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Estimates of Oseen kernels in weighted L^p spaces

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Abstract. We study convolutions with Oseen kernels (weakly singular and singular) in both two- and three-dimensional space. We give a detailed weighted L^p theory for $p \in (1; \infty]$ for anisotropic weights.

0. Introduction and basic notation.

This paper concerns convolution integrals whose kernels are given by the Oseen fundamental tensor $\mathcal{O}(\cdot\,;\lambda)$ and its first or second gradients $\nabla\mathcal{O}(\cdot\,;\lambda)$, $\nabla^2\mathcal{O}(\cdot\,;\lambda)$ as well as by the $\nabla\mathcal{E}(x)$ and $\nabla^2\mathcal{E}(x)$, $\mathcal{E}(x)$ being the fundamental solution to the Laplace equation, which play the role of the fundamental pressure. We derive estimates of these weakly and strongly singular integral operators in anisotropically weighted L^p spaces. Such estimates can be applied to the investigation of qualitative properties of solutions of the stationary compressible Navier–Stokes equations in exterior domains using the method of decomposition, see e.g. [**10**]. They can also be applied with some modifications to the case of stationary flows of certain non-Newtonian fluids, see e.g. [**13**] or [**11**], [**12**].

It is well known that the Oseen tensor exhibits various decay properties in various directions in $\boldsymbol{R}^N$, this is the mathematical reason for dealing with anisotropically weighted L^p spaces. Our work is based on the technique proposed by Farwig in [**2**], [**3**], where the volume potentials $\nabla^k\mathcal{O} * f$, $k = 0, 1, 2$ are studied in anisotropically weighted L^2 spaces with the weight function given by the formula $\eta_\beta^\alpha(x) = (1+|x|)^\alpha(1+s(x))^\beta$, $s(x) = |x| - x_1$. Similar results as those presented in [**3**] were obtained also by Kobayashi and Shibata by slightly different technique, see [**7**]. We also used the results of Kurtz and Wheeden concerning singular integrals in L^p weight spaces, see [**9**]. The aim of our work was to generalize Farwig's results on the case of L^p, $p \in (1; \infty]$ with the weight function

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η_β^α, and with the weight function $v_\beta^\alpha(x) = |x|^\alpha(1+s(x))^\beta$ and also to the two-dimensional case. In order to study the dependence of estimates on Reynolds number Re, the weight functions $v_\beta^\alpha(x;\lambda) = |x|^\alpha(1+s(\lambda x))^\beta$, $\eta_\beta^\alpha(x;\lambda) = \eta_\beta^\alpha(\lambda x;1) = (1+\lambda|x|)^\alpha(1+s(\lambda x))^\beta$, $\lambda = 2Re$ are also used.

Our paper is organized as follows. We first introduce the fundamental Oseen solution in both two- and three-dimensional cases and show their asymptotic properties. In Section 2 we calculate L^∞-weighted estimates of a certain convolution which plays an essential role in the next section where the L^∞-estimates of convolutions with Oseen kernels are studied. Applying this results we get, in Sections 4 and 5, the L^p-weighted theory of Oseen potentials (both weakly singular and singular kernels).

In this paper, we use the following notation

$c, c_0, c_1, \ldots, C, C_0, C_1, \ldots$—positive constants

$s(x) = |x| - x_1$, $|x| = (x_1^2 + x_2^2 + \cdots + x_N^2)^{1/2}$, $x \in \boldsymbol{R}^N$

$\mathscr{E}(x)$—fundamental solution of the Laplace equation

$(\mathscr{O}(x;\lambda), \mathscr{P}(x))$—fundamental solution of the Oseen problem

$(\mathscr{S}(x), \mathscr{P}(x))$—fundamental solution of the Stokes problem

$\eta_\beta^\alpha(x) = (1+|x|)^\alpha(1+s(x))^\beta$, $\sigma_\beta^\alpha(x) = |x|^\alpha s(x)^\beta$,

$v_\beta^\alpha(x) = |x|^\alpha(1+s(x))^\beta$, $\mu_\beta^{\alpha,\gamma}(x) = \eta_\beta^{\alpha-\gamma}(x)v_0^\gamma(x)$

$v_\beta^\alpha(x;\lambda) = |x|^\alpha(1+s(\lambda x))^\beta$, $\eta_\beta^\alpha(x;\lambda) = (1+|\lambda x|)^\alpha(1+s(\lambda x))^\beta$

$L^p(\Omega;w) = \{f; \|f\|_{p,(w),\Omega}^p = \int_\Omega |f|^p w\,\mathrm{d}x < +\infty\}$, $p > 1$, $w > 0$ (usually $w = \eta_\beta^\alpha$, σ_β^α, v_β^α), $\Omega \subseteq \boldsymbol{R}^N$, $N = 2, 3$

$B_r(a) = \{x \in \boldsymbol{R}^N, |x-a| < r\}$, $B^r(a) = \{x \in \boldsymbol{R}^N, |x-a| > r\}$, $\partial B_r(a) = \{x \in \boldsymbol{R}^N, |x-a| = r\}$, $r \in \boldsymbol{R}^1$, $a \in \boldsymbol{R}^N$, $|\cdot|$—norm in $\boldsymbol{R}^N$

1. Oseen fundamental solution.

In this section we recall some basic facts about the fundamental solution to the Oseen problem. Denote by $\mathscr{O}(\cdot;\lambda) = (\mathscr{O}_{ij}(\cdot;\lambda))$, $\mathscr{P} = (\mathscr{P}_i)$ its fundamental solution; it satisfies the identities

$$\begin{aligned} \partial_j \mathscr{O}_{ij} &= 0 \\ \Delta \mathscr{O}_{ij} + \partial_j \mathscr{P}_i - \lambda \partial_1 \mathscr{O}_{ij} &= \delta_{ij}\delta \end{aligned} \tag{1.1}$$

in the sense of distributions, where δ_{ij} denotes the Kronecker delta, while δ

denotes the Dirac delta-distribution. The latter is equivalent to

$$-\eta_i(x) = [\partial_k \mathcal{O}_{ij}(\cdot\,;\lambda) * \partial_k \eta_j + \mathcal{P}_i * \partial_j \eta_j - \lambda \mathcal{O}_{ij}(\cdot\,;\lambda) * \partial_1 \eta_j](x)$$

$$\forall \eta \in C_0^\infty(\boldsymbol{R}^N).$$

In particular, it holds,

$$\Delta \mathcal{O}_{ij}(x;\lambda) - \lambda \partial_1 \mathcal{O}_{ij}(x;\lambda) + \partial_j \mathcal{P}_i(x) = 0$$

pointwise in $\boldsymbol{R}^N \backslash \{\mathbf{0}\}$.

Now we shall study separately the three- and two-dimensional situations.

In three space dimensions we can easily verify (see e.g. [**6**]) that the fundamental solution can be written as

$$\mathcal{P}_i(x) = \partial_i \mathcal{E}(x) = \frac{1}{4\pi} \frac{x_i}{|x|^3} \tag{1.2}$$

$$\mathcal{O}_{ij}(x;\lambda) = (\delta_{ij}\Delta - \partial_i \partial_j)\varphi_{\mathcal{O}}(x;\lambda), \tag{1.3}$$

where

$$\varphi_{\mathcal{O}}(x;\lambda) = \frac{-1}{4\pi\lambda} \psi\left(\frac{\lambda s(x)}{2}\right) \tag{1.4}$$

with

$$\psi(z) = \int_0^z \frac{1-e^{-t}}{t}\,\mathrm{d}t = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i!i} z^i \tag{1.5}$$

and

$$s(x) = |x| - x_1. \tag{1.6}$$

The formulas (1.4)–(1.6) yield useful rescaling property

$$\lambda \mathcal{O}(\lambda x; 1) = \mathcal{O}(x;\lambda). \tag{1.7}$$

The integral representation (1.5) implies

$$\psi'(t) = (1 - e^{-t})/t, \quad \psi''(t) = (-1 + e^{-t} + te^{-t})/t^2,$$

$$\psi'''(t) = (2 - 2e^{-t} - 2te^{-t} - t^2 e^{-t})/t^3,$$

$$\psi^{(iv)}(t) = (-6 + 6e^{-t} + 6te^{-t} + 3t^2 e^{-t} + t^3 e^{-t})/t^4.$$

The representation by the sum in (1.5) yields,

$$\psi^{(k)}(t) = \frac{(-1)^{k+1}}{k} + O(t) \quad \text{as } t \to 0, \quad k = 1, 2, \ldots. \tag{1.8}$$

When differentiating (1.6), we obtain

$$\frac{\partial s(x)}{\partial x_i} = \frac{x_i}{|x|} - \delta_{1i}. \tag{1.9}$$

From here we get the estimates

$$\left|\frac{\partial s(x)}{\partial x_k}\right| \le \begin{cases} \dfrac{s(x)}{|x|} & (k = 1) \\ \sqrt{2}\sqrt{\dfrac{s(x)}{|x|}} & (k \neq 1) \end{cases} \qquad |D^\alpha s(x)| \le \frac{c(\alpha)}{|x|^{|\alpha|-1}}. \tag{1.10}$$

From (1.4)–(1.6) and (1.10) it is seen that $\mathcal{O}(\cdot\,;\cdot) \in C^\infty((\boldsymbol{R}^3\backslash\{0\}) \times \boldsymbol{R})$ and for fixed $x \neq \mathbf{0}$, $\mathcal{O}(x;\cdot)$ is an analytic function.

Now we calculate the derivatives of $\varphi_{\mathcal{O}}(\cdot;\lambda)$ in order to establish the asymptotic behaviour of $\mathcal{O}(\cdot;\lambda)$ and of its first and second derivatives near zero and at infinity.

$$\begin{aligned}
-\partial_i\varphi_{\mathcal{O}}(x;\lambda) &= (\lambda/8\pi)\psi'(\lambda s(x)/2)\partial_i s(x) \\
-\partial_r\partial_i\varphi_{\mathcal{O}}(x;\lambda) &= (1/16\pi)\psi''(\lambda s(x)/2)\partial_r s(x)\partial_i s(x) + (1/8\pi)\psi'(\lambda s(x)/2)\partial_r\partial_i s(x) \\
-\partial_k\partial_r\partial_i\varphi_{\mathcal{O}}(x;\lambda) &= (\lambda^2/32\pi)\psi'''(\lambda s(x)/2)\partial_k s(x)\partial_r s(x)\partial_i s(x) \\
&\quad + (\lambda/16\pi)\psi''(\lambda s(x)/2)[\partial_k\partial_r s(x)\partial_i s(x) + \partial_k\partial_i s(x)\partial_r s(x) \\
&\quad + \partial_r\partial_i s(x)\partial_k s(x)] + (1/8\pi)\psi'(\lambda s(x)/2)\partial_k\partial_r\partial_i s(x) \\
-\partial_l\partial_k\partial_r\partial_i\varphi_{\mathcal{O}}(x;\lambda) &= (\lambda^3/64\pi)\psi^{(iv)}(\lambda s(x)/2)\partial_l s(x)\partial_k s(x)\partial_r s(x)\partial_i s(x) \\
&\quad + (\lambda^2/32\pi)\psi'''(\lambda s(x)/2)[\partial_l\partial_k s(x)\partial_r s(x)\partial_i s(x) \\
&\quad + \partial_l\partial_r s(x)\partial_k s(x)\partial_i s(x) + \partial_l\partial_i s(x)\partial_k s(x)\partial_r s(x) \\
&\quad + \partial_k\partial_r s(x)\partial_i s(x)\partial_l s(x) + \partial_k\partial_i s(x)\partial_r s(x)\partial_l s(x) \\
&\quad + \partial_r\partial_i s(x)\partial_k s(x)\partial_l s(x)] + (\lambda/16\pi)\psi''(\lambda s(x)/2) \\
&\qquad \cdot [\partial_l\partial_k\partial_r s(x)\partial_i s(x) + \partial_l\partial_k\partial_i s(x)\partial_r s(x) \\
&\quad + \partial_l\partial_r\partial_i s(x)\partial_k s(x) + \partial_k\partial_r\partial_i s(x)\partial_l s(x) \\
&\quad + \partial_i\partial_k s(x)\partial_r\partial_l s(x) + \partial_r\partial_k s(x)\partial_i\partial_l s(x) \\
&\quad + \partial_i\partial_r s(x)\partial_k\partial_l s(x)] + (1/8\pi)\psi'(\lambda s(x)/2)\partial_l\partial_k\partial_r\partial_i s(x)
\end{aligned}$$

These formulas, together with (1.8), (1.10) and (1.3) yield

$$\begin{aligned}
&\mathcal{O}(x;\lambda) = \mathcal{S}(x) + \lambda O(1) \quad \text{as } \lambda|x| \to 0 \\
&\nabla\mathcal{O}(x;\lambda) = \nabla\mathcal{S}(x) + \lambda^2 O(1/\lambda|x|) \quad \text{as } \lambda|x| \to 0 \\
&\nabla^2\mathcal{O}(x;\lambda) = \nabla^2\mathcal{S}(x) + \lambda^3 O(1/\lambda^2|x|^2) \quad \text{as } \lambda|x| \to 0,
\end{aligned} \tag{1.11}$$

where $(\mathcal{S}, \mathcal{P})$ is the Stokes fundamental solution (see e.g. [**6**]),

$$\mathcal{S}_{ij}(x) = \frac{-1}{8\pi}\left[\frac{\delta_{ij}}{|x|} + \frac{x_i x_j}{|x|^3}\right]. \tag{1.12}$$

It can be shown (see e.g. [**6**] and also Section 5) that both the second derivative of $\mathcal{S}$ and $\nabla\mathcal{P}$ represent Calderón–Zygmund singular integral kernels.

In particular, for $\lambda \in (0;\lambda_0)$, $R > 0$ and $|\lambda x| \le R$

$$|\nabla^k \mathcal{O}(x;\lambda)| \le \frac{c(R;\lambda_0,k)}{|x|^{k+1}}. \tag{1.13}$$

For any $x \neq \mathbf{0}$ formulas (1.3) and (1.10) together with the properties of the function $s(x)$ give

$$\begin{aligned}
|\mathcal{O}(x;\lambda)| &\le \frac{c}{\lambda}\frac{1 - e^{-\lambda s(x)/2}}{s(x)|x|} \\
|\nabla\mathcal{O}(x;\lambda)| &\le \frac{c}{\lambda}\left[\frac{1 - e^{-\lambda s(x)/2}}{s(x)|x|^2} + \frac{1 - e^{-\lambda s(x)/2} - (\lambda s(x)/2)e^{-\lambda s(x)/2}}{s^{3/2}(x)|x|^{3/2}}\right] \\
|\nabla^2\mathcal{O}(x;\lambda)| &\le \frac{c}{\lambda}\left[\frac{1 - e^{-\lambda s(x)/2}}{s(x)|x|^3} + \frac{1 - e^{-\lambda s(x)/2} - (\lambda s(x)/2)e^{-\lambda s(x)/2}}{s^{2}(x)|x|^{2}}\right].
\end{aligned} \tag{1.14}$$

This yields for $|\lambda x| \ge R$ and any $\kappa > 0$

$$\begin{aligned}
|\mathcal{O}(x;\lambda)| &\le \frac{c(\kappa,R)}{|x|(\kappa + s(\lambda x))} \\
|\nabla\mathcal{O}(x;\lambda)| &\le \frac{c(\kappa,R)\lambda^{1/2}}{|x|^{3/2}(\kappa + s(\lambda x))^{3/2}} \\
|\nabla^2\mathcal{O}(x;\lambda)| &\le \frac{c(\kappa,R)\lambda}{|x|^{2}(\kappa + s(\lambda x))^{2}}.
\end{aligned} \tag{1.15}$$

Formulas (1.12) and (1.15) give us in particular that $\mathcal{O}$ and $\nabla\mathcal{O}$ are analogous to $\mathcal{P}$ weakly singular kernels while the second derivative of $\mathcal{O}$ can be written as a sum of a singular kernel $(\mathcal{S})$ and a weakly singular part.

