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Boundedness of maximal singular integral operators on spaces of homogeneous type and its applications

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Abstract. Some equivalent characterizations for boundedness of maximal singular integral operators on spaces of homogeneous type are given via certain norm inequalities on John-Strömberg sharp maximal functions and without resorting the boundedness of these operators themselves. As a corollary, the results of Grafakos on Euclidean spaces are generalized to spaces of homogeneous type. Moreover, applications to maximal Monge-Ampère singular integral operators and maximal Nagel-Stein singular integral operators on certain specific smooth manifolds are also presented.

1. Introduction.

It is well-known that the space of homogeneous type introduced by Coifman and Weiss [**5**] (see also [**6**]) is a natural setting for the Calderón-Zygmund theory of singular integrals. The main purpose of this paper is to establish some equivalent characterizations for the boundedness of maximal singular integral operators on spaces of homogeneous type via certain norm inequalities on John-Strömberg sharp maximal functions and without resorting the boundedness of these operators themselves. Part of the results are also new even on Euclidean spaces. As a corollary of this, we generalize the results of Grafakos in [**7**] on Euclidean spaces to spaces of homogeneous type. Moreover, using these results, we obtain the boundedness of maximal Monge-Ampère singular integral operators ([**4**]) and maximal Nagel-Stein singular integral operators on certain specific smooth manifolds ([**20**]) in L^p, $1 < p < \infty$, from L^1 to weak L^1 and from L^∞_c to BMO, where and in what follows, L^∞_c means the set of L^∞ functions with bounded support.

To be precise, we work on spaces of homogeneous type in the sense of Coifman and Weiss [**5**], [**6**]. A homogeneous-type space (X, d, μ) means that X is a set, d is a quasi-metric on X, namely, there exists a constant $A \ge 1$ such that for any x, y, $z \in X$,

$$d(x,y) \le A[d(x,z) + d(y,z)]. \tag{1.1}$$

Moreover, μ is a positive Borel regular measure and has the doubling property. Recall that a measure μ is said to be doubling, if there is a constant $C \ge 1$ such that for any $x \in X$ and $r > 0$,

$$\mu(B(x,2r)) \le C\mu(B(x,r)). \tag{1.2}$$

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In [**17**], Macías and Segovia proved that for any given quasi metric d, there is another quasi metric d', which is equivalent to d in the sense that there exists a constant $C > 0$ such that for all $x,\, y \in X$, $C^{-1}d(x,y) \le d'(x,y) \le Cd(x,y)$, and that the metric balls with respect to d' are open. Thus, throughout this paper, we always assume that all balls in X are open, and the measure of any ball is finite.

Let K be a locally integrable function on $X \times X \setminus \{(x,\, y) : x = y\}$ satisfying the following size condition and the standard Hörmander condition, that is, there exists a constant $C > 0$ such that for all $R > 0$, and all $y,\, y' \in X$,

$$\int_{R<d(x,y)\le 2R} [|K(x,y)| + |K(y,x)|]\, d\mu(x) \le C \tag{1.3}$$

and

$$\int_{d(x,y)\ge 2d(y,y')} \big[|K(x,y) - K(x,y')| + |K(y,x) - K(y',x)|\big]\, d\mu(x) \le C. \tag{1.4}$$

Associated to the above kernel K, we define a linear operator T by that for any $f \in L^\infty_c(X)$ and $\mu - a.\,e.\; x \notin \operatorname{supp} f$,

$$Tf(x) = \int_X K(x,y) f(y)\, d\mu(y). \tag{1.5}$$

We then define the truncated operator T_ϵ for any $\epsilon > 0$, and the maximal operator T^*, respectively, by

$$T_\epsilon f(x) = \int_{d(x,y)>\epsilon} K(x,y) f(y)\, d\mu(y)$$

and

$$T^* f(x) = \sup_{\epsilon>0} |T_\epsilon f(x)|, \tag{1.6}$$

where $x \in X$ and $f \in L^\infty_c(X)$.

The first result of this paper is the following equivalences for the boundedness of T^*.

Theorem 1.1. *Let $\mu(X) = \infty$ and T^* be the maximal operator as in* (1.6) *with K satisfying* (1.3) *and* (1.4). *Then the following statements are equivalent*:

(i) *for certain $r > 0$, there is a constant $C > 0$ such that for any $\lambda > 0$, ball B and bounded function f supported in B,*

$$\mu\big(\{x \in B : |T^* f(x)| > \lambda\}\big) \le C\lambda^{-r} \mu(B) \|f\|^r_{L^\infty(X)};$$

(ii) *for certain $\sigma > 0$, there is a constant $C > 0$ such that for any ball B and bounded function f supported in B,*

$$\frac{1}{\mu(B)} \int_B |T^* f(x)|^\sigma \, d\mu(x) \le C \|f\|^\sigma_{L^\infty(X)};$$

(iii) *T^* is bounded from $L^\infty_c(X)$ to* BMO(X);
(iv) *for any $1 < p < \infty$, T^* is bounded on $L^p(X)$;*
(v) *T^* is bounded from $L^1(X)$ to $L^{1,\infty}(X)$.*

To the extent that we know, the equivalence between (iii) and others of Theorem 1.1 are also new even when $X = \boldsymbol{R}^n$.

Remark 1.1. The proof of Theorem 1.1 also indicates that when $\mu(X) < \infty$, the implicity (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) and (iv) $\Longrightarrow$ (v) $\Longrightarrow$ (i) of Theorem 1.1 still hold. However, it is still unclear so far if the implicity (iii) $\Longrightarrow$ (iv) of Theorem 1.1 is true when $\mu(X) < \infty$.

When $\mu(X) < \infty$, instead of Theorem 1.1, we have the following conclusion.

Theorem 1.2. *Let $\mu(X) < \infty$ and T^* be the maximal operator as in* (1.6) *with K satisfying* (1.3) *and* (1.4). *The following statements are equivalent*:

(1) *T^* is bounded from $L^1(X)$ to $L^{1,\infty}(X)$;*
(2) *for any $1 < p < \infty$, T^* is bounded on $L^p(X)$.*

We remark that the conclusions of Theorem 1.2 when $\mu(X) = \infty$ are included in Theorem 1.1.

From Theorem 1.1 and Theorem 1.2, we can deduce the following conclusions, for which, the first and the second conclusions when $X = \boldsymbol{R}^n$ were obtained by Grafakos in [**7**].

Theorem 1.3. *Let T and T^* be the operators, respectively, as in* (1.5) *and* (1.6) *with K satisfying* (1.3) *and* (1.4). *If T is bounded on $L^2(X)$, then T^* is also bounded on $L^p(X)$ for any $p \in (1, \infty)$, bounded from $L^1(X)$ to $L^{1,\infty}(X)$, and bounded from $L^\infty_c(X)$ to* BMO(X).

It should be stressed that in the proof of Theorem 1.3, we employ some of the techniques which were inspired by the work of Rivière [**23**] and further developed by Grafakos in [**7**]; see also [**8**, pp. 305–309].

The organization of this paper is as follows. In Section 2, for any $p \in (0, \infty)$, we establish certain norm inequalities on the John-Strömberg sharp maximal function, which are used in the proof of Theorem 1.1 and have independent interest; see Theorem 2.1 and Theorem 2.2 below. The proofs of Theorem 1.1 and Theorem 1.2 are presented in Section 3, and Section 4 is devoted to the proof of Theorem 1.3. Finally, in Section 5, we present two applications of Theorem 1.1 through Theorem 1.3 to maximal Monge-Ampère singular integral operators in [**4**] and maximal Nagel-Stein singular integral operators on certain specific smooth manifolds in [**20**], respectively.

We now make some conventions. Throughout the paper, unless explicitly indicated, $\mu(X)$ can be finite or infinite. We always denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. Constant with subscript such as C_1, does not change in different occurrences. The notation $Y \lesssim Z$ means that there exists a constant $C > 0$ such that $Y \le CZ$. Let $D \subset X$ and we denote by χ_D the characteristic function of D. Given $\lambda > 0$ and a ball B, λB denotes the ball with the same center as B and whose radius is λ times that of B. For a local integrable function f on X and a ball B, f_B denotes the mean of f over B, namely,

$$f_B = \frac{1}{\mu(B)} \int_B f(y)\, d\mu(y).$$

2. Some maximal operators.

In this section, we consider the boundedness of some maximal operators, which are used in the proofs of Theorem 1.1 through Theorem 1.3 and are of independent interest. We begin with the following basic covering lemma in [**1**, p. 138].

