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On the structure of the group of Lipschitz homeomorphisms and its subgroups

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Abstract. We consider the group of Lipschitz homeomorphisms of a Lipschitz manifold and its subgroups. First we study properties of Lipschitz homeomorphisms and show the local contractibility and the perfectness of the group of Lipschitz homeomorphisms. Next using this result we can prove that the identity component of the group of equivariant Lipschitz homeomorphisms of a principal G-bundle over a closed Lipschitz manifold is perfect when G is a compact Lie group.

1. Introduction.

Let M be an m-dimensional topological manifold and d a metric on M. A continuous map $f: M \to M$ is said to be *Lipschitz* if there exists some $k \geq 0$ such that $d(f(x), f(y)) \leq kd(x, y)$ for all $x, y \in M$. The infimum of such $k \geq 0$ is called the Lipschitz constant and denoted by $lip(f)$. A homeomorphism $f: M \to M$ is called a Lipschitz homeomorphism if both f and f^{-1} are Lipschitz. We denote by $C(M, M)$ the space of all continuous maps from M to M with the topology induced by the sup-metric $\rho(f, g) = \sup\{d(f(x), g(x)) \mid x \in M\}$ and by $\mathscr{H}_{LIP}(M)$ the subspace of $C(M, M)$ which consists of Lipschitz homeomorphisms of M with compact support. Note that the space $\mathscr{H}_{LIP}(M)$ coincides with that endowed with the compact open topology. We consider only the identity component of $\mathscr{H}_{LIP}(M)$ which is also denoted by the same notation. For a manifold pair (M, N) (see §3 for the definition), let $\mathscr{H}_{LIP}(M, N)$ denote the subgroup consisting of the homeomorphisms of $\mathscr{H}_{LIP}(M)$ which map N to itself.

In §2, we consider the homologies of $\mathscr{H}_{LIP}(M)$ and $\mathscr{H}_{LIP}(M, N)$, that is, the homology groups of the groups $\mathscr{H}_{LIP}(M)$ and $\mathscr{H}_{LIP}(M, N)$ and show that the homologies of $\mathscr{H}_{LIP}(\boldsymbol{R}^m)$ and $\mathscr{H}_{LIP}(\boldsymbol{R}^m, \boldsymbol{R}^n)$ $(n > 0)$ vanish in all dimension > 0.

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These are Lipschitz versions of results of Fukui-Imanishi [**F-I**] and Fukui [**F**] which are generalizations of a result of Mather [**M**]. We show in §3 that $\mathscr{H}_{LIP}(M)$ and $\mathscr{H}_{LIP}(M,N)$ are perfect, *i.e.*, are equal to their own commutator subgroups, for a PL manifold pair (M,N).

In §4, we introduce a new fine topology on $\mathscr{H}_{LIP}(M)$ for a Lipschitz manifold, called the *compact open Lipschitz topology*. Then we study basic properties of $\mathscr{H}_{LIP}(M)$ and show the local contractibility and the perfectness of $\mathscr{H}_{LIP}(M)$, which may be well known. We remark that the results in §2 are valid for this topology.

In §5, we apply the above results to the groups of equivariant Lipschitz homeomorphisms of principal G-bundles over Lipschitz manifolds. Let G be a compact Lie group and M the total space of a principal G-bundle over a closed Lipschitz manifold B. Then we show that the identity component with respect to the compact open Lipschitz topology of the group of equivariant Lipschitz homeomorphisms of M, $\mathscr{H}_{LIP,G}(M)$, is perfect for $\dim B > 0$. This is a Lipschitz version of results of Banyaga [**B**] and Abe-Fukui [**A-F**].

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2. Homologies of $\mathscr{H}_{LIP}(\boldsymbol{R}^m)$ and $\mathscr{H}_{LIP}(\boldsymbol{R}^m,\boldsymbol{R}^n)$.

We recall that if G is any group, then there is a standard chain complex $C(G)$ whose homology is the homology of G.

Let $C_r(G)$ be the free abelian group on the set of all r-tuples $(g_1,\dots,g_r)$, where $g_i \in G$. The boundary operator $\partial : C_r(G) \to C_{r-1}(G)$ is defined by

$$\partial(g_1,\dots,g_r) = (g_1^{-1}g_2,\dots,g_1^{-1}g_r) + \sum_{i=1}^{r}(-1)^i(g_1,\dots,\hat{g}_i,\dots,g_r).$$

Then we have $\partial^2 = 0$. The symbol $H_r(G)$ will stand for the r-th homology group of the above chain complex.

