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On the gap between the first eigenvalues of the Laplacian on functions and 1-forms

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Abstract. We study the first positive eigenvalue $\lambda_1^{(p)}$ of the Laplacian on p-forms for oriented closed Riemannian manifolds. It is known that the inequality $\lambda_1^{(1)} \le \lambda_1^{(0)}$ holds in general. In the present paper, a Riemannian manifold is said to have the gap if the strict inequality $\lambda_1^{(1)} < \lambda_1^{(0)}$ holds. We show that any oriented closed manifold M with the first Betti number $b_1(M) = 0$ whose dimension is bigger than two, admits two Riemannian metrics, the one with the gap and the other without the gap.

1. Introduction.

Let (M, g) be an m-dimensional connected oriented closed Riemannian manifold. We denote by $\lambda_k^{(p)}(M, g)$ the k-th positive eigenvalue of the Laplacian on the spaces of p-forms. We compare $\lambda_1^{(1)}(M, g)$ with $\lambda_1^{(0)}(M, g)$. In gerenal, the commutativity of the Laplacian $\Delta = d\delta + \delta d$ and the exterior differential operator d implies the inequality $\lambda_1^{(1)}(M, g) \le \lambda_1^{(0)}(M, g)$. We are interested in a Riemannian manifold (M, g) satisfying $\lambda_1^{(1)}(M, g) < \lambda_1^{(0)}(M, g)$. For convenience, when a metric g satisfies $\lambda_1^{(1)}(M, g) < \lambda_1^{(0)}(M, g)$ (resp. $\lambda_1^{(1)}(M, g) = \lambda_1^{(0)}(M, g)$), we call it a metric with (resp. without) the gap.

First of all, we study which closed manifolds M admit metrics g with the gaps. While no 2-dimensional oriented closed manifold admits such a metric (cf. Proposition 2.4), we obtain the following two theorems.

Theorem 1.1. *For an m-dimensional connected oriented closed manifold M $(m \ge 3)$, there exists a metric g on M without the gap, namely, we have*

$$\lambda_1^{(1)}(M, g) = \lambda_1^{(0)}(M, g).$$

Theorem 1.2. *For an m-dimensional connected oriented closed manifold M $(m \ge 3)$ with the first Betti number $b_1(M) = 0$, there exists a metric g on M with the gap, namely, we have*

$$\lambda_1^{(1)}(M, g) < \lambda_1^{(0)}(M, g).$$

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We know some examples of closed manifolds with $b_1(M) \neq 0$ which admit metrics with the gaps, e.g., the m-dimensional tori $(m \geq 3)$ (cf. Theorem 0.2 in [**CC-90**]). We conjecture that our Theorem 1.2 is valid for any m-dimensional closed manifold $(m \geq 3)$.

Next, we study geometric properties of Riemannian manifolds whose metrics have the gaps. In the case of Einstein manifolds with positive Ricci curvature, we obtain the following.

THEOREM 1.3. *Let (M, g) be a connected oriented closed Einstein manifold with positive Ricci curvature, and* $\mathrm{Isom}(M, g)$ *the isometry group of (M, g). Suppose that (M, g) has the gap.*

(i) *If* $\dim \mathrm{Isom}(M, g) = 0$, *then the identity map is strongly stable as a harmonic map.*
(ii) *If* $\dim \mathrm{Isom}(M, g) \geq 1$, *then the identity map is weakly stable as a harmonic map.*

The structure of the present paper is as follows: In Section 2, we give a condition for a manifold to admit a metric with the gap, using the Hodge decomposition theorem. In Section 3, we prove Theorem 1.1 by constructing a one-parameter family of metrics including metrics without the gap. In Section 4, we prove Theorem 1.2. We first prove it in the case of the canonical spheres. For a general case, we do it by gluing this sphere to a given manifold. In Section 5, we prove Theorem 1.3. Furthermore, for all simply connected compact simple Lie groups and simply connected irreducible Riemannian symmetric spaces of compact type we completely determine whether or not their canonical metrics are metrics with the gaps.

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2. The decompositions of the eigenspaces.

Let (M, g) be an m-dimensional connected oriented closed Riemannian manifold. The Hodge decomposition theorem says