Lemma 2.1. *Let (X, d, μ) be a space of homogeneous type and $\mathscr{B} = \{\mathscr{B}_\alpha\}_{\alpha\in\Lambda}$ be a family of balls in X such that $U = \bigcup_{\alpha\in\Lambda} \mathscr{B}_\alpha$ is measurable and $\mu(U) < \infty$. Then there exists a disjoint sequence $\{B(x_j, r_j)\}_{j\in\boldsymbol{N}} \subset \mathscr{B}$, such that $U \subset \bigcup_{j\in\boldsymbol{N}} B(x_j, C_1 r_j)$ with C_1 a positive constant depending only on A. Moreover, for any $\alpha \in \Lambda$, B_α is contained in some $B(x_j, C_1 r_j)$.*

The first maximal operator we are concerned with is the operator $M_{0,s}$, $0 < s < 1$, defined by

$$M_{0,s} f(x) = \sup_{B\ni x} \inf \big\{t > 0 : \mu(\{y \in B : |f(y)| > t\}) < s\mu(B)\big\}$$

for any locally integrable function f and $x \in X$. This operator in the setting of Euclidean spaces was first introduced by John [**12**] and then rediscovered by Strömberg [**24**] and Lerner [**14**], [**15**]. For any locally integrable function f, let f^* be the nonincreasing rearrangement of f, namely,

$$f^*(t) = \inf \big\{s > 0 : \mu(\{x \in X : |f(x)| > s\}) < t\big\};$$

see, for example, [**13**]. Then, it is easy to see that

$$M_{0,s} f(x) = \sup_{B\ni x} (f\chi_B)^*(s\mu(B)).$$

Related to the operator $M_{0,s}$, there is a sharp maximal operator $M^\sharp_{0,s}$ which, for any locally integrable function f and $x \in X$, is defined by

$$M^\sharp_{0,s} f(x) = \sup_{B\ni x} \inf_{c\in\boldsymbol{C}} \inf \big\{t > 0 : \mu(\{y \in B : |f(y) - c| > t\}) < s\mu(B)\big\}.$$

It is easy to see that for any $s \in (0,1)$, locally integrable function f and $x \in X$,

$$M^{\sharp}_{0,\,s}f(x) \le M_{0,\,s}f(x).$$

A useful variant of $M_{0,\,s}$ is the central maximal operator $M^c_{0,\,s}$ which, for any locally integrable function f, is defined by

$$M^c_{0,\,s}f(x) = \sup_{r>0} \inf \big\{t > 0 :\, \mu(\{y \in B(x,\,r) :\, |f(y)| > t\}) < s\mu(B(x,\,r))\big\}.$$

Applying the doubling condition (1.2) of μ, we can verify that there is a constant $C_2 > 1$ depending on $(X,\,d,\,\mu)$ such that for any $x \in X$,

$$M^c_{0,\,s}f(x) \le M_{0,\,s}f(x) \le M^c_{0,\,C_2^{-1}s}f(x). \tag{2.1}$$

LEMMA 2.2. *Let $s \in (0,1)$. Then for any two locally integrable functions f_1 and f_2 and $x \in X$,*

$$M^c_{0,\,s}(f_1 + f_2)(x) \le M^c_{0,\,s/2}f_1(x) + M^c_{0,\,s/2}f_2(x). \tag{2.2}$$

PROOF. Let B be a ball and f be any locally integrable function. We define

$$m_{0,\,s,\,B}(f) = \inf \big\{t > 0 :\, \mu(\{y \in B :\, |f(y)| > t\}) < s\mu(B)\big\}. \tag{2.3}$$

For any fixed $\sigma > 0$, there are positive numbers t_1 and t_2 with

$$m_{0,\,s/2,\,B}(f_1) \le t_1 < m_{0,\,s/2,\,B}(f_1) + \frac{\sigma}{2},$$

and

$$m_{0,\,s/2,\,B}(f_2) \le t_2 < m_{0,\,s/2,\,B}(f_2) + \frac{\sigma}{2},$$

such that for $j = 1,\,2$,

$$\mu(\{y \in B :\, |f_j(y)| > t_j\}) < \left(\frac{s}{2}\right)\mu(B).$$

Thus,

$$\mu(\{y \in B :\, |f_1(y) + f_2(y)| > t_1 + t_2\}) < s\mu(B).$$

This in turn implies that

$$m_{0,\,s,\,B}(f_1 + f_2) \le t_1 + t_2 < m_{0,\,s/2,\,B}(f_1) + m_{0,\,s/2,\,B}(f_2) + \sigma,$$

which together with the arbitrariness of σ then gives (2.2) immediately. This finishes the proof of Lemma 2.2. $\square$

We now recall the Hardy-Littlewood maximal operator $\mathscr{M}$ defined by

$$\mathscr{M}f(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y)|\, d\mu(y),$$

where the supremum is taken over all balls containing x and of positive radius.

LEMMA 2.3. *Let $s \in (0, 1)$. Then for any locally integrable function f and any $\lambda > 0$,*

$$\{x \in X : |f(x)| > \lambda\} \subset \big\{x \in X : M^c_{0,\,s} f(x) > \lambda\big\}, \tag{2.4}$$

and

$$\mu(\{x \in X : M_{0,\,s} f(x) > \lambda\}) \le C_3 s^{-1} \mu(\{x \in X : |f(x)| > \lambda\}), \tag{2.5}$$

where $C_3 > 0$ is a constant independent of f, λ and s.

PROOF. Recall that μ is regular. For each fixed $\lambda > 0$, it follows from the Lebesgue differentiation theorem that

$$\begin{aligned}\mu(\{x \in X : |f(x)| > \lambda\}) &= \mu\big(\{x \in X : \chi_{\{y \in X :\, |f(y)| > \lambda\}}(x) = 1\}\big)\\ &\le \mu\big(\{x \in X : \mathscr{M}^c(\chi_{\{y \in X :\, |f(y)| > \lambda\}})(x) > s\}\big),\end{aligned}$$

where $\mathscr{M}^c$ is the central Hardy-Littlewood maximal operator defined by

$$\mathscr{M}^c f(x) = \sup_{r > 0} \frac{1}{\mu(B(x,\, r))} \int_{B(x,\, r)} |f(y)|\, d\mu(y).$$

We claim that

$$\big\{x \in X : \mathscr{M}^c(\chi_{\{y \in X :\, |f(y)| > \lambda\}})(x) > s\big\} \subset \big\{x \in X : M^c_{0,\,s} f(x) > \lambda\big\}. \tag{2.6}$$

In fact, if $\mathscr{M}^c(\chi_{\{y \in X :\, |f(y)| > \lambda\}})(x) > s$, then there is a ball $B(x,\, r)$ such that

$$\mu(\{y \in B(x,\, r) : |f(y)| > \lambda\}) > s\mu(B(x,\, r)). \tag{2.7}$$

Since for any $t \le \lambda$,

$$\mu(\{y \in B(x,\, r) : |f(y)| > \lambda\}) \le \mu(\{y \in B(x,\, r) : |f(y)| > t\}),$$

we have that

$$\inf\big\{t>0:\, \mu(\{y\in B(x,\,r):\,|f(y)|>t\})<s\mu(B(x,\,r))\big\}\geq\lambda. \tag{2.8}$$

Note that

$$\{y\in B(x,\,r):\,|f(y)|>\lambda\}=\bigcup_{k\geq1}\left\{y\in B(x,r):\,|f(y)|>\lambda+\frac{1}{k}\right\},$$

and

$$\mu(\{y\in B(x,\,r):\,|f(y)|>\lambda\})=\lim_{k\to\infty}\mu\bigg(\left\{y\in B(x,r):\,|f(y)|>\lambda+\frac{1}{k}\right\}\bigg),$$

which implies that if the equality in (2.8) holds, then for any $k\in\boldsymbol{N}$,

$$\mu\bigg(\left\{y\in B(x,r):\,|f(y)|>\lambda+\frac{1}{k}\right\}\bigg)\leq s\mu(B(x,\,r))$$

and therefore,

$$\mu(\{y\in B(x,r):\,|f(y)|>\lambda\})\leq s\mu(B(x,\,r)).$$

This contradicts (2.7). Thus, $\mathscr{M}^c(\chi_{\{y\in X:\,|f(y)|>\lambda\}})(x)>s$ implies that

$$M^c_{0,\,s}f(x)\geq\inf\big\{t>0:\, \mu(\{y\in B(x,\,r):\,|f(y)|>t\})<s\mu(B(x,\,r))\big\}>\lambda$$

and the inequality (2.6) holds.

The proof of (2.5) follows from the same argument as that used in [**15**, p. 2451]. In fact, note that for any $\lambda>0$ and $s\in(0,1)$,

$$\{x\in X:\, M_{0,\,s}f(x)>\lambda\}\subset\big\{x\in X:\,\mathscr{M}(\chi_{\{y\in X:\,|f(y)|>\lambda\}})(x)\geq s\big\},$$

which together with the fact that the operator $\mathscr{M}$ is bounded from $L^1(X)$ to weak $L^1(X)$ gives us (2.5). This finishes the proof of Lemma 2.3. □

The following good-λ inequality is an analog of the good-λ inequality on the Hardy-Littlewood maximal operator $\mathscr{M}$ and the Fefferman-Stein sharp maximal operator $M^\sharp$ defined by

$$M^\sharp f(x)=\sup_{B\ni x}\frac{1}{\mu(B)}\int_B|f(y)-f_B|\,d\mu(y);$$

see [**18**] for the details on the good-λ inequality related to $\mathscr{M}$ and $M^\sharp$.

LEMMA 2.4. *Let $0<s_1,\,s_2\leq1/2$, $\lambda>0$ and f be any locally integrable function. Suppose that B is a ball satisfying that there is a point $x_0\in B$ such that $M_{0,\,s_1}f(x_0)\leq\lambda$.*

Then there is a constant $C > 0$, which is independent of f, B, λ, s_1 and s_2, such that

$$\mu\left(\left\{x \in B:\, M^c_{0,\,s_1} f(x) > 3\lambda,\, M^\sharp_{0,\,s_2} f(x) \le \frac{\lambda}{4}\right\}\right) \le C s_1^{-1} s_2 \mu(B).$$

Before proving Lemma 2.4, we recall the definition of median values; see [**12**], [**24**], [**11**]. Let f be a real locally integrable function and B be a ball. The median value $m_f(B)$ of f over B is defined to be one of real numbers satisfying that

$$\mu(\{x \in B:\, f(x) > m_f(B)\}) \le \frac{\mu(B)}{2},$$

and

$$\mu(\{x \in B:\, f(x) < m_f(B)\}) \le \frac{\mu(B)}{2}.$$

If f is complex, we then define $m_f(B) = m_{\mathrm{Re}(f)}(B) + i m_{\mathrm{Im}(f)}(B)$, where $i^2 = -1$. It is easy to verify that for any $s \in (0, 1/2]$, ball B and locally integrable function f,

$$|m_f(B)| \le \sqrt{2} \inf_{x \in B} M_{0,\,1/2} f(x) \le \sqrt{2} \inf_{x \in B} M_{0,\,s} f(x), \tag{2.9}$$

and

$$\mu\left(\left\{y \in B:\, |f(y) - m_f(B)| > 2\sqrt{2} \inf_{x \in B} M^\sharp_{0,\,s} f(x)\right\}\right) \le s\mu(B); \tag{2.10}$$

see [**11**, p. 238] for (2.9), and [**11**, p. 236] and [**24**, p. 519] for (2.10) in the setting of Euclidean spaces.

