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# Isometries of weighted Bergman-Privalov spaces on the unit ball of $C^n$

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**Abstract.** Let $B$ denote the unit ball in $C^n$, and $v$ the normalized Lebesgue measure on $B$. For $\alpha > -1$, define $dv_\alpha(z) = \Gamma(n+\alpha+1)/\{\Gamma(n+1)\Gamma(\alpha+1)\}(1-|z|^2)^\alpha\, dv(z)$, $z \in B$. Let $H(B)$ denote the space of holomorphic functions in $B$. For $p \geq 1$, define

$$(AN)^p(v_\alpha) = \left\{ f \in H(B) : \|f\| \equiv \left[\int_B \{\log(1+|f|)\}^p\, dv_\alpha\right]^{1/p} < \infty \right\}.$$

$(AN)^p(v_\alpha)$ is an $F$-space with respect to the metric $\rho(f,g) \equiv \|f-g\|$. In this paper we prove that every linear isometry $T$ of $(AN)^p(v_\alpha)$ into itself is of the form $Tf = c(f \circ \psi)$ for all $f \in (AN)^p(v_\alpha)$, where $c$ is a complex number with $|c| = 1$ and $\psi$ is a holomorphic self-map of $B$ which is measure-preserving with respect to the measure $v_\alpha$.

## 1. Introduction.

Let $n \geq 1$ be a fixed integer. Let $H(B)$ denote the space of all holomorphic functions in the open unit ball $B \equiv B_n$ of the complex $n$-dimensional Euclidean space $C^n$. Let $v$ denote the normalized Lebesgue measure on $B$. For each $\alpha \in (-1, \infty)$, we set $c_\alpha = \Gamma(n+\alpha+1)/\{\Gamma(n+1)\Gamma(\alpha+1)\}$ and $dv_\alpha(z) = c_\alpha(1-|z|^2)^\alpha\, dv(z)$, $z \in B$. Note that $v_\alpha(B) = 1$. For each $\alpha \in (-1,\infty)$ and $p \in [1, \infty)$, we define the *weighted Bergman-Privalov space* $(AN)^p(v_\alpha)$ by

$$(AN)^p(v_\alpha) = \left\{ f \in H(B) : \|f\|_{(AN)^p(v_\alpha)} \equiv \left[\int_B \{\log(1+|f|)\}^p\, dv_\alpha\right]^{1/p} < \infty \right\}.$$

In [**9**], the *Privalov space* $N^p(B)$ $(1 < p < \infty)$ is defined by

$$N^p(B) = \left\{ f \in H(B) : \|f\|_{N^p(B)} \equiv \sup_{0 \leq r < 1} \left[\int_S \{\log(1+|f_r|)\}^p\, d\sigma\right]^{1/p} < \infty \right\},$$

where $\sigma$ is the normalized Euclidean measure on the unit sphere $S \equiv \partial B$ and

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$f_r(z) = f(rz)$ for $0 \le r < 1$, $z \in \boldsymbol{C}^n$ with $rz \in B$. In the case $n = 1$, the spaces $N^p(B_1)$ were firstly considered by I. I. Privalov in [**6**]. Their properties were studied in [**8**], [**5**] and [**9**]. Moreover, M. Stoll [**8**] (p. 157) defined the Bergman-Privalov spaces $(AN)^p(v)$ and mentioned their some properties. For $f \in H(B)$ and $p \in (1, \infty)$, it holds that $\lim_{\alpha\downarrow -1} \|f\|_{(AN)^p(v_\alpha)} = \|f\|_{N^p(B)}$. (See [**1**], §0.3 and p. 25.) Recently, Y. Iida-N. Mochizuki [**3**] (in the case $n = 1$) and A. V. Subbotin [**10**] (in any dimensional case $n \ge 1$) have determined the isometries of the Privalov spaces $N^p(B)$:

Theorem (Y. Iida-N. Mochizuki and A. V. Subbotin). *Let $1 < p < \infty$. Then every linear isometry $T$ of $N^p(B)$ into itself is of the form $Tf = \varphi(f \circ \psi)$ for all $f \in N^p(B)$, where $\varphi$ is an inner function in $B$ and $\psi$ is a holomorphic self-map of $B$ whose radial limit map $\psi^*$ is measure-preserving with respect to the measure $\sigma$. This means that it holds that $\int_S h \circ \psi^* \, d\sigma = \int_S h \, d\sigma$ for every bounded or positive Borel function $h$ on $S$.*

The purpose of the present paper is to prove an analogous result for the linear isometries of the Bergman-Privalov spaces $(AN)^p(v_\alpha)$.