$$A^p(M) = \boldsymbol{H}^p(M, g) \oplus dA^{p-1}(M) \oplus \delta A^{p+1}(M),$$

where $A^p(M)$ is the space of smooth p-forms on M and $\boldsymbol{H}^p(M, g)$ the space of harmonic p-forms on (M,g). Since Δ commutes d (resp. δ), the Laplacian leaves the space of exact forms $dA^{p-1}(M)$ (resp. the space of co-exact forms $\delta A^{p+1}(M)$) invariant. So we denote by $\lambda_k'^{(p)}(M,g)$ (resp. $\lambda_k''^{(p)}(M,g)$) the k-th eigenvalue of the Laplacian acting on $dA^{p-1}(M)$ (resp. $\delta A^{p+1}(M)$). For $\lambda > 0$, let us set $E^{(p)}(\lambda) = \{\omega \in A^p(M) \mid \Delta\omega = \lambda\omega\}$, $E'^{(p)}(\lambda) = E^{(p)}(\lambda) \cap dA^{p-1}(M)$, and $E''^{(p)}(\lambda) = E^{(p)}(\lambda) \cap \delta A^{p+1}(M)$. Then we have $E^{(p)}(\lambda) = E'^{(p)}(\lambda) \oplus E''^{(p)}(\lambda)$. The operators $(1/\sqrt{\lambda})d$ and $(1/\sqrt{\lambda})\delta$ induce the isomorphisms between the eigenspaces: $E'^{(p)}(\lambda) \simeq E''^{(p-1)}(\lambda)$. Especially, we obtain

$$\lambda_1'^{(p)}(M,g) = \lambda_1''^{(p-1)}(M,g). \tag{2.1}$$

From $\lambda_1^{(0)} = \lambda_1''^{(0)}$ and (2.1) for $p = 1$, we have the following.

Proposition 2.1. *For every connected oriended closed Riemannian manifold (M,g), we have $\lambda_1^{(1)}(M,g) = \min\{\lambda_1^{(0)}(M,g), \lambda_1''^{(1)}(M,g)\}$.*

Corollary 2.2. *A metric g has the gap if and only if $\lambda_1''^{(1)}(M,g) < \lambda_1^{(0)}(M,g)$.*

Remark 2.3. For general p-forms $(p \geq 2)$, the opposite inequality $\lambda_1^{(p)}(M,g) > \lambda_1^{(p-1)}(M,g)$ may hold. The reason why the case of 1-forms is an exception is that there exists no exact 0-form (except for the dual case $p = m$).

Moreover since the Hodge star operator commutes the Laplacian, all the eigenvalues on p-forms and $(m-p)$-forms coincide (the Hodge duality).

For any oriented 2-dimensional manifold (M,g), we have the duality $\lambda_1^{(0)}(M,g) = \lambda_1^{(2)}(M,g)$ and $\lambda_1''^{(1)}(M,g) = \lambda_1'^{(2)}(M,g) = \lambda_1^{(2)}(M,g)$ by (2.1) for $p = 2$. Hence, from Proposition 2.1, we have the following.

Proposition 2.4. *For any 2-dimensional connected oriented closed Riemannian manifold (M,g), we have $\lambda_1^{(1)}(M,g) = \lambda_1^{(0)}(M,g)$.*

3. Proof of Theorem 1.1.

3.1 The proof using the theorem of Anné and Colbois.

Let M be an m-dimensional connected oriented closed manifold $(m \geq 3)$. We construct a one-parameter family of metrics on M as follows: We remove an m-disk D_1 from the sphere S^m and glue $S^m - D_1$ to a cylinder $C = [0,1] \times S^{m-1}$ by identifying $\partial(S^m - D_1)$ with $\{0\} \times S^{m-1}$. Similarly, we remove an m-disk D_2 from M and glue $M - D_2$ to $(S^m - D_1) \cup C$ by identifying $\partial(M - D_2)$ with $\{1\} \times S^{m-1}$. Thus, we obtain the new manifold, denoted by $\overline{M}$, which is diffeomorphic to M and want to construct a family of metrics on $\overline{M}$ instead of M. Take a metric g_1 (resp. g_2) on S^m (resp. M) which is flat in a neighborhood of the disk D_1 (resp. D_2). We define a one-parameter family of continuous metrics g_ε on $\overline{M}$ such that

$$g_\varepsilon = \begin{cases} g_1 & \text{on } S^m - D_1, \\ dr^2 \oplus \varepsilon^2 h & \text{on } [0,1] \times S^{m-1}, \\ g_2 & \text{on } M - D_2, \end{cases}$$

where r is the canonical coordinate of $[0,1]$ and h is the canonical metric of the sphere $S^{m-1}(1)$ (see Figure 1).

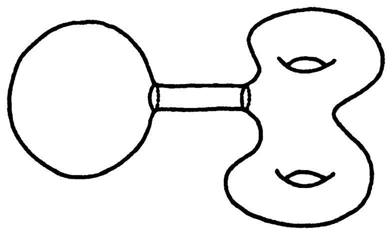

Figure 1: $(\overline{M}, g_\varepsilon)$

Although g_ε is not smooth, Anné and Colbois [**AC-95**] defined the Laplacian Δ_{g_ε} induced from $(\overline{M}, g_\varepsilon)$ and studied its eigenvalues. We denote by $\bar{\lambda}_1^{(p)}(\overline{M}, g_\varepsilon) \le \bar{\lambda}_2^{(p)}(\overline{M}, g_\varepsilon) \le \cdots$ the eigenvalues of the Laplacian on p-forms on $(\overline{M}, g_\varepsilon)$ and by $\bar{\mu}_1^{(p)} \le \bar{\mu}_2^{(p)} \le \cdots$ the union of the eigenvalues of the Laplacians on p-forms on $S^m(1)$, (M, g) and $[0,1]$. Here, the eigenvalues for $[0,1]$ should be understood with the relative boundary condition. The following theorem is a special case of Theorem B in [**AC-95**].