Let $\boldsymbol{R}^m = \{(x_1,\dots,x_m) \mid x_i \in \boldsymbol{R}\}$ be an m-dimensional Euclidean space and $\boldsymbol{R}^n$ the n-dimensional subspace $\{(x_1,\dots,x_n,0,\dots,0) \mid x_i \in \boldsymbol{R}\}$ of $\boldsymbol{R}^m$. Let U be an open rectangle in $\boldsymbol{R}^m$ such that $U \cap \boldsymbol{R}^n \neq \emptyset$. We put $\mathscr{H}_{LIP,U}(\boldsymbol{R}^m,\boldsymbol{R}^n) = \{f \in \mathscr{H}_{LIP}(\boldsymbol{R}^m,\boldsymbol{R}^n) \mid supp(f) \subset U\}$.

Let $\iota : \mathscr{H}_{LIP,U}(\boldsymbol{R}^m,\boldsymbol{R}^n) \to \mathscr{H}_{LIP}(\boldsymbol{R}^m,\boldsymbol{R}^n)$ denote the inclusion map, and let $\iota_* : H_r(\mathscr{H}_{LIP,U}(\boldsymbol{R}^m,\boldsymbol{R}^n)) \to H_r(\mathscr{H}_{LIP}(\boldsymbol{R}^m,\boldsymbol{R}^n))$ denote the induced homomorphism. By an argument similar to that in the proof of Lemma 2.2 of [**F-I**], we have the following lemma.

LEMMA 2.1. *ι_* is an isomorphism.*

THEOREM 2.2. *If $n > 0$, then the homology groups $H_r(\mathscr{H}_{LIP}(\boldsymbol{R}^m,\boldsymbol{R}^n)) = 0$ for $r > 0$.*

Proof. We put $U = (1,2) \times (-1,1)^{m-1} \subset \boldsymbol{R}^m$. Then we note $U \cap \boldsymbol{R}^n = (1,2) \times (-1,1)^{n-1}$. Take a diffeomorphism $\phi \in \mathscr{H}_{LIP}(\boldsymbol{R}^m, \boldsymbol{R}^n)$ given by $\phi(x) = (1/3)x$ for $x \in B(0,3) = \{x = (x_1, \dots, x_m) \in \boldsymbol{R}^m \mid (x_1)^2 + \cdots + (x_m)^2 < 9\}$. We set $U_j = \phi^j(U) = (1/3^j, 2/3^j) \times (-1/3^j, 1/3^j)^{m-1}$, $(j = 0, 1, 2, \dots)$. Note that $U_0 = U$. Then we have that $\bar{U}_j \cap \bar{U}_k = \emptyset$ if $j \neq k$ and $\{\bar{U}_j\}$ shrinks to the origin $0 \in \boldsymbol{R}^m$ as j goes to ∞.

For any $g \in \mathscr{H}_{LIP,U}(\boldsymbol{R}^m, \boldsymbol{R}^n)$ and $i = 0, 1$, we define $\psi_i(g)$ as follows;

$$\psi_i(g)(x) = \begin{cases} \phi^j g \phi^{-j}(x) & (x \in \bar{U}_j, j \geq i) \\ x & (x \notin \bigcup_{j \geq i} \bar{U}_j). \end{cases}$$

Since it is easy to see that $\psi_i(g)$ as well as $(\psi_i(g))^{-1} = \psi_i(g^{-1})$ is Lipschitz, the rest is proved as Theorem 2.1 of [**F-I**]. $\square$

The following is proved similarly.

Corollary 2.3. *The homology groups* $H_r(\mathscr{H}_{LIP}(\boldsymbol{R}^m)) = 0$ *for* $r > 0$.

Corollary 2.4. $\mathscr{H}_{LIP}(\boldsymbol{R}^m)$ *and* $\mathscr{H}_{LIP}(\boldsymbol{R}^m, \boldsymbol{R}^n)$ *are perfect groups for* $n > 0$.

Proof. These are immediate consequences of Theorem 2.2 and Corollary 2.3 because that $H_1(G) \cong G/[G,G]$ for any group G. $\square$

3. Commutators of $\mathscr{H}_{LIP}(M)$ and $\mathscr{H}_{LIP}(M,N)$.

A PL manifold pair (M,N) is by definition a pair of PL manifolds in a Euclidean space of sufficiently large dimension such that the PL structure on N is a PL substructure of that on M. In this section, we show that $\mathscr{H}_{LIP}(M,N)$ is perfect for a PL manifold pair (M,N) (possibly with $N = \emptyset$) for $\dim N \neq 0$.

From the results of Sullivan [**S**] and Siebenmann-Sullivan [**S-S**], we have the following fragmentation lemma.

