J. Math. Soc. Japan
Vol. 53, No. 3, 2001

On the singular solutions of nonlinear singular partial differential equations I

By Hidetoshi TAHARA

(Received Mar. 2, 2000)

Abstract. Let us consider the following nonlinear singular partial differential equation: $(t\partial_t)^m u = F(t, x, \{(t\partial_t)^j \partial_x^\alpha u\}_{j+|\alpha| \le m, j<m})$ in the complex domain. Denote by $\mathscr{S}_+$ [*resp.* $\mathscr{S}_{log}$] the set of all the solutions $u(t,x)$ with asymptotics $u(t,x) = O(|t|^a)$ [*resp.* $u(t,x) = O(1/|\log t|^a)$] (as $t \to 0$ uniformly in x) for some $a > 0$. Clearly $\mathscr{S}_{log} \supset \mathscr{S}_+$. The paper gives a sufficient condition for $\mathscr{S}_{log} = \mathscr{S}_+$ to be valid.

The paper deals with nonlinear singular partial differential equations of the form

$$(t\partial/\partial t)^m u = F(t, x, \{(t\partial/\partial t)^j (\partial/\partial x)^\alpha u\}_{j+|\alpha| \le m, j<m}) \tag{E}$$

in the complex domain. In Gérard-Tahara [**1**] the author has determined all the singular solutions $u(t,x)$ of (E) under the condition that $u(t,x) = O(|t|^a)$ (as $t \to 0$ uniformly in x) for some $a > 0$.

The present paper investigates singular solutions $u(t,x)$ of (E) under a weaker condition that $u(t,x) = O(1/|\log t|^a)$ (as $t \to 0$ uniformly in x) for some $a > 0$.

§1. Preliminaries.

Notations: $t \in \boldsymbol{C}$, $x = (x_1, \dots, x_n) \in \boldsymbol{C}^n$, $\boldsymbol{N} = \{0, 1, 2, \dots\}$, and $\boldsymbol{N}^* = \{1, 2, \dots\}$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \boldsymbol{N}^n$ we write $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and

$$\left(\frac{\partial}{\partial x}\right)^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

Let $m \in \boldsymbol{N}^*$, $N = \sharp\{(j,\alpha) \in \boldsymbol{N} \times \boldsymbol{N}^n; j + |\alpha| \le m, j < m\}$, and write the variable Z as

$$Z = \{Z_{j,\alpha}\}_{\substack{j+|\alpha| \le m \\ j<m}} \in \boldsymbol{C}^N.$$

2000 *Mathematics Subject Classification*. Primary 35A20; Secondary 35B40.

Key Words and Phrases. Nonlinear PDE, singular solutions.

This research was partially supported by Grant-in-Aid for Scientific Research (No. 12640189), The Ministry of Education, Culture, Sports, Science and Technology, Japan.

Let $F(t,x,Z)$ be a function in the variables (t,x,Z) defined in a neighborhood of the origin $(0,0,0) \in \boldsymbol{C}_t \times \boldsymbol{C}_x^n \times \boldsymbol{C}_Z^N$, and assume the following:

(A_1) $F(t,x,Z)$ is holomorphic near $(0,0,0)$;

(A_2) $F(0,x,0) \equiv 0$ near $x=0$;

(A_3) $\dfrac{\partial F}{\partial Z_{j,\alpha}}(0,x,0) \equiv 0$ near $x=0$, if $|\alpha|>0$.

In this paper we always assume the conditions (A_1), (A_2), (A_3), and we will consider the following nonlinear partial differential equation

$$\text{(E)} \qquad \left(t\frac{\partial}{\partial t}\right)^m u = F\left(t,x,\left\{\left(t\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\alpha u\right\}_{\substack{j+|\alpha|\le m\\ j<m}}\right)$$

with $u=u(t,x)$ as the unknown function.

For (E) we set

$$C(\lambda,x)=\lambda^m-\sum_{j<m}\frac{\partial F}{\partial Z_{j,0}}(0,x,0)\lambda^j$$

and denote by $\lambda_1(x),\ldots,\lambda_m(x)$ the roots of the equation $C(\lambda,x)=0$ in λ. These $\lambda_1(x),\ldots,\lambda_m(x)$ are called the *characteristic exponents* of (E).

The following is our basic problem:

Problem. Determine all kinds of local singularities which appear in the solutions of (E).

Let us recall the result in Gérard-Tahara [**1**]. Denote:

- $\mathscr{R}(\boldsymbol{C}\backslash\{0\})$ denotes the universal covering space of $\boldsymbol{C}\backslash\{0\}$;
- $S_\theta=\{t\in\mathscr{R}(\boldsymbol{C}\backslash\{0\});|\arg t|<\theta\}$;
- $S(\varepsilon(s))=\{t\in\mathscr{R}(\boldsymbol{C}\backslash\{0\});0<|t|<\varepsilon(\arg t)\}$, where $\varepsilon(s)$ is a positive-valued continuous function on $\boldsymbol{R}_s$;
- $D_r=\{x\in\boldsymbol{C}^n;|x|\le r\}$;
- $\boldsymbol{C}\{x\}$ denotes the ring of convergent power series in x, or equivalently the ring of germs of holomorphic functions at the origin of $\boldsymbol{C}^n$.

Definition 1. We denote by $\tilde{\mathcal{O}}_+$ the set of all $u(t,x)$ satisfying the following conditions i) and ii):

i) $u(t,x)$ is a holomorphic function on $S(\varepsilon(s))\times D_r$ for some positive-valued continuous function $\varepsilon(s)$ and some $r>0$;

ii) there is an $a>0$ such that for any $\theta>0$ we have

$$\max_{|x|\le r}|u(t,x)|=O(|t|^a) \quad (\text{as } t\to 0 \text{ in } S_\theta).$$

For the characteristic exponents $\lambda_1(x), \dots, \lambda_m(x)$, we set

$$\mu = \sharp\{i; \operatorname{Re}\lambda_i(0) > 0\}.$$

When $\mu = 0$, this is equivalent to the fact that $\operatorname{Re}\lambda_i(0) \le 0$ for all $i = 1, \dots, m$. When $\mu \ge 1$, by a renumeration we may assume

$$\begin{cases} \operatorname{Re}\lambda_i(0) > 0 & \text{for } 1 \le i \le \mu, \\ \operatorname{Re}\lambda_i(0) \le 0 & \text{for } \mu+1 \le i \le m. \end{cases} \tag{1.1}$$

Then we already have:

Theorem 1 (Gérard-Tahara [**1**]). *Denote by $\mathscr{S}_+$ the set of all $\tilde{\mathcal{O}}_+$-solutions of* (E). *Then we have*:

(I) *When $\mu = 0$, we have $\mathscr{S}_+ = \{u_0\}$ where $u_0 = u_0(t,x)$ is the unique holomorphic solution of* (E) *satisfying $u_0(0,x) \equiv 0$.*