THEOREM 3.1. *For all $k = 1, 2, \ldots,$ and $p = 0, 1,$ we have*

$$\lim_{\varepsilon \to 0} \bar{\lambda}_k^{(p)}(\overline{M}, g_\varepsilon) = \bar{\mu}_k^{(p)}.$$

Now we go to the proof of Theorem 1.1.

First, we consider the case of functions. It is easy to see that $\bar{\lambda}_1^{(0)}(\overline{M}, g_\varepsilon) = 0$ and $\bar{\lambda}_2^{(0)}(\overline{M}, g_\varepsilon) = \lambda_1^{(0)}(\overline{M}, g_\varepsilon) > 0$. Since $\bar{\lambda}_1^{(0)}(M, g) = 0$, $\bar{\lambda}_2^{(0)}(M, g) > 0$, $\bar{\lambda}_1^{(0)}(S^m(1)) = 0$, $\bar{\lambda}_2^{(0)}(S^m(1)) > 0$, and $\bar{\lambda}_1^{(0)}([0,1]) > 0$, we have

$$0 = \bar{\mu}_1^{(0)} = \bar{\mu}_2^{(0)} < \bar{\mu}_3^{(0)} \le \cdots.$$

Hence, by Theorem 3.1, we have

$$\lambda_1^{(0)}(\overline{M}, g_\varepsilon) \to 0, \quad \lambda_2^{(0)}(\overline{M}, g_\varepsilon) \to \bar{\mu}_3^{(0)} > 0 \tag{3.1}$$

as $\varepsilon \to 0$.

Similarly, we also consider the case of 1-forms. From the de Rham-Hodge theory, it follows that $\bar{\lambda}_1^{(1)}(\overline{M}, g_\varepsilon) = \cdots = \bar{\lambda}_{b_1}^{(1)}(\overline{M}, g_\varepsilon) = 0$ and $\bar{\lambda}_{b_1+1}^{(1)}(\overline{M}, g_\varepsilon) = \lambda_1^{(1)}(\overline{M}, g_\varepsilon) > 0$, where b_1 is the first Betti number of $\overline{M} \cong M$. Since $\bar{\lambda}_1^{(1)}(M, g) = \cdots = \bar{\lambda}_{b_1}^{(1)}(M, g) = 0$, $\bar{\lambda}_{b_1+1}^{(1)}(M, g) > 0$, $\bar{\lambda}_1^{(1)}([0,1]) = 0$, $\bar{\lambda}_2^{(1)}([0,1]) > 0$, and $\bar{\lambda}_1^{(1)}(S^m(1)) > 0$, we have

$$0 = \bar{\mu}_1^{(1)} = \cdots = \bar{\mu}_{b_1}^{(1)} = \bar{\mu}_{b_1+1}^{(1)} < \bar{\mu}_{b_1+2}^{(1)} \le \cdots.$$

Hence, by Theorem 3.1, we have

$$\lambda_1^{(1)}(\overline{M}, g_\varepsilon) \to 0, \quad \lambda_2^{(1)}(\overline{M}, g_\varepsilon) \to \bar{\mu}_{b_1+2}^{(1)} > 0, \tag{3.2}$$

as $\varepsilon \to 0$. Since $\lambda_1'^{(1)}(\overline{M}, g_\varepsilon) = \lambda_1^{(0)}(\overline{M}, g_\varepsilon) \to 0$ as $\varepsilon \to 0$ and (3.2), we have

$$\lambda_1''^{(1)}(\overline{M}, g_\varepsilon) \to \bar{\mu}_{b_1+l}^{(1)} > 0 \text{ (some } l \ge 2), \tag{3.3}$$

as $\varepsilon \to 0$.

Therefore, from (3.1) and (3.3), there is some $\varepsilon_0 > 0$ such that

$$\lambda_1^{(0)}(\overline{M}, g_{\varepsilon_0}) < \lambda_1''^{(1)}(\overline{M}, g_{\varepsilon_0}). \tag{3.4}$$

Since the eigenvalues of the Laplacian on the space of p-forms depend continuously on metrics with respect to the C^0-topology (cf. [**BU-83**], and [**MG-93**], p. 729), the strict inequality (3.4) still holds for a smooth metric $\tilde{g}_{\varepsilon_0}$ which is close to g_{ε_0}. Hence, by Proposition 2.1, we have $\lambda_1^{(0)}(\overline{M}, \tilde{g}_{\varepsilon_0}) = \lambda_1^{(1)}(\overline{M}, \tilde{g}_{\varepsilon_0})$. □