Proposition 3.1. *Let* (M,N) *be a PL manifold pair. For any* $f \in \mathscr{H}_{LIP}(M,N)$, *there exist* $f_i \in \mathscr{H}_{LIP}(M,N)$ $(i = 1, 2, \dots, k)$ *such that*

1) *the support of each* f_i *is contained in a small ball, and*
2) $f = f_k \circ f_{k-1} \circ \cdots \circ f_1$.

Theorem 3.2. *Let* (M,N) *be a PL manifold pair. Then* $\mathscr{H}_{LIP}(M,N)$ *is perfect for* $\dim N \neq 0$.

Proof. Let $f \in \mathscr{H}_{LIP}(M,N)$. From Proposition 3.1, we have $f = f_k \circ f_{k-1} \circ \cdots \circ f_1$, where $f_i \in \mathscr{H}_{LIP}(M,N)$ and each f_i is supported either in an open rectangle U_i with $U_i \cap N = \emptyset$ or in an open rectangle U_i with $U_i \cap N \neq \emptyset$. Hence we can assume that either $f_i \in \mathscr{H}_{LIP}(\boldsymbol{R}^m)$ or $f_i \in \mathscr{H}_{LIP}(\boldsymbol{R}^m, \boldsymbol{R}^n)$, where

$m = \dim M$ and $n = \dim N$. From Corollary 2.4, we have that each f_i is contained in the commutator subgroup of $\mathscr{H}_{LIP}(M,N)$ and hence so is f. Thus $\mathscr{H}_{LIP}(M,N)$ is perfect. This completes the proof. $\square$

Corollary 3.3. *Let M be a PL manifold with boundary ∂M. Then $\mathscr{H}_{LIP}(M,\partial M)$ is perfect for* $\dim M > 1$.

Remark 3.4. T. Tsuboi [**T**] proved that $\mathscr{H}_{LIP}([0,1])$ is uniformly perfect. It is known that the group $\mathscr{H}([0,1])$ of orientation preserving homeomorphisms of the interval $[0,1]$ is also perfect (cf. Lemma 4.4 of [**F-I**]). On the other hand, the group $\mathit{Diff}^{\infty}([0,1])$ of C^{∞}-orientation preserving diffeomorphisms of $[0,1]$ is not perfect. In fact, Theorem 1 of Fukui [**F1**] implies that $H_1(\mathit{Diff}^{\infty}([0,1])) \cong \boldsymbol{R}^2$.

4. Compact open Lipschitz topology on $\mathscr{H}_{LIP}(M)$.

In §2 and §3, we studied the group $\mathscr{H}_{LIP}(M)$ with the compact open topology. In the following section, we need the following alternative topology on $\mathscr{H}_{LIP}(M)$ for a Lipschitz manifold M.

First we recall the definitions of a Lipschitz manifold and a Lipschitz map. Let M be an m-dimensional topological manifold. By a Lipschitz atlas on M we mean a maximal family $\mathscr{S} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ of pairs $(U_\alpha, \varphi_\alpha)$ of open sets U_α in M and homeomorphisms φ_α of U_α to $\varphi_\alpha(U_\alpha)$ in $\boldsymbol{R}^m$ satisfying the following: (i) $\{U_\alpha\}_{\alpha\in A}$ covers M and (ii) If $U_\alpha \cap U_\beta \neq \emptyset$, the transition function $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ from an open set of $\boldsymbol{R}^m$ to an open set of $\boldsymbol{R}^m$ is Lipschitz. We call $(M, \mathscr{S})$ a Lipschitz manifold and simply write M instead of $(M, \mathscr{S})$.

Let M, N be two Lipschitz manifolds. A continuous map $f : M \to N$ is called a Lipschitz map if for any point p in M, there exist a local chart $(U_\alpha, \varphi_\alpha)$ of M around p and a local chart $(V_\lambda, \psi_\lambda)$ of N around $f(p)$ such that $f(U_\alpha) \subset V_\lambda$ and $\psi_\lambda \circ f \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \to \psi_\lambda(V_\lambda)$ is Lipschitz. We denote by $C_{LIP}(M,N)$ the set of all Lipschitz mappings from M to N.

Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ and $\{(V_\lambda, \psi_\lambda)\}_{\lambda\in\Lambda}$ be Lipschitz atlases on M and N, respectively. Let K_α be a compact subset of U_α for each $\alpha \in A$ such that the family $\{\mathit{int}\, K_\alpha\}_{\alpha\in A}$ covers M. Let $f \in C_{LIP}(M,N)$. We take a local chart $(V_\lambda, \psi_\lambda)$ on N such that $f(K_\alpha) \subset V_\lambda$. For $\varepsilon_\alpha > 0$, we let $\mathscr{N}^{LIP}(f, (U_\alpha, \varphi_\alpha), (V_\lambda, \psi_\lambda), \varepsilon_\alpha, K_\alpha)$ be the set of all $g \in C_{LIP}(M,N)$ such that $g(K_\alpha) \subset V_\lambda$ and $\mathit{lip}(f-g) < \varepsilon_\alpha$, where $\mathit{lip}(f-g) < \varepsilon_\alpha$ means that

$$\|\psi_\lambda \circ f \circ \varphi_\alpha^{-1}(x) - \psi_\lambda \circ g \circ \varphi_\alpha^{-1}(x)\| < \varepsilon_\alpha$$

and

$$\|(\psi_\lambda \circ f \circ \varphi_\alpha^{-1}(x) - \psi_\lambda \circ g \circ \varphi_\alpha^{-1}(x)) - (\psi_\lambda \circ f \circ \varphi_\alpha^{-1}(y) - \psi_\lambda \circ g \circ \varphi_\alpha^{-1}(y))\| < \varepsilon_\alpha \|x - y\|$$

for distinct $x, y \in \varphi_\alpha(K_\alpha)$. The sets $\mathcal{N}^{LIP}(f, (U_\alpha, \varphi_\alpha), (V_\lambda, \psi_\lambda), \varepsilon_\alpha, K_\alpha)$ form a subbasis for a topology on $C_{LIP}(M, N)$. We call this topology the *compact open Lipschitz topology*.

A homeomorphism $f : M \to M$ is called a Lipschitz homeomorphism if f and f^{-1} are Lipschitz. We denote by $\mathscr{H}_{LIP}(M)$ the group of all Lipschitz homeomorphisms of M with compact support (as a subspace of $C_{LIP}(M, M)$ endowed with the compact open Lipschitz topology).

Now we consider the case that M and N are Euclidean spaces. For $u \in C_{LIP}(\boldsymbol{R}^m, \boldsymbol{R})$ with compact support, we define a family of maps $\phi_t : \boldsymbol{R}^m \to \boldsymbol{R}^m$ $(0 \le t \le 1)$ by $\phi_t(x_1, \dots, x_m) = (x_1 + tu(x), x_2, \dots, x_m)$. Then we have the following.

Lemma 4.1. *If $lip(u) = k < 1$, then $\{\phi_t\}$ is a family of Lipschitz homeomorphisms.*

Proof. We can easily check that ϕ_t is Lipschitz. First we show that ϕ_t is a homeomorphism for each t. Suppose that there exist two distinct points $x^0 = (x_1^0, \dots, x_m^0)$, $y^0 = (y_1^0, \dots, y_m^0) \in \boldsymbol{R}^m$ with $\phi_t(x^0) = \phi_t(y^0)$. Then we have

$$\begin{aligned} |x_1^0 - y_1^0| &= t|u(x^0) - u(y^0)| \\ &\le tk\|x^0 - y^0\| \\ &= tk|x_1^0 - y_1^0| \\ &< |x_1^0 - y_1^0|. \end{aligned}$$

This is a contradiction. Thus ϕ_t is injective for each t. Since the support of u is compact, it is easy to see that ϕ_t is a homeomorphism.

Next we show that ϕ_t^{-1} is Lipschitz. Put $\phi_t^{-1}(x_1, \dots, x_m) = (x_1 + g_t(x), x_2, \dots, x_m)$. Then we have only to prove that g_t is Lipschitz for each $t \in [0, 1]$. Since $\phi_t \circ \phi_t^{-1} = 1$, we have $g_t(x) = -tu(x_1 + g_t(x), x_2, \dots, x_m)$. Thus for any $x, y \in \boldsymbol{R}^m$ and each t,

$$\begin{aligned} |g_t(x) - g_t(y)| &\le |u(x_1 + g_t(x), x_2, \dots, x_m) - u(y_1 + g_t(y), y_2, \dots, y_m)| \\ &\le k\|(x_1 + g_t(x), x_2, \dots, x_m) - (y_1 + g_t(y), y_2, \dots, y_m)\| \\ &\le k\|x - y\| + k|g_t(x) - g_t(y)|. \end{aligned}$$

Then we have $(1 - k)|g_t(x) - g_t(y)| \le k\|x - y\|$ and hence $|g_t(x) - g_t(y)| \le k/(1 - k)\|x - y\|$ for any $x, y \in \boldsymbol{R}^m$ and each t. This shows that $lip(g_t) \le k/(1 - k)$ and we have $lip(\phi_t^{-1}) \le 1/(1 - k)$, which completes the proof. □