(II) *When $\mu \ge 1$, under* (1.1) *and the following additional conditions*

1) $\lambda_i(0) \ne \lambda_j(0)$ *for* $1 \le i \ne j \le \mu$,
2) $C(1,0) \ne 0$,
3) $C(i + j_1\lambda_1(0) + \cdots + j_\mu\lambda_\mu(0), 0) \ne 0$ *for any* $(i,j) \in \boldsymbol{N} \times \boldsymbol{N}^\mu$ *satisfying* $i + |j| \ge 2$ (*where* $j = (j_1, \dots, j_\mu)$),

we have

$$\mathscr{S}_+ = \{U(\phi_1, \dots, \phi_\mu); (\phi_1, \dots, \phi_\mu) \in (\boldsymbol{C}\{x\})^\mu\},$$

where $U(\phi_1, \dots, \phi_\mu)$ is an $\tilde{\mathcal{O}}_+$-solution of (E) *determined by $(\phi_1, \dots, \phi_\mu) \in (\boldsymbol{C}\{x\})^\mu$ and having the expansion of the following form*:

$$\begin{aligned} U(\phi_1, \dots, \phi_\mu) = & \sum_{i \ge 1} u_i(x) t^i \\ & + \phi_1(x) t^{\lambda_1(x)} + \cdots + \phi_\mu(x) t^{\lambda_\mu(x)} \\ & + \sum_{\substack{i + 2m|j| \ge k + 2m \\ |j| \ge 1 \\ (i,|j|) \ne (0,1)}} \varphi_{i,j,k}(x) t^{i + j_1\lambda_1(x) + \cdots + j_\mu\lambda_\mu(x)} (\log t)^k. \end{aligned}$$

§2. Problems.

In Theorem 1 we have restricted ourselves to the study of singular solutions in $\tilde{\mathcal{O}}_+$. But, there seems to be a possibility that (E) has singular solutions which do not belong in the class $\tilde{\mathcal{O}}_+$, as is seen in the following example.

Example 1. The equation

$$t\frac{\partial u}{\partial t} = u\left(\frac{\partial u}{\partial x}\right)^k$$

(where $(t,x) \in \boldsymbol{C}^2$ and $k \in \boldsymbol{N}^*$) has a family of singular solutions

$$u(t,x) = \left(\frac{1}{k}\right)^{1/k} \frac{x+\alpha}{(c-\log t)^{1/k}}, \quad \alpha, c \in \boldsymbol{C},$$

which do not belong in the class $\tilde{\mathcal{O}}_+$.

In order to include this kind of singular solutions in our framework, we introduce the following new class of singular solutions:

DEFINITION 2. We denote by $\tilde{\mathcal{O}}_{log}$ the set of all $u(t,x)$ satisfying the following conditions i) and ii):

i) $u(t,x)$ is a holomorphic function on $S(\varepsilon(s)) \times D_r$ for some positive-valued continuous function $\varepsilon(s)$ and some $r > 0$;

ii) there is an $a > 0$ such that for any $\theta > 0$ we have

$$\max_{|x| \le r} |u(t,x)| = O\left(\frac{1}{|\log t|^a}\right) \quad (\text{as } t \to 0 \text{ in } S_\theta).$$

Clearly we have $\tilde{\mathcal{O}}_{log} \supset \tilde{\mathcal{O}}_+$. Therefore, if we denote by $\mathcal{S}_{log}$ the set of all $\tilde{\mathcal{O}}_{log}$-solutions of (E), we have $\mathcal{S}_{log} \supset \mathcal{S}_+$. Hence, our next problems can be set up as follows:

PROBLEM 1. When does $\mathcal{S}_{log} = \mathcal{S}_+$ hold?

PROBLEM 2. When does $\mathcal{S}_{log} \neq \mathcal{S}_+$ hold?

The purpose of this paper is to give a partial answer and a conjecture on the problem 1. The problem 2 will be discussed in the forthcoming paper.

§3. A result and a conjecture.

In this section we will give a result on the problem 1 in a general form.

A function $\mu(t)$ on $(0,T)$ is called *a weight function* if it satisfies the following conditions $\mu_1)\sim\mu_3)$:

$\mu_1)$ $\mu(t) \in C^0((0,T))$,

$\mu_2)$ $\mu(t) > 0$ on $(0,T)$ and $\mu(t)$ is increasing in t,

$\mu_3)$ $\displaystyle\int_0^T \frac{\mu(s)}{s} ds < \infty$.

By $\mu_2)$ and $\mu_3)$ the condition $\mu(t) \to 0$ (as $t \to +0$) is clear. In this paper we impose the additional condition on $\mu(t)$:

$$\mu(t) \in C^1((0,T)) \quad \text{and} \quad \left(t\frac{d\mu}{dt}\right)(t) = o(\mu(t)) \quad (\text{as } t \to +0). \tag{3.1}$$

The following functions are typical examples:

$$\mu(t) = \frac{1}{(-\log t)^b}, \quad \frac{1}{(-\log t)(\log(-\log t))^c}$$

with $b > 1$, $c > 1$. Note that the function $\mu(t) = t^d$ with $d > 0$ does not satisfy the condition (3.1).

Definition 3. Let $\mu(t)$ be a weight function.

(1) For $a > 0$ we denote by $\tilde{\mathcal{O}}_a(\mu(t))$ the set of all $u(t,x)$ satisfying the following conditions i) and ii):

i) $u(t,x)$ is a holomorphic function on $S(\varepsilon(s)) \times D_r$ for some positive-valued continuous function $\varepsilon(s)$ and some $r > 0$;

ii) for any $\theta > 0$ we have

$$\max_{|x| \le r} |u(t,x)| = O(\mu(|t|)^a) \quad (\text{as } t \to 0 \text{ in } S_\theta).$$

(2) We define $\tilde{\mathcal{O}}_+(\mu(t))$ by

$$\tilde{\mathcal{O}}_+(\mu(t)) = \bigcup_{a>0} \tilde{\mathcal{O}}_a(\mu(t)).$$

Lemma 1. (1) *$\tilde{\mathcal{O}}_{log} = \tilde{\mathcal{O}}_+(\mu(t))$ if $\mu(t) = 1/(-\log t)^b$ with $b > 1$.*

(2) *If $\mu(t)$ satisfies* (3.1) *we have $\tilde{\mathcal{O}}_+ \subset \tilde{\mathcal{O}}_1(\mu(t))$ $(\subset \tilde{\mathcal{O}}_+(\mu(t)))$.*

Proof. (1) is clear. (2) is verified as follows. By (3.1), for any $\varepsilon > 0$ there is a $\delta > 0$ such that $t\mu_t'(t) \le \varepsilon\mu(t)$ holds on $(0,\delta]$ and therefore we have

$$\frac{d}{dt}(t^{-\varepsilon}\mu(t)) \le 0 \quad \text{for } 0 < t \le \delta.$$

Integrating this from t to δ we have

$$\delta^{-\varepsilon}\mu(\delta) \le t^{-\varepsilon}\mu(t) \quad \text{for } 0 < t \le \delta$$

and so

$$\left(\frac{\mu(\delta)}{\delta^\varepsilon}\right) t^\varepsilon \le \mu(t) \quad \text{for } 0 < t \le \delta. \tag{3.2}$$

Since $\varepsilon > 0$ is arbitrary, (3.2) leads us to the conclusion of (2). □

Denote by $\mathcal{S}_+(\mu(t))$ (resp. $\mathcal{S}_a(\mu(t))$) the set of all $\tilde{\mathcal{O}}_+(\mu(t))$-solutions of (E) (resp. $\tilde{\mathcal{O}}_a(\mu(t))$-solutions of (E)). By (2) of Lemma 1 we have

$$\mathcal{S}_+ \subset \mathcal{S}_1(\mu(t)) \subset \mathcal{S}_+(\mu(t)).$$

The following theorem gives a sufficient condition for $\mathcal{S}_+(\mu(t)) = \mathcal{S}_+$ to be valid.