3.2. The proof using the theorem of Gentile and Pagliara.

Here we give an alternative proof of Theorem 1.1. We consider the family of metrics according to [**GP-95**]. We use the notation as in Subsection 3.1. Take a metric g on $\overline{M}$ such that $g = dr^2 \oplus h$ on $C = [0,1] \times S^{m-1}$. A one-parameter family of metrics g_t $(t > 0)$ is defined on $\overline{M}$ as

$$g_t := \begin{cases} g & \text{on } \overline{M} - \Omega, \\ t^2 dr^2 \oplus h & \text{on } \Omega, \end{cases}$$

where $\Omega := [1/3, 2/3] \times S^{m-1}$ in C. Note that g_t is smooth. Finally, we set

$$\bar{g}_t := \mathrm{vol}(\overline{M}, g_t)^{-2/m} g_t$$

so that the volume is one. Then, as $t \to \infty$, we see that $\lambda_1^{(0)}(\overline{M}, \bar{g}_t) \to 0$ by [**Tk-98**] and that $\lambda_1''^{(1)}(\overline{M}, \bar{g}_t) = \lambda_1'^{(2)}(\overline{M}, \bar{g}_t) \to \infty$ by [**GP-95**]. Hence, by Proposition 2.1, for a sufficiently large $t_0 > 0$, a metric $\bar{g}_{t_0}$ has no gap. □

4. Proof of Theorem 1.2.

We first construct a metric with the gap on the sphere. Note that the canonical metric on the sphere has no gap (cf. Theorem 5.7). Throughout this section, we use the notation as in Subsection 3.1.

Lemma 4.1. *The sphere S^m $(m \ge 3)$ admits a metric with the gap.*

Proof. First, we discuss the case of the odd dimensional sphere $S^{2n+1} (n \geq 1)$. Let g_t be the Berger metric on S^{2n+1}, that is,

$$g_t = (t^2 - 1)\eta \odot \eta + g \quad (t > 0),$$

where $\eta \odot \eta$ is the symmetric product of the dual 1-form η of a unit Killing vector field with respect to the canonical metric g. Tanno (cf. [**Tn-79**], [**Tn-83**]) verified that

$$\lambda_1^{(p)}(S^{2n+1}, g_t) \to \begin{cases} 4(n+1) & (p = 0), \\ 0 & (p = 1), \end{cases} \tag{4.1}$$

as $t \to 0$. Hence, if t is sufficiently small, g_t is a metric with the gap.

Next, we discuss the case of the even dimensional sphere $M = S^{2n} (n \geq 2)$. Then $\overline{M}$ is diffeomorphic to S^{2n}. So we may construct a metric with the gap on $\overline{M}$. We define a one-parameter family of metrics h_t on $\overline{M}$ such that

$$h_t = \begin{cases} \alpha^2 dr^2 \oplus \beta g_t & \text{on } \Omega = [1/3, 2/3] \times S^{2n-1}, \\ dr^2 \oplus \beta g_t & \text{on } C - \Omega, \end{cases}$$

where g_t is the Berger metric on S^{2n-1} and α and β are certain positive constants to be specified later. Note that h_t is smooth. Moreover, we may assume that h_t is invariant under the reflection T of $\overline{M}$ with respect to $\{1/2\} \times S^{2n-1}$. For a sufficiently small $\varepsilon > 0$, we can take a sequence of smooth functions $\{F_{\varepsilon,i}\}_{i=1}^{\infty}$ on $\overline{M}$ such that for every $i = 1, 2, \ldots, \varepsilon \leq F_{\varepsilon,i} \leq 1$ and

$$F_{\varepsilon,i} = \begin{cases} 1 & \text{on } \Omega = \left[\frac{1}{3}, \frac{2}{3}\right] \times S^{2n-1}, \\ \varepsilon & \text{on } \overline{M} - \left[\frac{1}{3} - s_i, \frac{2}{3} + s_i\right] \times S^{2n-1}, \end{cases}$$

where $\{s_i\}_{i=1}^{\infty}$ is a sequence of positive numbers satisfying that $s_i \to 0$ as $i \to \infty$. Furthermore, we may also assume that $F_{\varepsilon,i}$ depends only on r on C and that $F_{\varepsilon,i}$ is invariant under T. Then, we can define a family of metrics on $\overline{M}$ by

$$h_{\varepsilon,i,t} := F_{\varepsilon,i} h_t.$$

Now, we estimate $\lambda_1^{(1)}(\overline{M}, h_{\varepsilon,i,t})$ from above such as (4.3). Let ω_t be the first eigen 1-form on (S^{2n-1}, g_t). Since g_t is a metric with the gap for a sufficiently small $t > 0$ by (4.1), ω_t is co-exact. We set

$$\tilde{\omega}_t = \begin{cases} \varphi \omega_t & \text{on } C = [0,1] \times S^{2n-1}, \\ 0 & \text{on } \overline{M} - C, \end{cases}$$

where φ is a smooth cut-off function on $\overline{M}$ depending only on r such that

$$\varphi = \begin{cases} 1 & \text{on } \Omega, \\ 0 & \text{on } \overline{M} - C, \end{cases}$$