Let M be a compact Lipschitz manifold of dimension m. We take a local chart $(U_\alpha, \varphi_\alpha)$ on M and identify U_α with an open set in $\boldsymbol{R}^m$ via φ_α, and take relatively compact open subsets W_1, W_2 of U_α such that $\overline{W}_2 \subset W_1$. Then the metric on $\overline{W}_1$ may be considered as the Euclidean metric. Furthermore we take a Lipschitz function $\mu_\alpha : U_\alpha \to [0,1]$ such that $\mu_\alpha = 1$ on $\overline{W}_2$ and $\mu_\alpha = 0$ outside of W_1. Then for any $f \in \mathcal{N}^{LIP}(1_M, (U_\alpha, \varphi_\alpha), (U_\alpha, \varphi_\alpha), \varepsilon, \overline{W}_1)$, we define a map $f_\alpha : M \to M$ by

$$f_\alpha(x) = \begin{cases} x + \mu_\alpha(x)(f(x) - x) & \text{for } x \in U_\alpha \\ x & \text{for } x \notin U_\alpha. \end{cases}$$

Then we have the following.

Proposition 4.2. *If* $m(1 + lip(\mu_\alpha))\varepsilon < 1$, *then* f_α *is a Lipschitz homeomorphism which is isotopic to* 1_M *through Lipschitz homeomorphisms, that is,* $f_\alpha \in \mathcal{H}_{LIP}(M)$.

Proof. For $x, y \in \overline{W}_1$, we have

$$\begin{aligned} &\|\mu_\alpha(x)(f(x) - x) - \mu_\alpha(y)(f(y) - y)\| \\ &\quad\le \|\mu_\alpha(x)(f(x) - x) - \mu_\alpha(x)(f(y) - y)\| \\ &\qquad + \|\mu_\alpha(x)(f(y) - y) - \mu_\alpha(y)(f(y) - y)\| \\ &\quad= |\mu_\alpha(x)| \cdot \|f(x) - x - (f(y) - y)\| \\ &\qquad + |\mu_\alpha(x) - \mu_\alpha(y)| \cdot \|f(y) - y\| \\ &\quad\le \varepsilon \cdot \|x - y\| + lip(\mu_\alpha)\varepsilon \cdot \|x - y\|. \end{aligned}$$

Putting $\mu_\alpha(x)(f(x) - x) = (u_1(x), \dots, u_m(x))$, we have $lip(u_i) < 1/m$ for each i. We define maps $f_\alpha^i : U_\alpha \to U_\alpha$ $(i = 1, \dots, m)$ by $f_\alpha^1(x_1, \dots, x_m) = (x_1 + u_1(x), \dots, x_m)$ and $f_\alpha^i(x_1, \dots, x_m) = (x_1, \dots, x_{i-1}, x_i + u_i((f_\alpha^{i-1} \circ \cdots \circ f_\alpha^1)^{-1}(x)), x_{i+1}, \dots, x_m)$ for $i = 2, \dots, m$. We prove by induction that each f_α^i is a Lipschitz homeomorphism which is isotopic to 1_M through Lipschitz homeomorphisms. By using Lemma 4.1 f_α^1 is a Lipschitz homeomorphism which is isotopic to 1_M through Lipschitz homeomorphisms and $lip((f_\alpha^1)^{-1}) < m/(m-1)$. For the inductive step, we assume that $lip(u_i((f_\alpha^{i-1} \circ \cdots \circ f_\alpha^1)^{-1})) < 1/(m-i+1)$ and hence f_α^i is a Lipschitz homeomorphism which is isotopic to 1_M through Lipschitz homeomorphisms and, $lip((f_\alpha^i)^{-1}) < (m-i+1)/(m-i)$ $(i < m)$. Then we have that $lip(u_{i+1}((f_\alpha^i \circ \cdots \circ f_\alpha^1)^{-1})) < 1/(m-i)$ and hence, by Lemma 4.1 f_α^{i+1} is a Lipschitz homeomorphism which is isotopic to 1_M through Lipschitz homeomorphisms and $lip((f_\alpha^{i+1})^{-1}) < (m-i)/(m-i-1)$. Since $f_\alpha = f_\alpha^m \circ f_\alpha^{m-1} \circ \cdots \circ f_\alpha^1$, we have $f_\alpha \in \mathcal{H}_{LIP}(M)$. This completes the proof. □

Corollary 4.3 (fragmentation lemma). *For any* $f \in \mathcal{H}_{LIP}(M)$, *there are* $f_i \in \mathcal{H}_{LIP}(M)$ $(i = 1, 2, \dots, k)$ *such that* (1) $f = f_k \circ f_{k-1} \circ \cdots \circ f_1$ *and* (2) *the support of each* f_i *is contained in a small ball.*

Proof. This follows from Proposition 4.2 because of the compactness of M. $\square$

Corollary 4.4. *$\mathscr{H}_{LIP}(M)$ is locally contractible.*

Proof. This is an immediate consequence of Proposition 4.2 and Corollary 4.3. $\square$

Remark 4.5. By Corollary 4.3 and 4.4, the connected component of $\mathscr{H}_{LIP}(M)$ coincides with the path component of $\mathscr{H}_{LIP}(M)$.

