if } x < x_0, \end{cases} \tag{4.2}$$

where $x = \cos(\gamma/\alpha)$. We now claim that $\rho_\gamma(1) \le \rho_0(1) = (2\alpha+\beta)/3$ holds for any γ with $x = \cos(\gamma/\alpha) < x_0$. The claim is equivalent to $2\alpha x\sqrt{1-(1-\alpha^2/4)x^2} \le 2\alpha - \alpha^2\beta x^2$. We compute

$$\begin{aligned}&(2-\alpha\beta x^2)^2 - \big(2x\sqrt{1-(1-\alpha^2/4)x^2}\big)^2 \\ &\quad = (4-\alpha^2+\alpha^2\beta^2)x^4 - 4(1+\alpha\beta)x^2 + 4 \\ &\quad = (4-\alpha^2+\alpha^2\beta^2)\left(x^2 - \frac{2(1+\alpha\beta)}{4-\alpha^2+\alpha^2\beta^2}\right)^2 + \frac{4(3-\alpha^2-2\alpha\beta)}{4-\alpha^2+\alpha^2\beta^2}.\end{aligned}$$

The last term is obviously non-negative, and thus, the claim has been confirmed.

Case 3: $2/3 < \mu < 1$. Choose positive numbers s and t so that $s + t = 1$ and $\mu = 2s/3 + t$. By Lemma 2.1 and the previous two cases, we conclude that

$$\rho_\gamma(\mu) = \rho_\gamma\left(\frac{2s}{3+t}\right) \le s\rho_\gamma\left(\frac{2}{3}\right) + t\rho_\gamma(1) \le \frac{2\alpha+\beta}{3}.$$

Thus, we have shown that $\rho_\gamma(\mu) \le (2\alpha+\beta)/3$ for $2/3 \le \mu \le 1$ and for $\gamma \in (-\pi\alpha/2, \pi\alpha/2)$. The last assertion in the theorem is a direct consequence of the relation $\rho_0(\mu) = (2\alpha+\beta)/3$. $\square$

We end the article with the remark that the quantity $\rho_\gamma(1)$ is not necessarily a monotone function of $x = \cos(\gamma/\alpha)$ even when $\beta = 1$ as one can check it by (4.2). The lack of monotonicity in γ seems to cause difficulty in verification of the inequality $\rho_\gamma(\mu) \le \rho_0(\mu)$ for an arbitrary μ.

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