and that φ is invariant under T. Since $b_1(\overline{M}) = b_1(S^{2n}) = 0$, there is no non-trivial harmonic 1-form on $(\overline{M}, h_{\varepsilon,i,t})$. Hence, by using the min-max principle, we obtain

$$\lambda_1^{(1)}(\overline{M}, h_{\varepsilon,i,t}) \leq \frac{\|d\tilde{\omega}_t\|^2_{L^2(\overline{M}, h_{\varepsilon,i,t})}}{\|\tilde{\omega}_t\|^2_{L^2(\overline{M}, h_{\varepsilon,i,t})}}, \tag{4.2}$$

where we use the fact that $\tilde{\omega}_t$ is co-closed. Indeed, from $\langle dF_{\varepsilon,i}, \tilde{\omega}_t\rangle_{h_t} = 0$, it follows that

$$\begin{aligned} \delta_{h_{\varepsilon,i,t}}(\tilde{\omega}_t) &= F_{\varepsilon,i}^{-1}\delta_{h_t}(\tilde{\omega}_t) + (1-n)F_{\varepsilon,i}^{-2}\langle dF_{\varepsilon,i}, \tilde{\omega}_t\rangle_{h_t} \\ &= F_{\varepsilon,i}^{-1}\varphi\beta^{-1}\delta_{g_t}\omega_t = 0. \end{aligned}$$

We compute the right-hand side of (4.2). First, the numerator is

$$\begin{aligned} \|d\tilde{\omega}_t\|^2_{L^2(\overline{M}, h_{\varepsilon,i,t})} &= \|d\omega_t\|^2_{L^2(\Omega, h_{\varepsilon,i,t})} + \|d(\varphi\omega_t)\|^2_{L^2(C-\Omega, h_{\varepsilon,i,t})} \\ &= \|d\omega_t\|^2_{L^2(\Omega, h_t)} + 2\|d(\varphi\omega_t)\|^2_{L^2([0,1/3]\times S^{2n-1}, h_{\varepsilon,i,t})}. \end{aligned}$$

Since ω_t is a co-exact eigen 1-form on (S^{2n-1}, g_t), the first term is

$$\begin{aligned} \|d\omega_t\|^2_{L^2(\Omega, h_t)} &= \frac{\alpha}{3}\|d\omega_t\|^2_{L^2(S^{2n-1}, \beta g_t)} \\ &= \frac{\alpha}{3}\lambda_1^{(1)}(S^{2n-1}, \beta g_t)\|\omega_t\|^2_{L^2(S^{2n-1}, \beta g_t)}. \end{aligned}$$

The second term is

$$\begin{aligned} &2\|d(\varphi\omega_t)\|^2_{L^2([0,1/3]\times S^{2n-1}, h_{\varepsilon,i,t})} \\ &= 2\int_0^{1/3}\int_{S^{2n-1}} F_{\varepsilon,i}^{n-2}|d(\varphi\omega_t)|^2_{h_t}\, dr d\mu_{\beta g_t} \\ &\leq 4\int_0^{1/3}\int_{S^{2n-1}} \{|d\varphi\wedge\omega_t|^2_{h_t} + |\varphi\, d\omega_t|^2_{h_t}\}\, dr d\mu_{\beta g_t} \\ &\leq 4\{\|\varphi'\|^2_{L^2(0,1/3)}\|\omega_t\|^2_{L^2(S^{2n-1}, \beta g_t)} + \|\varphi\|^2_{L^2(0,1/3)}\|d\omega_t\|^2_{L^2(S^{2n-1}, \beta g_t)}\} \\ &= 4\{\|\varphi'\|^2_{L^2(0,1/3)} + \lambda_1^{(1)}(S^{2n-1}, \beta g_t)\|\varphi\|^2_{L^2(0,1/3)}\}\|\omega_t\|^2_{L^2(S^{2n-1}, \beta g_t)}. \end{aligned}$$

On the other hand, the denominator of the right-hand side of (4.2) is

$$\begin{aligned}\|\tilde{\omega}_t\|^2_{L^2(\overline{M}, h_{\varepsilon,i,t})} &\geq \|\tilde{\omega}_t\|^2_{L^2(\Omega, h_{\varepsilon,i,t})} \\ &= \frac{\alpha}{3}\|\omega_t\|^2_{L^2(S^{2n-1}, \beta g_t)}.\end{aligned}$$

Hence, substituting the above facts for (4.2), we obtain

$$\lambda_1^{(1)}(\overline{M}, h_{\varepsilon,i,t}) \leq \left(1 + \frac{12}{\alpha}\|\varphi\|^2_{L^2(0,1/3)}\right)\lambda_1^{(1)}(S^{2n-1}, \beta g_t) + \frac{12}{\alpha}\|\varphi'\|^2_{L^2(0,1/3)}. \tag{4.3}$$