Proof of Lemma 2.4. Let r_B be the radius of B and

$$E = \left\{x \in B:\, M^c_{0,\,s_1} f(x) > 3\lambda,\, M^\sharp_{0,\,s_2} f(x) \le \frac{\lambda}{4}\right\}.$$

To prove the lemma, it suffices to consider the case when $E \ne \varnothing$. For each fixed $x \in E$, we have that $M^c_{0,\,s_1} f(x) > 3\lambda$, which means that there exists a ball $B(x, r_x)$ such that

$$\inf\left\{t > 0:\, \mu(\{y \in B(x, r_x):\, |f(y)| > t\}) < s_1 \mu(B(x, r_x))\right\} > 3\lambda.$$

Since $x_0 \notin B(x, r_x)$ (for otherwise, $M_{0,\,s_1} f(x_0) > \lambda$), by (1.1) together with x_0, $x \in B$, we then have $r_x < 2Ar_B$ and $B(x, r_x) \subset 4A^2 B$. This in turn implies that

$$M^c_{0,\,s_1}(f\chi_{4A^2B})(x) > 3\lambda$$

and so

$$E \subset \left\{ x \in B : M^c_{0,\, s_1}(f\chi_{4A^2B})(x) > 3\lambda,\, M^\sharp_{0,\, s_2}(f)(x) \le \frac{\lambda}{4} \right\}.$$

Lemma 2.2 together with (2.9) and the definition of $M^c_{0,\, s_1/2}$ tell us that

$$\begin{aligned}
&M^c_{0,\, s_1}(f\chi_{4A^2B})(x)\\
&\quad \le M^c_{0,\, s_1/2}\big((f - m_f(4A^2B))\chi_{4A^2B}\big)(x) + M^c_{0,\, s_1/2}(m_f(4A^2B)\chi_{4A^2B})(x)\\
&\quad \le M^c_{0,\, s_1/2}\big((f - m_f(4A^2B))\chi_{4A^2B}\big)(x) + |m_f(4A^2B)|\\
&\quad \le M^c_{0,\, s_1/2}\big((f - m_f(4A^2B))\chi_{4A^2B}\big)(x) + 2 \inf_{y \in 4A^2B} M_{0,\, s_1} f(y)\\
&\quad \le M^c_{0,\, s_1/2}\big((f - m_f(4A^2B))\chi_{4A^2B}\big)(x) + 2M_{0,\, s_1} f(x_0).
\end{aligned}$$

Therefore, by the assumption that $M_{0,\, s_1} f(x_0) \le \lambda$, Lemma 2.3 and $E \ne \varnothing$, which implies that $\lambda \ge 4 \inf_{z \in 4A^2B} M^\sharp_{0,\, s_2} f(z)$, together with (2.10), we have

$$\begin{aligned}
\mu(E) &\le \mu\big(\big\{x \in B : M^c_{0,\, s_1/2}(|f - m_f(4A^2B)|\chi_{4A^2B})(x) > \lambda\big\}\big)\\
&\lesssim s_1^{-1} \mu\big(\{y \in 4A^2B : |f(y) - m_f(4A^2B)| > \lambda\}\big)\\
&\lesssim s_1^{-1} \mu\Big(\Big\{y \in 4A^2B : |f(y) - m_f(4A^2B)| > 4 \inf_{z \in 4A^2B} M^\sharp_{0,\, s_2} f(z)\Big\}\Big)\\
&\lesssim s_1^{-1} s_2 \mu(B),
\end{aligned}$$

which completes the proof of Lemma 2.4. □

Lemma 2.5. *Let $0 < s_1,\, s_2 \le 1/2$. Then there exist constants $C_4 > 0$ and $C > 0$, which depend only on X, such that for any locally integrable function f and any $\lambda > 0$,*

$$\begin{aligned}
&\mu\big(\{x \in X : M^c_{0,\, s_1} f(x) > 3\lambda\}\big)\\
&\quad \le C_4 s_1^{-2} s_2 \mu(\{x \in X : |f(x)| > \lambda\}) + C\mu\left(\left\{x \in X : M^\sharp_{0,\, s_2} f(x) > \frac{\lambda}{4}\right\}\right),
\end{aligned} \tag{2.11}$$

provided that $\mu(\{x \in X : |f(x)| > \lambda\}) < C_3^{-1} s_1 \mu(X)$.

Proof. For each fixed $\lambda > 0$, set

$$V_\lambda = \{x \in X : M_{0,\, s_1} f(x) > \lambda\},$$

and

$$W_\lambda = \left\{x \in X : M^c_{0,\, s_1} f(x) > 3\lambda,\, M^\sharp_{0,\, s_2} f(x) \le \frac{\lambda}{4}\right\}.$$

We see by (2.5) that

$$\mu(V_\lambda) \le C_3 s_1^{-1} \mu(\{y \in X : |f(y)| > \lambda\}) < \mu(X).$$

Thus, $X \backslash V_\lambda$ is not an empty set. Note that $W_\lambda \subset V_\lambda$ and V_λ is an open set. For each $x \in V_\lambda$, denote by r_x the distance of x and the set $X \backslash V_\lambda$, that is,

$$r_x = \inf_{y \in X \backslash V_\lambda} d(x, y).$$

Let C_1 be the same as in Lemma 2.1. Obviously, we can assume that $C_1 \ge 1$. It is also easy to see that $r_x > 0$ and

$$V_\lambda = \bigcup_{x \in V_\lambda} B\left(x, \frac{r_x}{2C_1}\right).$$

Applying Lemma 2.1, we can find a sequence of non-overlapping balls $\{B(x_j, r_j/(2C_1))\}$ such that

$$V_\lambda = \bigcup_j B\left(x_j, \frac{4r_j}{5}\right), \; B\left(x_j, \frac{5r_j}{4}\right) \cap (X \backslash V_\lambda) \ne \varnothing.$$

By Lemma 2.4 and (1.2), we thus have that

$$\begin{aligned}
\mu(W_\lambda) &\le \sum_j \mu\left(\left\{x \in B\left(x_j, \frac{5r_j}{4}\right) : M^c_{0,\, s_1} f(x) > 3\lambda,\; M^\sharp_{0,\, s_2} f(x) \le \frac{\lambda}{4}\right\}\right) \\
&\lesssim s_1^{-1} s_2 \sum_j \mu\left(B\left(x_j, \frac{r_j}{2C_1}\right)\right) \\
&\lesssim s_1^{-1} s_2 \mu(V_\lambda).
\end{aligned}$$

This together with (2.5) in Lemma 2.3 in turn gives us that

$$\begin{aligned}
&\mu\big(\{x \in X : M^c_{0,\, s_1} f(x) > 3\lambda\}\big) \\
&\quad \le \mu(W_\lambda) + \mu\left(\left\{x \in X : M^\sharp_{0,\, s_2} f(x) > \frac{\lambda}{4}\right\}\right) \\
&\quad \lesssim s_1^{-1} s_2 \mu(V_\lambda) + \mu\left(\left\{x \in X : M^\sharp_{0,\, s_2} f(x) > \frac{\lambda}{4}\right\}\right) \\
&\quad \lesssim s_1^{-2} s_2 \mu(\{x \in X : |f(x)| > \lambda\}) + \mu\left(\left\{x \in X : M^\sharp_{0,\, s_2} f(x) > \frac{\lambda}{4}\right\}\right). \qquad \square
\end{aligned}$$

We can then formulate the main results of this section.

THEOREM 2.1. *Let $\mu(X) = \infty$, $p \in (0, \infty)$ and $C_4 > 0$ be the same as in Lemma 2.5. Then there exists a constant $C > 0$ such that for any $s \in (0, 1/2]$ and $s < (2^2 3^p C_4)^{-1}$, and any locally integrable function f,*

$$\sup_{\lambda>0} \lambda^p \mu(\{x \in X : |f(x)| > \lambda\}) \le C \sup_{\lambda>0} \lambda^p \mu\big(\{x \in X : M^{\sharp}_{0,\,s} f(x) > \lambda\}\big),$$

provided that

$$\sup_{\lambda>0} \lambda^p \mu(\{x \in X : |f(x)| > \lambda\}) < \infty. \tag{2.12}$$

PROOF. By (2.4) and Lemma 2.5 with $0 < s_2 = s \le 1/2 = s_1$, we see that for any $s \in (0, 1/2]$, $\lambda > 0$ and locally integrable function f,

$$\begin{aligned}
&\mu(\{x \in X : |f(x)| > 3\lambda\})\\
&\quad \le \mu\big(\{x \in X : M^{c}_{0,\,1/2} f(x) > 3\lambda\}\big)\\
&\quad \le 2^2 C_4 s \mu(\{x \in X : |f(x)| > \lambda\}) + C\mu\left(\left\{x \in X : M^{\sharp}_{0,\,s} f(x) > \frac{\lambda}{4}\right\}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&(3\lambda)^p \mu(\{x \in X : |f(x)| > 3\lambda\})\\
&\quad \le 2^2 3^p C_4 s \lambda^p \mu(\{x \in X : |f(x)| > \lambda\}) + C\lambda^p \mu\left(\left\{x \in X : M^{\sharp}_{0,\,s} f(x) > \frac{\lambda}{4}\right\}\right).
\end{aligned}$$