Theorem 2. *Let $\mu(t)$ be a weight function satisfying* (3.1). *Then, $\mathcal{S}_+(\mu(t)) = \mathcal{S}_+$ is valid if*

$$\operatorname{Re} \lambda_i(0) < 0 \quad \textit{for all } i = 1, \ldots, m \tag{3.3}$$

or if

$$\mathrm{Re}\,\lambda_i(0) > 0 \quad \textit{for all } i = 1, \ldots, m. \tag{3.4}$$

In case (3.3), by Theorem 1 we have $\mathscr{S}_+ = \{u_0\}$ and therefore the condition $\mathscr{S}_+(\mu(t)) = \mathscr{S}_+$ is equivalent to the fact that the local uniqueness of the solution is valid in $\mathscr{S}_+(\mu(t))$ which is already proved in Tahara [**4**], [**5**].

In case (3.4) the proof of Theorem 2 consists of the following two parts:

C_1) if $u \in \mathscr{S}_+(\mu(t))$ we have $u \in \mathscr{S}_m(\mu(t))$;

C_2) if $u \in \mathscr{S}_m(\mu(t))$ we have $u \in \mathscr{S}_+$.

The part C_1) will be proved in §4 and the part C_2) will be proved in §5.

Corollary. *If* (3.3) *or* (3.4) *holds, we have* $\mathscr{S}_{log} = \mathscr{S}_+$.

Remark. The author believes that the following conjecture is true, though at present he has no idea to prove this conjecture:

Conjecture. $\mathscr{S}_{log} = \mathscr{S}_+$ *is valid if*

$$\mathrm{Re}\,\lambda_i(0) \neq 0 \quad \textit{for all } i = 1, \ldots, m. \tag{3.5}$$

§4. Proof of C_1).

The assertion C_1) comes from the following proposition.

Proposition 1. *Let* $\mu(t)$ *be a weight function satisfying* (3.1). *Assume the condition* (3.4). *Then, if* $u(t,x) \in \tilde{\mathscr{O}}_+(\mu(t))$ *is a solution of* (E) *we have* $u(t,x) \in \tilde{\mathscr{O}}_m(\mu(t))$.

First we note:

Lemma 2. *Let* $\delta > 0$, U *be a compact neighborhood of the origin of* $\boldsymbol{C}_x^n$, $\lambda(x) \in C^0(U)$, $u(t,x) \in C^1((0,\delta], C^0(U))$ *and* $f(t,x) \in C^0((0,\delta] \times U)$. *Assume* $\varepsilon > 0$, $h > 0$, $C > 0$, $a > 0$ *and assume the following* i)$\sim$iv):

i) $t\mu_t'(t) \le \varepsilon\mu(t)$ *on* $(0,\delta]$,

ii) $\mathrm{Re}\,\lambda(x) \ge h$ *on* U,

iii) $|f(t,x)| \le C\mu(t)^a$ *on* $(0,\delta] \times U$,

iv) $(t\partial/\partial t - \lambda(x))u = f$ *on* $(0,\delta] \times U$.

Then, if $a\varepsilon < h$ *holds we have*

$$|u(t,x)| \le \left(\frac{|u(\delta,x)|}{\mu(\delta)^a} + \frac{C}{h - a\varepsilon}\right)\mu(t)^a \quad on \ (0,\delta] \times U. \tag{4.1}$$

Proof. By solving the equation iv) we see that $u(t,x)$ is expressed by

$$u(t,x) = \left(\frac{t}{\delta}\right)^{\lambda(x)} u(\delta,x) - \int_t^\delta \left(\frac{t}{\tau}\right)^{\lambda(x)} f(\tau,x)\frac{d\tau}{\tau}$$

and by ii) and iii) we have

$$|u(t,x)| \le \left(\frac{t}{\delta}\right)^h |u(\delta,x)| + C\int_t^\delta \left(\frac{t}{\tau}\right)^h \mu(\tau)^a \frac{d\tau}{\tau} \quad \text{on } (0,\delta] \times U.$$

Therefore, to show (4.1) it is sufficient to prove the following inequalities:

$$\left(\frac{t}{\delta}\right)^h \le \left(\frac{\mu(t)}{\mu(\delta)}\right)^a \quad \text{on } (0,\delta], \tag{4.2}$$

$$\int_t^\delta \left(\frac{t}{\tau}\right)^h \mu(\tau)^a \frac{d\tau}{\tau} \le \frac{1}{h-a\varepsilon}\mu(t)^a \quad \text{on } (0,\delta]. \tag{4.3}$$

The proofs of (4.2) and (4.3) are as follows. Recall that the condition i) implies (3.2) and so

$$\left(\frac{t}{\delta}\right)^\varepsilon \le \frac{\mu(t)}{\mu(\delta)} \quad \text{on } (0,\delta].$$

Since $0 < a\varepsilon < h$ is assumed, we have

$$\left(\frac{t}{\delta}\right)^h \le \left(\frac{t}{\delta}\right)^{a\varepsilon} \le \left(\frac{\mu(t)}{\mu(\delta)}\right)^a \quad \text{on } (0,\delta]$$

which proves (4.2). Moreover, by the integration by parts and using the condition i) we have

$$\begin{aligned}\int_t^\delta \frac{1}{\tau^{h+1}}\mu(\tau)^a\,d\tau &= \left[\frac{-1}{h}\frac{1}{\tau^h}\mu(\tau)^a\right]_t^\delta + \frac{a}{h}\int_t^\delta \frac{1}{\tau^h}\mu(\tau)^{a-1}\mu'_\tau(\tau)\,d\tau \\ &\le \frac{1}{h}\frac{1}{t^h}\mu(t)^a + \frac{a}{h}\int_t^\delta \frac{1}{\tau^{h+1}}\mu(\tau)^{a-1}(\varepsilon\mu(\tau))\,d\tau \\ &= \frac{1}{h}\frac{1}{t^h}\mu(t)^a + \frac{a\varepsilon}{h}\int_t^\delta \frac{1}{\tau^{h+1}}\mu(\tau)^a\,d\tau\end{aligned}$$

and therefore we obtain

$$\int_t^\delta \frac{1}{\tau^{h+1}}\mu(\tau)^a\,d\tau \le \frac{1}{h-a\varepsilon}\frac{1}{t^h}\mu(t)^a \quad \text{on } (0,\delta]$$

which leads us to (4.3). □

Next let us consider

$$C\left(t\frac{\partial}{\partial t}, x\right)u = f. \tag{4.4}$$

Since $\lambda_1(x), \ldots, \lambda_m(x)$ are solutions of $C(\lambda, x) = 0$ in λ, the equation (4.4) is written as

$$\left(t\frac{\partial}{\partial t}-\lambda_1(x)\right)\cdots\left(t\frac{\partial}{\partial t}-\lambda_m(x)\right)u=f.$$

Therefore, applying Lemma 2 m-times to this equation we obtain

Lemma 3. *Assume the condition* (3.4), *and assume that* $u,f\in\tilde{\mathcal{O}}_+(\mu(t))$ *satisfy the equation* (4.4). *Then, if* $f\in\tilde{\mathcal{O}}_a(\mu(t))$ *holds for some* $a>0$ *we have* $u\in\tilde{\mathcal{O}}_a(\mu(t))$.