With $(\mathcal{O}, \mathcal{P})$ at hand, we can write explicitly a C^∞-solution of the problem

$$-\Delta u_i + \lambda \partial_1 u_i + \partial_i \Pi = f_i$$

$$\nabla \cdot u = g$$

with $f, g \in C_0^\infty(\boldsymbol{R}^3)$. Namely

$$u_i = -\mathcal{O}_{ij}(\cdot\,; \lambda) * f_j + \mathcal{P}_i * g$$

$$\Pi = \mathcal{P}_j * f_j + g - \lambda \partial_1(\mathcal{E} * g) = \mathcal{P}_j * f_j + g - \lambda \mathcal{P}_1 * g.$$

In the case of $f = \operatorname{div} \mathcal{F}$ we have

$$u_i = -\partial_s \mathcal{O}_{ij} * \mathcal{F}_{js} + \mathcal{P}_i * g$$

$$\Pi = \partial_s \mathcal{P}_j * \mathcal{F}_{js} + g - \lambda \mathcal{P}_1 * g + c_{js} \mathcal{F}_{js},$$

where c_{js}, $j, s = 1, \ldots, N$ are constants.

The two-dimensional case is a bit more complicated. Therefore, we shall study this case in more detail. Let $K_0(z)$ be the modified Bessel function, i.e. $K_0(z)$ solves the modified Bessel equation

$$z^2 K_0''(z) + z K_0'(z) - z^2 K_0(z) = 0 \tag{1.16}$$

for $z \neq 0$ and it is singular at 0, i.e.

$$K_0(z) = -\ln(z) + \ln 2 - \gamma - \left(\ln \frac{z}{2}\right) \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{z}{2}\right)^{2k}$$

$$+ \sum_{k=1}^{\infty} \frac{1}{(k!)^2} \left(\sum_{j=1}^{k} \frac{1}{j} - \gamma \right) \left(\frac{z}{2}\right)^{2k} \tag{1.17}$$

for $z \neq 0$ sufficiently small (see e.g. [**8**]); here γ is the Euler constant. We put

$$\Phi(x) = \frac{1}{4\pi\lambda} \int_{\infty}^{-x_1} \{\log \sqrt{\tau^2 + x_2^2} + K_0(\lambda \sqrt{\tau^2 + x_2^2}) e^{-\lambda \tau}\} \, \mathrm{d}\tau. \tag{1.18}$$

The problem consists in the right choice of the constants which in turn corresponds to the right choice of lower bound for the integral in (1.18). The lower bound is, at least formally, required to be equal to ∞. Unfortunately, then the integral in (1.18) does not converge, as K_0 behaves regularly at infinity (see (1.27)). We calculate formally the derivatives of (1.18) and put

$$G(x; 2\lambda) = \frac{1}{4\pi\lambda} (\log \sqrt{x_1^2 + x_2^2} + K_0(\lambda \sqrt{x_1^2 + x_2^2}) e^{\lambda x_1}), \tag{1.19}$$

$$H(x;2\lambda)=\frac{1}{4\pi\lambda}\int_{\infty}^{-x_1}\left[\frac{\tau^2-x_2^2}{(\tau^2+x_2^2)^2}+\left(K_0''(\lambda\sqrt{\tau^2+x_2^2})\frac{\lambda^2x_2^2}{\tau^2+x_2^2}\right.\right.$$
$$\left.\left.+K_0'(\lambda\sqrt{\tau^2+x_2^2})\frac{\lambda\tau^2}{(\tau^2+x_2^2)^{3/2}}\right)e^{-\lambda\tau}\right]\mathrm{d}\tau, \tag{1.20}$$

i.e. G is formally taken derivative of (1.18) with respect to x_1 multiplied by -1, H the second derivative of (1.18) with respect to x_2^2. First we express (1.20) in a more appropriate way (without the integrals).

We formally calculate

$$\frac{\partial^2\Phi(x)}{\partial x_2^2}=\frac{1}{4\pi\lambda}\int_{\infty}^{-x_1}\left[\frac{\tau^2-x_2^2}{(\tau^2+x_2^2)^2}+\left(K_0''(\lambda\sqrt{\tau^2+x_2^2})\frac{\lambda^2x_2^2}{\tau^2+x_2^2}\right.\right.$$
$$\left.\left.+K_0'(\lambda\sqrt{\tau^2+x_2^2})\frac{\lambda\tau^2}{(\tau^2+x_2^2)^{3/2}}\right)e^{-\lambda\tau}\right]\mathrm{d}\tau$$
$$=\frac{1}{4\pi\lambda}\left\{\frac{x_1}{x_1^2+x_2^2}+\left[K_0'(\lambda\sqrt{x_1^2+x_2^2})\frac{\lambda x_1}{\sqrt{x_1^2+x_2^2}}-K_0(\lambda\sqrt{x_1^2+x_2^2})\lambda\right]e^{\lambda x_1}\right\}.$$

Following [**6**] it can be verified that $\mathcal{O}_{ij}(x;\lambda)=(\delta_{ij}\Delta-\partial_i\partial_j)\Phi(x)$, where the derivative are calculated formally—the integral in (1.18) is not finite. Nevertheless we shall verify that the formally deduced function is really the fundamental solution being sought. We put ($r=\sqrt{x_1^2+x_2^2}$)

$$\mathcal{O}_{11}(x;2\lambda)=H(x;2\lambda)=\frac{1}{4\pi\lambda}\left\{\frac{x_1}{r^2}+\left[K_0'(\lambda r)\frac{\lambda x_1}{r}-K_0(\lambda r)\lambda\right]e^{\lambda x_1}\right\}$$
$$\mathcal{O}_{12}(x;2\lambda)=\mathcal{O}_{21}(x;2\lambda)=\frac{\partial}{\partial x_2}G(x,2\lambda)=\frac{1}{4\pi\lambda}\left[\frac{x_2}{r^2}+K_0'(\lambda r)\frac{\lambda x_2}{r}e^{\lambda x_1}\right] \tag{1.21}$$
$$\mathcal{O}_{22}(x;2\lambda)=-\frac{\partial}{\partial x_1}G(x;2\lambda)=-\frac{1}{4\pi\lambda}\left\{\frac{x_1}{r^2}+\left[K_0'(\lambda r)\frac{\lambda x_1}{r}+K_0(\lambda r)\lambda\right]e^{\lambda x_1}\right\}$$

$$\mathcal{P}_i(x)=\frac{1}{2\pi}\frac{x_i}{r^2}=\partial\mathcal{E}(x). \tag{1.22}$$

From (1.21) and (1.17) we get for small $\lambda|x|$

$$\begin{aligned}\mathcal{O}_{ii}(x;2\lambda)&=\mathcal{S}_{ii}(x)-(1/4\pi)\log(1/\lambda)+O(1)\\ \mathcal{O}_{ij}(x;2\lambda)&=\mathcal{S}_{ij}(x)+O(\lambda r\log(\lambda r))\quad i\neq j\\ \nabla\mathcal{O}_{ij}(x;2\lambda)&=\nabla\mathcal{S}_{ij}(x)+\lambda O(\log(\lambda r))\\ \nabla^2\mathcal{O}_{ij}(x;2\lambda)&=\nabla^2\mathcal{S}_{ij}(x)+\lambda^2O(1/\lambda r),\end{aligned} \tag{1.23}$$

where $\mathscr{S}_{ij}(x) = -(1/4\pi)[\delta_{ij}\log(1/r) + x_i x_j/r^2]$ is the fundamental Stokes tensor (see [**6**]).

The explicit formulas (1.21) also imply the following homogeneity property

$$\mathcal{O}_{ij}(x; 2\lambda) = \mathcal{O}_{ij}(2\lambda x; 1). \tag{1.24}$$

We are going to verify that $\mathcal{O}, \mathscr{P}$ solve (1.1) in the sense of distributions. Let us observe that

$$\begin{aligned}
&\mathcal{O}_{11}(x; 2\lambda) = (1/2\lambda)(\partial\mathscr{E}/\partial x_1)(x) - (1/2\lambda)(\partial\Phi_1/\partial x_1)(x; 2\lambda) + \Phi_1(x; 2\lambda)\\
&\mathcal{O}_{12}(x; 2\lambda) = \mathcal{O}_{21}(x; 2\lambda) = (1/2\lambda)(\partial\mathscr{E}/\partial x_2)(x) - (1/2\lambda)(\partial\Phi_1/\partial x_2)(x; 2\lambda)\\
&\mathcal{O}_{22}(x; 2\lambda) = -(1/2\lambda)(\partial\mathscr{E}/\partial x_1)(x) + (1/2\lambda)(\partial\Phi_1/\partial x_1)(x; 2\lambda),
\end{aligned} \tag{1.25}$$

where $\Phi_1(x) = -(1/2\pi)K_0(\lambda|x|)\mathrm{e}^{\lambda x_1}$. We therefore easily see that $\mathcal{O}_{ij}, \mathscr{P}_j$ solves $(1.1)_1$ for $x \neq \mathbf{0}$. Moreover

$$\frac{\partial\mathcal{O}_{21}}{\partial x_1} + \frac{\partial\mathcal{O}_{22}}{\partial x_2} = 0 \quad \text{for } x \neq \mathbf{0}$$

and

$$\frac{\partial\mathcal{O}_{11}}{\partial x_1} + \frac{\partial\mathcal{O}_{12}}{\partial x_2} = \frac{-1}{4\pi\lambda}\left(K_0''(\lambda r)\lambda^2 + K_0'(\lambda r)\frac{\lambda}{r} - \lambda^2 K_0(\lambda r)\right)\mathrm{e}^{\lambda x_1} = 0$$

for $x \neq \mathbf{0}$. Moreover from (1.23) we have that $|\nabla\mathcal{O}(x; 1)| \leq C/|x|$ for $|x|$ small and therefore

$$\frac{\partial\mathcal{O}_{ij}}{\partial x_j} = 0 \quad \text{in } \mathscr{D}'(\boldsymbol{R}^2).$$

It remains to verify that

$$(\Delta - \lambda\partial_1)\mathcal{O}_{ij} + \partial_j\mathscr{P}_i = \delta_{ij}\delta.$$

Due to the asymptotic behaviour of $\mathcal{O}$ and due to the fact that

$$\lim_{\varepsilon\to 0^+}\int_{\partial B^\varepsilon(x)} \mathscr{P}_i(x-y)n_j(y)F_j(y)\,\mathrm{d}_yS = \frac{1}{2}F_j(x)\delta_{ij}$$

it is enough to verify that

$$\lim_{\varepsilon\to 0^+}\int_{\partial B^\varepsilon(x)} \frac{\partial O_{ij}(x-y)}{\partial n}F_j(y)\,\mathrm{d}_yS = \frac{1}{2}F_j(x)\delta_{ij}. \tag{1.26}$$

Namely, then we easily get (n is the outer normal to $B^\varepsilon(x)$, i.e. the inner normal to $B_\varepsilon(x)$)

$$\int_{\boldsymbol{R}^2}\left[\mathcal{O}_{ij}(x-y;\lambda)\left[\left(\Delta-\lambda\frac{\partial}{\partial y_1}\right)F_j(y)\right]+\mathcal{P}_i(x-y)\frac{\partial F_j}{\partial y_j}(y)\right]\mathrm{d}y$$

$$=\text{v.p.}\int_{\boldsymbol{R}^2}\left[\left(\Delta-\lambda\frac{\partial}{\partial x_1}\right)\mathcal{O}_{ij}(x-y;\lambda)+\frac{\partial\mathcal{P}_i}{\partial x_j}(x-y)\right]F_j(y)\,\mathrm{d}y$$

$$+\lim_{\varepsilon\to 0^+}\left[\left[\int_{\partial B^\varepsilon(x)}\frac{\partial\mathcal{O}_{ij}(x-y;\lambda)}{\partial n}F_j(y)\,\mathrm{d}_yS-\lambda\int_{\partial B^\varepsilon(x)}\mathcal{O}_{ij}(x-y;\lambda)n_i(y)F_j(y)\,\mathrm{d}_yS\right.\right.$$

$$\left.+\int_{\partial B^\varepsilon(x)}\mathcal{P}_i(x-y)n_j(y)F_j(y)\,\mathrm{d}_yS\right]$$

$$=\lim_{\varepsilon\to 0^+}\int_{\partial B^\varepsilon(x)}\left[\frac{\partial\mathcal{O}_{ij}}{\partial n}(x-y;\lambda)+\mathcal{P}_i(x-y)n_j(y)\right]F_j(y)\,\mathrm{d}_yS=F_j(x)\delta_{ij}.$$

From (1.22) and (1.17) we find after a bit tedious but straightforward calculations that

$$\frac{\partial\mathcal{O}_{11}}{\partial n}=-\frac{1}{4\pi}\frac{1}{r}+O(\ln r)$$
$$\frac{\partial\mathcal{O}_{12}}{\partial n}=\frac{\partial\mathcal{O}_{21}}{\partial n}=O(\ln r)$$
$$\frac{\partial\mathcal{O}_{22}}{\partial n}=-\frac{1}{4\pi}\frac{1}{r}+O(\ln r)$$

and therefore (1.26) follows.

The next part is devoted to the asymptotic properties when $\lambda r\to\infty$. Unlike the Stokes fundamental tensor we get anisotropic structure. First, let us recall that for $z\to\infty$ (see e.g. [**8**]):

$$\begin{aligned}
K_0(z)&=(\pi/(2z))^{1/2}e^{-z}[1-1/(8z)+9/(2!(8z)^2)+O(z^{-3})]\\
K_0'(z)&=(\pi/(2z))^{1/2}e^{-z}[-1-3/(8z)+15/(128z^2)+O(z^{-3})]\\
K_0''(z)&=(\pi/(2z))^{1/2}e^{-z}[1+7/(8z)+57/(128z^2)+O(z^{-3})]\\
K_0'''(z)&=(\pi/(2z))^{1/2}e^{-z}[-1-11/(8z)-225/(128z^2)+O(z^{-3})].
\end{aligned}\tag{1.27}$$

Using the polar coordinates $x_1=r\cos\varphi$, $x_2=r\sin\varphi$ we get from (1.6)

$$s(x)=(r-x_1)=r(1-\cos\varphi)$$

and (1.21) yields the following asymptotic expansion of $\mathcal{O}$:

$$\begin{aligned}
\mathcal{O}_{11}(x;2\lambda) &= (1/4\pi\lambda r)\cos\varphi - (1/4\sqrt{2\pi\lambda r})e^{-\lambda s}[\cos\varphi + 1 \\
&\quad + (1/\lambda r)((3/8)\cos\varphi - 1/8) + v(\lambda r)] \\
\mathcal{O}_{12}(x;2\lambda) &= (1/4\pi\lambda r)\sin\varphi - (1/4\sqrt{2\pi\lambda r})e^{-\lambda s}\sin\varphi[1 + 3/(8\lambda r) + v(\lambda r)] \qquad (1.28) \\
\mathcal{O}_{22}(x;2\lambda) &= -(1/4\pi\lambda r)\cos\varphi + (1/4\sqrt{2\pi\lambda r})e^{-\lambda s}[(\cos\varphi - 1) \\
&\quad + (1/\lambda r)((3/8)\cos\varphi + 1/8) + v(\lambda r)].
\end{aligned}$$

Using several straightforward properties of the function $s(x)$

$$\frac{\partial s}{\partial x_1} = \frac{-s}{r}, \quad \frac{\partial s}{\partial x_2} = \frac{x_2}{r} = \sin\varphi, \tag{1.29}$$

$$\frac{\partial \varphi}{\partial x_2} = \frac{\cos\varphi}{r}, \quad \frac{\partial \varphi}{\partial x_1} = -\frac{\sin\varphi}{r} \tag{1.30}$$

$$s \sim r \quad \text{for } x_1 \le 0 \quad \text{but } s \sim \frac{(x_2)^2}{r} \quad \text{for } x_1 > 0, \tag{1.31}$$

$$e^{-2\lambda s}\sin^2\varphi\lambda r = \lambda s e^{-2\lambda s}(1 + \cos\varphi) \le e^{-1}, \tag{1.32}$$

we get the following uniform behaviour

$$\left.\begin{aligned} |\mathcal{O}_{12}(x;2\lambda)|, |\mathcal{O}_{22}(x;2\lambda)| &\le C/(\lambda r) \\ |\mathcal{O}_{11}(x;2\lambda)| &\le C/\sqrt{\lambda r} \end{aligned}\right. \quad \text{as } \lambda r \to \infty. \tag{1.33}$$

Moreover, from $(1.28)_1$ we may deduce the following anisotropic structure

$$|\mathcal{O}_{11}(x;2\lambda)| \le \frac{C}{\sqrt{\lambda r}\cdot\sqrt{1+\lambda s}}. \tag{1.34}$$

Now we could calculate all the derivatives and get the asymptotic expansions of them. But we are interested only in the estimates of the type (1.33)–(1.34). The formulas (1.29)–(1.32) yield

$$\begin{aligned}
|\partial_2\mathcal{O}_{11}(x;2\lambda)| &\le C/(r(1+\lambda s)) \\
|\partial_2\mathcal{O}_{12}(x;2\lambda)|, |\partial_1\mathcal{O}_{11}(x;2\lambda)| &\le C/(r\sqrt{\lambda r(1+\lambda s)}) \qquad (1.35) \\
|\partial_1\mathcal{O}_{12}(x;2\lambda)|, |\partial_1\mathcal{O}_{22}(x;2\lambda)|, |\partial_2\mathcal{O}_{22}(x;2\lambda)| &\le C/(\lambda r^2).
\end{aligned}$$

For higher derivatives we do not need such precise estimates. We therefore only observe that

$$|\partial_2\partial_2\mathcal{O}_{11}(x;2\lambda)| \le C\sqrt{\lambda}/(r^{3/2}(1+\lambda s)^{3/2}), \tag{1.36}$$

while for the other terms we have the following uniform estimate

$$|\tilde{D}^2 \mathcal{O}(x; 2\lambda)| \leq \frac{C}{r^2}, \tag{1.37}$$

where $\tilde{D}^2\mathcal{O}$ contains the second gradient of $\mathcal{O}$ except of $\partial_2\partial_2\mathcal{O}_{11}$. For higher derivatives we get analogously

$$|\nabla^k \mathcal{O}(x; 2\lambda)| \leq \frac{C\lambda^{k/2-1/2}}{r^{k/2+1/2}} \quad \text{for } k \geq 3. \tag{1.38}$$

The formulas (1.33)–(1.38) are valid for $\lambda r > 1$.