Taking supremum over the last inequality gives us that for any $R > 0$,

$$\begin{aligned}
&\sup_{0<\lambda<3R} \lambda^p \mu(\{x \in X : |f(x)| > \lambda\})\\
&\quad \le 2^2 3^p C_4 s \sup_{0<\lambda<R} \lambda^p \mu(\{x \in X : |f(x)| > \lambda\}) + C \sup_{\lambda>0} \lambda^p \mu\big(\{x \in X : M^{\sharp}_{0,\,s} f(x) > \lambda\}\big),
\end{aligned}$$

which together with the assumptions that $s < (2^2 3^p C_4)^{-1}$ and (2.12) completes the proof of Theorem 2.1. □

THEOREM 2.2. *Let $p \in (0, \infty)$ and $C_4 > 0$ be the same as in Lemma 2.5. There exists a constant $C > 0$ such that for any $s \in (0, 1/2]$ and $s < (2^2 3^p C_4)^{-1}$, and any locally integrable function f,*

(i) *if $\mu(X) = \infty$, and for some $p_0 \in (0, p)$,*

$$\sup_{t>0} t^{p_0} \mu(\{x \in X : |f(x)| > t\}) < \infty, \tag{2.13}$$

then

$$\|f\|_{L^p(X)} \le C\big\|M^\sharp_{0,\,s}f\big\|_{L^p(X)};$$

(ii) *if* $\mu(X) < \infty$, *then*

$$\|f\|_{L^p(X)} \le C\big\|M^\sharp_{0,\,s}f\big\|_{L^p(X)} + C\|f\|_{L^{1,\,\infty}(X)},$$

where $\|f\|_{L^{1,\,\infty}(X)}$ *is the* $L^{1,\infty}(X)$ *norm of* f *defined by*

$$\|f\|_{L^{1,\,\infty}(X)} = \sup_{\tau>0} \tau\mu(\{x \in X : |f(x)| > \tau\}).$$

Proof. We first consider the case that $\mu(X) = \infty$. As in the proof of Theorem 2.1, by (2.4) and Lemma 2.5 with $0 < s_2 = s \le 1/2 = s_1$, we see that for any $s \in (0, 1/2]$, $R > 0$ and locally integrable function f,

$$\begin{aligned}
&\int_0^{3R} \mu(\{x \in X : |f(x)| > \lambda\})\lambda^{p-1}\,d\lambda \\
&\quad\le 3^p \int_0^R \mu\big(\{x \in X : M^c_{0,\,1/2}f(x) > 3\lambda\}\big)\lambda^{p-1}\,d\lambda \\
&\quad\le 2^2 3^p C_4 s \int_0^R \mu(\{x \in X : |f(x)| > \lambda\})\lambda^{p-1}\,d\lambda + C\big\|M^\sharp_{0,\,s}f\big\|^p_{L^p(X)}.
\end{aligned}$$

Note that if $\mu(X) = \infty$ and f satisfies (2.13), it then follows that

$$\int_0^R \mu(\{x \in X : |f(x)| > \lambda\})\lambda^{p-1}\,d\lambda < \infty.$$

Taking $s \in (0, 1/2]$ such that $s < (2^2 3^p C_4)^{-1}$, we then have that for any $R > 0$,

$$\int_0^{3R} \mu(\{x \in X : |f(x)| > \lambda\})\lambda^{p-1}\,d\lambda \lesssim \big\|M^\sharp_{0,\,s}f\big\|^p_{L^p(X)}.$$

Letting $R \to \infty$ leads to the desired result.

Now let $\mu(X) < \infty$. If $\lambda > \lambda_{f,\,X} = 2C_3\|f\|_{L^{1,\,\infty}(X)}[\mu(X)]^{-1}$, we know from Lemma 2.5 that

$$\begin{aligned}
&\mu(\{x \in X : |f(x)| > 3\lambda\}) \\
&\quad\le 2^2 C_4 s\mu(\{x \in X : |f(x)| > \lambda\}) + C\mu\left(\left\{x \in X : M^\sharp_{0,\,s}f(x) > \frac{\lambda}{4}\right\}\right).
\end{aligned}$$

For any $R > \lambda_{f,\,X}$, integrating the last estimate then yields

$$\begin{aligned}
&\int_{3\lambda_{f,\,X}}^{3R} \mu(\{x \in X : |f(x)| > \lambda\})\lambda^{p-1}\,d\lambda \\
&\quad \le C\big\|M_{0,\,s}^{\sharp} f\big\|_{L^p(X)}^p + 2^2 3^p C_4 s \int_{\lambda_{f,\,X}}^{R} \mu(\{x \in X : |f(x)| > \lambda\})\lambda^{p-1}\,d\lambda.
\end{aligned}$$

This in turn implies that

$$\begin{aligned}
&\int_0^{3R} \mu(\{x \in X : |f(x)| > \lambda\})\lambda^{p-1}\,d\lambda \\
&= \int_0^{3\lambda_{f,\,X}} \mu(\{x \in X : |f(x)| > \lambda\})\lambda^{p-1}\,d\lambda + \int_{3\lambda_{f,\,X}}^{3R} \mu(\{x \in X : |f(x)| > \lambda\})\lambda^{p-1}\,d\lambda \\
&\le C\lambda_{f,\,X}^p \mu(X) + C\big\|M_{0,\,s}^{\sharp} f\big\|_{L^p(X)}^p + 2^2 3^p C_4 s \int_0^{R} \mu(\{x \in X : |f(x)| > \lambda\})\lambda^{p-1}\,d\lambda,
\end{aligned}$$

and the desired conclusion follows immediately. $\square$

3. Proofs of Theorem 1.1 and Theorem 1.2.

To prove Theorem 1.1, we first recall the following Calderón-Zygmund decomposition theorem on spaces of homogeneous type in [**5**, pp. 72–74].

Lemma 3.1. *Let $f \in L^1(X)$ and $\lambda > \|f\|_{L^1(X)}\,[\mu(X)]^{-1}$, there exist a family of balls $\{B_j\}_{j\in\Lambda}$, and constants $C > 0$ and $L \ge 1$, such that*

- (a$_1$) $f = g + b$ *with* $b = \sum_{j\in\Lambda} b_j$,
- (a$_2$) $|g(x)| \le C\lambda$ *for almost all* $x \in X$,
- (a$_3$) $\operatorname{supp} b_j \subset B_j$, $\int_{B_j} b_j(x)\,d\mu(x) = 0$, *and for all* $x \in X$, $\sum_{j\in\Lambda} \chi_{B_j}(x) \le L$,
- (a$_4$) $\|b_j\|_{L^1(X)} \le C\lambda\mu(B_j)$,
- (a$_5$) $\sum_{j\in\Lambda} \mu(B_j) \le C\lambda^{-1}\|f\|_{L^1(X)}$.

We also need the following interpolation theorem of operators, which can be proved by the same argument as that used in the proof of Marcinkiewicz interpolation theorem. We omit the details for brevity. The advantage of this interpolation theorem is that the operators included are even not necessary to be sub-linear.

Lemma 3.2. *Let $1 \le p_1 < p_0 < p_2 \le \infty$, and T, T_1 and T_2 be three operators. Assume that*

(1) *there is a constant $C > 0$ such that for any $f_1,\, f_2 \in \bigcup_{p\ge1} L^p(X)$,*

$$|T(f_1 + f_2)(x)| \le C(|T_1(f_1)(x)| + |T_2(f_2)(x)|);$$

(2) *T_1 and T_2 are bounded, respectively, from $L^{p_1}(X)$ to $L^{p_1,\,\infty}(X)$ and from $L^{p_2}(X)$ to $L^{p_2,\,\infty}(X)$.*

Then T is also bounded on $L^{p_0}(X)$.

Lemma 3.3. *Let $\{B_j = B(x_j, r_j)\}_{j\in\Lambda}$ be a sequence of balls in X such that for some constant $C \geq 1$,*

$$\sum_{j\in\Lambda} \chi_{B_j}(x) \leq C.$$

Furthermore, let $\lambda > 0$ and $\{b_j\}_{j\in\Lambda}$ be a sequence of functions such that for some fixed constant $C_5 > 0$, $\operatorname{supp} b_j \subset B_j$, $\int_X b_j(x)\, d\mu(x) = 0$ and $\int_{B_j} |b_j(x)|\, d\mu(x) \leq C_5\lambda\mu(B_j)$. With the same assumptions as in Theorem 1.1, then there are constants $C > 0$ and $C_6 > 2$, which are independent of b_j and λ, such that

$$\mu\biggl(\biggl\{x \in X \setminus \bigcup_j B_j^* : T^*\biggl(\sum_{j\in\Lambda} b_j\biggr)(x) > C_6\lambda\biggr\}\biggr) \leq C \sum_{j\in\Lambda} \mu(B_j),$$

where $B_j^* = A(4A^2 + 1)B_j$ with A same as in (1.1).