## 2. Preliminaries.

In order to prove our main result (Theorem 1 in §3) we need several lemmas. From now on, till the end of this paper, we fix $\alpha \in (-1, \infty)$ and $p \in [1, \infty)$.

Lemma 1. *Suppose $f \in H(B)$ and $z \in B$. Then*

$$\log(1 + |f(z)|) \le \left(\frac{1 + |z|}{1 - |z|}\right)^{(n+1+\alpha)/p} \|f\|_{(AN)^p(v_\alpha)}.$$

Proof. Let $\varphi_z$ be the biholomorphic involution of $B$ described in [**7**], p. 25. Put $u = \{\log(1 + |f \circ \varphi_z|)\}^p$ in $B$. Then $u$ is a positive plurisubharmonic function in $B$. We therefore have

$$\begin{aligned} u(0) &= \int_B u(0) \, dv_\alpha = 2nc_\alpha \int_0^1 r^{2n-1}(1 - r^2)^\alpha u(0) \, dr \\ &\le 2nc_\alpha \int_0^1 r^{2n-1}(1 - r^2)^\alpha \, dr \int_S u(r\zeta) \, d\sigma(\zeta) = \int_B u \, dv_\alpha. \end{aligned}$$

That is,

$$\begin{aligned} \{\log(1 + |f(z)|)\}^p &\le c_\alpha \int_B \{\log(1 + |f(\varphi_z(w))|)\}^p (1 - |w|^2)^\alpha \, dv(w) \\ &= c_\alpha \int_B \{\log(1 + |f(w)|)\}^p (J_{\boldsymbol{R}}\varphi_z)(w)(1 - |\varphi_z(w)|^2)^\alpha \, dv(w). \qquad (1) \end{aligned}$$

By [7], Theorems 2.2.2 and 2.2.6, for $w \in B$,

$$\begin{aligned}
&(J_{\boldsymbol{R}}\varphi_z)(w)(1-|\varphi_z(w)|^2)^\alpha \\
&\quad = \left(\frac{1-|z|^2}{|1-\langle z,w\rangle|^2}\right)^{n+1}\left\{\frac{(1-|z|^2)(1-|w|^2)}{|1-\langle z,w\rangle|^2}\right\}^\alpha \\
&\quad = \frac{(1-|z|^2)^{n+1+\alpha}(1-|w|^2)^\alpha}{|1-\langle z,w\rangle|^{2(n+1+\alpha)}} \le \left(\frac{1+|z|}{1-|z|}\right)^{n+1+\alpha}(1-|w|^2)^\alpha. \qquad (2)
\end{aligned}$$

The lemma follows from (1) and (2). □

Lemma 2. (a) *Let* $\{f,g\} \subset H(B)$ *and* $c \in \boldsymbol{C}$. *Then*

$$\|f+g\|_{(AN)^p(v_\alpha)} \le \|f\|_{(AN)^p(v_\alpha)} + \|g\|_{(AN)^p(v_\alpha)},$$
$$\|fg\|_{(AN)^p(v_\alpha)} \le \|f\|_{(AN)^p(v_\alpha)} + \|g\|_{(AN)^p(v_\alpha)},$$
$$\min\{1,|c|\}\|f\|_{(AN)^p(v_\alpha)} \le \|cf\|_{(AN)^p(v_\alpha)} \le \max\{1,|c|\}\|f\|_{(AN)^p(v_\alpha)}.$$

(b) *Define* $\rho_{(AN)^p(v_\alpha)}(f,g) = \|f-g\|_{(AN)^p(v_\alpha)}$ *for* $\{f,g\} \subset (AN)^p(v_\alpha)$. *Then* $\rho_{(AN)^p(v_\alpha)}$ *is a complete metric on the space* $(AN)^p(v_\alpha)$.