Let us summarize the asymptotic behaviour of $\mathcal{O}$ and its derivatives using the weights ν_β^α, $\mu_\beta^{\alpha,\gamma}$ and η_β^α, introduced in Section 0. Moreover, we assume $\lambda = 1$ and for $\lambda \neq 1$ we may use the homogeneity properties (1.7) and (1.24). Then we have for $N = 3$ and $x \in \boldsymbol{R}^3 \backslash \{\mathbf{0}\}$

$$\begin{aligned}
|\mathcal{O}(x;1)| &\leq C\nu_{-1}^{-1}(x) \\
|\nabla\mathcal{O}(x;1)| &\leq C\mu_{-3/2}^{-3/2,-2}(x) \\
|\partial_1\mathcal{O}(x;1)| &\leq C\nu_{-1}^{-2}(x) \\
|\nabla^2\mathcal{O}(x;1) - \nabla^2\mathcal{S}(x)| &\leq C\nu_{-1}^{-2}(x)
\end{aligned} \tag{1.39}$$

and for $N = 2$ and $x \in \boldsymbol{R}^3 \backslash \{\mathbf{0}\}$

$$\begin{aligned}
|\mathcal{O}_{11}(x;1)| &\leq C\eta_{-1/2}^{-1/2}(x)\log(2 + 1/|x|) \\
|\mathcal{O}_{ij}(x;1)| &\leq C\eta_0^{-1}(x)\log(2 + 1/|x|) \quad i, j > 1 \\
|\partial_2\mathcal{O}_{11}(x;1)| &\leq C\nu_{-1}^{-1}(x) \\
\left.\begin{array}{r}
|\partial_1\mathcal{O}_{11}(x;1)| \\
|\partial_2\mathcal{O}_{12}(x;1)| \\
|\partial_2\partial_2\mathcal{O}_{11}(x;1) - \partial_2\partial_2\mathcal{S}_{11}(x)|
\end{array}\right\} &\leq C\mu_{-1/2}^{-3/2,-1}(x) \\
\left.\begin{array}{r}
|\partial_1\mathcal{O}_{12}(x;1)| \\
|\partial_1\mathcal{O}_{22}(x;1)| \\
|\partial_2\mathcal{O}_{22}(x;1)| \\
|\partial_k\partial_l\mathcal{O}_{mn}(x;1) - \partial_k\partial_l\mathcal{S}_{mn}(x)| \\
(k,l,m,n) \neq (2,2,1,1)
\end{array}\right\} &\leq C\mu_0^{-2,-1}(x).
\end{aligned} \tag{1.40}$$

2. L^∞-estimates of a convolution in $\boldsymbol{R}^N$.

This section is devoted to the study of an auxiliary problem—the L^∞ estimates of certain convolution which will play a fundamental role in the following sections. Our aim is to give conditions on a, b, c, d, e, f such that

$$(\eta_{-b}^{-a} * \eta_{-d}^{-c})(x) \le K\eta_{-f}^{-e}(x), \quad x \in \boldsymbol{R}^N. \tag{2.1}$$

We shall calculate the estimates for $N \ge 2$. Since we study the physically interesting cases $N = 2, 3$, the results will be summarized in Tab. 1–Tab. 4 only in these situations. Nevertheless, the calculations will be performed for general dimensions and the results can be easily read from the integrals $I_0 - I_{15}$.

Before calculating estimates of the type (2.1), we shall first study the asymptotic behaviour of the function

$$s(x) = |x| - x_1.$$

In what follows, by $f(x) \sim g(x)$ as $|x| \to A$ we mean the following: there exist C_1, $C_2 > 0$ and $U(A)$, a neighbourhood of A, such that

$$C_1 f(x) \le g(x) \le C_2 f(x) \tag{2.2}$$

for all x such that $|x|$ is from $U(A)$. If we write $f(x) \sim g(x)$, $x \in M \subseteq \boldsymbol{R}^N$ then the inequality (2.2) holds for any $x \in M$ with constants independent of x.

Let us denote by x' the vector of the last $N-1$ components of x, i.e. $x = (x_1, x')$. We have

LEMMA 2.1. *If $x_1 > 0$ then $s(x) \sim |x'|^2/|x|$; otherwise $s(x) \sim |x|$.*

PROOF. Introducing the generalized spherical coordinates ($N \ge 3$)

$$\begin{aligned} x_1 &= R\cos\theta_1 \\ x_2 &= R\sin\theta_1\cos\theta_2 \\ &\cdots \\ x_{N-1} &= R\sin\theta_1\cdots\sin\theta_{N-2}\cos\theta_{N-1} \\ x_N &= R\sin\theta_1\cdots\sin\theta_{N-2}\sin\theta_{N-1}, \end{aligned} \tag{2.3}$$

where $\theta_1, \dots, \theta_{N-2} \in (0; \pi)$, $\theta_{N-1} \in (0; 2\pi)$, we have

$$s = R(1 - \cos\theta_1) = 2\frac{(R\sin\theta_1)^2}{R}\left(\frac{\sin(\theta_1/2)}{\sin\theta_1}\right)^2.$$

For $x_1 > 0$ we have

$$\theta_1 \in (0;\pi/2), \quad \text{i.e. } 2\left(\frac{\sin(\theta_1/2)}{\sin\theta_1}\right)^2 \in (1/2;1)$$

which implies

$$\frac{1}{2}\frac{|x'|^2}{R} \le s(x) \le \frac{|x'|^2}{R}.$$

Analogously we proceed for $x_1 \le 0$ where $\theta_1 \in [\pi/2;\pi]$ and $|x| \le s(x) \le 2|x|$. If $N = 2$ we use the polar coordinates and the only change consists in the fact that $\varphi = \theta_1 \in (-\pi/2;\pi/2)$ for $x_1 > 0$ and $\varphi \in [\pi/2; 3/2\pi]$ for $x_1 \le 0$. $\square$

Next we study the integral of $\eta_{-b}^{-a}(x)$ over the sphere for sufficiently large $R = |x|$.

Lemma 2.2. *Let $N \ge 2$. Then for the exponents $a, b \in \mathbf{R}$ we have*

$$\int_{\partial B_R} \eta_{-b}^{-a}(x)\,dS \sim R^{N-1-a-\min((N-1)/2,b)} \cdot (\ln R \text{ if } b = (N-1)/2) \tag{2.4}$$

as $R \to \infty$. Consequently, $\int_{\mathbf{R}^N} \eta_{-b}^{-a}(x)\,\mathrm{d}x < \infty \Leftrightarrow a + \min((N-1)/2, b) > N$.

Proof. Using the generalized spherical coordinates (if $N \ge 3$—see (2.3)) or the polar ones ($N = 2$) we get

$$\begin{aligned}\int_{\partial B_R} \eta_{-b}^{-a}(x)\,\mathrm{d}S &= C\int_0^\pi (1+R)^{-a}(1+s)^{-b}R^{N-1}(\sin\theta_1)^{N-2}\,\mathrm{d}\theta_1\\ &= C\int_0^\pi (1+R)^{-a}(1+R(1-\cos\theta_1))^{-b}R^{N-1}\sin\theta_1^{N-2}\,\mathrm{d}\theta_1.\end{aligned}$$

Changing the variables $s = R(1-\cos\theta_1)$ we estimate the last integral by

$$C(1+R)^{1-a}\int_0^{2R}(1+s)^{-b}(\sqrt{2sR - s^2})^{N-3}\,\mathrm{d}s. \tag{2.5}$$

We estimate the integral (2.5) over three subintervals. Let us also note that for $N = 3$ it can be calculated explicitly. We have

$$\int_0^1 (1+s)^{-b}(2sR-s^2)^{(N-3)/2}\,\mathrm{d}s \sim R^{(N-3)/2}\int_0^1 s^{(N-3)/2}\,\mathrm{d}s \sim R^{(N-3)/2},$$

$$\begin{aligned}\int_1^R (1+s)^{-b}(2sR-s^2)^{(N-3)/2}\,\mathrm{d}s &\sim R^{(N-3)/2}\int_1^R s^{-b+(N-3)/2}\,\mathrm{d}s\\ &\sim R^{N-2-\min(b,(N-1)/2)} \cdot (\ln R \text{ if } b = (N-1)/2),\end{aligned}$$

$$\int_R^{2R} (1+s)^{-b}(2sR-s^2)^{(N-3)/2}\,\mathrm{d}s \sim R^{N-2-b}$$

which implies (2.4). As $\eta_{-b}^{-a}(\cdot) \in C(\boldsymbol{R}^N)$, the condition implying global integrability follows trivially. □

We can start to deal with the convolution (2.1). Recall that similar estimates were for the first time studied by Finn (see [**4**], [**5**]) in the three-dimensional case (but only for special values of c, d) and by Smith (see [**14**]) in the two-dimensional case. A generalization of their approach due to Farwig (see [**2**]) for the three-dimensional case was adapted to the two-dimensional case by Dutto (see [**1**]). We shall repeat their calculation in N dimensions where $N \geq 2$, arbitrary.

Finally note that the estimate (2.1) remains true if we replace the kernel $\eta_{-b}^{-a}(x-y)$ by

$$K(z) \sim \mu_{-d}^{-c,-\gamma}(z), \quad \gamma < N, z \in \boldsymbol{R}^N \tag{2.6}$$

(see also I_1 below).

In the sequel we shall use the following notation

$$\begin{array}{lll} x = (x_1, x') & y = (y_1, y') & \\ R = |x| & r = |y| & \tilde{r} = |x-y| \\ s = s(x) & t = y_1 & \tilde{t} = x_1 - y_1 \\ & \varrho = |y'| & \tilde{\varrho} = |x' - y'|. \end{array} \tag{2.7}$$

In order to capture the anisotropic structure of the function $\eta_{-b}^{-a}(\cdot)$ we shall study the convolution (2.1) in four different situations:

A) $R \leq R_0$

B) $x_1 > 0,\ |x'| \leq \sqrt{x_1},\ R > R_0$

C) $x_1 > 0,\ |x'| = (1/2)R^{1/2+\sigma},\ R > R_0,\ \sigma \in [0; 1/2]$

D) $x_1 > 0,\ |x'| \geq R/2,\ R > R_0$ or $x_1 < 0,\ R > R_0$.

Using Lemma 2.1 we easily verify that

$$\begin{gathered} \eta_{-b}^{-a}(y) \sim \begin{cases} 1, & r \leq 1 \\ r^{-a}, & r > 1, \quad t > 0, \quad \varrho < \sqrt{t} \\ r^{-a+b}\varrho^{-2b}, & r > 1, \quad t > 0, \quad \varrho \geq \sqrt{t} \\ r^{-a-b}, & r > 1, \quad t \leq 0 \end{cases} \\ \eta_{-d}^{-c}(x-y) \sim \begin{cases} 1, & \tilde{r} \leq 1 \\ \tilde{r}^{-c}, & \tilde{r} > 1, \quad \tilde{t} > 0, \quad \tilde{\varrho} < \sqrt{\tilde{t}} \\ \tilde{r}^{-c+d}\tilde{\varrho}^{-2d}, & \tilde{r} > 1, \quad \tilde{t} > 0, \quad \tilde{\varrho} \geq \sqrt{\tilde{t}} \\ \tilde{r}^{-c-d}, & \tilde{r} > 1, \quad \tilde{t} \leq 0\ . \end{cases} \end{gathered} \tag{2.8}$$

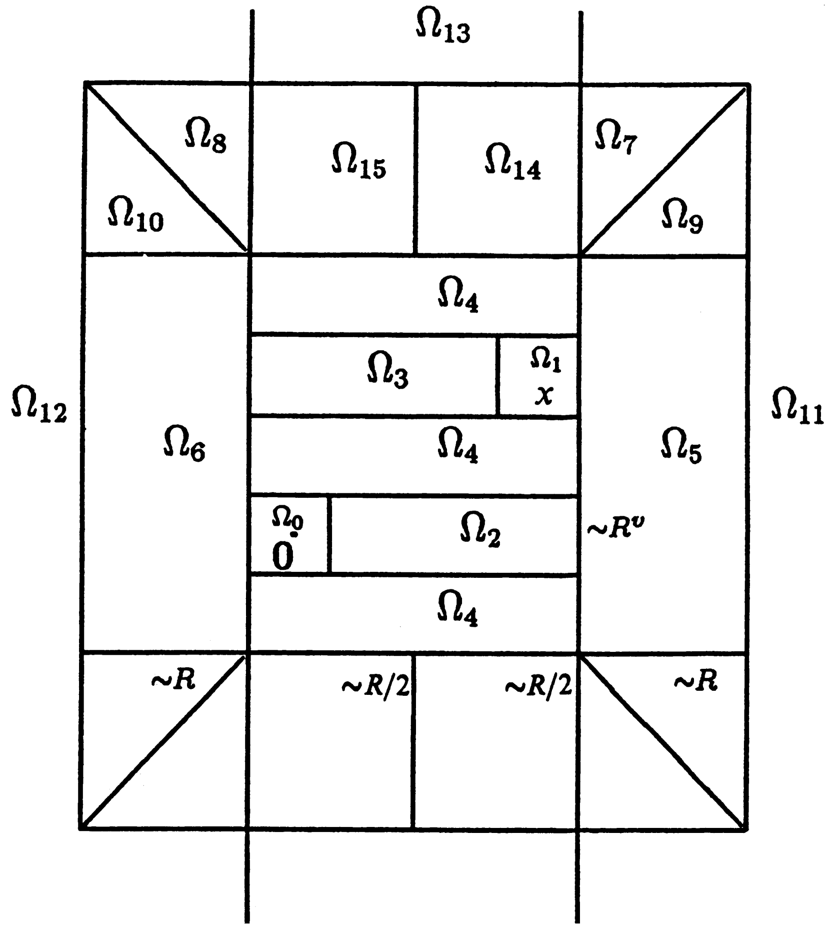

Fig. 1.

For notational convenience we denote $v = \sigma + 1/2$ and $b^* = \min((N-1)/2, b)$; analogously $d^* = \min((N-1)/2, d)$.