Proof. To prove this lemma, we invoke some ideas from [**7**]. Let $b = \sum_{j\in\Lambda} b_j$, and set

$$\mathrm{E}_1(x) = \sum_{j\in\Lambda} \int_{B_j} |K(x,y) - K(x,x_j)||b_j(y)|\, d\mu(y)$$

and

$$\mathrm{E}_2(x) = \lambda \sum_{j\in\Lambda} \int_{B_j} |K(x,y) - K(x,x_j)|\, d\mu(y),$$

where x_j is the center of B_j. If we can prove that there is a constant $C_7 > 0$ such that for $x \in X\setminus \bigcup_j B_j^*$,

$$T^*b(x) \leq 2\mathrm{E}_1(x) + C_5\mathrm{E}_2(x) + C_7\lambda, \tag{3.1}$$

it then follows by (1.4) that

$$\begin{aligned}
&\mu\biggl(\biggl\{x \in X \setminus \bigcup_j B_j^* : T^*(b)(x) > (C_7+2)\lambda\biggr\}\biggr)\\
&\quad\leq \mu\biggl(\biggl\{x \in X \setminus \bigcup_j B_j^* : \mathrm{E}_1(x) > \frac{\lambda}{2}\biggr\}\biggr) + \mu\biggl(\biggl\{x \in X \setminus \bigcup_j B_j^* : \mathrm{E}_2(x) > \frac{\lambda}{C_5}\biggr\}\biggr)\\
&\quad\lesssim \lambda^{-1}\bigl\{\|\mathrm{E}_1\|_{L^1(X\setminus\bigcup_j B_j^*)} + \|\mathrm{E}_2\|_{L^1(X\setminus\bigcup_j B_j^*)}\bigr\}
\end{aligned}$$

$$\lesssim \lambda^{-1}\sum_{j\in\Lambda}\int_{B_j}\left\{\int_{X\setminus B_j^*}|K(x,y)-K(x,x_j)|\,d\mu(x)\right\}|b_j(y)|\,d\mu(y)$$
$$+\sum_{j\in\Lambda}\int_{B_j}\int_{X\setminus B_j^*}|K(x,y)-K(x,x_j)|\,d\mu(x)\,d\mu(y)$$
$$\lesssim \lambda^{-1}\sum_{j\in\Lambda}\|b_j\|_{L^1(X)}+\sum_{j\in\Lambda}\mu(B_j),$$

which together with the assumption that $\|b_j\|_{L^1(X)}\lesssim \lambda\mu(B_j)$ gives us the desired estimate.

We now prove (3.1). For each fixed $x\in X\setminus\bigcup_j B_j^*$ and $\epsilon>0$, set

$$\mathrm{I}_1(x,\,\epsilon)=\{j\in\Lambda:\ \text{for all } y\in B_j,\, d(x,y)\le\epsilon\},$$
$$\mathrm{I}_2(x,\,\epsilon)=\{j\in\Lambda:\ \text{for all } y\in B_j,\, d(x,y)>\epsilon\},$$

and

$$\mathrm{I}_3(x,\,\epsilon)=\{j\in\Lambda: B_j\cap\{y\in X:\,d(x,\,y)>\epsilon\}\ne\varnothing$$
$$\text{and } B_j\cap\{y\in X:\,d(x,\,y)\le\epsilon\}\ne\varnothing\}.$$

Then

$$|T_\epsilon b(x)|\le\left|T_\epsilon\Bigg(\sum_{j\in\mathrm{I}_2(x,\,\epsilon)}b_j\Bigg)(x)\right|+\left|T_\epsilon\Bigg(\sum_{j\in\mathrm{I}_3(x,\,\epsilon)}b_j\Bigg)(x)\right|.$$

Applying the vanishing moment hypothesis of b_j, we know that

$$\left|T_\epsilon\Bigg(\sum_{j\in\mathrm{I}_2(x,\,\epsilon)}b_j\Bigg)(x)\right|=\left|\sum_{j\in\mathrm{I}_2(x,\,\epsilon)}\int_{B_j}[K(x,y)-K(x,x_j)]b_j(y)\,d\mu(y)\right|\le\mathrm{E}_1(x),$$

and

$$\left|T_\epsilon\Bigg(\sum_{j\in\mathrm{I}_3(x,\,\epsilon)}b_j\Bigg)(x)\right|=\left|\sum_{j\in\mathrm{I}_3(x,\,\epsilon)}\int_{B_j}K(x,y)b_j(y)\chi_{B(x,\epsilon)^{\complement}}(y)\,d\mu(y)\right|$$
$$\le\left|\sum_{j\in\mathrm{I}_3(x,\,\epsilon)}\int_{B_j}K(x,y)\big[b_j(y)\chi_{B(x,\epsilon)^{\complement}}(y)-D_j(\epsilon)\big]\,d\mu(y)\right|$$
$$+\left|\sum_{j\in\mathrm{I}_3(x,\,\epsilon)}\int_{B_j}K(x,y)D_j(\epsilon)\,d\mu(y)\right|,$$

where $B(x,\,\epsilon)^{\complement}=X\setminus B(x,\,\epsilon)$ and $D_j(\epsilon)=[\mu(B_j)]^{-1}\int_X b_j(y)\chi_{B(x,\epsilon)^{\complement}}(y)\,d\mu(y)$. Obviously, $|D_j(\epsilon)| \le C_5\lambda$ for any $j \in \Lambda$ by the assumption on b_j. Thus

$$\begin{aligned}
&\left|T_\epsilon\Big(\sum_{j\in \mathrm{I}_3(x,\,\epsilon)} b_j\Big)(x)\right|\\
&\quad\le \left|\sum_{j\in \mathrm{I}_3(x,\,\epsilon)} \int_{B_j} [K(x,y)-K(x,x_j)]\big[b_j(y)\chi_{B(x,\epsilon)^\complement}(y)-D_j(\epsilon)\big]\,d\mu(y)\right|\\
&\qquad+\left|\sum_{j\in \mathrm{I}_3(x,\,\epsilon)} D_j(\epsilon)\int_{B_j} K(x,y)\,d\mu(y)\right|\\
&\quad\le \mathrm{E}_1(x)+C_5\mathrm{E}_2(x)+C_5\lambda\sum_{j\in \mathrm{I}_3(x,\,\epsilon)}\int_{B_j}|K(x,y)|\,d\mu(y).
\end{aligned}$$

For each $j \in \mathrm{I}_3(x,\,\epsilon)$, by its definition, we can choose $y_j^1,\, y_j^2 \in B_j$ such that $d(x,\, y_j^1) > \epsilon$ and $d(x,\, y_j^2) \le \epsilon$. Note that $d(x, x_j) \ge A(4A^2+1)r_j$ by $x \notin B_j^*$,

$$\epsilon \ge d\big(x,\, y_j^2\big) \ge \frac{1}{A}d(x,\, x_j) - d\big(x_j,\, y_j^2\big) \ge (4A^2+1)r_j - r_j = 4A^2 r_j. \tag{3.2}$$

Therefore, for any $y \in B_j$ with $j \in \mathrm{I}_3(x,\,\epsilon)$, we have that

$$d(x,y) \le A\big(d(x,y_j^2)+d(y_j^2,\, y)\big) \le A(\epsilon+2Ar_j) \le \left(A+\frac{1}{2}\right)\epsilon$$

and

$$d(x,y) \ge \frac{1}{A}d\big(x,\, y_j^1\big) - d\big(y_j^1,\, y\big) \ge \frac{\epsilon}{2A}.$$

This in turn implies

$$\bigcup_{j\in \mathrm{I}_3(x,\,\epsilon)} B_j \subset B\left(x,\, \left(A+\frac{1}{2}\right)\epsilon\right) \setminus B\left(x,\, \frac{\epsilon}{2A}\right),$$

and thus by the hypothesis that $\sum_{j\in\Lambda}\chi_{B_j}(x) \le C$ and (1.3),

$$\begin{aligned}
\sum_{j\in \mathrm{I}_3(x,\,\epsilon)}\int_{B_j}|K(x,\,y)|\,d\mu(y) &\le \int_{B(x,(A+1/2)\epsilon)\setminus B(x,\,\epsilon/(2A))} |K(x,y)| \sum_{j\in \mathrm{I}_3(x,\,\epsilon)} \chi_{B_j}(y)\,d\mu(y)\\
&\le C.
\end{aligned}$$

Therefore, (3.1) holds for $C_7 = CC_5$, which completes the proof of Lemma 3.3. □

Proof of Theorem 1.1. (i) ⇒ (ii). Choose $\sigma \in (0, r)$ in (ii). For any fixed ball

B and bounded function f supported in B, by (i), we have

$$\begin{aligned}
&\int_B |T^*f(x)|^\sigma \, d\mu(x) \\
&\quad \le \sigma\mu(B) \int_0^{\|f\|_{L^\infty(X)}} t^{\sigma-1}dt + \sigma \int_{\|f\|_{L^\infty(X)}}^\infty \mu(\{x \in B : T^*f(x) > t\})t^{\sigma-1}dt \\
&\quad \lesssim \mu(B)\|f\|^\sigma_{L^\infty(X)},
\end{aligned}$$

namely, (ii) holds.

(ii) $\Rightarrow$ (iii). Without loss of generality, we may assume that $\sigma < 1$ in (ii). Observe that for any $f \in L^\infty_c(X)$, $(T^*f)^\sigma$ is locally integrable, and thus T^*f is finite almost everywhere. By the characterization of the space $\mathrm{BMO}(X)$ of Long and Yang [**16**, p. 700], to prove (iii), it suffices to verify that for any $f \in L^\infty_c(X)$,

$$\sup_B \inf_{c\in \boldsymbol{C}} \frac{1}{\mu(B)} \int_B |T^*f(x) - c|^\sigma \, d\mu(x) \lesssim \|f\|^\sigma_{L^\infty(X)}. \tag{3.3}$$

To this end, for any fixed ball B and $f \in L^\infty_c(X)$, decompose f into

$$f(x) = f(x)\chi_{B^*}(x) + f(x)\chi_{X\setminus B^*}(x) = f_1(x) + f_2(x),$$

where $B^* = A(4A^2 + 1)B$. It then follows that

$$\begin{aligned}
&\inf_{c\in \boldsymbol{C}} \frac{1}{\mu(B)} \int_B |T^*f(x) - c|^\sigma \, d\mu(x) \\
&\quad \le \frac{1}{\mu(B)} \int_B |T^*f_1(x)|^\sigma \, d\mu(x) + \inf_{c\in \boldsymbol{C}} \frac{1}{\mu(B)} \int_B |T^*f_2(x) - c|^\sigma \, d\mu(x).
\end{aligned}$$