(c) *The space* $(AN)^p(v_\alpha)$ *equipped with the metric* $\rho_{(AN)^p(v_\alpha)}$ *is an F-algebra with respect to pointwise addition and multiplication.*

Proof. We can prove this lemma in the same way that is used to prove the corresponding one for the Privalov space $N^p(B)$. See [8], Theorem 4.2 and [9], Theorem 3. The completeness of the metric space $((AN)^p(v_\alpha), \rho_{(AN)^p(v_\alpha)})$ follows from Lemma 1. □

Lemma 3. *Every* $f \in (AN)^p(v_\alpha)$ *satisfies* $\lim_{r\uparrow 1}\|f_r - f\|_{(AN)^p(v_\alpha)} = 0$.

Proof (cf. [2], §6.1). Fix $\varepsilon > 0$. Since $f \in (AN)^p(v_\alpha)$, there exists an $r_1 \in (0,1)$ such that $\int_{B\setminus r_1B}\{\log(1+|f|)\}^p\,dv_\alpha < \varepsilon$, where $r_1B = \{z \in \boldsymbol{C}^n : |z| < r_1\}$. Noting that $\{\log(1+|f|)\}^p$ is plurisubharmonic in $B$, we have for any $r \in (0,1)$

$$\begin{aligned}
&\int_{B\setminus r_1B}\{\log(1+|f_r|)\}^p\,dv_\alpha \\
&\quad = 2nc_\alpha\int_{r_1}^1 t^{2n-1}(1-t^2)^\alpha\,dt\int_S\{\log(1+|f(tr\zeta)|)\}^p\,d\sigma(\zeta) \\
&\quad \le 2nc_\alpha\int_{r_1}^1 t^{2n-1}(1-t^2)^\alpha\,dt\int_S\{\log(1+|f(t\zeta)|)\}^p\,d\sigma(\zeta) \\
&\quad = \int_{B\setminus r_1B}\{\log(1+|f|)\}^p\,dv_\alpha < \varepsilon.
\end{aligned}$$

Choose $\varepsilon_0 \in (0,\infty)$ so that $\{\log(1+\varepsilon_0)\}^p = \varepsilon$. Since $f$ is continuous on the compact set $r_1\bar{B}$, there exists a $\delta \in (0,1)$ such that $|f(z)-f(w)| < \varepsilon_0$ if $\{z,w\} \subset r_1\bar{B}$ and $|z-w| < \delta$. If $1-\delta < r < 1$, then

$$\begin{aligned}
\|f_r - f\|^p_{(AN)^p(v_\alpha)} &= \int_B \{\log(1+|f_r - f|)\}^p \, dv_\alpha \\
&= \left(\int_{r_1 B} + \int_{B\setminus r_1 B}\right)\{\log(1+|f_r - f|)\}^p \, dv_\alpha \\
&< \{\log(1+\varepsilon_0)\}^p + \int_{B\setminus r_1 B}\{\log(1+|f_r|) + \log(1+|f|)\}^p \, dv_\alpha \\
&\le \varepsilon + 2^{p-1}\left[\int_{B\setminus r_1 B}\{\log(1+|f_r|)\}^p \, dv_\alpha + \int_{B\setminus r_1 B}\{\log(1+|f|)\}^p \, dv_\alpha\right] \\
&< (1+2^p)\varepsilon.
\end{aligned}$$

This completes the proof. □

LEMMA 4. *Let $T$ be a linear isometry of $(AN)^p(v_\alpha)$ into itself. Then $T(A^p(v_\alpha)) \subset A^p(v_\alpha)$ and the restriction of $T$ to $A^p(v_\alpha)$ is a linear isometry of $A^p(v_\alpha)$ into itself. Here $A^p(v_\alpha)$ is the weighted Bergman space*:

$$A^p(v_\alpha) = \left\{ f \in H(B) : \|f\|_{A^p(v_\alpha)} \equiv \left(\int_B |f|^p \, dv_\alpha\right)^{1/p} < \infty \right\}.$$