We start with the case A). Applying Lemma 2.2 to the halfspaces $y_1 > 0$ and $y_1 < 0$ we find that the convolution is uniformly bounded if

$$a + b^* + c + d > N \quad a + b + c + d^* > N. \tag{2.9}$$

Next we continue with the most complicated case C). We follow Farwig (see [**2**]) and divide $\boldsymbol{R}^N$ into 16 subdomains as shown in Fig. 1. If $N = 2$ the subdomains are plane, otherwise they are cylindrical.

We calculate the convolutions separately on each subdomain. For the reader's convenience, the results are summarized in Tab. 1, Tab. 2 (for $N = 3$) and Tab. 3, Tab. 4 (for $N = 2$). We denote by I_k the corresponding part of the integral (2.1) over Ω_k, $k = 0, 1, \ldots, 15$. We shall get

$$I_i(x) \le KR^{-e_i - 2\sigma f_i} \sim \eta_{-f_i}^{-e_i}(x).$$

Unfortunately in many cases additional logarithmic terms will appear which will cause some losses in the weighted estimates later on.

Tab. 1 $N=3$

Dom.	t	ϱ	r	$\tilde t$	$\tilde\varrho$	$\tilde r$	$\eta^{-a}_{-b}(y)$	$\eta^{-c}_{-d}(x-y)$	e	f	*log. factors*
Ω_0	$(-(1/8)R^\upsilon;$ $(1/8)R^\upsilon)$	$(0;(1/8)R^\upsilon)$	$(0;(1/8)R^\upsilon)$	$\sim x_1$ R	$\sim\lvert x'\rvert$ R^υ	$\sim\lvert x\rvert$ R	$r^{-a}(1+s(y))^{-b}$	$R^{-c-2\sigma d}$	$c+1/2\cdot$ $\min(0,a+b^*-3)$	$d+1/2\cdot$ $\min(0,a+b^*-3)$	$\ln R(b=1\wedge a<2)\vee$ $(b\neq1\wedge a+b^*=3)$ $\ln^2 R(b=1\wedge a=2)$
Ω_1	$\sim x_1$ R	$\sim\lvert x'\rvert$ R^υ	$\sim\lvert x\rvert$ R	$(-(1/8)R^\upsilon;$ $(1/8)R^\upsilon)$	$(0;(1/8)R^\upsilon)$	$(0;(1/8)R^\upsilon)$	$R^{-a-2\sigma b}$	$\tilde r^{-c}(1+s(x-y))^{-d}$	$a+1/2\cdot$ $\min(0,c+d^*-3)$	$b+1/2\cdot$ $\min(0,c+d^*-3)$	$\ln R(d=1\wedge c<2)\vee$ $(d\neq1\wedge c+d^*=3)$ $\ln^2 R(c=2\wedge d=1)$
Ω_2	$\sim r$ $(R^\upsilon;R)$	$(0;(1/8)R^\upsilon)$	$\sim t$ $(R^\upsilon;R)$	$(-(1/8)R^\upsilon;$ $R-R^\upsilon)$	$\sim\lvert x'\rvert$ R^υ	$R+R^\upsilon-r$	r^{-a} $\varrho<\sqrt{t}$ $r^{-a+b}\varrho^{-2b}$ $\varrho>\sqrt{t}$	$\tilde r^{-c+d}R^{-2d\upsilon}$	$a+c-2+1/2\cdot$ $\min(0,1+b^*-a,1+d-c)$	$b^*+d-1-1/2\cdot$ $\min(0,1-b'-a,1+d-c)$	$\ln(R/(1+s))(\min(1+b^*-a,$ $1+d-c)=0\wedge b\neq1)$ $\ln_+ s\cdot\ln(R/(1+s))$ $(b=1\wedge1+d=0)$ $(\ln_+ s\ \ a<2)(\ln R\ \ a>2)$ $(\ln R\ln(R/(1+s))a=2)\wedge$ $b=1$
Ω_3	$(-(1/8)R^\upsilon;$ $R-R^\upsilon)$	$\sim\lvert x'\rvert$ R^υ	$R+R^\upsilon-\tilde r$	$\sim\tilde r$ $(R^\upsilon;R)$	$(0;(1/8)R^\upsilon)$	$\sim\tilde t$ $(R^\upsilon;R)$	$r^{-a+b}R^{-2b\upsilon}$	$\tilde r^{-c}\ \ \tilde\varrho<\sqrt{\tau}$ $\tilde r^{-c+d}\tilde\varrho^{-2d}\ \ \tilde\varrho>\sqrt{\tau}$	$a+c-2+1/2\cdot$ $\min(0,1+b-a,1+d^*-c)$	$b+d^*-1-1/2\cdot$ $\min(0,1+b-a,1+d^*-c)$	$\ln(R/(1+s))(\min(1+b-a,$ $1+d^*-c)=0\wedge d\neq1)$ $\ln_+ s\cdot\ln(R/(1+s))$ $(d=1\wedge1+b-a=0)$ $(\ln_+ s\ \ c<2)(\ln R\ \ c>2)$ $(\ln R\ln(R/(1+s))c=2)\wedge$ $d=1$
Ω_4									see Ω_2,Ω_3	see Ω_2,Ω_3	see Ω_2,Ω_3
Ω_5	$\sim r$ R	$(0;R^\upsilon)$	$\sim t$ R	$\sim-\tilde r$ $(-R;-R^\upsilon)$	$(0;R^\upsilon)$	$\sim\lvert\tilde t\rvert$ $(R^\upsilon;R)$	R^{-a} $\varrho<\sqrt{t}$ $R^{-a+b}\varrho^{-2b}$ $\varrho>\sqrt{t}$	$\lvert\tilde t\rvert^{-c-d}$	$a+c+d-2+1/2\cdot$ $\min(0,1-c-d)$	$b^*-1-1/2\cdot$ $\min(0,1-c-d)$	$(\ln_+ s\ \ b=1)\cdot$ $(\ln(R/(1+s))c+d=1)$
Ω_6	$\sim-r$ $(-R;-R^\upsilon)$	$(0;R^\upsilon)$	$\sim\lvert t\rvert$ $(R^\upsilon;R)$	$\sim\tilde r$ R	$(0;R^\upsilon)$	$\sim\tilde t$ R	$\lvert t\rvert^{-a-b}$	$R^{-c}\ \ \tilde\varrho<\sqrt{\tilde t}$ $R^{-c+d}\tilde\varrho^{-2d}\ \ \tilde\varrho>\sqrt{\tilde t}$	$a+b+c-2+1/2\cdot$ $\min(0,1-a-b)$	$d^*-1-1/2\cdot$ $\min(0,1-a-b)$	$(\ln_+ s\ \ d=1)\cdot$ $(\ln(R/(1+s))a+b=1)$
Ω_7	R	$\sim\tilde\varrho$ $(\lvert\tilde t\rvert;R)$	R	$(-R;-R^\upsilon)$	$\sim\varrho,\tilde r$ $(\lvert\tilde t\rvert;R)$	$\sim\tilde\varrho$ $(\lvert\tilde t\rvert;R)$	$R^{-a+b}\varrho^{-2b}$	ϱ^{-c-d}	$a+b+c+d-3+1/2\cdot$ $\min(0,3-2b-c-d)$	$-1/2\cdot$ $\min(0,3-2b-c-d)$	$\ln(R/(1+s))$ $2b+c+d=3$
Ω_8	$(-R;-R^\upsilon)$	$\sim\tilde\varrho,r$ $(\lvert t\rvert;R)$	$\sim\varrho$ $(\lvert t\rvert;R)$	R	$\sim\varrho$ $(\lvert t\rvert;R)$	R	$\tilde\varrho^{-a-b}$	$R^{-c+d}\tilde\varrho^{-2d}$	$a+b+c+d-3+1/2\cdot$ $\min(0,3-a-b-2d)$	$-1/2\cdot$ $\min(0,3-a-b-2d)$	$\ln(R/(1+s))$ $a+b+2d=3$

Tab. 2 $N = 3$

Dom.	t	ϱ	r	$\tilde{t}$	$\tilde{\varrho}$	$\tilde{r}$	$\eta^{-a}_{-b}(y)$	$\eta^{-c}_{-d}(x-y)$	e	f	*log. factors*
Ω_9	$\sim r$ R	$\sim\tilde{\varrho}$ $(R^{\upsilon};\lvert\tilde{t}\rvert)$	$\sim\tilde{t}$ R	$\sim -\tilde{r}$ $(-R;-R^{\upsilon})$	$\sim\varrho$ $(R^{\upsilon};\lvert\tilde{t}\rvert)$	$\sim\lvert\tilde{t}\rvert$ $(R^{\upsilon};R)$	$R^{-a+b}\varrho^{-2b}$	$\lvert\tilde{t}\rvert^{-c-d}$	$b>1$ see Ω_5 $b<1$ see Ω_7 $b=1$ see Ω_5	$b>1$ see Ω_5 $b<1$ see Ω_7 $b=1$ see Ω_5	$((\ln(R/(1+s))c+d<1)$ $(\ln^2(R/(1+s))c+d=1))\wedge b=1$
Ω_{10}	$\sim -r$ $(-R;$ $-R^{\upsilon})$	$\sim\tilde{\varrho}$ $(R^{\upsilon};\lvert t\rvert)$	$\sim\lvert t\rvert$ $(R^{\upsilon};R)$	$\sim\tilde{r}$ R	$\sim\varrho$ $(R^{\upsilon};\lvert t\rvert)$	$\sim\tilde{t}$ R	$\lvert t\rvert^{-a-b}$	$R^{-c+d}\tilde{\varrho}^{-2d}$	$d>1$ see Ω_6 $d<1$ see Ω_8 $d=1$ see Ω_6	$d>1$ see Ω_6 $d<1$ see Ω_8 $d=1$ see Ω_6	$((\ln(R/(1+s))a+b<1)$ $(\ln^2(R/(1+s))a+b=1))\wedge d=1$
Ω_{11}	$(R;\infty)$	$(0;\infty)$	$\sim\tilde{r}$ $(R;\infty)$	$(-\infty;-R^{\upsilon})$	$(0;\infty)$	$\sim r$ $(R;\infty)$	$(1+r)^{-a}(1+s(y))^{-b}$	r^{-c-d}	$a+b^*+c+d-3>0$	0	$\ln R\;\; b=1$
Ω_{12}	$(-\infty;$ $-R^{\upsilon})$	$(0;\infty)$	$\sim\tilde{r}$ $(R;\infty)$	$(R;\infty)$	$(0;\infty)$	$\sim r$ $(R;\infty)$	$\tilde{r}^{-a-b}$	$(1+\tilde{r})^{-c}(1+s(x-y))^{-d}$	$a+b+c+d^*-3>0$	0	$\ln R\;\; d=1$
Ω_{13}		$\sim r,\tilde{r}$ $(R;\infty)$	$\sim\tilde{r}$ $(R;\infty)$		$\sim\varrho$ $(R;\infty)$	$\sim r$ $(R;\infty)$			see Ω_{11} and Ω_{12}	see Ω_{11} and Ω_{12}	see Ω_{11} and Ω_{12}
Ω_{14}	$\sim r$ R	$\sim\tilde{\varrho}$ $(R^{\upsilon};R)$	$\sim t$ R	$(-(1/8)R^{\upsilon};$ $R/2)$	$\sim\varrho$ $(R^{\upsilon};R)$	$\sim\lvert\tilde{t}\rvert+\tilde{\varrho}$	$R^{-a+b}\varrho^{-2b}$	$(\tilde{t}+\tilde{\varrho})^{-c+d}\cdot$ $\tilde{\varrho}^{-2d}\tilde{t}>0$ $\tilde{\varrho}^{-c-d}\tilde{t}<0$	see Ω_2,Ω_3 Ω_7 Ω_2 $a+b+c+d-3$	$b+d>1\wedge 1+d-c>0$ $1+d-c<0$ $1+d-c=0\wedge b+d>1$ 0 otherwise	$\ln(R/(1+s))b+d=$ $1\wedge 1+d-c>0$ $\ln^2(R/(1+s))b+d=$ $1\wedge 1+d-c=0$
Ω_{15}	$(-R^{\upsilon};$ $R/2)$	$\sim\tilde{\varrho}$ $(R^{\upsilon};R)$	$\sim\lvert t\rvert+\varrho$	$\sim r$ R	$\sim\varrho$ $(R^{\upsilon};R)$	$\sim t$ R	$(t+\varrho)^{-a+b}$ $\varrho^{-2b}\;\; t>0$ $\varrho^{-a-b}\;\; t<0$	$R^{-c+d}\tilde{\varrho}^{-2d}$	see Ω_2,Ω_3 Ω_8 Ω_3 $a+b+c+d-3$	$b+d>1\wedge 1+b-a>0$ $1+b-a<0$ $1+b-a=0\wedge b+d>1$ 0 otherwise	$\ln(R/(1+s))b+d=$ $1\wedge 1+b-a>0$ $\ln^2(R/(1+s))b+d=$ $1\wedge 1+b-a=0$