Since $\|\varphi'\|^2_{L^2(0,1/3)}$ is independent of all the parameters $\alpha, \beta, t, \varepsilon$ and i by the choice of φ, we can take two positive numbers α and β such that

$$\frac{12}{\alpha}\|\varphi'\|^2_{L^2(0,1/3)} \times 4 < \min\left\{\frac{9\pi^2}{\alpha^2}, \frac{4n}{\beta}\right\}. \tag{4.4}$$

Moreover, there exists some $t_0 > 0$ such that for all $t < t_0$,

$$\min\left\{\frac{9\pi^2}{\alpha^2}, \frac{4n}{\beta}\right\} = \nu_1^{(0)}(\Omega, h_t), \tag{4.5}$$

where $\nu_1^{(0)}(\Omega, h_t)$ is the first eigenvalue on functions on (Ω, h_t) with the Neumann condition. In fact, by Proposition 4.2 in [**Tn-79**], there exists some $t_0 > 0$ such that for all $t < t_0$, $\lambda_1^{(0)}(S^{2n-1}, g_t) \equiv 4n$. Thus, by using the product formula for the eigenvalues, we have

$$\begin{aligned}\nu_1^{(0)}(\Omega, h_t) &= \min\left\{\nu_1^{(0)}\left(\left[\frac{1}{3}, \frac{2}{3}\right], \alpha^2 dr^2\right), \lambda_1^{(0)}(S^{2n-1}, \beta g_t)\right\} \\ &= \min\left\{\frac{9\pi^2}{\alpha^2}, \frac{4n}{\beta}\right\}.\end{aligned}$$

Since $\lambda_1^{(1)}(S^{2n-1}, \beta g_t) \to 0$ as $t \to 0$, there exists $t_1 > 0$ $(0 < t_1 < t_0)$ such that

$$\left(1 + \frac{12}{\alpha}\|\varphi\|^2_{L^2(0,1/3)}\right)\lambda_1^{(1)}(S^{2n-1}, \beta g_{t_1}) \leq \frac{12}{\alpha}\|\varphi'\|^2_{L^2(0,1/3)}. \tag{4.6}$$

Hence, substituting (4.6) for (4.3), we have

$$\lambda_1^{(1)}(\overline{M}, h_{\varepsilon,i,t_1}) \leq \frac{12}{\alpha}\|\varphi'\|^2_{L^2(0,1/3)} \times 2 \tag{4.7}$$

for all ε and i.

On the other hand, according to Theorem III.1 in [**CV-86**] (see also Lemma 2 in [**D-94**]), there exist a sufficiently small $\varepsilon_1 > 0$ and a sufficiently large integer i_1

such that

$$\nu_1^{(0)}(\Omega, h_{t_1}) - \frac{12}{\alpha}\|\varphi'\|^2_{L^2(0,1/3)} < \lambda_1^{(0)}(\overline{M}, h_{\varepsilon_1, i_1, t_1}). \tag{4.8}$$

Therefore, we obtain

$$\begin{aligned} \lambda_1^{(0)}(\overline{M}, h_{\varepsilon_1, i_1, t_1}) - \lambda_1^{(1)}(\overline{M}, h_{\varepsilon_1, i_1, t_1}) &> \nu_1^{(0)}(\Omega, h_{t_1}) - \frac{12}{\alpha}\|\varphi'\|^2_{L^2(0,1/3)} \times 3 \\ &\quad (\text{by } (4.7), (4.8)) \\ &> \frac{12}{\alpha}\|\varphi'\|^2_{L^2(0,1/3)} > 0 \\ &\quad (\text{by } (4.4), (4.5)). \end{aligned}$$

Hence, $h_{\varepsilon_1, i_1, t_1}$ is a metric with the gap on $S^{2n} \cong \overline{M}$. □

Proof of Theorem 1.2. Now we give a proof of Theorem 1.2. Let M be an m-dimensional connected oriented closed manifold ($m \geq 3$) with $b_1(M) = 0$. We construct a metric with the gap on the manifold $\overline{M}$ in Section 3. By Lemma 4.1, we can take a metric g_1 with the gap on S^m. Then $A := \lambda_1^{(0)}(S^m, g_1) - \lambda_1^{(1)}(S^m, g_1)$ is positive. We consider the disk D_1 as the geodesic ball $B(x_1, r)$ with the radius $r > 0$ centered at $x_1 \in S^m$. Set $U_r := S^m - B(x_1, r)$.