Our hypothesis says that

$$\frac{1}{\mu(B)} \int_B |T^*f_1(x)|^\sigma \, d\mu(x) \lesssim \|f\|^\sigma_{L^\infty(X)}. \tag{3.4}$$

Since $|T^*f_2|^\sigma$ is locally integrable, we can take $x_B \in B$ such that $T^*f_2(x_B) < \infty$. For any $x \in B$, a standard computation together with (1.3) and (1.4) leads to that

$$\begin{aligned}
&|T^*f_2(x) - T^*f_2(x_B)| \\
&\quad \le \sup_{\epsilon>0} |T_\epsilon f_2(x) - T_\epsilon f_2(x_B)| \\
&\quad \le \sup_{\epsilon>0} \int_X |K(x, z) - K(x_B, z)||f_2(z)| \, d\mu(z) \\
&\qquad + \sup_{\epsilon>0} \int_X |K(x_B, z)|\left|\chi_{\{d(x_B, z)>\epsilon\}}(z) - \chi_{\{d(x, z)>\epsilon\}}(z)\right||f_2(z)| \, d\mu(z)
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \|f\|_{L^\infty(X)} + \sup_{\epsilon>0} \int_{\substack{d(x_B,\, z)\le\epsilon \\ d(x,\, z)>\epsilon}} |K(x_B, z)||f_2(z)|\, d\mu(z) \\
&\quad + \sup_{\epsilon>0} \int_{\substack{d(x_B,\, z)>\epsilon \\ d(x,\, z)\le\epsilon}} |K(x_B, z)||f_2(z)|\, d\mu(z) \\
&\lesssim \|f\|_{L^\infty(X)} \left\{ 1 + \sup_{\epsilon>0} \int_{\epsilon/(2A)<d(x_B,\, z)\le\epsilon} |K(x_B,\, z)|\, d\mu(z) \right. \\
&\qquad\qquad \left. + \sup_{\epsilon>0} \int_{\epsilon<d(x_B,\, z)\le(A+1/2)\epsilon} |K(x_B,\, z)|\, d\mu(z) \right\} \\
&\lesssim \|f\|_{L^\infty(X)},
\end{aligned}
$$

which together with (3.4) then gives us the inequality (3.3). Here, in the second-to-last inequality, we used the fact that $4A^2 r_B \le \epsilon$ by (3.2), where r_B is the radius of B. Hence, (iii) holds.

(iii) $\Rightarrow$ (iv). We first claim that for any $0 < s < 1$ and $f_1, f_2 \in \bigcup_{p\ge1} L^p(X)$,

$$
M^\sharp_{0,\, s}[T^*(f_1 + f_2)](x) \le M^\sharp_{0,\, s/2}(T^* f_1)(x) + M_{0,\, s/2}(T^* f_2)(x). \tag{3.5}
$$

In fact, for each fixed ball B, let $m_{0,\, s,\, B}(f)$ be the same as in (2.3) and

$$
\widetilde{m_{0,\, s,\, B}}(f) = \inf_{c\in C} \inf \big\{ t > 0 :\, \mu(\{y \in B :\, |f(y) - c| > t\}) < s\mu(B) \big\}.
$$

For each fixed $\sigma > 0$, there exist constants $c \in \boldsymbol{C}$, $t_1 \ge 0$ and $t_2 \ge 0$ with $t_1 < \widetilde{m_{0,\, s/2,\, B}}(T^* f_1) + \sigma/2$ and $t_2 < m_{0,\, s/2,\, B}(T^* f_2) + \sigma/2$ such that

$$
\mu\big(\{y \in B :\, |T^* f_1(y) - c| > t_1\}\big) < \left(\frac{s}{2}\right)\mu(B),
$$

and

$$
\mu\big(\{y \in B :\, T^* f_2(y) > t_2\}\big) < \left(\frac{s}{2}\right)\mu(B).
$$

Note that

$$
|T^*(f_1 + f_2)(y) - c| \le |T^* f_1(y) - c| + |T^* f_2(y)|.
$$

It is easy to see that

$$
\begin{aligned}
&\mu\big(\{y \in B :\, |T^*(f_1 + f_2)(y) - c| > t_1 + t_2\}\big) \\
&\qquad \le \mu\big(\{y \in B :\, |T^* f_1(y) - c| > t_1\}\big) + \mu\big(\{y \in B :\, T^* f_2(y) > t_2\}\big).
\end{aligned}
$$

Therefore,

$$\widetilde{m_{0,\,s,\,B}}[T^*(f_1+f_2)] \le \widetilde{m_{0,\,s/2,\,B}}(T^*f_1) + m_{0,\,s/2,\,B}(T^*f_2) + \sigma,$$

which together with the arbitrariness of σ and the definitions of $M^\sharp_{0,s}$ and $M_{0,s}$ yields (3.5).

We now prove that for any $s \in (0,1)$, there is some constant $C_8 > 0$ such that for any $f \in L^1(X) \cap L^2(X)$ with bounded support,

$$\mu\big(\{x \in X :\, M^\sharp_{0,\,s}(T^*f)(x) > (C_6 + C_8 s^{-1})\lambda\}\big) \lesssim s^{-1}\lambda^{-1}\|f\|_{L^1(X)}, \tag{3.6}$$

where C_6 is the same as in Lemma 3.3. Since T^* is bounded from $L^\infty_c(X)$ to $\mathrm{BMO}(X)$, then for any $f \in L^\infty_c(X)$,

$$\|M^\sharp(T^*f)\|_{L^\infty(X)} \lesssim \|f\|_{L^\infty(X)}, \tag{3.7}$$

where $M^\sharp$ is the Fefferman-Stein sharp function; see [**18**] for the details about $M^\sharp$. It is obvious that for any locally integrable function f and $x \in X$,

$$M^\sharp_{0,\,s} f(x) \le s^{-1} M^\sharp f(x). \tag{3.8}$$

Note that $\mu(X) = \infty$. For each fixed $f \in L^1(X) \cap L^2(X)$ with bounded support, and $\lambda > 0$, by Lemma 3.1, we can obtain a family of balls $\{B_j\}_{j\in\Lambda}$ and a constant $C > 0$ such that f can be decomposed into $f = g + b$ as in Lemma 3.1. The hypothesis that T^* is bounded from $L^\infty_c(X)$ to $\mathrm{BMO}(X)$ together with (3.8), (3.7) and (a_2) in Lemma 3.1 states that for some constant $C_8 > 0$,

$$\big\{x \in X :\, M^\sharp_{0,\,s/2}(T^*g)(x) > C_8 s^{-1}\lambda\big\} = \varnothing.$$

This along with the inequalities (3.5) and (2.5) leads to that

$$\begin{aligned}
&\mu\big(\{x \in X :\, M^\sharp_{0,\,s}(T^*f)(x) > (C_6 + C_8 s^{-1})\lambda\}\big)\\
&\quad\le \mu\big(\{x \in X :\, M^\sharp_{0,\,s/2}(T^*g)(x) > C_8 s^{-1}\lambda\}\big) + \mu\big(\{x \in X :\, M_{0,\,s/2}(T^*b)(x) > C_6\lambda\}\big)\\
&\quad\lesssim s^{-1}\mu\big(\{x \in X :\, T^*b(x) > C_6\lambda\}\big).
\end{aligned}$$

On the other hand, by Lemma 3.3 and Lemma 3.1 (a_5), we have that

$$\begin{aligned}
\mu\big(\{x \in X :\, T^*b(x) > C_6\lambda\}\big) &\le \mu\Big(\bigcup_j B_j^*\Big) + \mu\bigg(\Big\{x \in X \setminus \bigcup_j B_j^* :\, T^*b(x) > C_6\lambda\Big\}\bigg)\\
&\lesssim \lambda^{-1}\|f\|_{L^1(X)}.
\end{aligned}$$

Combining the estimates above yields (3.6).

Let $L^\infty_{c,0}(X)$ be the set of functions $f \in L^\infty_c(X)$ with $\int_X f(x)\,d\mu(x) = 0$. If we can verify that for any $f \in L^\infty_{c,0}(X)$,

$$\sup_{\lambda>0} \lambda\mu(\{x \in X : T^*f(x) > \lambda\}) < \infty, \tag{3.9}$$

then by Theorem 2.1 (with $s < (2^2 3^p C_4)^{-1}$) and (3.6), we see

$$\begin{aligned}\sup_{\lambda>0} \lambda\mu(\{x \in X : T^*f(x) > \lambda\}) &\lesssim \sup_{\lambda>0} \lambda\mu\big(\{x \in X : M^\sharp_{0,s}(T^*f)(x) > \lambda\}\big)\\ &\lesssim \|f\|_{L^1(X)}.\end{aligned}$$

This via a standard density argument states that T^* is bounded from $L^1(X)$ to $L^{1,\infty}(X)$, and so is $M_{0,s}T^*$ for any $s \in (0, 1/2]$ and $s < (2^2 3^p C_4)^{-1}$ by this fact together with (2.5) in Lemma 2.3. By the last fact and the fact that $M^\sharp_{0,s}T^*$ is bounded from $L^\infty_c(X)$ to $L^\infty(X)$ together with (3.5) and Lemma 3.2, we know that $M^\sharp_{0,s}T^*$ is bounded on $L^p(X)$ for any $p \in (1, \infty)$. An application of Theorem 2.2 then yields that for any $p \in (1, \infty)$ and $f \in L^\infty_{c,0}(X)$,

$$\|T^*f\|_{L^p(X)} \lesssim \|M^\sharp_{0,s}(T^*f)\|_{L^p(X)} \lesssim \|f\|_{L^p(X)}.$$

The density argument then gives us the desired $L^p(X)$-boundedness of T^* with $p \in (1, \infty)$.