Denote

$$R[u]=F\left(t,x,\left\{\left(t\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\alpha u\right\}_{\substack{j+|\alpha|\le m\\ j<m}}\right)-\sum_{j<m}\frac{\partial F}{\partial Z_{j,0}}(0,x,0)\left(t\frac{\partial}{\partial t}\right)^j u.$$

The equation (E) is written as

$$C\left(t\frac{\partial}{\partial t},x\right)u=R[u]. \tag{4.5}$$

Moreover we have

Lemma 4. *If* $u\in\tilde{\mathcal{O}}_a(\mu(t))$ *holds for some* $a>0$ *we have* $R[u]\in\tilde{\mathcal{O}}_b(\mu(t))$ *for any* b *with* $0<b\le\min\{2a,m\}$.

Proof. By [**5**, Lemma 11] we know that

$$\mu(t+ct)=O(\mu(t))\quad(\text{as } t\to+0)$$

for some $c>0$ and hence we can see that $u\in\tilde{\mathcal{O}}_a(\mu(t))$ implies

$$\left(t\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\alpha u\in\tilde{\mathcal{O}}_a(\mu(t)),\quad j+|\alpha|\le m\ \text{and}\ j<m$$

(see the proof of [**5**, Theorem 3]).

Therefore, by (A_1), (A_2) and (A_3) we have

$$\begin{aligned}
R[u]&=F(t,x,0)\\
&\quad+\sum_{j<m}\left(\frac{\partial F}{\partial Z_{j,0}}(t,x,0)-\frac{\partial F}{\partial Z_{j,0}}(0,x,0)\right)\left(t\frac{\partial}{\partial t}\right)^j u\\
&\quad+\sum_{\substack{j+|\alpha|\le m\\ |\alpha|>0}}\frac{\partial F}{\partial Z_{j,\alpha}}(t,x,0)\left(t\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\alpha u\\
&\quad+\sum_{\substack{j+|\alpha|\le m\\ j<m}}\sum_{\substack{k+|\beta|\le m\\ k<m}}O\left(\left(t\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\alpha u\times\left(t\frac{\partial}{\partial t}\right)^k\left(\frac{\partial}{\partial x}\right)^\beta u\right)\\
&=O(|t|)+O(|t|)O(\mu(|t|)^a)+O(O(\mu(|t|)^a)\times O(\mu(|t|)^a)).
\end{aligned}$$

Since $|t| = O(\mu(|t|)^m)$ (as $t \to +0$) is already proved in (3.2) with $\varepsilon = 1/m$, we obtain the conclusion of Lemma 4. □

Now, by using Lemmas 3 and 4 let us prove Proposition 1.

Proof of Proposition 1. Let $u \in \tilde{\mathcal{O}}_+(\mu(t))$ be a solution of (E). Then, by the definition of $\tilde{\mathcal{O}}_+(\mu(t))$ we have $u \in \tilde{\mathcal{O}}_a(\mu(t))$ for some $a > 0$. Choose a sequence $a_0, a_1, \dots, a_N$ such that

i) $a_0 = a < a_1 < a_2 < \cdots < a_N = m$, and

ii) $a_{i+1} \le \min\{2a_i, m\}$ for $i = 0, 1, \dots, N-1$.

Since $u \in \tilde{\mathcal{O}}_{a_0}(\mu(t))$ is known, by Lemma 4 we have $R[u] \in \tilde{\mathcal{O}}_{a_1}(\mu(t))$ and therefore by applying Lemma 3 to the equation $C(t\partial/\partial t, x)u = R[u]$ we have $u \in \tilde{\mathcal{O}}_{a_1}(\mu(t))$. Then, by Lemma 4 we have $R[u] \in \tilde{\mathcal{O}}_{a_2}(\mu(t))$ and so applying Lemma 3 again to $C(t\partial/\partial t, x)u = R[u] \in \tilde{\mathcal{O}}_{a_2}(\mu(t))$ we have $u \in \tilde{\mathcal{O}}_{a_2}(\mu(t))$.

Thus, by repeating the same argument as above we obtain $u \in \tilde{\mathcal{O}}_{a_N}(\mu(t))$. Since $a_N = m$, this completes the proof of Proposition 1. □

§5. Proof of C_2).

The assertion C_2) comes from the following proposition.

Proposition 2. *Let $\mu(t)$ be a weight function satisfying*

$$\mu(t) \in C^1((0,T)) \quad \textit{and} \quad \left(t\frac{d\mu}{dt}\right)(t) = O(\mu(t)) \quad (\textit{as } t \to +0). \tag{5.1}$$

Assume the condition (3.4). *Then, if $u \in \tilde{\mathcal{O}}_m(\mu(t))$ is a solution of* (E) *we have $u \in \tilde{\mathcal{O}}_+$.*

We will prove this proposition from now. By (5.1) we have

$$t\mu_t'(t) \le A\mu(t) \quad \text{on } (0,T) \tag{5.2}$$

for some $A > 0$. Also, by (3.4) we can find $h > 0$ and $R > 0$ such that

$$\mathrm{Re}\,\lambda_i(x) \ge 2h > 0 \quad \text{on } D_R, \quad i = 1, \dots, m. \tag{5.3}$$

Without loss of generality we may assume that $0 < h < 1$ holds.

Let $u \in \tilde{\mathcal{O}}_m(\mu(t))$ be a solution of (E), and assume that $u(t,x)$ is holomorphic on $S(\varepsilon(s)) \times D_{2R}$ where $\varepsilon(s)$ is a positive-valued continuous function and $R > 0$ is sufficiently small. Since the condition (5.1) is assumed, by [**5**, Lemma 11] we have $\mu(t + ct) = O(\mu(t))$ (as $t \to +0$) for some $c > 0$ and by the same argument as in the proof of [**5**, Theorem 3] we have

$$\left(t\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u \in \tilde{\mathcal{O}}_m(\mu(t)) \quad \text{for } j + |\alpha| \le m \text{ and } j < m.$$

Therefore, for any $\theta_0 > 0$ we can find $\delta > 0$ and $M > 0$ such that

(5.4) $$\left|\left(t\frac{\partial}{\partial t}\right)^{j}\left(\frac{\partial}{\partial x}\right)^{\alpha}u(t,x)\right| \le M\mu(|t|)^{m} \quad \text{on } S_{\theta_0}(\delta)\times D_R$$
$$\text{for } j+|\alpha|\le m \text{ and } j<m$$

where $S_{\theta_0}(\delta)=\{t\in S_{\theta_0}; 0<|t|\le\delta\}$.