First, from Theorem 2 in [**A-87**] (see also Lemma 3 in [**D-94**]), it follows that

$$\lim_{r \to 0} \nu_1^{(0)}(U_r, g_1) = \lambda_1^{(0)}(S^m, g_1). \tag{4.9}$$

Furthermore, from p. 193 and (4.2) in [**AC-93**], it follows that

$$\lim_{r \to 0} \tilde{\mu}_1^{(1)}(U_r, g_1) = \lambda_1^{(1)}(S^m, g_1), \tag{4.10}$$

where $\tilde{\mu}_1^{(1)}(U_r, g_1)$ is the first positive eigenvalue of the Laplacian on 1-forms on U_r satisfying the boundary condition (III) in [**AC-93**], p. 191, i.e. vanishing at boundary. The spectrum under this boundary condition has no 0-eigenvalue by [**A-89**]. By (4.9) and (4.10), there exists an $\varepsilon > 0$ such that

$$\lambda_1^{(0)}(S^m, g_1) - \frac{1}{4}A < \nu_1^{(0)}(U_\varepsilon, g_1), \tag{4.11}$$

$$\tilde{\mu}_1^{(1)}(U_\varepsilon, g_1) < \lambda_1^{(1)}(S^m, g_1) + \frac{1}{4}A. \tag{4.12}$$

Next, by Theorem III.1 in [**CV-86**] (see also Lemma 2 in [**D-94**]), there exists a metric h on $\overline{M}$ such that

$$h = g_1 \quad \text{on} \quad U_\varepsilon, \tag{4.13}$$

$$\nu_1^{(0)}(U_\varepsilon, g_1) - \frac{1}{4}A < \lambda_1^{(0)}(\overline{M}, h). \tag{4.14}$$

By the inclusion of the Sobolev spaces $H_0^1(\Lambda^1 T^* U_\varepsilon, g_1) \subset H^1(\Lambda^1 T^* \overline{M}, h)$ via the 0-extension and the assumption $b_1(\overline{M}) = 0$, the min-max principle implies

$$\lambda_1^{(1)}(\overline{M}, h) \leq \tilde{\mu}_1^{(1)}(U_\varepsilon, g_1). \tag{4.15}$$

Therefore, we obtain

$$\begin{aligned}\lambda_1^{(0)}(\overline{M}, h) - \lambda_1^{(1)}(\overline{M}, h) &> \nu_1^{(0)}(U_\varepsilon, g_1) - \frac{1}{4}A - \tilde{\mu}_1^{(1)}(U_\varepsilon, g_1) \\ &\qquad (\text{by } (4.14), (4.15)) \\ &> \lambda_1^{(0)}(S^m, g_1) - \lambda_1^{(1)}(S^m, g_1) - \frac{3}{4}A \\ &\qquad (\text{by } (4.11), (4.12)) \\ &> \frac{1}{4}A > 0.\end{aligned}$$

Hence, h is a metric with the gap on $\overline{M}$. □

5. Gap and stability of Einstein manifolds.

First of all, let us recall the definition of the stability of the identity map of Riemannian manifolds (see [**EL-83**], [**U-87**]). Let (M, g) be a connected oriented closed Riemannian manifold. Since the identity map $id : (M, g) \to (M, g)$ is a harmonic map, the Jacobi operator J_{id} acting on vector fields can be defined.

DEFINITION 5.1. The identity map is unstable (resp. weakly stable, or strongly stable) if the first eigenvalue $\lambda_1(J_{id})$ of J_{id} is negative (resp. non-negative, or positive).

From now on, by the duality of vector fields and 1-forms with respect to g, we can regard for J_{id} to act on the spase of 1-forms. The following lemma immediately follows.

LEMMA 5.2. *Let (M, g) be a connected oriented closed Einstein manifold with Einstein constant α. For any 1-form ω, we have $J_{id}(\omega) = \Delta\omega - 2\alpha\omega$. Hence, $\lambda_1'(J_{id}) = \lambda_1^{(0)} - 2\alpha$ and $\lambda_1''(J_{id}) = \lambda_1''^{(1)} - 2\alpha$, where $\lambda_1'(J_{id})$ (resp. $\lambda_1''(J_{id})$) is the first* *eigenvalue of the Jacobi operator acting on the space of exact* (*resp. co-exact*) 1-*forms*.

The following lemma due to Nagano (cf. [**N-61**]) is very crucial in our argument.

Lemma 5.3. *Let* (M, g) *be a connected oriented closed Einstein manifold with positive Einstein constant* α. *Then* $\lambda_1''^{(1)}(M, g) \geq 2\alpha$. *Moreover, the equality holds if and only if the isometry group* $\mathrm{Isom}(M, g)$ *is not discrete.*

Proof. We give a proof which is different from Nagano's original proof by using the results in [**GM-75**] and [**TY-80**]. From the Weitzenböck formula, it follows that for any 1-form ω,

$$(\Delta\omega, \omega)_{L^2} = \|\nabla\omega\|_{L^2}^2 + \int_M \mathrm{Ric}(\omega^\sharp, \omega^\sharp)\, d\mu_g$$
$$= \|\nabla\omega\|_{L^2}^2 + \alpha\|\omega\|_{L^2}^2. \tag{5.1}$$

On the other hand, Lemme 6.8 in [**GM-75**] showed that for any p-form ω on M,

$$|\nabla\omega|^2 \geq \frac{1}{p+1}|d\omega|^2 + \frac{1}{m-p+1}|\delta\omega|^2. \tag{5.2}$$

Furthermore, Lemma 2.5 in [**TY-80**] showed that the equality in (5.2) holds if and only if ω is a conformal Killing. Especially, if ω is a co-closed 1-form, this condition is equivalent to that ω is a Killing 1-form.