To see (3.9), let $f \in L^\infty_{c,0}(X)$ and B be a ball such that $\operatorname{supp} f \subset B$. It is obvious that

$$\lambda\mu(\{x \in B^* : T^*f(x) > \lambda\}) \le \|T^*f\|_{L^1(B^*)} < \infty,$$

since $T^*f \in \mathrm{BMO}(X)$ and T^*f is locally integrable. On the other hand, by Lemma 3.3, we see that for any $\lambda > 0$,

$$\lambda\mu(\{x \in X \setminus B^* : T^*f(x) > \lambda\}) \lesssim \|f\|_{L^1(X)} < \infty.$$

The estimate (3.9) then holds. Thus, (iv) is true.

(iv) $\Rightarrow$ (v). It is easy to see that this is true by Lemma 3.1 and Lemma 3.3.

(v) $\Rightarrow$ (i). This is obvious if we choose $r = 1$ in (i), which completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. As we have pointed out in Remark 1.1, it suffices to show that when $\mu(X) < \infty$, then (1) $\Longrightarrow$ (2). Recall that T^* is bounded from $L^1(X)$ to $L^{1,\infty}(X)$ implies that T^* is bounded from $L^\infty_c(X)$ to $\mathrm{BMO}(X)$. For each fixed $s \in (0, 1/2)$, by Lemma 2.3, we know that $M_{0,s}$ is bounded from $L^{1,\infty}(X)$ to itself. If (1) is true, then the composition operator $M^\sharp_{0,s}T^*$ is bounded on $L^\infty_c(X)$, and $M_{0,s}T^*$ is bounded from $L^1(X)$ to $L^{1,\infty}(X)$. Thus by the estimate (3.5) and Lemma 3.2, we know that $M^\sharp_{0,s}T^*$ is bounded on $L^p(X)$ for any $p \in (1, \infty)$. Take $s < (2^2 3^p C_4)^{-1}$. An application of Theorem 2.2 then leads to that

$$\begin{aligned}\|T^* f\|_{L^p(X)} &\le C\|M^{\sharp}_{0,\,s}(T^* f)\|_{L^p(X)} + C\|T^* f\|_{L^{1,\,\infty}(X)}\\ &\le C\|f\|_{L^p(X)} + C\|T^* f\|_{L^{1,\,\infty}(X)}\\ &\le C\|f\|_{L^p(X)},\end{aligned}$$

which completes the proof of Theorem 1.2. □

4. Proof of Theorem 1.3.

By Theorem 1.1 and Theorem 1.2, we only need to prove that T^* is bounded from $L^1(X)$ to $L^{1,\infty}(X)$. To this end, we need the following Cotlar-type inequality.

LEMMA 4.1. *Under the hypothesis of Theorem* 1.3, *there exists a constant* $C > 0$ *such that for any* $f \in L^2(X) \cap L^\infty(X)$ *and* μ*-a.e.* $x \in X$,

$$T^* f(x) \le \mathscr{M}(Tf)(x) + C\|f\|_{L^\infty(X)}.$$

PROOF. We proceed with the proof as in that of Theorem 1 in [**7**]. For any fixed $f \in L^2(X) \cap L^\infty(X)$, since T is bounded on $L^2(X)$, $|Tf(x)|$ is finite almost everywhere. For each fixed x such that $|Tf(x)| < \infty$ and $\epsilon > 0$, decompose f into

$$\begin{aligned}f(y) &= f(y)\chi_{B(x,\,\epsilon)}(y) + f(y)\chi_{X\setminus B(x,\,A(4A^2+1)\epsilon)}(y) + f(y)\chi_{B(x,\,A(4A^2+1)\epsilon)\setminus B(x,\,\epsilon)}(y)\\ &= f_1(y) + f_2(y) + f_3(y).\end{aligned}$$

Observe that for any $y \in B(x,\,\epsilon)$, by (1.3),

$$\begin{aligned}|T_\epsilon f(x)| &= |T_\epsilon f_2(x)| + |T_\epsilon f_3(x)|\\ &= |Tf_2(x)| + |Tf_3(x)|\\ &\lesssim |Tf_2(x) - Tf_2(y)| + |Tf(y)| + |Tf_1(y)| + \|f\|_{L^\infty(X)}.\end{aligned}$$

As in (ii)$\Rightarrow$(iii) in the proof of Theorem 1.1, we know that for any $y \in B(x,\,\epsilon)$,

$$|Tf_2(x) - Tf_2(y)| \lesssim \|f\|_{L^\infty(X)}.$$

Therefore, by the $L^2(X)$-boundedness of T,

$$\begin{aligned}|T_\epsilon f(x)| &\lesssim \|f\|_{L^\infty(X)} + \frac{1}{\mu(B(x,\,\epsilon))}\int_{B(x,\,\epsilon)} |Tf(y)|\,d\mu(y)\\ &\quad + \left\{\frac{1}{\mu(B(x,\,\epsilon))}\int_{B(x,\,\epsilon)} |Tf_1(y)|^2\,d\mu(y)\right\}^{1/2}\\ &\lesssim \|f\|_{L^\infty(X)} + \mathscr{M}(Tf)(x).\end{aligned}$$

Taking supremum for all $\epsilon > 0$ gives the conclusion of Lemma 4.1. □

Proof of Theorem 1.3. By Theorem 1.1 and Theorem 1.2, it suffices to prove that T^* is bounded from $L^1(X)$ to weak $L^1(X)$. For any fixed $\lambda > [\mu(X)]^{-1}\|f\|_{L^1(X)}$, applying Lemma 3.1 to f at level λ, we can decompose f into $f = g + b$ with g and b same as in Lemma 3.1. By Lemma 3.3, we see that

$$\mu(\{x \in X : T^*b(x) > \lambda\}) \lesssim \lambda^{-1}\|f\|_{L^1(X)}.$$

On the other hand, for some constant $C_9 > 0$, which is large enough and independent of f and λ, Lemma 4.1 together with the $L^2(X)$-boundedness of $\mathscr{M}$ and Lemma 3.1 tells us that

$$\begin{aligned}\mu(\{x \in X : T^*g(x) > C_9\lambda\}) &\le \mu(\{x \in X : \mathscr{M}(Tg)(x) > \lambda\})\\ &\le \lambda^{-2}\|\mathscr{M}(Tg)\|_{L^2(X)}\\ &\lesssim \lambda^{-1}\|g\|_{L^1(X)}\\ &\lesssim \lambda^{-1}\|f\|_{L^1(X)}.\end{aligned}$$

This completes the proof of Theorem 1.3. □

5. Some applications.

This section is devoted to some applications of Theorem 1.1 through Theorem 1.3. We first consider the Monge-Ampère singular integral operators in [**4**].

5.1. Monge-Ampère singular integral operators.

For $x \in \boldsymbol{R}^n$ and $t > 0$, denote by $S(x, t)$ certain open and bounded convex set containing x. We call $\mathscr{F} = \{S(x, t) : x \in \boldsymbol{R}^n, t > 0\}$ a family of sections if $\{S(x, t) : x \in \boldsymbol{R}^n, t > 0\}$ is monotone increasing in t, i. e., $S(x, t) \subset S(x, t')$ for $t \le t'$, and satisfies the following three conditions:

(a) There exist positive constants K_1, K_2, K_3, ϵ_1 and ϵ_2 such that given two sections $S(x_0, t_0)$ and $S(x, t)$ with $t \le t_0$ satisfying $S(x_0, t_0) \cap S(x, t) \ne \varnothing$, and given an affine transformation T that "normalizes" $S(x_0, t_0)$, i. e.,

$$B\left(0, \frac{1}{n}\right) \subset T(S(x_0, t_0)) \subset B(0, 1),$$

there exists $z \in B(0, K_3)$ depending on $S(x_0, t_0)$ and $S(x, t)$ such that

$$B\left(z, K_2\left(\frac{t}{t_0}\right)^{\epsilon_2}\right) \subset T(S(x, t)) \subset B\left(z, K_1\left(\frac{t}{t_0}\right)^{\epsilon_1}\right)$$

and $T(x) \in B(z, \frac{1}{2}K_2(\frac{t}{t_0})^{\epsilon_2})$. Here and in what follows, $B(x,t)$ denotes the Euclidean open ball centered at x with radius $t > 0$.

(b) There exists a constant $\sigma > 0$ such that for any given section $S(x, t)$ and $y \notin$

$S(x, t)$, if T is an affine transformation that normalizes $S(x, t)$,

$$B(T(y), \epsilon^\sigma) \cap T(S(x, (1-\epsilon)t)) = \varnothing$$

for any $\epsilon \in (0, 1)$.

(c) $\bigcap_{t>0} S(x, t) = \{x\}$ and $\bigcup_{t>0} S(x, t) = \boldsymbol{R}^n$.

In addition we assume that a positive Borel regular measure μ which is finite on compact sets is given, $\mu(\boldsymbol{R}^n) = \infty$, and satisfies the following doubling condition

$$\mu(S(x, 2t)) \le C\mu(S(x, t)), \tag{5.1}$$

where $C > 0$ is independent of x and t.