Our purpose is to show the following: *if $R_1>0$ is sufficiently small, for any $\theta_0>0$ we can find $\delta_1>0$ and $M_1>0$ such that*

(5.5) $$|u(t,x)|\le M_1|t|^{h} \quad \textit{on } S_{\theta_0}(\delta_1)\times D_{R_1}.$$

The rest part of this section is used to prove this estimate.

Denote

$$\Theta_0=1,$$
$$\Theta_1=\left(t\frac{\partial}{\partial t}-\lambda_1(0)\right),$$
$$\Theta_2=\left(t\frac{\partial}{\partial t}-\lambda_2(0)\right)\left(t\frac{\partial}{\partial t}-\lambda_1(0)\right),$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$
$$\Theta_m=\left(t\frac{\partial}{\partial t}-\lambda_m(0)\right)\left(t\frac{\partial}{\partial t}-\lambda_{m-1}(0)\right)\cdots\left(t\frac{\partial}{\partial t}-\lambda_1(0)\right).$$

Since $u\in\tilde{\mathcal{O}}_m(\mu(t))$ is a solution of (E), we have

(5.6) $$\begin{aligned}\Theta_m u &= F(t,x,0)\\ &\quad+\sum_{j<m}\left(\frac{\partial F}{\partial Z_{j,0}}(t,x,0)-\frac{\partial F}{\partial Z_{j,0}}(0,0,0)\right)\left(t\frac{\partial}{\partial t}\right)^{j}u\\ &\quad+\sum_{\substack{j+|\alpha|\le m\\|\alpha|>0}}\frac{\partial F}{\partial Z_{j,\alpha}}(t,x,0)\left(t\frac{\partial}{\partial t}\right)^{j}\left(\frac{\partial}{\partial x}\right)^{\alpha}u\\ &\quad+\sum_{\substack{j+|\alpha|\le m\\ j<m}}\sum_{\substack{k+|\beta|\le m\\ k<m}}O\left(\left(t\frac{\partial}{\partial t}\right)^{j}\left(\frac{\partial}{\partial x}\right)^{\alpha}u\times\left(t\frac{\partial}{\partial t}\right)^{k}\left(\frac{\partial}{\partial x}\right)^{\beta}u\right)\\ &=F(t,x,0)+\sum_{j<m}a_j(t,x)\Theta_j u+\sum_{\substack{j+|\alpha|\le m\\ j<m}}b_{j,\alpha}(t,x)\Theta_j\left(\frac{\partial}{\partial x}\right)^{\alpha}u,\end{aligned}$$

where $a_j(t,x)$ $(j<m)$ are holomorphic functions in a neighborhood of $(0,0)$ satisfying $a_j(0,0)=0$, and $b_{j,\alpha}(t,x)$ $(j+|\alpha|\le m, j<m)$ are functions in $\tilde{\mathcal{O}}_m(\mu(t))$. Note that $a_j(t,x)$ $(j<m)$ are independent of u, but $b_{j,\alpha}(t,x)$ $(j+|\alpha|\le m, j<m)$ depend on u.

Introduce the following notation. For a formal power series $f(t,x)$ in x with coefficients in $C^0((0,T))$ of the form

$$f(t,x)=\sum_{\alpha\in\boldsymbol{N}^n} f_\alpha(t)x^\alpha, \quad f_\alpha(t)\in C^0((0,T))$$

we write

$$\|f(t)\|_\rho=\sum_{\alpha\in\boldsymbol{N}^n}|f_\alpha(t)|\frac{\alpha!}{|\alpha|!}\rho^{|\alpha|}$$

(which is a formal power series in ρ with coefficients in $C^0((0,T))$). In case $f(t,x)$ is a function on $(0,T)\times D_R$ continuous in t and holomorphic in x, by using the Taylor expansion of $f(t,x)$ in x we can define $\|f(t)\|_\rho$ in the same way. Note that the following majorant relation holds:

$$\left\|\left(\frac{\partial}{\partial x_i}\right)f(t)\right\|_\rho \ll \frac{\partial}{\partial\rho}\|f(t)\|_\rho, \quad i=1,\dots,n.$$

Take any $\theta_0>0$. Let $R>0$ and $\delta>0$ be the ones in (5.4). Note that δ depends on θ_0 but R is independent of θ_0. For $(j,k)\in\boldsymbol{N}\times\boldsymbol{N}$ satisfying $j+k\le m-1$ we set

$$\psi_{j,k}(t,\rho,\theta)=\mu(t)^k\times\sum_{|\alpha|=k}\left\|\Theta_j\left(\frac{\partial}{\partial x}\right)^\alpha u(te^{\sqrt{-1}\theta})\right\|_\rho, \tag{5.7}$$

$$\begin{aligned}\phi_{j,k}(t,\rho,\theta)=\int_t^\delta\left(\frac{t}{\tau}\right)^{\operatorname{Re}\lambda_{j+1}(0)}&\mu(\tau)^k\\ &\times\left\{\sum_{|\alpha|=k}\left\|\Theta_{j+1}\left(\frac{\partial}{\partial x}\right)^\alpha u(\tau e^{\sqrt{-1}\theta})\right\|_\rho\right.\\ &\left.\quad+kA\sum_{|\alpha|=k}\left\|\Theta_j\left(\frac{\partial}{\partial x}\right)^\alpha u(\tau e^{\sqrt{-1}\theta})\right\|_\rho\right\}\frac{d\tau}{\tau}.\end{aligned} \tag{5.8}$$

Then, by the argument similar to the proof of [**4**, Lemma 3] we have

Lemma 5. *$\psi_{j,k}(t,\rho,\theta)$ $(j+k \le m-1)$ and $\phi_{j,k}(t,\rho,\theta)$ $(j+k \le m-1)$ are well-defined in $C^0([0,\delta]\times[0,R]\times(-\theta_0,\theta_0))$ and satisfy the following properties (1)$\sim$(4) on $\{(t,\rho,\theta); 0<t\le\delta, 0\le\rho\le R$ and $|\theta|<\theta_0\}$:*

(1) *For any (j,k) we have*

$$\psi_{j,k}(t,\rho,\theta) \le \left(\frac{t}{\delta}\right)^{2h}\psi_{j,k}(\delta,\rho,\theta) + \phi_{j,k}(t,\rho,\theta).$$

(2) *When $k>0$, we have*

$$\begin{aligned}&\left(-t\frac{\partial}{\partial t}+2h\right)\phi_{j,k}(t,\rho,\theta)\\ &\qquad \le n\mu(t)\frac{\partial}{\partial\rho}\psi_{j+1,k-1}(t,\rho,\theta) + nkA\mu(t)\frac{\partial}{\partial\rho}\psi_{j,k-1}(t,\rho,\theta).\end{aligned}$$

(3) *When $k=0$ and $j=0,1,\dots,m-2$, we have*

$$\left(-t\frac{\partial}{\partial t}+2h\right)\phi_{j,0}(t,\rho,\theta) \le \psi_{j+1,0}(t,\rho,\theta).$$