Tab. 3 $N=2$

Dom.	t	ϱ	r	$\tilde{t}$	$\tilde{\varrho}$	$\tilde{r}$	$\eta^{-a}_{-b}(y)$	$\eta^{-c}_{-d}(x-y)$	e	f	*log. factors*
Ω_0	$(-(1/8)R^{\upsilon};$ $(1/8)R^{\upsilon})$	$(0;(1/8)R^{\upsilon})$	$(0;(1/8)R^{\upsilon})$	$\sim x_1$ R	$\sim\lvert x_2\rvert$ R^{υ}	$\sim\lvert x\rvert$ R	$r^{-a}(1+s(y))^{-b}$	$R^{-c-2\sigma d}$	$c+1/2\cdot$ $\min(0,a+b^*-2)$	$d+1/2\cdot$ $\min(0,a+b^*-2)$	$\ln R(b=(1/2)\wedge a<(3/2))\vee$ $(b\neq(1/2)\wedge a+b^*=2)$ $\ln^2 R(b=(1/2)\wedge a=3/2)$
Ω_1	$\sim x_1$ R	$\sim\lvert x_2\rvert$ R^{υ}	$\sim\lvert x\rvert$ R	$(-(1/8)R^{\upsilon};$ $(1/8)R^{\upsilon})$	$(0;(1/8)R^{\upsilon})$	$(0;(1/8)R^{\upsilon})$	$R^{-a-2\sigma b}$	$\tilde{r}^{-c}(1+s(x-y))^{-d}$	$a+1/2\cdot$ $\min(0,c+d^*-2)$	$b+1/2\cdot$ $\min(0,c+d^*-2)$	$\ln R(d=(1/2)\wedge c<(3/2))\vee$ $(d\neq(1/2)\wedge c+d^*=2)$ $\ln^2 R(c=(3/2)\wedge d=1/2)$
Ω_2	$\sim r$ $(R^{\upsilon};R)$	$(0;(1/8)R^{\upsilon})$	$\sim t$ $(R^{\upsilon};R)$	$(-(1/8)R^{\upsilon};$ $R-R^{\upsilon})$	$\sim\lvert x'\rvert$ R^{υ}	$R+R^{\upsilon}-r$	$r^{-a}\quad \varrho<\sqrt{t}$ $r^{-a+b}\varrho^{-2b}$ $\varrho>\sqrt{t}$	$\tilde{r}^{-c+d}R^{-2d\upsilon}$	$a+c-3/2+1/2\cdot$ $\min(0,1+b^*-a,$ $1+d-c)$	$b^*+d-1/2-1/2\cdot$ $\min(0,1+b^*-a,$ $1+d-c)$	$\ln(R/(1+s))(\min(1+b^*-a,$ $1+d-c)=0\wedge b\neq 1/2)$ $\ln_+ s\cdot\ln(R/(1+s))$ $(b=1/2\wedge 1+d=c)$ $(\ln_+ s\ \ a<3/2)$ $(\ln R\ \ a>3/2)$ $(\ln R\ln(R/(1+s))\ \ a=$ $3/2)\wedge b=1/2$
Ω_3	$(-(1/8)R^{\upsilon};$ $R-R^{\upsilon})$	$\sim\lvert x_2\rvert$ R^{υ}	$R+R^{\upsilon}-\tilde{r}$	$\sim\tilde{r}$ $(R^{\upsilon};R)$	$(0;(1/8)R^{\upsilon})$	$\sim\tilde{t}$ $(R^{\upsilon};R)$	$r^{-a+b}R^{-2b\upsilon}$	$\tilde{r}^{-c}\quad\tilde{\varrho}<\sqrt{\tau}$ $\tilde{r}^{-c+d}\tilde{\varrho}^{-2d}$ $\tilde{\varrho}>\sqrt{\tau}$	$a+c-3/2+1/2\cdot$ $\min(0,1+b-a,$ $1+d^*-c)$	$b+d^*-1/2-1/2\cdot$ $\min(0,1+b-a,$ $1+d^*-c)$	$\ln(R/(1+s))(\min(1+b-a,$ $1+d^*-c)=0\wedge d\neq 1)$ $\ln_+ s\cdot\ln(R/(1+s))$ $(d=1/2\wedge 1+b-a=0)$ $(\ln_+ s\ \ c<3/2)(\ln R\ \ c>3/2)$ $(\ln R\ln(R/(1+s))\ \ c=3/2)\wedge$ $d=1/2$
Ω_4									see Ω_2,Ω_3	see Ω_2,Ω_3	see Ω_2,Ω_3
Ω_5	$\sim r$ R	$(0;R^{\upsilon})$	$\sim t$ R	$\sim-\tilde{r}$ $(-R;-R^{\upsilon})$	$(0;R^{\upsilon})$	$\sim\lvert\tilde{t}\rvert$ $(R^{\upsilon};R)$	$R^{-a}\quad\varrho<\sqrt{t}$ $R^{-a+b}\varrho^{-2b}$ $\varrho>\sqrt{t}$	$\lvert\tilde{t}\rvert^{-c-d}$	$a+c+d-3/2+1/2\cdot$ $\min(0,1-c-d)$	$b^*-1/2-1/2$ $\min(0,1-c-d)$	$(\ln_+ s\ \ b=1/2)\cdot$ $(\ln(R/(1+s))\ \ c+d=1)$
Ω_6	$\sim-r$ $(-R;-R^{\upsilon})$	$(0;R^{\upsilon})$	$\sim\lvert t\rvert$ $(R^{\upsilon};R)$	$\sim\tilde{r}$ R	$(0;R^{\upsilon})$	$\sim\tilde{t}$ R	$\lvert t\rvert^{-a-b}$	$R^{-c}\quad\tilde{\varrho}<\sqrt{\tilde{t}}$ $R^{-c+d}\tilde{\varrho}^{-2d}$ $\tilde{\varrho}>\sqrt{\tilde{t}}$	$a+b+c-3/2+1/2\cdot$ $\min(0,1-a-b)$	$d^*-1/2-1/2\cdot$ $\min(0,1-a-b)$	$(\ln_+ s\ \ d=1/2)\cdot$ $(\ln(R/(1+s))\ \ a+b=1)$
Ω_7	R	$\sim\tilde{\varrho}$ $(\lvert\tilde{t}\rvert;R)$	R	$(-R;-R^{\upsilon})$	$\sim\varrho,\tilde{r}$ $(\lvert\tilde{t}\rvert;R)$	$\sim\tilde{\varrho}$ $(\lvert\tilde{t}\rvert;R)$	$R^{-a+b}\varrho^{-2b}$	ϱ^{-c-d}	$a+b+c+d-2+1/2\cdot$ $\min(0,2-2b-c-d)$	$-1/2\cdot$ $\min(0,2-2b-c-d)$	$\ln(R/(1+s))\ \ 2b+c+d=2$
Ω_8	$(-R;-R^{\upsilon})$	$\sim\tilde{\varrho},r$ $(\lvert t\rvert;R)$	$\sim\varrho$ $(\lvert t\rvert;R)$	R	$\sim\varrho$ $(\lvert t\rvert;R)$	R	$\tilde{\varrho}^{-a-b}$	$R^{-c+d}\tilde{\varrho}^{-2d}$	$a+b+c+d-2+1/2\cdot$ $\min(0,2-a-b-2d)$	$-1/2\cdot$ $\min(0,2-a-b-2d)$	$\ln(R/(1+s))\ \ a+b+2d=2$

Tab. 4 $N=2$

Dom.	t	ϱ	r	$\tilde{t}$	$\tilde{\varrho}$	$\tilde{r}$	$\eta^{-a}_{-b}(y)$	$\eta^{-c}_{-d}(x-y)$	e	f	*log. factors*
Ω_9	$\sim r$ R	$\sim\tilde{\varrho}$ $(R^\nu;\lvert\tilde{t}\rvert)$	$\sim t$ R	$\sim -\tilde{r}$ $(-R;-R^\nu)$	$\sim\varrho$ $(R^\nu;\lvert\tilde{t}\rvert)$	$\sim\lvert\tilde{t}\rvert$ $(R^\nu;R)$	$R^{-a+b}\varrho^{-2b}$	$\lvert\tilde{t}\rvert^{-c-d}$	$b>1/2$ see Ω_5 $b<1/2$ see Ω_7 $b=1/2$ see Ω_5	$b>1/2$ see Ω_5 $b<1/2$ see Ω_7 $b=1/2$ see Ω_5	$((\ln(R/(1+s))c+d<1)$ $(\ln^2(R/(1+s))c+d=1))\wedge$ $b=1/2$
Ω_{10}	$\sim -r$ $(-R;$ $-R^\nu)$	$\sim\tilde{\varrho}$ $(R^\nu;\lvert t\rvert)$	$\sim\lvert t\rvert$ $(R^\nu;R)$	$\sim\tilde{r}$ R	$\sim\varrho$ $(R^\nu;\lvert t\rvert)$	$\sim\tilde{t}$ R	$\lvert t\rvert^{-a-b}$	$R^{-c+d}\tilde{\varrho}^{-2d}$	$d>1/2$ see Ω_6 $d<1/2$ see Ω_8 $d=1/2$ see Ω_6	$d>1/2$ see Ω_6 $d<1/2$ see Ω_8 $d=1/2$ see Ω_6	$((\ln(R/(1+s))a+b<1)$ $(\ln^2(R/(1+s))a+b=1))\wedge$ $d=1/2$
Ω_{11}	$(R;\infty)$	$(0;\infty)$	$\sim\tilde{r}$ $(R;\infty)$	$(-\infty;-R^\nu)$	$(0;\infty)$	$\sim r$ $(R;\infty)$	$(1+r)^{-a}(1+s(y))^{-b}$	r^{-c-d}	$a+b^*+c+d-2>0$	0	$\ln R$ $b=1/2$
Ω_{12}	$(-\infty;$ $-R^\nu)$	$(0;\infty)$	$\sim\tilde{r}$ $(R;\infty)$	$(R;\infty)$	$(0;\infty)$	$\sim r$ $(R;\infty)$	$\tilde{r}^{-a-b}$	$(1+\tilde{r})^{-c}(1+s(x-y))^{-d}$	$a+b+c+d^*-2>0$	0	$\ln R$ $d=1/2$
Ω_{13}		$\sim r,\tilde{r}$ $(R;\infty)$	$\sim\tilde{r}$ $(R;\infty)$		$\sim\varrho$ $(R;\infty)$	$\sim r$ $(R;\infty)$			see Ω_{11} and Ω_{12}	see Ω_{11} and Ω_{12}	see Ω_{11} and Ω_{12}
Ω_{14}	$\sim r$ R	$\sim\tilde{\varrho}$ $(R^\nu;R)$	$\sim t$ R	$(-(1/8)R^\nu;$ $R/2)$	$\sim\varrho$ $(R^\nu;R)$	$\sim\lvert\tilde{t}\rvert+\tilde{\varrho}$	$R^{-a+b}\varrho^{-2b}$	$(\tilde{t}+\tilde{\varrho})^{-c+d}\tilde{\varrho}^{-2d}\tilde{t}>0$ $\tilde{\varrho}^{-c-d}\tilde{t}<0$	see Ω_2,Ω_3 Ω_7 Ω_2 $a+b+c+d-2$	$b+d>1/2\wedge 1+d-c>0$ $1+d-c<0$ $1+d-c=0\wedge b+d>1/2$ 0 otherwise	$\ln(R/(1+s))b+d=$ $1/2\wedge 1+d-c>0$ $\ln^2(R/(1+s))b+d=$ $1/2\wedge 1+d-c=0$
Ω_{15}	$(-R^\nu;$ $R/2)$	$\sim\tilde{\varrho}$ $(R^\nu;R)$	$\sim\lvert t\rvert+\varrho$	$\sim r$ R	$\sim\varrho$ $(R^\nu;R)$	$\sim t$ R	$(t+\varrho)^{-a+b}$ ϱ^{-2b} $t>0$ ϱ^{-a-b} $t<0$	$R^{-c+d}\tilde{\varrho}^{-2d}$	see Ω_2,Ω_3 Ω_8 Ω_3 $a+b+c+d-2$	$b+d>1/2\wedge 1+b-a>0$ $1+b-a<0$ $1+b-a=0\wedge b+d>1/2$ 0 otherwise	$\ln(R/(1+s))b+d=$ $1/2\wedge 1+b-a>0$ $\ln^2(R/(1+s))b+d=$ $1/2\wedge 1+b-a=0$

REMARK 2.1. Let A be a positive function. We denote

$$\ln_+ A = \max(\ln A, 1).$$

We start by estimating the convolutions over the sixteen subdomains.

$\underline{\mathrm{I}_0}$ We have $\Omega_0 = \{y \in \mathbf{R}^N : |t| \le (1/8)R^\nu; \varrho \le (1/8)R^\nu\}$ and therefore $\eta^{-a}_{-b}(y) \sim r^{-a}(1+s(y))^{-b}$, $\eta^{-a}_{-b}(x-y) \sim R^{-c-2d\sigma}$. Applying Lemma 2.2 we get

$$I_0 \sim R^{-c-2\sigma d} \int_0^{R^\nu} (1+r)^{N-1-a-b^*} \cdot (\ln_+ r, \text{ if } b = (N-1)/2)\, \mathrm{d}r$$

$$\sim R^{-(c+(1/2)\min(0,a+b^*-N))-2\sigma(d+\min(0,a+b^*-N))}$$

$$\cdot \begin{cases} \ln R & \text{if } a+b^* = N, b \ne (N-1)/2 \\ & \text{or } a+b^* < N, b = (N-1)/2 \\ \ln^2 R & \text{if } a+b^* = N, b = (N-1)/2. \end{cases}$$

The results are summarized in Tab. 1–Tab. 4.

$\underline{\mathrm{I}_1}$ The integral can be estimated in the same way by exchanging a, b for c, d. Assuming the kernel (2.6) instead of η^{-c}_{-d} we have

$$\tilde{I}_1(x) = \int_{B_1(\mathbf{0})} K(x-y)\eta^{-a}_{-b}(x)\,\mathrm{d}x \sim R^{-a-2\sigma b}$$

since $\gamma < N$. Again, the summarized results can be found in Tab. 1–Tab. 4.

$\underline{\mathrm{I}_2}$ We have in Ω_2 that $r \sim t \in (R^\nu; R)$. So $\eta^{-a}_{-b}(y) \sim r^{-a}$ for $\varrho < \sqrt{t}$ and $\eta^{-a}_{-b}(y) \sim r^{-a+b}\varrho^{-2b}$ for $\varrho > \sqrt{t}$. Further $\tilde{\varrho} \sim R^\nu$ and $\tilde{r} \sim R + R^\nu - r$; therefore $\eta^{-c}_{-d}(x-y) \sim \tilde{r}^{-c+d}R^{-2\nu d}$. Thus

$$I_2 \sim R^{-2d\nu} \int_{R^\nu}^{R} \mathrm{d}r (R+R^\nu-r)^{d-c} \left[r^{-a} \int_0^{\sqrt{r}} \varrho^{N-2}\,\mathrm{d}\varrho + r^{b-a} \int_{\sqrt{r}}^{R^\nu} \varrho^{N-2-2b}\,\mathrm{d}\varrho \right]$$

$$= R^{-2\nu d} \int_{R^\nu}^{R} \mathrm{d}r (R+R^\nu-r)^{d-c}$$

$$\cdot [r^{N/2-(1/2)-a} + r^{b-a}(R^{\nu(N-1-2b)} - r^{N/2-1/2-b})]$$

$$\cdot (\ln R^\nu/\sqrt{r} \text{ if } b = (N-1)/2)$$

$$\sim R^{-2\nu(b^*-(N-1)/2+d)} \int_{R^\nu}^{R} \mathrm{d}r (R+R^\nu-r)^{d-c} r^{b^*-a}$$

$$\cdot (\ln R^\nu/\sqrt{r} \text{ if } b = (N-1)/2) \equiv J.$$

In order to verify the last equivalence it is enough to consider first $b \geq (N-1)/2$, i.e. $b^* = (N-1)/2$, and then $b < (N-1)/2$, i.e. $b^* = b$, and estimate the integrals.

Let us divide J into two parts—the integral over $(R^\nu; R/2)$ and the integral over $(R/2; R)$. We estimate these two parts separately.

$$R^{-c+d-2\nu(b^*-(N-1)/2+d)} \int_{R^\nu}^{R/2} \mathrm{d}r r^{b^*-a} \cdot (\ln R^\nu/\sqrt{r} \text{ if } b = (N-1)/2)$$

$$\sim R^{-(c+a-(N+1)/2)-2\sigma(b^*-(N-1)/2+d)+(\sigma-1/2)\min(0,b^*-a+1)}$$

$$\cdot (\ln_+ R/(1+s) \text{ if } b \neq (N-1)/2,\ a = b^*+1)$$

$$\cdot ((\ln_+ s, a < (N+1)/2) \cdot (\ln R, a > (N+1)/2)$$

$$\cdot (\ln R \ln_+ R/(1+s),\ a = (N+1)/2) \text{ if } b = (N-1)/2).$$

We used the fact that $s(x) \sim \frac{|x'|^2}{|x|} \sim R^{2\sigma}$ and $\ln_+ R/R^\nu = (1/2)\ln_+ R/R^{2\sigma} \sim \ln_+ R/(1+s)$. Analogously

$$R^{-2\nu(b^*-(N-1)/2+d)+b^*-a} \int_{R/2}^{R} \mathrm{d}r (R+R^\nu-r)^{d-c} \cdot (\ln R^\nu/\sqrt{r} \text{ if } b = (N-1)/2)$$

$$\sim R^{-(c+a-(N+1)/2)-2\sigma(b^*-(N-1)/2+d)+(\sigma-1/2)\min(0,d-c+1)}$$

$$\cdot (\ln_+ R/(1+s) \text{ if } c = d+1) \cdot (\ln_+ s \text{ if } b = (N-1)/2).$$

The results may again be found in Tab. 1–Tab. 4.

$\underline{I_3}$ We proceed analogously as for I_2 exchanging a, b for c, d.

$\underline{I_4}$ Ω_4 can be considered as a subset of Ω_2 and Ω_3. Therefore I_4 can be estimated by I_2 and I_3.

$\underline{I_5}$ We have in Ω_5 $t \sim r \sim R$, so $\eta_{-b}^{-a}(y) \sim R^{-a}$ $(\varrho < \sqrt{t})$ or $\eta_{-b}^{-a}(y) \sim R^{-a+b}\varrho^{-2b}$ $(\varrho > \sqrt{t})$ where ϱ varies between 0 and R^ν. Further $\tilde{r} \sim |\tilde{t}| \in (R^\nu; R)$. As $\tilde{t} < 0$ we have $\eta_{-d}^{-c}(x-y) \sim |\tilde{t}|^{-c-d}$.

$$I_5 \sim \int_{R^\nu}^{R} \mathrm{d}\tau \tau^{-c-d} \left(R^{-a} \int_0^{\sqrt{R}} \varrho^{N-2}\,\mathrm{d}\varrho + R^{b-a} \int_{\sqrt{R}}^{R^\nu} \varrho^{N-2-2b}\,\mathrm{d}\varrho \right)$$

$$\sim R^{1-c-d+(\sigma-1/2)\min(0,1-c-d)}[R^{(N-1)/2-a} + (R^{(N-2-2b)\nu} - R^{(N-2-2b)/2})$$

$$\cdot R^{b-a}(\ln_+ s,\ \text{if } b = (N-1)/2)] \cdot (\ln R/R^\nu \text{ if } c+d = 1)$$

$$\sim R^{1-c-d+(N-1)/2-a+(\sigma-1/2)\min(0,1-c-d)+2\sigma((N-1)/2-b^*)}$$

$$\cdot (\ln_+ s \text{ if } b = (N-1)/2) \cdot (\ln_+ R/(1+s) \text{ if } c+d = 1).$$

I$_6$ It is sufficient to interchange a, b by c, d and use the result for Ω_5.