PROOF. By adopting the way to prove Lemma 2 in [**3**], we can easily show this lemma. (See also [**10**], §3.) □

LEMMA 5 (C. J. Kolaski [**4**]). *Let $0 < q < \infty$, $q \neq 2$ and $-1 < \beta < \infty$. Let $T$ be a linear isometry of $A^q(v_\beta)$ into itself. Then there exists a holomorphic self-map $\psi$ of $B$ with the following two properties*:

(a) *$Tf = g(f \circ \psi)$ for every $f \in A^q(v_\beta)$, where $g = T1 \in A^q(v_\beta)$.*

(b) *$\int_B (h \circ \psi)|g|^q \, dv_\beta = \int_B h \, dv_\beta$ for every bounded or positive Borel function $h$ in $B$.*

PROOF. See [**4**], Theorem 1 and §4. □

LEMMA 6. Let $0 < q < \infty$. *Then there exists a bounded continuous function $\theta_q$ in $[0,\infty)$ such that*

$$x^q - \{\log(1+x)\}^q = \frac{q}{2}x^{q+1} - x^{q+2}\theta_q(x) \ge 0 \quad (0 \le x < \infty).$$

*In particular, $\theta_2 \ge 0$ in $[0,\infty)$.*

Proof. See [**3**], Lemma 1 and p. 299. □

Lemma 7. *Let $T$ be a linear isometry of $(AN)^2(v_\alpha)$ into itself. Then the restriction of $T$ to $A^3(v_\alpha)$ is a linear isometry of $A^3(v_\alpha)$ into itself.*

Proof (cf. [**3**], p. 299). By Lemma 4, the restriction of $T$ to $A^2(v_\alpha)$ is a linear isometry of $A^2(v_\alpha)$ into itself. Let $f \in A^3(v_\alpha)$ and put $g = Tf$. Then $\{f, g\} \subset A^2(v_\alpha)$. For all $t \in (0, \infty)$, by Lemma 6,

$$\begin{aligned}\int_B \{|g|^3 - t|g|^4\theta_2(|tg|)\}\,dv_\alpha &= \int_B \left[\frac{1}{t}|g|^2 - \frac{1}{t^3}\{\log(1+|tg|)\}^2\right] dv_\alpha \\ &= \int_B \left[\frac{1}{t}|f|^2 - \frac{1}{t^3}\{\log(1+|tf|)\}^2\right] dv_\alpha \\ &= \int_B \{|f|^3 - t|f|^4\theta_2(|tf|)\}\,dv_\alpha.\end{aligned}$$

By Lemma 6, in $B$,

$$0 \le |f|^3 - t|f|^4\theta_2(|tf|) \le |f|^3, \quad \lim_{t\downarrow 0}\{|f|^3 - t|f|^4\theta_2(|tf|)\} = |f|^3,$$

$$0 \le |g|^3 - t|g|^4\theta_2(|tg|) \le |g|^3, \quad \lim_{t\downarrow 0}\{|g|^3 - t|g|^4\theta_2(|tg|)\} = |g|^3.$$

It follows from Fatou's lemma and Lebesgue's dominated convergence theorem that

$$\begin{aligned}\int_B |g|^3\,dv_\alpha &\le \liminf_{t\downarrow 0} \int_B \{|g|^3 - t|g|^4\theta_2(|tg|)\}\,dv_\alpha \\ &= \lim_{t\downarrow 0} \int_B \{|f|^3 - t|f|^4\theta_2(|tf|)\}\,dv_\alpha = \int_B |f|^3\,dv_\alpha < \infty.\end{aligned}$$

Hence $g \in A^3(v_\alpha)$ and $\|g\|_{A^3(v_\alpha)} = \|f\|_{A^3(v_\alpha)}$. This completes the proof. □

## 3. Main results.

The proofs of our main theorems are essentially the same as those of Y. Iida-N. Mochizuki-A. V. Subbotin theorems. For the sake of completeness, however, we describe them.