(4) *When $k=0$ and $j=m-1$, we have*

$$\begin{aligned}&\left(-t\frac{\partial}{\partial t}+2h\right)\phi_{m-1,0}(t,\rho,\theta)\\ &\qquad \le Kt + (a(t,\rho)+b(t,\rho))\sum_{j<m}\psi_{j,0}(t,\rho,\theta)\\ &\qquad\quad + B\mu(t)\frac{\partial}{\partial\rho}\sum_{j+k\le m-1}\psi_{j,k}(t,\rho,\theta)\end{aligned}$$

for some $K>0$, $B>0$, $a(t,\rho)\in C^0([0,\delta]\times[0,R])$ with $a(0,0)=0$, and $b(t,\rho)\in C^0([0,\delta]\times[0,R])$ with $b(t,\rho)=O(\mu(t)^m)$ (as $t\to+0$ uniformly in $\rho\in[0,R]$). Moreover, by (5.6) *we see that K and $a(t,\rho)$ are independent of θ_0.*

Proof. Set

$$u_{j,k}(t,x) = \mu(t)^k\Theta_j\left(\frac{\partial}{\partial x}\right)^{\alpha} u(te^{\sqrt{-1}\theta},x).$$

Then we have

$$\left(t\frac{\partial}{\partial t}-\lambda_{j+1}(0)\right)u_{j,k}(t,x)$$
$$=\mu(t)^k\Theta_{j+1}\left(\frac{\partial}{\partial x}\right)^{\alpha}u(te^{\sqrt{-1}\theta},x)+kt\mu_t'(t)\mu(t)^{k-1}\Theta_j\left(\frac{\partial}{\partial x}\right)^{\alpha}u(te^{\sqrt{-1}\theta},x)$$

and by integrating this from t to δ we have

$$u_{j,k}(t,x)=\left(\frac{t}{\delta}\right)^{\lambda_{j+1}(0)}u_{j,k}(\delta,x)$$
$$-\int_t^{\delta}\left(\frac{t}{\tau}\right)^{\lambda_{j+1}(0)}\left\{\mu(\tau)^k\Theta_{j+1}\left(\frac{\partial}{\partial x}\right)^{\alpha}u(\tau e^{\sqrt{-1}\theta},x)\right.$$
$$\left.+k\tau\mu_\tau'(\tau)\mu(\tau)^{k-1}\Theta_j\left(\frac{\partial}{\partial x}\right)^{\alpha}u(\tau e^{\sqrt{-1}\theta},x)\right\}\frac{d\tau}{\tau}.$$

Therefore by taking the norm and by using (5.2) and (5.3) we obtain

$$\mu(t)^k\left\|\Theta_j\left(\frac{\partial}{\partial x}\right)^{\alpha}u(te^{\sqrt{-1}\theta})\right\|_\rho=\|u_{j,k}(t)\|_\rho$$
$$\le\left(\frac{t}{\delta}\right)^{\mathrm{Re}\,\lambda_{j+1}(0)}\|u_{j,k}(\delta)\|_\rho$$
$$+\int_t^{\delta}\left(\frac{t}{\tau}\right)^{\mathrm{Re}\,\lambda_{j+1}(0)}\left\{\mu(\tau)^k\left\|\Theta_{j+1}\left(\frac{\partial}{\partial x}\right)^{\alpha}u(\tau e^{\sqrt{-1}\theta})\right\|_\rho\right.$$
$$\left.+kA\mu(\tau)\mu(\tau)^{k-1}\left\|\Theta_j\left(\frac{\partial}{\partial x}\right)^{\alpha}u(\tau e^{\sqrt{-1}\theta})\right\|_\rho\right\}\frac{d\tau}{\tau}$$
$$\le\left(\frac{t}{\delta}\right)^{2h}\mu(\delta)^k\left\|\Theta_j\left(\frac{\partial}{\partial x}\right)^{\alpha}u(\delta e^{\sqrt{-1}\theta})\right\|_\rho$$
$$+\int_t^{\delta}\left(\frac{t}{\tau}\right)^{\mathrm{Re}\,\lambda_{j+1}(0)}\mu(\tau)^k\left\{\left\|\Theta_{j+1}\left(\frac{\partial}{\partial x}\right)^{\alpha}u(\tau e^{\sqrt{-1}\theta})\right\|_\rho\right.$$
$$\left.+kA\left\|\Theta_j\left(\frac{\partial}{\partial x}\right)^{\alpha}u(\tau e^{\sqrt{-1}\theta})\right\|_\rho\right\}\frac{d\tau}{\tau}$$

which leads us to the property (1).

Denote: $e_1=(1,0,\dots,0),\dots,e_n=(0,\dots,0,1)\in\boldsymbol{N}^n$. If $|\alpha|>0$ we have

$$\left(\frac{\partial}{\partial x}\right)^{\alpha}=\left(\frac{\partial}{\partial x_i}\right)\left(\frac{\partial}{\partial x}\right)^{\alpha-e_i}$$

for some $i = i_\alpha$ and

$$\left\| \Theta_l \left(\frac{\partial}{\partial x}\right)^{\alpha} u(te^{\sqrt{-1}\theta}) \right\|_\rho \le \frac{\partial}{\partial \rho} \left\| \Theta_l \left(\frac{\partial}{\partial x}\right)^{\alpha - e_i} u(te^{\sqrt{-1}\theta}) \right\|_\rho \tag{5.9}$$

for any $l = 0, 1, \ldots, m$ and any $\rho \in [0, R]$.

When $k > 0$, by using (5.3) and (5.9) we can verify the property (2) as follows:

$$\begin{aligned}
&\left(-t\frac{\partial}{\partial t} + 2h\right)\phi_{j,k}(t,\rho,\theta) \\
&\quad \le \left(-t\frac{\partial}{\partial t} + \operatorname{Re}\lambda_{j+1}(0)\right)\phi_{j,k}(t,\rho,\theta) \\
&\quad = \mu(t)^k \Biggl\{ \sum_{|\alpha|=k} \left\| \Theta_{j+1} \left(\frac{\partial}{\partial x}\right)^{\alpha} u(te^{\sqrt{-1}\theta}) \right\|_\rho \\
&\qquad\qquad + kA \sum_{|\alpha|=k} \left\| \Theta_j \left(\frac{\partial}{\partial x}\right)^{\alpha} u(te^{\sqrt{-1}\theta}) \right\|_\rho \Biggr\} \\
&\quad \le \mu(t)^k \Biggl\{ \sum_{|\alpha|=k} \frac{\partial}{\partial \rho} \left\| \Theta_{j+1} \left(\frac{\partial}{\partial x}\right)^{\alpha - e_i} u(te^{\sqrt{-1}\theta}) \right\|_\rho \\
&\qquad\qquad + kA \sum_{|\alpha|=k} \frac{\partial}{\partial \rho} \left\| \Theta_j \left(\frac{\partial}{\partial x}\right)^{\alpha - e_i} u(te^{\sqrt{-1}\theta}) \right\|_\rho \Biggr\} \\
&\quad \le n\mu(t)\frac{\partial}{\partial \rho}\psi_{j+1,k-1}(t,\rho,\theta) + nkA\mu(t)\frac{\partial}{\partial \rho}\psi_{j,k-1}(t,\rho,\theta).
\end{aligned}$$