I$_7$ Denoting $\tau = |\tilde{t}| \in (R^\nu, R)$ we have in Ω_7 that $t \sim r \sim R$, $\varrho \sim \tilde{\varrho} \sim \tilde{r} \in (\tau; R)$. Therefore $\eta_{-b}^{-a}(y) \sim R^{-a+b}\rho^{-2b}$, $\eta_{-d}^{-c}(x-y) \sim \rho^{-c-d}$ and

$$
\begin{aligned}
I_7 &\sim R^{-a+b} \int_{R^\nu}^{R} \mathrm{d}\tau \left(\int_{\tau}^{R} \varrho^{N-2-d-c-2b} \, \mathrm{d}\varrho \right) \\
&\sim R^{b-a} \int_{R^\nu}^{R} \mathrm{d}\tau (R^{N-1-c-d-2b} - \tau^{N-1-c-d-2b}) \\
&\quad \cdot (\ln R/\tau \ \text{ if } \ c+d+2b = N-1) \\
&\sim R^{N-a-b-c-d} + R^{N-a-b-c-d+(\sigma-1/2)\min(0,N-c-d-2b)} \\
&\quad \cdot (\ln R/(1+s) \ \text{ if } \ c+d+2b = N) \\
&\sim R^{N-a-b-c-d+(\sigma-1/2)\min(0,N-c-d-2b)} \cdot (\ln R/(1+s) \ \text{ if } \ c+d+2b = N).
\end{aligned}
$$

I$_8$ We get the result by interchanging a, b by c, d and using the result for Ω_7.

I$_9$ Analogously as in Ω_7 we have $t \sim r \sim R$, $\varrho \sim \tilde{\varrho} \in (R^\nu; \tau)$, $\tilde{r} \sim \tau = |\tilde{t}| \in (R^\nu, R)$; so $\eta_{-b}^{-a}(y) \sim R^{-a+b}\rho^{-2b}$, $\eta_{-d}^{-c}(x-y) \sim \rho^{-c-d}$ and

$$
I_9 \sim R^{-a+b} \int_{R^\nu}^{R} \mathrm{d}\tau \tau^{-d-c} \left(\int_{R^\nu}^{\tau} \varrho^{N-2-2b} \, \mathrm{d}\varrho \right).
$$

If $b > (N-1)/2$ the significant term in the inner integral will be the lower bound and we can use I_5. If $b < (N-1)/2$ the significant term in the inner integral will be the upper bound and we can use I_7. If $b = (N-1)/2$ then

$$
I_9 \sim R^{-a+(N-1)/2} \int_{R^\nu}^{R} \tau^{-d-c} \ln \frac{\tau}{R^\nu} \mathrm{d}\tau.
$$

In comparison with I_5 we get some additional logarithmic factors

$$
\begin{aligned}
b = (N-1)/2 : &(\ln_+ R/(1+s) \ \text{ if } \ c+d < 1) \\
&\cdot (\ln_+^2 R/(1+s) \ \text{ if } \ c+d = 1).
\end{aligned}
$$

I$_{10}$ As in I_9, we may use I_6 for $d > (N-1)/2$, I_8 for $d > (N-1)/2$ and get some additional logarithmic factors to I_8 for $d = (N-1)/2$.

I$_{11}$ The domain Ω_{11} is unbounded. We have $\tilde{r} \sim r \in (R; \infty)$. Therefore $\eta_{-d}^{-c}(x-y) \sim r^{-c-d}$, $\eta_{-b}^{-a}(y) \sim r^{-a}(1+s(y))^{-b}$ and applying Lemma 2.2 under the assumption $a+b^*+c+d > N$ we get

$$I_{11} \sim \int_R^\infty \mathrm{d}r r^{N-1-a-b^*-c-d} \cdot (\ln r \text{ if } b = (N-1)/2)$$
$$\sim R^{N-a-b^*-c-d} \cdot (\ln R \text{ if } b = (N-1)/2).$$

$\underline{I_{12}}$ We proceed as in the previous case and under the assumption $a+b+c+d^* > N$ we get

$$I_{12} \sim R^{N-a-b-c-d^*} \cdot (\ln R \text{ if } d = (N-1)/2).$$

$\underline{I_{13}}$ The domain Ω_{13} can be considered as a subset of Ω_{11} and Ω_{12}. Therefore I_{13} can be bounded by I_{11} and I_{12}.

$\underline{I_{14}}$ In this subdomain we have $r \sim R$, $\varrho \sim \tilde{\varrho} \in (R^\nu; R)$. Moreover $\tilde{r} \sim |\tilde{t}| + \tilde{\varrho}$ where $\tilde{t} \in ((-1/8)R^\nu; R/2)$. Then $\eta^{-a}_{-b}(y) \sim R^{-a+b}\varrho^{-2b}$, $\eta^{-c}_{-d}(x-y) \sim (\tilde{t}+\tilde{\varrho})^{-c+d}\varrho^{-2d}$ if $\tilde{t} > 0$ and $\eta^{-c}_{-d}(x-y) \sim \tilde{\varrho}^{-c-d}$ if $\tilde{t} < 0$. Let us note that the strip $\tilde{t} \in ((-1/8)R; 0)$ has no influence on the asymptotic behaviour since $\tilde{\varrho} > |\tilde{t}|$ there.

$$I_{14} \sim R^{b-a} \int_{R^\nu}^R \mathrm{d}\varrho \varrho^{N-2-2b-2d} \int_0^{R/2} (\tilde{t}+\rho)^{d-c} \, \mathrm{d}\tilde{t}$$

$$\sim R^{b-a} \int_{R^\nu}^R \mathrm{d}\varrho \varrho^{N-2-2b-2d} \cdot \begin{cases} R^{1+d-c} & 1+d-c > 0 \\ \varrho^{1+d-c} & 1+d-c < 0 \\ \ln R/\varrho & 1+d-c = 0. \end{cases}$$

Now we distinguish three cases.

a) $1+d-c > 0$

If $b+d \le (N-1)/2$ then

$$I_{14} \sim R^{N-a-b-c-d} \cdot (\ln_+ R/(1+s) \text{ if } b+d = (N-1)/2),$$

while for $b+d > (N-1)/2$ we have

$$I_{14} \sim R^{-a-c+(N+1)/2+2\sigma((N-1)/2-b-d)} \qquad (\text{see } I_2, I_3).$$

b) $1+d-c < 0$

$$I_{14} \sim R^{b-a} \int_{R^\nu}^R \varrho^{-2b-c-d+N-1} \, \mathrm{d}\varrho$$

and the integral can be estimated by I_7.

c) $1+d-c = 0$

$$I_{14} \sim R^{b-a} \int_{R^\nu}^R \varrho^{N-2-2b-2d} \ln(R/\varrho) \, \mathrm{d}\varrho$$

$$= R^{N-a-b-c-d} \int_1^{R^{1-\nu}} z^{2b+2d-N} \ln z \, \mathrm{d}z.$$

Now for $b+d \le (N-1)/2$

$$I_{14} \sim R^{N-a-b-c-d} \cdot (\ln_+^2 R/(1+s) \text{ if } b+d = (N-1)/2)$$

and for $b+d > (N-1)/2$

$$I_{14} \sim R^{(N+1)/2-a-c-2\sigma(b+d-(N-1)/2)} \ln_+ R/(1+s)$$

which can be estimated by I_2.

$\underline{\mathrm{I}_{15}}$ Interchanging a, b by c, d we can use the results from I_{14}.

We have now completed investigation of the situation C). The results are summarized in Tab. 1, 2 ($N=3$) and Tab. 3, 4 ($N=2$).

The situation D) is almost trivial since we are left with subdomains of the type Ω_1, Ω_2, Ω_{11}, Ω_{12} and Ω_{13}. The integrals can be estimated by the corresponding integrals in C) taking $\sigma = 1/2$, i.e. $v = 1$.

Finally in the case B) we proceed as in case C) but the subdomains Ω_2, Ω_3 and Ω_4 coincide. The other integrals can again be estimated by the corresponding ones from the part C) taking $\sigma = 0$, i.e. $v = 1/2$.

The study of the convolution (2.1) is therefore completed. In the next section, we shall apply the results in the study of L^∞-estimates of convolutions with the Oseen kernels.

3. L^∞-estimates for weakly singular Oseen kernels.

Now we intend to get estimates for the functions v_β^α, $\mu_\beta^{\alpha,\gamma}$ by analogy with the preceding section,

$$v_\beta^\alpha(x) = v_\beta^\alpha(x;1) = |x|^\alpha (1+s(x))^\beta, \quad \mu_\beta^{\alpha,\gamma}(x) = \eta_\beta^{\alpha-\gamma}(x) v_0^\gamma(x).$$

This is the aim of Lemma 3.1 and Lemma 3.2. We will formulate these lemmas in the case of $\boldsymbol{R}^N$, $N \in \boldsymbol{N}$, $N \ge 2$. Afterward, we apply the lemmas in the study of L^∞-estimates of convolutions with Oseen kernels. Let $I_{\alpha,\gamma}(x)$ denote the following integral

$$I_{\alpha,\gamma}^{(N)}(x) = \int_{B_1(\mathbf{0})} v_0^{-\alpha}(y) v_0^{-\gamma}(x-y)\,\mathrm{d}y = \int_{B_1(x)} v_0^{-\alpha}(x-y') v_0^{-\gamma}(y')\,\mathrm{d}y',$$

where $x, y, y' \in \boldsymbol{R}^N$. For the notational convenience we also denote

$$\ln_-|x| := \max(1, -\ln|x|)$$

for $x \neq \mathbf{0}$.

Lemma 3.1. *For $\alpha < N$, $\gamma < N$ there exists a positive constant C_1 such that for $x \in B_2(\mathbf{0}) \backslash \{\mathbf{0}\} \subset \boldsymbol{R}^N$*

$$I_{\alpha,\gamma}^{(N)}(x) \le C_1 \begin{cases} v_0^{-(\alpha+\gamma-N)}(x), & \text{if } \alpha+\gamma > N \\ \ln_-|x|, & \text{if } \alpha+\gamma = N \\ 1, & \text{if } \alpha+\gamma < N. \end{cases}$$

Morerover, there exists a positive constant C_2 such that for $x \in B^2(\mathbf{0}) \subset \boldsymbol{R}^N$

$$I_{\alpha,\gamma}^{(N)}(x) \le C_2 v_0^{-\gamma}(x).$$

Proof. We divide the proof into two parts:

a) First we assume $|x| \le 2$. We will estimate integrals over sets, whose union contains unit ball $B_1(\mathbf{0})$.

$$B_1(\mathbf{0}) \subset B_{|x|/2}(\mathbf{0}) \cup B_{|x|/2}(x) \cup \{B_{2|x|}(\mathbf{0}) \backslash (B_{|x|/2}(\mathbf{0}) \cup B_{|x|/2}(x))\} \cup \{B_4(\mathbf{0}) \backslash B_{2|x|}(\mathbf{0})\} \equiv M_1 \cup M_2 \cup M_3 \cup M_4$$

$$\int_{M_1} \frac{1}{|y|^\alpha} \frac{1}{|x-y|^\gamma} \, dy \le \frac{c_1}{|x|^\gamma} \int_0^{|x|/2} \frac{1}{r^\alpha} r^{N-1} \, dr \le \frac{c_2}{|x|^{\alpha+\gamma-N}}, \quad \alpha < N$$

$$\int_{M_2} \frac{1}{|y|^\alpha} \frac{1}{|x-y|^\gamma} \, dy \le \frac{c_3}{|x|^\alpha} \int_0^{|x|/2} \frac{1}{r^\gamma} r^{N-1} \, dr \le \frac{c_4}{|x|^{\alpha+\gamma-N}}, \quad \gamma < N$$

$$\int_{M_3} \frac{1}{|y|^\alpha} \frac{1}{|x-y|^\gamma} \, dy \le \frac{c_5}{|x|^{\alpha+\gamma}} \int_0^{2|x|} r^{N-1} \, dr \le \frac{c_6}{|x|^{\alpha+\gamma-N}}$$

$$\int_{M_4} \frac{1}{|y|^\alpha} \frac{1}{|x-y|^\gamma} \, dy \le c_7 \int_{2|x|}^4 \frac{r^{-\alpha} r^{N-1}}{|r-|x||^\gamma} \, dr$$

$$\le c_8 \int_{2|x|}^4 \left(\frac{r}{|r-|x||} \right)^\gamma r^{-\alpha-\gamma+N-1} \, dr \le c_9 \int_{2|x|}^4 r^{-\alpha-\gamma+N-1} \, dr;$$

here we use the inequality $r/|r-|x|| \le 2$.

The last integral can be estimated by $c_{10}|x|^{-\alpha-\gamma+N}$ if $\alpha+\gamma > N$, by $c_{11} \ln_-|x|$ if $\alpha+\gamma = N$ and by some constant if $\alpha+\gamma < N$.

b) Now we assume $|x| \ge 2$.

$$\int_{B_1(\mathbf{0})} \frac{1}{|y|^\alpha} \frac{1}{|x-y|^\gamma} \, dy \le \frac{c_{11}}{|x|^\gamma} \int_{B_1(\mathbf{0})} \frac{1}{|y|^\alpha} \, dy \le \frac{c_{12}}{|x|^\gamma}.$$

The assertion of Lemma 3.1 follows from these five estimates of convolution integrals. $\square$

We define for pairs of real numbers $[a,b] \le [c,d] : a \le c$ and $a+b \le c+d$. It is evident that $\eta_b^a(x) \le 2^{c-a} \eta_d^c(x), \ x \in \boldsymbol{R}^N$ if $[a,b] \le [c,d]$.

In the formulation of Lemma 3.2 we use functions $\mu_\beta^{\alpha,\gamma}(x) = \eta_\beta^{\alpha-\gamma}(x)v_0^\gamma(x)$. Let us note that $\mu_\beta^{\alpha,\alpha}(x) = \eta_\beta^0(x)v_0^\alpha(x) = v_\beta^\alpha(x)$, $\mu_\beta^{\alpha,\gamma}(x) \sim \eta_\beta^\alpha(x)$, $x \in B^1(\mathbf{0})$, $\mu_\beta^{\alpha,\gamma}(x) \sim v_\beta^\gamma(x) \sim v_0^\gamma(x)$, $x \in B_1(\mathbf{0})\backslash\{\mathbf{0}\}$.

Lemma 3.2. *Let $a, b, c, d, e, f \in \boldsymbol{R}$ and positive constant C be such that for all $x \in \boldsymbol{R}^N$: $\int_{\boldsymbol{R}^N} \eta_{-d}^{-c}(x-y)\eta_{-b}^{-a}(y)\,dy \le C\eta_{-f}^{-e}(x)$, $N \in \boldsymbol{N}$, $N \ge 2$.*

Let $g < N$, $h < N$, $[e,f] \le [a,b]$, $[e,f] \le [c,d]$. Then there exists a positive constant C' such that the following inequality is satisfied for $x \in \boldsymbol{R}^N\backslash\{\mathbf{0}\}$:

$$\int_{\boldsymbol{R}^N} \mu_{-d}^{-c,-h}(x-y)\mu_{-b}^{-a,-g}(y)\,dy \le C' \begin{cases} \mu_{-f}^{-e,-g-h+N}(x) & g+h > N \\ \mu_{-f}^{-e,-\delta}(x), \delta > 0 & g+h = N \\ \mu_{-f}^{-e,0}(x) \equiv \eta_{-f}^{-e}(x) & g+h < N. \end{cases}$$

Proof. Evidently, for α, $\beta \in \boldsymbol{R}^1$ there exist positive constants c_1, c_2, c_3 and c_4 such that

a) $c_1\eta_\beta^\alpha(x) \le v_\beta^\alpha(x) \le c_2\eta_\beta^\alpha(x)$ for all $x \in B^1(\mathbf{0})$

b) $c_3\eta_\beta^\alpha(x) \le \eta_\beta^\alpha(y) \le c_4\eta_\beta^\alpha(x)$ for all $x \in \boldsymbol{R}^N$, $y \in B_1(x)$.