Theorem 4.6. *Let M be a compact Lipschitz manifold. Then $\mathscr{H}_{LIP}(M)$ is perfect.*

Proof. This follows from Corollary 4.3 and Corollary 2.4 as in the proof of Theorem 3.2. $\square$

5. Commutators of equivariant Lipschitz homeomorphisms.

In this section we consider commutators of equivariant Lipschitz homeomorphisms of principal G-bundles. Let G be a compact Lie group of dimension q and M the total space of a principal G-bundle over a closed $(m-q)$-dimensional Lipschitz manifold B such that each transition function is Lipschitz. We denote by the same letter $\mathscr{H}_{LIP}(M)$ the identity component of the space of all Lipschitz homeomorphisms of M. Moreover we denote by $\mathscr{H}_{LIP,G}(M)$ the identity component of the group of equivariant Lipschitz homeomorphisms of M (as a subspace of $\mathscr{H}_{LIP}(M)$).

Then we have the following.

Theorem 5.1. *$\mathscr{H}_{LIP,G}(M)$ is perfect for* $\dim B > 0$.

For any $f \in \mathscr{H}_{LIP,G}(M)$, there is the following commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ B & \xrightarrow{\bar{f}} & B, \end{array}$$

where π is the projection of the principal G-bundle. Then we easily see the following.

Lemma 5.2. $\bar{f} \in \mathscr{H}_{LIP}(B)$.

We define a map $P : \mathscr{H}_{LIP,G}(M) \to \mathscr{H}_{LIP}(B)$ by $P(f) = \bar{f}$. Then we have the following.

Lemma 5.3. *P is an epimorphism.*

PROOF. It is clear that P is a homomorphism. We show that P is surjective. Take any $\bar{f} \in \mathscr{H}_{LIP}(B)$. We may assume that $\bar{f}$ is close to 1_B. From Corollary 4.3, there exist $\bar{f_i} \in \mathscr{H}_{LIP}(B)$ $(i = 1, 2, \dots, k)$ such that the support of $\bar{f_i}$ is in a small ball and $\bar{f} = \bar{f_k} \circ \bar{f}_{k-1} \circ \cdots \circ \bar{f_1}$. Thus we may assume that each $\bar{f_i}$ is supported in a small coordinate neighborhood U_i over which π is trivial. We identify $\pi^{-1}(U_i)$ with $U_i \times G$. Then we define a map $f_i : U_i \times G \to U_i \times G$ by

$$f_i(x, g) = (\bar{f_i}(x), g)$$

for $x \in U_i, g \in G$.

We can extend f_i to M defining to be the identity outside of $\pi^{-1}(U_i)$. Then we easily see that $f_i \in \mathscr{H}_{LIP,G}(M)$. Putting $f = f_k \circ f_{k-1} \circ \cdots \circ f_1$, we have $P(f) = \bar{f}$. This completes the proof. □

Let $e : \boldsymbol{R}^{m-q} \to G$ be the constant mapping defined by $e(x) = 1$ for $x \in \boldsymbol{R}^{m-q}$. For a map $h : \boldsymbol{R}^{m-q} \to G$, put $supp(h) = \overline{h^{-1}(G - \{1\})}$. Take a local chart (V, ψ) of G around 1 such that $\psi : V \to \boldsymbol{R}^q$ maps 1 to the origin. Let $h : \boldsymbol{R}^{m-q} \to G$ be a Lipschitz map close to e. Then we may assume $h(\boldsymbol{R}^{m-q}) \subset V$.

Let $\xi : \boldsymbol{R}^{m-q} \to [0, 1]$ be a Lipschitz function such that for $(x_1, \dots, x_{m-q}) \in \boldsymbol{R}^{m-q}$,

$$\xi(x_1, \dots, x_{m-q}) = \begin{cases} 1 & \text{for } x_1^2 + \cdots + x_{m-q}^2 \le 1 \\ 0 & \text{for } x_1^2 + \cdots + x_{m-q}^2 \ge 2. \end{cases}$$

We define a map $h' : \boldsymbol{R}^{m-q} \to G$ by

$$h'(x_1, \dots, x_{m-q}) = \psi^{-1}(\psi \circ h(x_1, \dots, x_{m-q}) \cdot \xi(x_1, \dots, x_{m-q})).$$

Then we easily see the following.