When $k = 0$ and $j = 0, 1, \ldots, m-2$, the property (3) is verified by:

$$\begin{aligned}
\left(-t\frac{\partial}{\partial t} + 2h\right)\phi_{j,0}(t,\rho,\theta) &\le \left(-t\frac{\partial}{\partial t} + \operatorname{Re}\lambda_{j+1}(0)\right)\phi_{j,0}(t,\rho,\theta) \\
&= \|\Theta_{j+1} u(te^{\sqrt{-1}\theta})\|_\rho = \psi_{j+1,0}(t,\rho,\theta).
\end{aligned}$$

When $k = 0$ and $j = m-1$ we have

$$\begin{aligned}
&\left(-t\frac{\partial}{\partial t} + 2h\right)\phi_{m-1,0}(t,\rho,\theta) \\
&\quad \le \left(-t\frac{\partial}{\partial t} + \operatorname{Re}\lambda_m(0)\right)\phi_{m-1,0}(t,\rho,\theta) = \left\| \Theta_m u(te^{\sqrt{-1}\theta}) \right\|_\rho.
\end{aligned} \tag{5.10}$$

On the other hand, by (5.6) we know that the equation (E) is written as

$$\begin{aligned}\Theta_m u &= F(t,x,0) + \sum_{j<m}(a_j(t,x) + b_{j,0}(t,x))\Theta_j u \\ &\quad + \sum_{\substack{j+|\alpha|\le m \\ |\alpha|>0}} b_{j,\alpha}(t,x)\Theta_j\left(\frac{\partial}{\partial x}\right)^{\alpha} u \\ &= O(|t|) + \sum_{j<m}(O(|t|+|x|) + O(\mu(|t|)^m))\Theta_j u \\ &\quad + \sum_{\substack{j+|\alpha|\le m \\ |\alpha|>0}} O(\mu(|t|)^m)\Theta_j\left(\frac{\partial}{\partial x}\right)^{\alpha} u.\end{aligned}$$

Therefore, by taking the norm and by using (5.9) we have

$$\begin{aligned}&\|\Theta_m u(te^{\sqrt{-1}\theta})\|_\rho \\ &\quad \le Kt + (O(t+\rho) + O(\mu(t)^m))\sum_{j<m}\|\Theta_j u(te^{\sqrt{-1}\theta})\|_\rho \\ &\qquad + \sum_{\substack{j+|\alpha|\le m \\ |\alpha|>0}} O(\mu(t)^m)\frac{\partial}{\partial\rho}\left\|\Theta_j\left(\frac{\partial}{\partial x}\right)^{\alpha-e_i} u(te^{\sqrt{-1}\theta})\right\|_\rho \\ &\quad \le Kt + (O(t+\rho) + O(\mu(t)^m))\sum_{j<m}\psi_{j,0}(t,\rho,\theta) \\ &\qquad + O(\mu(t))\frac{\partial}{\partial\rho}\sum_{j+k\le m-1}\psi_{j,k}(t,\rho,\theta).\end{aligned}$$

Hence, combining this with (5.10) we obtain the property (4). □

Next, we choose $\sigma_j > 0$ $(j = 0, 1, \dots, m-1)$ so that

$$\frac{\sigma_j}{\sigma_{j+1}} < \frac{h}{2}, \quad j = 0, 1, \dots, m-2 \tag{5.11}$$

hold and then we choose $\delta_2 > 0$ and $R_2 > 0$ sufficiently small so that

$$\frac{\sigma_{m-1}}{\sigma_j} a(t,\rho) < \frac{h}{4}, \quad j = 0, 1, \dots, m-1, \tag{5.12}$$

$$(5.13) \qquad \frac{\sigma_{m-1}}{\sigma_j} b(t,\rho) < \frac{h}{4}, \quad j = 0, 1, \ldots, m-1$$

hold on $\{(t,\rho); 0 \le t \le \delta_2, 0 \le \rho \le R_2\}$. Since $a(t,\rho)$ is independent of θ_0 we may assume that $R_2 > 0$ is also independent of θ_0.

Set

$$\Psi(t,\rho,\theta) = \sum_{j+k \le m-1} \psi_{j,k}(t,\rho,\theta),$$

$$\Phi(t,\rho,\theta) = \sum_{j<m} \sigma_j \phi_{j,0}(t,\rho,\theta) + \sum_{\substack{j+k \le m-1 \\ k>0}} \phi_{j,k}(t,\rho,\theta).$$

Then we have:

LEMMA 6. *There are* $C_1 > 0$ *and* $C_2 > 0$ *such that*

$$(5.14) \qquad \left(-t\frac{\partial}{\partial t} + h\right)\Phi(t,\rho,\theta) \le \sigma_{m-1}Kt + C_1\left(\frac{t}{\delta}\right)^{2h}\left(1 + \mu(t)\frac{\partial}{\partial\rho}\right)\Psi(\delta,\rho,\theta) + C_2\mu(t)\frac{\partial}{\partial\rho}\Phi(t,\rho,\theta)$$

holds on $\{(t,\rho,\theta); 0 < t \le \delta_2, 0 \le \rho \le R_2$ *and* $|\theta| < \theta_0\}$.

PROOF. By using (2)$\sim$(4) of Lemma 5 we have

$$\begin{aligned}
&\left(-t\frac{\partial}{\partial t} + 2h\right)\Phi(t,\rho,\theta) \\
&\quad \le \sum_{j \le m-2} \sigma_j \psi_{j+1,0}(t,\rho,\theta) \\
&\qquad + \sigma_{m-1}Kt + \sigma_{m-1}(a(t,\rho) + b(t,\rho)) \sum_{j<m} \psi_{j,0}(t,\rho,\theta) \\
&\qquad + C_3\mu(t)\frac{\partial}{\partial\rho} \sum_{j+k \le m-1} \psi_{j,k}(t,\rho,\theta)
\end{aligned}$$

for some $C_3 > 0$, and therefore by (1) of Lemma 5, (5.11), (5.12) and (5.13) we obtain

$$
\begin{aligned}
&\left(-t\frac{\partial}{\partial t}+2h\right)\Phi(t,\rho,\theta)\\
&\quad\le \sum_{j\le m-2}\frac{h}{2}\sigma_{j+1}\left[\left(\frac{t}{\delta}\right)^{2h}\psi_{j+1,0}(\delta,\rho,\theta)+\phi_{j+1,0}(t,\rho,\theta)\right]+\sigma_{m-1}Kt\\
&\qquad+\sum_{j<m}\left(\frac{h}{4}+\frac{h}{4}\right)\sigma_j\left[\left(\frac{t}{\delta}\right)^{2h}\psi_{j,0}(\delta,\rho,\theta)+\phi_{j,0}(t,\rho,\theta)\right]\\
&\qquad+C_3\mu(t)\frac{\partial}{\partial\rho}\sum_{j+k\le m-1}\left[\left(\frac{t}{\delta}\right)^{2h}\psi_{j,k}(\delta,\rho,\theta)+\phi_{j,k}(t,\rho,\theta)\right]\\
&\quad\le\left(\frac{h}{2}+\frac{h}{4}+\frac{h}{4}\right)\Phi(t,\rho,\theta)+\sigma_{m-1}Kt\\
&\qquad+C_1\left(\frac{t}{\delta}\right)^{2h}\left(1+\mu(t)\frac{\partial}{\partial\rho}\right)\Psi(\delta,\rho,\theta)+C_2\mu(t)\frac{\partial}{\partial\rho}\Phi(t,\rho,\theta)
\end{aligned}
$$