Now we will prove the assertion of the Lemma 3.2:

$$\begin{aligned}(\mu_{-d}^{-c,-h} * \mu_{-b}^{-a,-g})(x) &= \int_{\boldsymbol{R}^N} \mu_{-d}^{-c,-h}(x-y)\mu_{-b}^{-a,-g}(y)\,\mathrm{d}y \\ &\le C_1 \int_{B_1(x)} v_0^{-h}(x-y)\mu_{-b}^{-a,-g}(y)\,\mathrm{d}y \\ &\quad + C_2 \int_{B^1(x)} \eta_{-d}^{-c}(x-y)\mu_{-b}^{-a,-g}(y)\,\mathrm{d}y.\end{aligned}$$

We will study these two integrals separately. By using Lemma 3.1 we get the estimate of $\int_{B_1(x)} v_0^{-h}(x-y)\mu_{-b}^{-a,-g}(y)\,\mathrm{d}y = \int_{B_1(\mathbf{0})} \mu_{-b}^{-a,-g}(x-y)v_0^{-h}(y)\,\mathrm{d}y$.

$$\begin{aligned}&\int_{B_1(\mathbf{0})} \mu_{-b}^{-a,-g}(x-y)v_0^{-h}(y)\,\mathrm{d}y \\ &\le \max_{y \in B_1(x)} \eta_{-b}^{-a+g}(y) \int_{B_1(\mathbf{0})} v_0^{-g}(x-y)v_0^{-h}(y)\,\mathrm{d}y \\ &\le C_3\eta_{-b}^{-a+g}(x) \begin{cases} \left.\begin{array}{ll} v_0^{-g-h+N}(x), & g+h > N \\ \ln_-|x|, & g+h = N \\ 1, & g+h < N \end{array}\right\} x \in B_1\backslash\{\mathbf{0}\} \\ \eta_0^{-g}(x) \qquad\qquad\qquad\quad x \in B^1 \end{cases}\end{aligned}$$

$$\leq C_5 \begin{cases} \mu_{-b}^{-a,-g-h+N}(x), & g+h>N \\ \mu_{-b}^{-a,-\delta}(x), \delta>0, & g+h=N \\ \mu_{-b}^{-a,0}(x), & g+h<N \end{cases} \leq C_6 \begin{cases} \mu_{-f}^{-e,-g-h+N}(x), & g+h>N \\ \mu_{-f}^{-e,-\delta}(x), \delta>0, & g+h=N \\ \mu_{-f}^{-e,0}(x) \equiv \eta_{-f}^{-e}(x), & g+h<N. \end{cases}$$

In the second inequality we use Lemma 3.1 and the relation b). In the last inequality we take into account the assumption $[e,f] \leq [a,b]$.

We estimate the remaining integral $\int_{B^1(x)} \eta_{-d}^{-c}(x-y)\mu_{-b}^{-a,-g}(y)\,\mathrm{d}y$ for $x \in \boldsymbol{R}^N \setminus \{\mathbf{0}\}$ in the following way:

$$\int_{B^1(x)} \eta_{-d}^{-c}(x-y)\mu_{-b}^{-a,-g}(y)\,\mathrm{d}y \leq \int_{\boldsymbol{R}^N} \eta_{-d}^{-c}(x-y)\mu_{-b}^{-a,-g}(y)\,\mathrm{d}y$$

$$= \int_{B_1} \eta_{-d}^{-c}(x-y)\nu_0^{-g}(y)\,\mathrm{d}y + \int_{B^1} \eta_{-d}^{-c}(x-y)\eta_{-b}^{-a}(y)\,\mathrm{d}y$$

$$\leq C_5 \max_{y \in B_1(x)} \eta_{-d}^{-c}(y) \int_{B_1(\mathbf{0})} \nu_0^{-g}(y)\,\mathrm{d}y + C_6\eta_{-f}^{-e}(x) \leq C_7\eta_{-f}^{-e}(x).$$

The proof of Lemma 3.2 follows from these estimates. $\square$

We now formulate main results of this section. We will use the following notation

$$\bar{\eta}_F^E(x;\lambda) = \eta_F^E(x;\lambda) \quad \text{if no logarithmic factor appears, and}$$

$$\bar{\eta}_F^E(x;\lambda) = \eta_F^E(x;\lambda) \cdot \begin{cases} P(\ln_+^{-1}|\lambda x|) \\ P(\ln_+^{-1} s(\lambda x)) \end{cases} \quad \text{if there are logarithmic factors,}$$

where function $P(\cdot)$ is a polynomial of the first or the second order, see also Remark 3.1. Similarly we define $\bar{\nu}_F^E(\cdot\,;\lambda)$. Then we have

Theorem 3.1. *Let $A+B^* > 1$. Let $f \in L^\infty(\boldsymbol{R}^3, \eta_B^A(\cdot\,;\lambda))$. Then we have $|\mathcal{O}| * f \in L^\infty(\boldsymbol{R}^3, \bar{\eta}_F^E(\cdot\,;\lambda))$, where*

$$E = \begin{cases} A-1 & \textit{for } A \leq B^*+1 \\ (A+B^*-1)/2 & \textit{for } A \geq B+1, A+B \leq 3 \\ 1 & \textit{for } A+B^* \geq 3 \end{cases} \tag{i}$$

$$E+F = \begin{cases} A+B^*-1 & \textit{for } A+B^* \leq 3 \\ 2 & \textit{for } A+B^* \geq 3 \end{cases} \tag{ii}$$

with logarithmic factors

$$\ln_+(\lambda|x|) \quad \textit{for} \begin{cases} A + B^* = 3 \\ A = B + 1, 0 \le B \le 1 \end{cases} \tag{iii}$$

$$\ln_+(\lambda s(x)) \quad \textit{for} \ A + B < 3, B \le 1, \tag{iv}$$

(*see Remark* 3.1). *Moreover we have*

$$\| |\mathcal{O}(\cdot\,;\lambda)| * f\|_{\infty,(\bar{\eta}_F^E(\cdot\,;\lambda)),\boldsymbol{R}^3} \le C\lambda^{-2}\|f\|_{\infty,(\eta_B^A(\cdot\,;\lambda)),\boldsymbol{R}^3}. \tag{3.1}$$

Let in addition for A, B *the following conditions be satisfied*

$$1 \le A < 3, \quad B > 0, \quad \textit{or} \ A \le B + 5, \quad 1 < A + B \le 3, \quad B \le 0. \tag{v}$$

Then for $f \in L^\infty(\boldsymbol{R}^3, v_B^A(\cdot\,;\lambda))$ *we have* $|\mathcal{O}(\cdot\,;\lambda)| * f \in L^\infty(\boldsymbol{R}^3, \bar{v}_F^E(\cdot\,;\lambda))$ *and*

$$\| |\mathcal{O}(\cdot\,;\lambda)| * f\|_{\infty,(\bar{v}_F^E(\cdot\,;\lambda)),\boldsymbol{R}^3} \le C\lambda^{-2+A-E}\|f\|_{\infty,(v_B^A(\cdot\,;\lambda)),\boldsymbol{R}^3}. \tag{3.2}$$

Remark 3.1. The inequalities (3.1), (3.2) must be understood in the following sense. If no logarithmic terms appear then

$$\| |\mathcal{O}(\cdot\,;\lambda)| * f\|_{\infty,(\eta_F^E(\cdot\,;\lambda)),\boldsymbol{R}^3} \le C\lambda^{-2}\|f\|_{\infty,(\eta_B^A(\cdot\,;\lambda)),\boldsymbol{R}^3}.$$

Analogously for the weight $v_F^E(\cdot\,;\lambda)$. But for $A + B^* = 3$ or $A = B + 1$, $0 \le B \le 1$ we have

$$\| |\mathcal{O}(\cdot\,;\lambda)| * f\|_{\infty,(\eta_F^E(\cdot\,;\lambda)P(\ln_+^{-1}(\lambda|\cdot|))),\boldsymbol{R}^3} \le C\lambda^{-2}\|f\|_{\infty,(\eta_B^A(\cdot\,;\lambda)),\boldsymbol{R}^3} \tag{3.3}$$

and for $A + B < 3$, $B \le 1$

$$\| |\mathcal{O}(\cdot\,;\lambda)| * f\|_{\infty,(\eta_F^E(\cdot\,;\lambda)P(\ln_+^{-1}(s(\lambda\cdot)))),\boldsymbol{R}^3} \le C\lambda^{-2}\|f\|_{\infty,(\eta_B^A(\cdot\,;\lambda)),\boldsymbol{R}^3}, \tag{3.4}$$

where $P(\cdot)$ is a polynomial. The order of the polynomial can be traced out from the proof of Theorem 3.1 using Tab. 1,2. Analogously for the weights $v_F^E(\cdot\,;\lambda)$. We can use instead of (3.3), (3.4) for $\varepsilon > 0$

$$\| |\mathcal{O}(\cdot\,;\lambda)| * f\|_{\infty,(\eta_F^{E-\varepsilon}(\cdot\,;\lambda)),\boldsymbol{R}^3} \le C\lambda^{-2}\|f\|_{\infty,(\eta_B^A(\cdot\,;\lambda)),\boldsymbol{R}^3}, \tag{3.3'}$$

$$\| |\mathcal{O}(\cdot\,;\lambda)| * f\|_{\infty,(\eta_{F-\varepsilon}^E(\cdot\,;\lambda)),\boldsymbol{R}^3} \le C\lambda^{-2}\|f\|_{\infty,(\eta_B^A(\cdot\,;\lambda)),\boldsymbol{R}^3}, \tag{3.4'}$$

respectively.

Finally, in the case of $f = 0$ in $B_{1/2}(\boldsymbol{0})$ (this is usually the case for $\Omega \subset \boldsymbol{R}^N$, exterior domain) we can get for the weight $v_B^A(\cdot\,;\lambda)$

$$\| |\mathcal{O}(\cdot\,;\lambda)| * f\|_{\infty,(v_F^{E-\varepsilon}(\cdot\,;\lambda)),\boldsymbol{R}^3} \le C\lambda^{-2+A-E+\varepsilon}\|f\|_{\infty,(v_B^A(\cdot\,;\lambda)),\boldsymbol{R}^3}.$$

The proof of Theorem 3.1 is similar to the proof of the other theorems in this section. We therefore give only the proof of Theorem 3.2 which is in applications the most often used one. The other theorems in both two- and three-dimensional cases can be shown by analogy with this proof.

Theorem 3.2. *Let $A+B>1/2$, $A>-1$. Then for $f\in L^\infty(\boldsymbol{R}^3,\eta_B^A(\cdot\,;\lambda))$ we have $|\nabla\mathcal{O}|*f\in L^\infty(\boldsymbol{R}^3,\bar{\eta}_F^E(\cdot\,;\lambda))$, where*

$$E=\begin{cases} A-1/2 & \text{for } A\le 2, A\le B+1, B\ge 0\\ 3/2 & \text{for } A+B^*\ge 3\\ A+B-1/2 & \text{for } B<0, A+B\le 1\\ (A+B)/2 & \text{for } B\le A-1, 1\le A+B\le 3\end{cases} \tag{i}$$

$$E+F=\begin{cases} A+B^* & \text{for } A\le 2, B\ge 3/2\\ A+B-1/2 & \text{for } A+B\le 7/2\\ 3 & \text{for } A+B\ge 7/2, A\ge 2\end{cases} \tag{ii}$$

with logarithmic factors

$$\ln_+(\lambda|x|)\quad \text{for}\ \begin{cases} A+B^*=3\\ A=B+1, 0\le B\le 1\\ A+B=1, B\le 0\\ B=-1/2, 3/4\le A\le 5/4.\end{cases} \tag{iii}$$

Moreover we have

$$\|\,|\nabla\mathcal{O}(\cdot\,;\lambda)|*f\|_{\infty,(\bar{\eta}_F^E(\cdot\,;\lambda)),\boldsymbol{R}^3}\le C\lambda^{-1}\|f\|_{\infty,(\eta_B^A(\cdot\,;\lambda)),\boldsymbol{R}^3}. \tag{3.5}$$

Let in addition for A, B the following conditions be satisfied

$$\frac{1}{2}\le A\le\frac{5}{2},\quad B\ge-\frac{1}{2},\quad A\le B+2. \tag{iv}$$

Then for $f\in L^\infty(\boldsymbol{R}^3,\nu_B^A(\cdot\,;\lambda))$ we have $|\nabla\mathcal{O}(\cdot\,;\lambda)|*f\in L^\infty(\boldsymbol{R}^3,\bar{\nu}_F^E(\cdot\,;\lambda))$ and

$$\|\,|\nabla\mathcal{O}(\cdot\,;\lambda)|*f\|_{\infty,(\bar{\nu}_F^E(\cdot\,;\lambda)),\boldsymbol{R}^3}\le C\lambda^{-1+A-E}\|f\|_{\infty,(\nu_B^A(\cdot\,;\lambda)),\boldsymbol{R}^3}. \tag{3.6}$$

Theorem 3.3. *Let $A+B^*>0$. Let $R=|\nabla^2\mathcal{O}-\nabla^2\mathcal{S}|$ or $R=|\partial_1\mathcal{O}|$. Then for $f\in L^\infty(\boldsymbol{R}^3,\eta_B^A(\cdot\,;\lambda))$ we have $R*f\in L^\infty(\boldsymbol{R}^3,\bar{\eta}_F^E(\cdot\,;\lambda))$, where*

$$E=\begin{cases} A & \text{for } -1<A\le 2, A\le B+1, B\ge 0\\ 2 & \text{for } A+B^*\ge 3\\ A+B & \text{for } B\le 0, 0<A+B<1\\ (A+B+1)/2 & \text{for } B\le A-1, 1\le A+B\le 3\end{cases} \tag{i}$$

$$E + F = \begin{cases} A + B^* & \textit{for } A + B^* \le 3 \\ 3 & \textit{for } A + B^* \ge 3 \end{cases} \tag{ii}$$

with logarithmic factors

$$\ln_+(\lambda|x|) \quad \textit{for } A + B^* \le 3. \tag{iii}$$

Moreover we have

$$\| |\nabla^2 \mathcal{O}(\cdot\,;\lambda) - \nabla^2 \mathcal{S}(\cdot)| * f\|_{\infty,(\bar{\eta}_F^E(\cdot\,;\lambda)),\boldsymbol{R}^3} \le C\|f\|_{\infty,(\eta_B^A(\cdot\,;\lambda)),\boldsymbol{R}^3}, \tag{3.7}$$

$$\| |\partial_1 \mathcal{O}(\cdot\,;\lambda)| * f\|_{\infty,(\bar{\eta}_F^E(\cdot\,;\lambda)),\boldsymbol{R}^3} \le C\lambda^{-1}\|f\|_{\infty,(\eta_B^A(\cdot\,;\lambda)),\boldsymbol{R}^3}. \tag{3.8}$$

Let in addition for A, B the following conditions be satisfied

$$0 \le A < 3, \quad B \ge -1, \quad A < B + 3. \tag{iv}$$

Then for $f \in L^\infty(\boldsymbol{R}^3, v_B^A(\cdot\,;\lambda))$ *we have* $R * f \in L^\infty(\boldsymbol{R}^3, \bar{v}_F^E(\cdot\,;\lambda))$ *and*

$$\| |\nabla^2 \mathcal{O}(\cdot\,;\lambda) - \nabla^2 \mathcal{S}(\cdot)| * f\|_{\infty,(\bar{v}_F^E(\cdot\,;\lambda)),\boldsymbol{R}^3} \le C\lambda^{A-E}\|f\|_{\infty,(v_B^A(\cdot\,;\lambda)),\boldsymbol{R}^3}, \tag{3.9}$$

$$\| |\partial_1 \mathcal{O}(\cdot\,;\lambda)| * f\|_{\infty,(\bar{v}_F^E(\cdot\,;\lambda)),\boldsymbol{R}^3} \le C\lambda^{-1+A-E}\|f\|_{\infty,(v_B^A(\cdot\,;\lambda)),\boldsymbol{R}^3}. \tag{3.10}$$