Theorem 1. *Every linear isometry $T$ of $(AN)^p(v_\alpha)$ into itself is of the form $Tf = c(f \circ \psi)$ for all $f \in (AN)^p(v_\alpha)$, where $c$ is a complex number with $|c| = 1$ and $\psi$ is a holomorphic self-map of $B$ such that $\int_B (h \circ \psi)\,dv_\alpha = \int_B h\,dv_\alpha$ for every bounded or positive Borel function $h$ in $B$.*

Proof. Let $T$ be a linear isometry of $(AN)^p(v_\alpha)$ into itself. Put $q = p$ if $p \neq 2$, and $q = 3$ if $p = 2$. Then $1 \leq q < \infty$ and $q \neq 2$. By Lemma 4 and Lemma 7, the restriction of $T$ to $A^q(v_\alpha)$ is a linear isometry of $A^q(v_\alpha)$ into itself. By Lemma 5, there exists a holomorphic self-map $\psi$ of $B$ with the two properties (a) and (b) in the statement of Lemma 5. Since $g = T1 \in A^q(v_\alpha)$ and $v_\alpha(B) = 1$, by Hölder's inequality we have

$$1 = \|1\|_{A^q(v_\alpha)} = \|g\|_{A^q(v_\alpha)} = \|g\|_{L^q(v_\alpha)} \leq \|g\|_{L^{q+1}(v_\alpha)}.$$

And as in the proof of Lemma 7, we have

$$\begin{aligned}\int_B \frac{q}{2}|g|^{q+1}\,dv_\alpha &\leq \liminf_{t\downarrow 0}\int_B \left\{\frac{q}{2}|g|^{q+1} - t|g|^{q+2}\theta_q(|tg|)\right\} dv_\alpha \\ &= \lim_{t\downarrow 0}\int_B \left\{\frac{q}{2} - t\theta_q(t)\right\} dv_\alpha = \int_B \frac{q}{2}\,dv_\alpha = \frac{q}{2}.\end{aligned}$$

Hence $\|g\|_{L^q(v_\alpha)} = \|g\|_{L^{q+1}(v_\alpha)} = 1$, and so $|g| = 1$ in $B$. Since $g \in H(B)$, $g \equiv c$ in $B$ where $c \in \boldsymbol{C}$ with $|c| = 1$.

Now let $f \in (AN)^p(v_\alpha)$. By Lemma 3,

$$\lim_{r\uparrow 1}\|T(f_r) - Tf\|_{(AN)^p(v_\alpha)} = \lim_{r\uparrow 1}\|f_r - f\|_{(AN)^p(v_\alpha)} = 0.$$

Since $\{f_r : 0 \leq r < 1\} \cup \{T(f_r) : 0 \leq r < 1\} \subset A^q(v_\alpha)$, (a) and Lemma 1 give

$$Tf = \lim_{r\uparrow 1} T(f_r) = \lim_{r\uparrow 1} c(f_r \circ \psi) = c(f \circ \psi) \quad \text{in } B.$$

Since $|g| = 1$ in $B$, (b) implies that the self-map $\psi$ of $B$ is measure-preserving with respect to the measure $v_\alpha$.

Conversely, if $T$ is a mapping of the form described in the statement of the present theorem, it is easily shown that $T$ is a linear isometry of $(AN)^p(v_\alpha)$ into itself. □

Theorem 2. *Every linear isometry $T$ of $(AN)^p(v_\alpha)$ onto itself is of the form $Tf = c(f \circ U)$ for all $f \in (AN)^p(v_\alpha)$, where $c$ is a complex number with $|c| = 1$ and $U$ is a unitary operator on $\boldsymbol{C}^n$.*

Proof. Let $T$ be a linear isometry of $(AN)^p(v_\alpha)$ onto itself. Then, by Theorem 1, there exists a $c \in \boldsymbol{C}$ with $|c| = 1$ and a holomorphic self-map $\psi$ of $B$ such that $\int_B (h \circ \psi)\,dv_\alpha = \int_B h\,dv_\alpha$ for every bounded or positive Borel function $h$ in $B$. This property of $\psi$ yields $\psi(0) = 0$. Since $T^{-1}$ is also a linear isometry of $(AN)^p(v_\alpha)$ onto itself, it follows that $\psi$ is biholomorphic. Hence $\psi$ is a unitary operator on $\boldsymbol{C}^n$. □

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