For a co-exact 1-form ω, from (5.1) and (5.2), it follows that

$$(\Delta\omega, \omega)_{L^2} \geq \frac{1}{2}\|d\omega\|_{L^2}^2 + \alpha\|\omega\|_{L^2}^2.$$

Considering that $\|d\omega\|_{L^2}^2 = (\Delta\omega, \omega)_{L2}$, we have $(\Delta\omega, \omega)_{L2} \geq 2\alpha\|\omega\|_{L^2}^2$, that is, $\lambda_1''^{(1)}(M, g) \geq 2\alpha$.

Finally, the equality holds if and only if (M, g) has a non-trivial Killing vector field, that is, $\dim \mathrm{Isom}(M, g) \geq 1$. □

Proposition 5.4. *Let* (M, g) *be as in Lemma 5.3.*

(i) *In the case of* $\dim \mathrm{Isom}(M, g) = 0$, *the metric* g *has the gap only if* $\lambda_1'(J_{id}) > 0$.

(ii) *In the case of* $\dim \mathrm{Isom}(M, g) \geq 1$, *the metric* g *has the gap if and only if* $\lambda_1'(J_{id}) > 0$.

Proof. From Lemma 5.3, if $\dim \mathrm{Isom}(M, g) = 0$ (resp. $\dim \mathrm{Isom}(M, g) \geq 1$), we have $\lambda_1''^{(1)}(M, g) > 2\alpha$ (resp. $\lambda_1''^{(1)}(M, g) = 2\alpha$). Since $\lambda_1'(J_{id}) = \lambda_1^{(0)} - 2\alpha$ by Lemma 5.2, we see Proposition 5.4, immediately. □

Similarly, we have the following proposition.

Proposition 5.5. *Let (M, g) be as in Lemma 5.3.*
(i) *If* $\dim \mathrm{Isom}(M, g) = 0$, *we have* $\lambda_1''(J_{id}) > 0$.
(ii) *If* $\dim \mathrm{Isom}(M, g) \geq 1$, *we have* $\lambda_1''(J_{id}) = 0$.

Since $\alpha > 0$, there is no non-trivial harmonic 1-form by the Bochner vanishing theorem. Hence, $\lambda_1(J_{id}) = \min\{\lambda_1'(J_{id}), \lambda_1''(J_{id})\}$. Therefore, by using Propositions 5.4 and 5.5, we can prove Theorem 1.3.

Finally, we determine whether or not the metrics of well-known special Einstein manifolds have the gaps.

Proposition 5.6. *Let G be a simply connected compact simple Lie group and let g be the bi-invariant metric induced from the Killing form. The metric g on G has the gap if and only if $\lambda_1(G, g) > 1/2$, i.e. G is one of the following types*: B_l $(l \geq 3)$, D_l $(l \geq 4)$, E_l $(l = 6, 7, 8)$, F_4.

Proof. From Lemma 5.3 and $\alpha = 1/4$, it follows that $\lambda_1^{\prime\prime(1)}(G, g) = 1/2$. On the other hand, the first eigenvalues on functions are computed in Table *A*.1 [**U-86**]. Hence, by Corollary 2.2, we obtain Proposition 5.6. □

Proposition 5.7. *Let $(G/K, g)$ be a simply connected irreducible Riemannian symmetric space of compact type with the canonical metric. The metric g on G/K is one with the gap if and only if $\lambda_1(G/K, g) > 1$, i.e. G/K is one of the following types*:

AI	$SU(q+1)/SO(q+1)$ $(q \geq 2)$
BI	$SO(2l+1)/SO(2l+1-q) \times SO(q)$ $(l \geq q \geq 3)$
DI	$SO(2l)/SO(2l-q) \times SO(q)$ $(l \geq q \geq 3)$
EI	$(E_6/Sp(4))^\sim$
EII	$(E_6/SU(2) \cdot SU(6))^\sim$
EV	$(E_7/SU(8))^\sim$
EVI	$(E_7/SO(12) \cdot SU(2))^\sim$
EVIII	$E_8/SO(16)$
EIX	$(E_8/E_7 \cdot SU(2))^\sim$
FI	$(F_4/Sp(3) \cdot SU(2))^\sim$
G	$G_2/SU(2) \times SU(2)$

Here, $M^\sim$ means the universal covering of M.

Proof. We can prove this proposition by the same way as Proposition 5.6. Namely, since $\alpha = 1/2$, we have $\lambda_1''^{(1)}(G/K, g) = 1$. On the other hand, we also know $\lambda_1^{(0)}(G/K, g)$ due to Table A.2 in [**U-86**]. Here, the first eigenvalue of the symmetric space of type *EIII* was dropped in the table but it is 1 because the symmetric space of type *EIII* is hermitian. Hence, by Corollary 2.2, we obtain Proposition 5.7. □

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