LEMMA 5.4. *h' is a Lipschitz map such that* (1) *$h' = h$ on the unit $(m-q)$-ball,* (2) *h' is close to e and* (3) *$supp(h')$ is contained in the $(m-q)$-ball of radius* 2.

COROLLARY 5.5 (equivariant fragmentation lemma). *For any $f \in \mathscr{H}_{LIP,G}(M)$, there are $f_i \in \mathscr{H}_{LIP,G}(M)$ $(i = 1, 2, \dots, k)$ such that* (1) *$f = f_k \circ f_{k-1} \circ \cdots \circ f_1$ and* (2) *the image of the support of each f_i by π is contained in a small ball in B.*

PROOF. For $f \in \mathscr{H}_{LIP,G}(M)$, put $P(f) = \bar{f}$. We may assume that f is close to 1_M. As in the proof of Lemma 5.3, we can find $f_i \in \mathscr{H}_{LIP,G}(M)$ $(i = 1, 2, \dots, s)$ such that each f_i is close to 1_M and $P(f_s \circ f_{s-1} \circ \cdots \circ f_1) = \bar{f}$. Then $f \circ (f_s \circ f_{s-1} \circ \cdots \circ f_1)^{-1}$ has the following form on each coordinate neighborhood U:

$$f \circ (f_s \circ f_{s-1} \circ \cdots \circ f_1)^{-1}(x, g) = (x, g \cdot h(x))$$

for $x \in U, g \in G$, where h is a Lipschitz map of U to G which is close to e. Thus the rest of the proof follows from Lemma 5.4. □

Now we consider the trivial G-bundle $\pi : \boldsymbol{R}^{m-q} \times G \to \boldsymbol{R}^{m-q}$. We denote by $\mathscr{H}_{LIP,G}(\boldsymbol{R}^{m-q} \times G)$ the group of equivariant Lipschitz homeomorphisms of $\boldsymbol{R}^{m-q} \times G$ which are isotopic to the identity through compactly supported equivariant Lipschitz homeomorphisms.

Then $P : \mathscr{H}_{LIP,G}(\boldsymbol{R}^{m-q} \times G) \to \mathscr{H}_{LIP}(\boldsymbol{R}^{m-q})$ is a homomorphism given by $P(f)(x) = \pi(f(x,1))$, for $f \in \mathscr{H}_{LIP,G}(\boldsymbol{R}^{m-q} \times G)$ and $x \in \boldsymbol{R}^{m-q}$.

Let G_0 be the identity component of G. Let $C_{LIP}(\boldsymbol{R}^{m-q}, G_0)$ denote the space of all Lipschitz maps from $\boldsymbol{R}^{m-q}$ to G_0 which are homotopic to e through compactly supported Lipschitz maps.

We define a homomorphism $L : \ker P \to C_{LIP}(\boldsymbol{R}^{m-q}, G_0)$ by $f(x,1) = (x, L(f)(x)^{-1})$ for $f \in \ker P$ and $x \in \boldsymbol{R}^{m-q}$. Note that if f is close to the identity, then $L(f)$ is close to e.

For $\delta > 0$, put $B_\delta = \{x \in \boldsymbol{R}^m \mid \|x\| \leq \delta\}$. Then we see the following.

Lemma 5.6. *For any $u \in C_{LIP}(\boldsymbol{R}^m, \boldsymbol{R})$ with $supp(u) \subset B_\delta$ and $lip(u) = k < 1$, there exist $v \in C_{LIP}(\boldsymbol{R}^m, \boldsymbol{R})$ with $supp(v) \subset B_{5\delta}$ and $\phi \in \mathscr{H}_{LIP}(\boldsymbol{R}^m)$ such that $u = v \circ \phi - v$.*

Proof. We define a map $\phi : \boldsymbol{R}^m \to \boldsymbol{R}^m$ by

$$\phi(x_1, \dots, x_m) = (x_1 + u(x), x_2, \dots, x_m)$$

for $x = (x_1, \dots, x_m) \in \boldsymbol{R}^m$. Then Lemma 4.1 implies $\phi \in \mathscr{H}_{LIP}(\boldsymbol{R}^m)$ since $k < 1$. Note that $supp(\phi) \subset B_\delta$.