for some $C_1>0$ and $C_2>0$. This immediately leads us to (5.14). □

Now, let us complete the proof of Proposition 2. Set

$$M_2=\sigma_{m-1}K+\frac{C_1}{\delta^{2h}}\sup_{\substack{0\le\rho\le R_2\\|\theta|<\theta_0}}\left(\left(1+\mu(\delta_2)\frac{\partial}{\partial\rho}\right)\Psi(\delta,\rho,\theta)\right).$$

Then, by Lemma 6 we have

$$\left(-t\frac{\partial}{\partial t}+h-C_2\mu(t)\frac{\partial}{\partial\rho}\right)\Phi(t,\rho,\theta)\le M_2(t+t^{2h}) \tag{5.15}$$

on $\{(t,\rho,\theta);0<t\le\delta_2,0\le\rho\le R_2 \text{ and } |\theta|<\theta_0\}$.

Completion of the proof of Proposition 2. Take any R_1 such that $0<R_1<R_2$, and then choose $\delta_1>0$ so that $0<\delta_1<\delta_2$ and

$$R_1+C_2\int_0^{\delta_1}\frac{\mu(s)}{s}\,ds\le R_2.$$

Define the function $\rho(t)$ by

$$\rho(t)=R_1+C_2\int_0^{t}\frac{\mu(s)}{s}\,ds\quad\text{for } 0\le t\le\delta_1.$$

Then, $R_1 \le \rho(t) \le R_2$ for $0 \le t \le \delta_1$, $t(d\rho/dt) = C_2\mu(t)$, and $\rho(t)$ is increasing in t. Moreover we have

$$[0,\delta_1] \times [0,R_1] \subset \{(t,\rho); 0 \le t \le \delta_1, 0 \le \rho \le \rho(t)\}. \tag{5.16}$$

Set

$$\varphi(t,\theta) = \Phi(t,\rho(t),\theta) \quad \text{for } 0 \le t \le \delta_1 \text{ and } |\theta| < \theta_0. \tag{5.17}$$

By (5.15) we have

$$\begin{aligned} \left(-t\frac{\partial}{\partial t} + h\right)\varphi(t,\theta) &= \left(-t\frac{\partial}{\partial t} + h\right)\Phi - \frac{\partial \Phi}{\partial \rho} t \frac{d\rho(t)}{dt} \\ &= \left(-t\frac{\partial}{\partial t} + h - C_2\mu(t)\frac{\partial}{\partial \rho}\right)\Phi \\ &\le M_2(t + t^{2h}), \end{aligned}$$

that is

$$\left(-t\frac{\partial}{\partial t} + h\right)\varphi(t,\theta) \le M_2(t + t^{2h}), \quad 0 < t \le \delta_1 \text{ and } |\theta| < \theta_0$$

which is equivalent to

$$-\frac{\partial}{\partial t}(t^{-h}\varphi(t,\theta)) \le M_2\left(\frac{1}{t^h} + \frac{1}{t^{1-h}}\right), \quad 0 < t \le \delta_1 \text{ and } |\theta| < \theta_0.$$

Since $0 < h < 1$ is assumed, by integrating this from t to δ_1 we have

$$t^{-h}\varphi(t,\theta) \le \delta_1^{-h}\varphi(\delta_1,\theta) + M_2\left(\frac{\delta_1^{1-h}}{1-h} + \frac{\delta_1^h}{h}\right)$$

and hence

$$\varphi(t,\theta) \le M_3 t^h, \quad 0 < t \le \delta_1 \text{ and } |\theta| < \theta_0 \tag{5.18}$$

where

$$M_3 = \frac{1}{\delta_1^h} \sup_{\substack{0 \le \rho \le R_2 \\ |\theta| < \theta_0}} (\Phi(\delta_1,\rho,\theta)) + M_2\left(\frac{\delta_1^{1-h}}{1-h} + \frac{\delta_1^h}{h}\right).$$

Thus, if we notice the fact that $\Phi(t,\rho,\theta)$ is increasing in ρ, by (5.16), (5.17) and (5.18) we obtain

$$\Phi(t,\rho,\theta) \le M_3 t^h \tag{5.19}$$

on $\{(t, \rho, \theta); 0 < t \le \delta_1, 0 \le \rho \le R_1 \text{ and } |\theta| < \theta_0\}$.

Finally, let us show that the estimate (5.5) follows from (5.19). Note that

$$\psi_{0,0}(t, \rho, \theta) = \|u(te^{\sqrt{-1}\theta})\|_\rho$$

holds. Therefore, by (5.19) and (1) of Lemma 5 we have

$$\begin{aligned}\|u(te^{\sqrt{-1}\theta})\|_\rho &\le \left(\frac{t}{\delta}\right)^{2h} \|u(\delta e^{\sqrt{-1}\theta})\|_\rho + \phi_{0,0}(t, \rho, \theta) \\ &\le \left(\frac{t}{\delta}\right)^{2h} \sup_{\substack{0 \le \rho \le R_1 \\ |\theta| < \theta_0}} \|u(\delta e^{\sqrt{-1}\theta})\|_\rho + \frac{M_3}{\sigma_0} t^h\end{aligned}$$

on $\{(t, \rho, \theta); 0 < t \le \delta_1, 0 \le \rho \le R_1 \text{ and } |\theta| < \theta_0\}$. This implies (5.5).

Since $R_1 > 0$ is chosen independently of θ_0, this completes the proof of Proposition 2. □

References

[1] R. Gérard and H. Tahara, Solutions holomorphes et singulières d'équations aux dérivées partielles singulières non linéaires, Publ. RIMS, Kyoto Univ., **29** (1993), 121–151.

[2] R. Gérard and H. Tahara, Singular nonlinear partial differential equations, Aspects of Mathematics, E **28**, Vieweg, 1996.

[3] H. Tahara, Removable singularities of solutions of nonlinear singular partial differential equations, Banach Center Publications, **33** (1996), 395–399.

[4] H. Tahara, Uniqueness of the solution of non-linear singular partial differential equations, J. Math. Soc. Japan, **48** (1996), 729–744.

[5] H. Tahara, On the uniqueness theorem for nonlinear singular partial differential equations, J. Math. Sci. Univ. Tokyo, **5** (1998), 477–506.

Hidetoshi TAHARA

Department of Mathematics
Sophia University
Kioicho, Chiyoda-ku
Tokyo 102-8554
Japan