The definition of sections was introduced by Caffarelli and Gutiérrez [**3**] to establish a real variable theory associated to the Monge-Ampère equation. Caffarelli and Gutiérrez [**3**] established a Besicovitch type covering lemma for $\mathscr{F}$, a family of sections. In terms of sections, they set up a variant of the Calderón-Zygmund decomposition by applying this covering lemma. As applications of this decomposition, Caffarelli and Gutiérrez introduced the Hardy-Litttlewood maximal operator $\mathscr{M}$ and the space $\mathrm{BMO}_{\mathscr{F}}$ associated to a family of sections and the above given doubling measure, and obtained some important results on the maximal operator $\mathscr{M}$ and this $\mathrm{BMO}_{\mathscr{F}}$ space. Recently, there are several papers concerning the real analysis associated to the Monge-Ampère equation. Aimar, Forzani and Toledano [**2**] proved that the properties (a) and (b) imply the following engulfing properties of sections: there is a constant $\theta > 1$, depending only on σ, K_1 and ϵ_1, such that for any $x, y \in \boldsymbol{R}^n$ and $t > 0$, $y \in S(x, t)$ implies that $S(y, t) \subset S(x, \theta t)$ and $S(x, t) \subset S(y, \theta t)$. Moreover, they introduced the function

$$d(x, y) = \inf\{t > 0 : x \in S(y, t) \text{ and } y \in S(x, t)\}$$

and proved that d is a quasi metric satisfying that for all $x, z, y \in \boldsymbol{R}^n$,

$$d(x,y) \le \theta(d(x,z) + d(z,y)),$$

and also that for any $x \in \boldsymbol{R}^n$ and $t > 0$, $S(x, t/(2\theta)) \subset B_d(x, t) \subset S(x, t)$, where $B_d(x, t) = \{y \in \boldsymbol{R}^n : d(x, y) < t\}$. From this and (5.1), it is easy deduce that

$$\mu(B_d(x, 2t)) \le (C_0)^{[\log_2(4\theta)]+1} \mu(B_d(x, t)),$$

where $[\log_2(4\theta)]$ is the largest integer no more than $\log_2(4\theta)$. Thus $(\boldsymbol{R}^n, d, \mu)$ is a space of homogeneous type in the sense of Coifman and Weiss [**5**]; see also [**2**]. Incognito [**10**] introduced another "metric" ρ associated to the sections:

$$\rho(x, y) = \inf\{t > 0 : y \in S(x, t)\}$$

and proved that $\rho(x, y) \le \theta\rho(y, x)$ and

$$\rho(x, y) \le \theta^2(\rho(x, z) + \rho(z, y))$$

for any $x, y, z \in \boldsymbol{R}^n$. With the aid of the function ρ, Incognito discussed the boundedness of following Monge-Ampère singular integrals.

For each fixed $y \in \boldsymbol{R}^n$ and $j \in \boldsymbol{Z}$, denote by $S_j(y)$ the section $S(y, 2^j)$. Let $\{K_j\}_j$ be a sequence of functions on $\boldsymbol{R}^n \times \boldsymbol{R}^n$ such that for any $x \in \boldsymbol{R}^n$, $\operatorname{supp} K_j(x, \cdot) \subset S_j(x)$,

$$\int_{\boldsymbol{R}^n} K_j(x, y)\, d\mu(y) = 0,$$

$$\sup_j \int_{\boldsymbol{R}^n} |K_j(x, y)|\, d\mu(y) \lesssim 1,$$

if T is an affine transformation that normalizes the section $S_j(y)$, then for $\alpha \in (0, 1]$,

$$|K_j(x, u) - K_j(x, v)| \lesssim \frac{1}{\mu(S_j(x))} |Tu - Tv|^\alpha,$$

and all still hold with x and y interchanged. Let $K = \sum_j K_j$. The operator defined by

$$Tf(x) = \int_{\boldsymbol{R}^n} K(x, y) f(y)\, d\mu(y) \tag{5.2}$$

is called the Monge-Ampère singular integral operator. Caffarelli and Gutiérrez [**4**] proved that for $\alpha = 1$, the operator T is bounded on $L^2(\boldsymbol{R}^n, \mu)$. Incognito [**10**] proved that for $0 < \alpha \le 1$, the operator T is bounded on $L^2(\boldsymbol{R}^n, \mu)$ and also that

$$\int_{\rho(x, y) \le 4\theta^2 \rho(y', y)} |K(x, y) - K(x, y')|\, d\mu(x) \lesssim 1,$$

from which he further deduced that T is bounded from $L^1(\boldsymbol{R}^n, \mu)$ to $L^{1,\infty}(\boldsymbol{R}^n, \mu)$. Note that $\theta^{-1} d(x, y) \le \rho(x, y) \le d(x, y)$; see [**9**]. As in the proof of Lemma 1 in [**10**], we can verify that

$$\int_{d(x, y) \ge 4\theta^3 d(y, y')} \big(|K(x, y) - K(x, y')| + |K(y, x) - K(y', x)|\big)\, d\mu(x) \lesssim 1.$$

Therefore, by Theorem 1.3, we have

Theorem 5.1. *Let T be the Monge-Ampère singular integral operator as in* (5.2) *and T^* be the associated maximal operator defined by*

$$T^* f(x) = \sup_{\epsilon > 0} \left| \int_{d(x, y) \ge \epsilon} K(x, y) f(y)\, d\mu(y) \right|.$$

If, in addition, the kernel K satisfies

$$\sup_{R>0}\int_{R<d(x,\,y)\le 2R}(|K(x,\,y)|+K(y,\,x)|)\,d\mu(x)<\infty,$$

then T^* is bounded on $L^p(\boldsymbol{R}^n,\,\mu)$ for any $p\in(1,\infty)$, bounded from $L^1(\boldsymbol{R}^n,\,\mu)$ to weak $L^1(\boldsymbol{R}^n,\,\mu)$ and bounded from $L^\infty_c(\boldsymbol{R}^n,\mu)$ to $\mathrm{BMO}(\boldsymbol{R}^n,\mu)$.

5.2. Singular integral operators of Nagel and Stein.

In this subsection, we apply Theorem 1.3 to obtain the boundedness of maximal singular integral operators of Nagel and Stein in [**20**]. Such singular integral operators naturally appear in the study on solutions of the Kohn-Laplacian for certain unbounded model polynomial domains in several complex variables in [**21**]. To be precise, we consider two specific settings as in [**20**]:

(A) Let M be a compact connected C^∞-manifold of dimension at least 3. Suppose that there exist smooth vector fields $\{X_1,\,\dots,\,X_k\}$ on M, which together with their commutators of order $\le m$ span the tangent space to M at each point.

(B) Let $\Omega=\{(z,w)\in\boldsymbol{C}^2:\ \Im m[w]>P(z)\}$, where P is a real, subharmonic, non-harmonic polynomial of degree m. Then $M=\partial\Omega$ can be identified with $\boldsymbol{C}\times\boldsymbol{R}=\{(z,t):\ z\in\boldsymbol{C},\ t\in\boldsymbol{R}\}$. The basic $(0,1)$ Levi vector field is then $\bar Z=\frac{\partial}{\partial\bar z}-i\frac{\partial P}{\partial\bar z}\frac{\partial}{\partial t}$, and we write $\bar Z=X_1+iX_2$. The real vector fields $\{X_1,\ X_2\}$ and their commutators of orders $\le m$ span the tangent space at each point.

In both cases, we endow M with the control, or Carnot-Carathéodory, metric d determined by the given smooth real vector fields ([**22**], [**19**], [**20**]); and in the compact situation (A), we endow M with any fixed smooth measure μ of strictly positive density, and in the situation (B), we endow M with the Lebesgue measure μ. Then by the results in [**22**], [**19**] (see also Proposition 3.1.1 in [**21**]), $(M,\,d,\,\mu)$ is a space of homogeneous type in the sense of Coifman and Weiss; see [**20**] for more details.

A function φ on M is said to be a bump function associated to a ball $B(x_0,\,\delta)$, if it is supported in that ball, and satisfies the differential inequalities $|\partial^a_X\varphi|\lesssim\delta^{-a}$ for all monomials ∂_X in $X_1,\dots,X_k$ of degree a and all $a\ge 0$.

We now recall the definition of singular integral operators of Nagel and Stein as below; see [**20**, pp. 546–547]. Let T be a linear mapping from $C^\infty_0(M)$ to $C^\infty(M)$, where $C^\infty_0(M)$ is the space of C^∞ functions on M of compact support. Suppose the operator T has a distribution kernel K which is C^∞ away from the diagonal of $M\times M$ and satisfies the following four properties:

(I-1) If $\varphi,\,\psi\in C^\infty_0(M)$ have disjoint supports, then

$$\langle T\varphi,\,\psi\rangle=\int_{M\times M}K(x,\,y)\varphi(y)\psi(x)\,d\mu(x)\,d\mu(y).$$

(I-2) If φ is a normalized bump function associated to a ball of radius r, then $|\partial^a_X T\varphi|\lesssim r^{-a}$. More precisely, for each integer $a\ge 0$, there is another integer $b\ge 0$ and a constant $M_{a,\,b}$ so that whenever φ is a C^∞ function supported in a ball $B(x_0,\,r)$, then

$$\sup_{x\in M} r^a\left|(\partial_X^a T\varphi)(x)\right| \le M_{a,\,b}\sup_{c\le b}\sup_{x\in B(x_0,\,r)} r^c\left|\partial_X^c\varphi\right|.$$

(I-3) If $x\neq y$, then for every $a\geq 0$, $|\partial^a_{X,\,Y}K(x,\,y)|\lesssim d(x,\,y)^{-a}V(x,\,y)^{-1}$.

(I-4) Properties (I-1) through (I-3) also hold with x and y interchanged. That is, these properties also hold for the adjoint operator T^t defined by $\langle T^t\varphi,\,\psi\rangle=\langle T\psi,\,\varphi\rangle$.

Nagel and Stein [**20**] proved that each singular integral operator T satisfying the conditions (I-1) through (I-4) extends to a bounded operator on $L^p(M)$ for any $p\in(1,\infty)$. From this and Theorem 1.3, it immediately follows the following conclusion.

Theorem 5.2. *Let T be a singular integral operator satisfying the conditions* (I-1) *through* (I-4). *Define*

$$T^*f(x)=\sup_{\epsilon>0}\left|\int_{d(x,\,y)>\epsilon}K(x,\,y)f(y)\,d\mu(y)\right|.$$

Then T^* is bounded on $L^p(M)$ for any $p\in(1,\infty)$, bounded from $L^1(M)$ to weak $L^1(M)$ and bounded from $L^\infty_c(M)$ to BMO(M).

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