Let $\xi : \boldsymbol{R} \to \boldsymbol{R}$ be a Lipschitz function such that $\xi(x) = x$ for $\|x\| \leq 2\delta$, $|\xi(x)| \leq 3\delta$ for $2\delta \leq \|x\| \leq 3\delta$ and $\xi(x) = 0$ for $\|x\| \geq 3\delta$. Let $\mu : \boldsymbol{R}^{m-1} \to [0,1]$ be a Lipschitz function such that for $(x_1, \dots, x_{m-1}) \in \boldsymbol{R}^{m-1}$,

$$\mu(x_1, \dots, x_{m-1}) = \begin{cases} 1 & \text{for } x_1^2 + \dots + x_{m-1}^2 \leq \delta^2 \\ 0 & \text{for } x_1^2 + \dots + x_{m-1}^2 \geq 9\delta^2. \end{cases}$$

We define a Lipschitz function $v : \boldsymbol{R}^m \to \boldsymbol{R}$ by $v(x_1, \dots, x_m) = \xi(x_1)\mu(x_2, \dots, x_m)$. Note that v is supported in $B_{5\delta}$. Then we have $u = v \circ \phi - v$. This completes the proof. □

Let $L(G_0)$ be the Lie algebra of G and let $\{X_1, \dots, X_q\}$ be a basis of $L(G_0)$. Let $\Psi : L(G_0) \to G_0$ be the mapping defined by $\Psi(a_1X_1 + \dots + a_qX_q) = (\exp a_1X_1) \cdots (\exp a_qX_q)$. Then there exist an open ball V in $L(G_0)$ of radius ε, centered at 0 and a neighborhood W of 1 in G_0 such that Ψ is a diffeomorphism of V onto W.

Lemma 5.7. *Let $u : \boldsymbol{R}^{m-q} \to W$ be a mapping in $C_{LIP}(\boldsymbol{R}^{m-q}, G_0)$ which is close to e and the support of u is contained in a ball of radius ε.*

Then there exist $v_i \in C_{LIP}(\boldsymbol{R}^{m-q}, G_0)$ *and* $\phi_i \in \mathscr{H}_{LIP}(\boldsymbol{R}^m)$, $i = 1, 2, \dots, q$, *such that*

$$u = (v_1^{-1} \cdot (v_1 \circ \phi_1)) \cdot \cdots \cdot (v_q^{-1} \cdot (v_q \circ \phi_q)).$$

Proof. Using Lemma 5.6, it is proved by the same argument as in the proof of Lemma 4 of [**A-F**]. □

For any $\phi \in \mathscr{H}_{LIP}(\boldsymbol{R}^{m-q})$, we define $\bar{\phi} \in \mathscr{H}_{LIP,G}(\boldsymbol{R}^{m-q} \times G)$ by $\bar{\phi}(x, g) = (\phi(x), g)$ for $x \in \boldsymbol{R}^{m-q}, g \in G$.

Lemma 5.8. *Let* $h \in \ker P$ *and* $u = L(h)$. *For any* $\phi \in \mathscr{H}_{LIP}(\boldsymbol{R}^{m-q})$, *we have*

$$L(h^{-1} \circ \bar{\phi}^{-1} \circ h \circ \bar{\phi}) = u^{-1} \cdot (u \circ \phi).$$

Proof. An easy computation. □

Using Corollary 5.5, Lemma 5.7 and 5.8, we have the following.

Proposition 5.9. $\ker P = [\ker P, \mathscr{H}_{LIP,G}(\boldsymbol{R}^{m-q} \times G)]$.

Proof of Theorem 5.1. The proof is similar to the proof of Theorem of [**A-F**]. Since $1 \to \ker P \to \mathscr{H}_{LIP,G}(\boldsymbol{R}^{m-q} \times G) \to \mathscr{H}_{LIP}(\boldsymbol{R}^{m-q}) \to 1$ is exact, we have the following exact sequence:

$$\begin{aligned} &\ker P/[\ker P, \mathscr{H}_{LIP,G}(\boldsymbol{R}^{m-q} \times G)] \\ &\qquad \to H_1(\mathscr{H}_{LIP,G}(\boldsymbol{R}^{m-q} \times G)) \to H_1(\mathscr{H}_{LIP}(\boldsymbol{R}^{m-q})) \to 0. \end{aligned}$$

By Corollary 2.4 and Proposition 5.9, we have $H_1(\mathscr{H}_{LIP,G}(\boldsymbol{R}^{m-q} \times G)) = 0$, hence $\mathscr{H}_{LIP,G}(\boldsymbol{R}^{m-q} \times G)$ is perfect. Using Corollary 5.5, we see that $\mathscr{H}_{LIP,G}(M)$ is perfect. This completes the proof. □

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