THEOREM 3.4. *Let* $A + B^* > 1$. *Then for* $f \in L^\infty(\boldsymbol{R}^3, \eta_B^A(\cdot\,;\lambda))$ *we have* $|\mathcal{P}| * f \in L^\infty(\boldsymbol{R}^3, \bar{\eta}_F^E(\cdot\,;\lambda))$, *where*

$$E = \begin{cases} 2 & \textit{for } A + B^* \ge 3, A \ge 5/2 \\ A - 1/2 & \textit{for } A \le 5/2, B \ge 1/2 \\ A + B^* - 1 & \textit{for } B \le 1/2, A + B^* \le 3 \end{cases} \tag{i}$$

$$E + F = \begin{cases} 2 & \textit{for } A + B^* \ge 3 \\ A + B^* - 1 & \textit{for } A + B^* \le 3 \end{cases} \tag{ii}$$

with logarithmic factors

$$\ln_+(\lambda|x|) \quad \textit{for } \begin{cases} B = 1/2, 1/2 < A < 5/2 \\ B = 1, 2 \le A \le 5/2 \\ A + B^* = 3, A \ge 5/2 \end{cases} \tag{iii}$$

$$\ln_+(\lambda s(x)) \quad \textit{for } \begin{cases} A + B^* = 3, 2 \le A < 5/2 \\ B = 1, 0 < A < 2. \end{cases} \tag{iv}$$

Moreover we have

$$\| |\mathcal{P}| * f\|_{\infty,(\bar{\eta}_F^E(\cdot\,;\lambda)),\boldsymbol{R}^3} \le C\lambda^{-1}\|f\|_{\infty,(\eta_B^A(\cdot\,;\lambda)),\boldsymbol{R}^3}. \tag{3.11}$$

Let in addition for A, B the following conditions be satisfied

$$\frac{1}{2} \leq A < 3, \quad B \geq 0. \tag{v}$$

Then for $f \in L^\infty(\boldsymbol{R}^3, \nu_B^A(\cdot\,;\lambda))$ *we have* $|\mathscr{P}| * f \in L^\infty(\boldsymbol{R}^3, \bar{\nu}_F^E(\cdot\,;\lambda))$ *and*

$$\|\,|\mathscr{P}| * f\|_{\infty,(\bar{\nu}_F^E(\cdot\,;\lambda)),\boldsymbol{R}^3} \leq C\lambda^{-1+A-E}\|f\|_{\infty,(\nu_B^A(\cdot\,;\lambda)),\boldsymbol{R}^3}. \tag{3.12}$$

In the next part of this section we give formulations of the theorems in the two-dimensional case:

Theorem 3.5. *Let* $A + B^* > 1$. *Then for* $f \in L^\infty(\boldsymbol{R}^2, \eta_B^A(\cdot\,;\lambda))$ *we have* $|\mathcal{O}_{11}| * f \in L^\infty(\boldsymbol{R}^2, \bar{\eta}_F^E(\cdot\,;\lambda))$, *where*

$$E = \begin{cases} A-1 & \text{for } A \leq B^* + 1 \\ (1/2)(A+B-1) & \text{for } A \geq B+1, A+B \leq 2 \\ 1/2 & \text{for } A + B^* \geq 2 \end{cases} \tag{i}$$

$$E + F = \begin{cases} 1 & \text{for } A + B^* \geq 2 \\ A + B^* - 1 & \text{for } A + B^* \leq 2 \end{cases} \tag{ii}$$

with logarithmic factors

$$\ln_+(\lambda|x|) \quad \text{for} \quad \begin{cases} A + B^* = 2 \\ A = B + 1, 0 < B \leq 1/2 \end{cases} \tag{iii}$$

$$\ln_+(\lambda s(x)) \quad \text{for} \quad \{A + B < 2, B \leq 1/2. \tag{iv}$$

Moreover we have

$$\|\,|\mathcal{O}_{11}(\cdot\,;\lambda)| * f\|_{\infty,(\bar{\eta}_F^E(\cdot\,;\lambda)),\boldsymbol{R}^2} \leq C\lambda^{-2}\|f\|_{\infty,(\eta_B^A(\cdot\,;\lambda)),\boldsymbol{R}^2}. \tag{3.13}$$

Let in addition for A the following conditions be satisfied

$$1 \leq A < 2. \tag{v}$$

Then for $f \in L^\infty(\boldsymbol{R}^2, \nu_B^A(\cdot\,;\lambda))$ *we have* $|\mathcal{O}_{11}(\cdot\,;\lambda)| * f \in L^\infty(\boldsymbol{R}^2, \bar{\nu}_F^E(\cdot\,;\lambda))$ *and*

$$\|\,|\mathcal{O}_{11}(\cdot\,;\lambda)| * f\|_{\infty,(\bar{\nu}_F^E(\cdot\,;\lambda)),\boldsymbol{R}^2} \leq C\lambda^{-2+A-E}\|f\|_{\infty,(\nu_B^A(\cdot\,;\lambda)),\boldsymbol{R}^2}. \tag{3.14}$$

Theorem 3.6. *Let* $A + B^* > 1$, $i, j = 1, 2$, $i \cdot j \neq 1$, $R = |\mathcal{O}_{ij}|$ *or* $R = |\mathscr{P}|$. *Then for* $f \in L^\infty(\boldsymbol{R}^2, \eta_B^A(\cdot\,;\lambda))$ *we have* $R * f \in L^\infty(\boldsymbol{R}^2, \bar{\eta}_F^E(\cdot\,;\lambda))$, *where*

$$E = E + F = \begin{cases} 1 & \text{for } A + B^* \geq 2 \\ A + B^* - 1 & \text{for } A + B^* \leq 2 \end{cases} \tag{i}$$

with logarithmic factors

$$\ln_+(\lambda|x|) \quad \textit{for} \ \begin{cases} A+B^* = 2 \\ A+B^* \le 2, B \ge 1/2. \end{cases} \tag{ii}$$

Moreover, we have

$$\| \, |\mathcal{O}_{ij}(\cdot\,;\lambda)| * f \|_{\infty,(\bar{\eta}_F^E(\cdot\,;\lambda)),\boldsymbol{R}^2} \le C\lambda^{-2} \|f\|_{\infty,(\eta_B^A(\cdot\,;\lambda)),\boldsymbol{R}^2}, \tag{3.15}$$

$$\| \, |\mathcal{P}| * f \|_{\infty,(\bar{\eta}_F^E(\cdot\,;\lambda)),\boldsymbol{R}^2} \le C\lambda^{-1} \|f\|_{\infty,(\eta_B^A(\cdot\,;\lambda)),\boldsymbol{R}^2}. \tag{3.16}$$

Let in addition for A, B the following conditions be satisfied

$$A < 2, \quad B^* \ge 0 \quad (\textit{if} \ \ R = |\mathcal{P}|) \quad \textit{or} \quad A < 2, \quad (\textit{if} \ \ R = |\mathcal{O}_{ij}|). \tag{iii}$$

Then for $f \in L^\infty(\boldsymbol{R}^2, v_B^A(\cdot\,;\lambda))$ *we have* $R * f \in L^\infty(\boldsymbol{R}^2, \bar{v}_F^E(\cdot\,;\lambda))$ *and*

$$\| \, |\mathcal{O}_{ij}(\cdot\,;\lambda)| * f \|_{\infty,(\bar{v}_F^E(\cdot\,;\lambda)),\boldsymbol{R}^2} \le C\lambda^{-2+A-E} \|f\|_{\infty,(v_B^A(\cdot\,;\lambda)),\boldsymbol{R}^2}, \tag{3.17}$$

$$\| \, |\mathcal{P}| * f \|_{\infty,(\bar{v}_F^E(\cdot\,;\lambda)),\boldsymbol{R}^2} \le C\lambda^{-1+A-E} \|f\|_{\infty,(v_B^A(\cdot\,;\lambda)),\boldsymbol{R}^2}. \tag{3.18}$$

Theorem 3.7. *Let* $A + B^* > 0$ *and* $A + B > 1/2$. *Then for* $f \in L^\infty(\boldsymbol{R}^2, \eta_B^A(\cdot\,;\lambda))$ *we have* $|\partial_2 \mathcal{O}_{11}(\cdot\,;\lambda)| * f \in L^\infty(\boldsymbol{R}^2, \bar{\eta}_F^E(\cdot\,;\lambda))$, *where*

$$E = \begin{cases} 1 & \textit{for} \ \ A + B^* \ge 2 \\ A - 1/2 & \textit{for} \ \ -1/2 < A \le 3/2, B \ge 0, A \le B+1 \\ (A+B^*)/2 & \textit{for} \ \ 1 \le A + B \le 2, A \ge B + 1 \\ A + B - 1/2 & \textit{for} \ \ B \le 0, A + B \le 1 \end{cases} \tag{i}$$

$$E + F = \begin{cases} 2 & \textit{for} \ \ A + B \ge 5/2, A \ge 3/2 \\ A + B^* & \textit{for} - 1/2 < A \le 3/2, B \ge 1 \\ A + B - 1/2 & \textit{for} \ \ A + B \le 5/2, B \le 1 \end{cases} \tag{ii}$$

with logarithmic factors

$$\ln_+(\lambda|x|) \quad \textit{for} \ \begin{cases} A + B^* = 2 \\ B = A - 1, 0 \le B \le 1/2 \\ A + B = 1, B \le 0 \end{cases} \tag{iii}$$

$$\ln_+(\lambda s(x)) \quad \textit{for} \ \ A + B = 1, 0 < B \le 1. \tag{iv}$$

Moreover, we have

$$\| \, |\partial_2 \mathcal{O}_{11}(\cdot\,;\lambda)| * f \|_{\infty,(\bar{\eta}_F^E(\cdot\,;\lambda)),\boldsymbol{R}^2} \le C\lambda^{-1} \|f\|_{\infty,(\eta_B^A(\cdot\,;\lambda)),\boldsymbol{R}^2}. \tag{3.19}$$

Let in addition for A, B the following conditions be satisfied

$$\frac{1}{2} < A < 2, \quad B \ge -\frac{1}{2}, \quad B \ge A - 2. \tag{v}$$

Then for $f \in L^\infty(\boldsymbol{R}^2, v_B^A(\cdot\,;\lambda))$ *we have* $|\partial_2\mathcal{O}_{11}(\cdot\,;\lambda)| * f \in L^\infty(\boldsymbol{R}^2, \bar{v}_F^E(\cdot\,;\lambda))$ *and*

$$\| |\partial_2\mathcal{O}_{11}(\cdot\,;\lambda)| * f\|_{\infty,(\bar{v}_F^E(\cdot\,;\lambda)),\boldsymbol{R}^2} \le C\lambda^{-1+A-E}\|f\|_{\infty,(v_B^A(\cdot\,;\lambda)),\boldsymbol{R}^2}. \tag{3.20}$$

Theorem 3.8. *Let* $A + B^* > 0$ *and* $R = |\nabla^2\mathcal{O} - \nabla^2\mathcal{S}|$ *or* $R = |\partial_k\mathcal{O}_{ij}|$, $i, j, k = 1, 2$, *except* $|\partial_2\mathcal{O}_{11}|$. *Then for* $f \in L^\infty(\boldsymbol{R}^2, \eta_B^A(\cdot\,;\lambda))$ *we have* $R * f \in L^\infty(\boldsymbol{R}^2, \bar{\eta}_F^E(\cdot\,;\lambda))$, *where*

$$E = \begin{cases} 3/2 & \text{for } A + B^* \ge 2 \\ A & \text{for } A \le 3/2, B \ge 0, A \le B + 1 \\ (A + B + 1)/2 & \text{for } 1 \le A + B \le 2, A \ge B + 1 \\ A + B & \text{for } B \le 0, A + B \le 1 \end{cases} \tag{i}$$

$$E + F = \begin{cases} 2 & \text{for } A + B^* \ge 2 \\ A + B^* & \text{for } A + B^* \le 2, \end{cases} \tag{ii}$$

with logarithmic factors

$$\ln_+(\lambda|x|) \quad \text{for } A + B^* \le 2. \tag{iii}$$

Moreover, we have

$$\| |\nabla^2\mathcal{O}(\cdot\,;\lambda) - \nabla^2\mathcal{S}(\cdot)| * f\|_{\infty,(\bar{\eta}_F^E(\cdot\,;\lambda)),\boldsymbol{R}^2} \le C\|f\|_{\infty,(\eta_B^A(\cdot\,;\lambda)),\boldsymbol{R}^2}, \tag{3.21}$$

$$\| |\partial_k\mathcal{O}_{ij}(\cdot\,;\lambda)| * f\|_{\infty,(\bar{\eta}_F^E(\cdot\,;\lambda)),\boldsymbol{R}^2} \le C\lambda^{-1}\|f\|_{\infty,(\eta_B^A(\cdot\,;\lambda)),\boldsymbol{R}^2}. \tag{3.22}$$

Let in addition for A, B *the following conditions be satisfied*

$$0 \le A < 2, \quad B \ge -1. \tag{iv}$$

Then for $f \in L^\infty(\boldsymbol{R}^2, v_B^A(\cdot\,;\lambda))$ *we have* $R * f \in L^\infty(\boldsymbol{R}^2, \bar{v}_F^E(\cdot\,;\lambda))$ *and*

$$\| |\nabla^2\mathcal{O}(\cdot\,;\lambda) - \nabla^2\mathcal{S}(\cdot)| * f\|_{\infty,(\bar{v}_F^E(\cdot\,;\lambda)),\boldsymbol{R}^2} \le C\lambda^{A-E}\|f\|_{\infty,(v_B^A(\cdot\,;\lambda)),\boldsymbol{R}^2}, \tag{3.23}$$

$$\| |\partial_k\mathcal{O}_{ij}(\cdot\,;\lambda)| * f\|_{\infty,(\bar{v}_F^E(\cdot\,;\lambda)),\boldsymbol{R}^2} \le C\lambda^{-1+A-E}\|f\|_{\infty,(v_B^A(\cdot\,;\lambda)),\boldsymbol{R}^2}. \tag{3.24}$$

Proof of Theorem 3.2. *Let* $f \in L^\infty(\boldsymbol{R}^3, \eta_B^A(\cdot\,;1))$. Recalling that $|\nabla\mathcal{O}(x - y; 1)| \le C_0\mu_{-3/2}^{-3/2,-2}(x - y; 1)$ we have

$$\begin{aligned} | |\nabla\mathcal{O}(\cdot\,;1)| * f(x)| &\le C_1\mu_{-3/2}^{-3/2,-2}(\cdot\,;1) * \eta_{-B}^{-A}(\cdot\,;1)(x) \\ &\le C_2\eta_{-3/2}^{-3/2}(\cdot\,;1) * \eta_{-B}^{-A}(\cdot\,;1)(x). \end{aligned} \tag{3.25}$$

We have therefore to study the convolution (3.25); we apply Tab. 1 and Tab. 2 with $c = d = 3/2$, $a = A$, $b = B$ and we get, under the condition $A + B^* > 1$, that (we skip the logarithmic factors, for the moment)

$$(\eta^{-3/2}_{-3/2}(\cdot\,;1) * \eta^{-A}_{-B}(\cdot\,;1))(x) \le C\eta^{-E}_{-F}(x;1)$$

with

$$\begin{aligned} E &\le \min(3/2, (A+B^*)/2, A-1/4, A-1/2, A+B-1/2, (A+B)/2, A+B^*) \\ &= \min(3/2, (A+B^*)/2, A-1/2, A+B-1/2) \qquad (3.26) \end{aligned}$$

$$E + F \le \min\left(3, A+B^*, A+B-\frac{1}{2}\right).$$

We therefore easily get (i) and (ii), see Fig. 2 and Fig. 3 below:

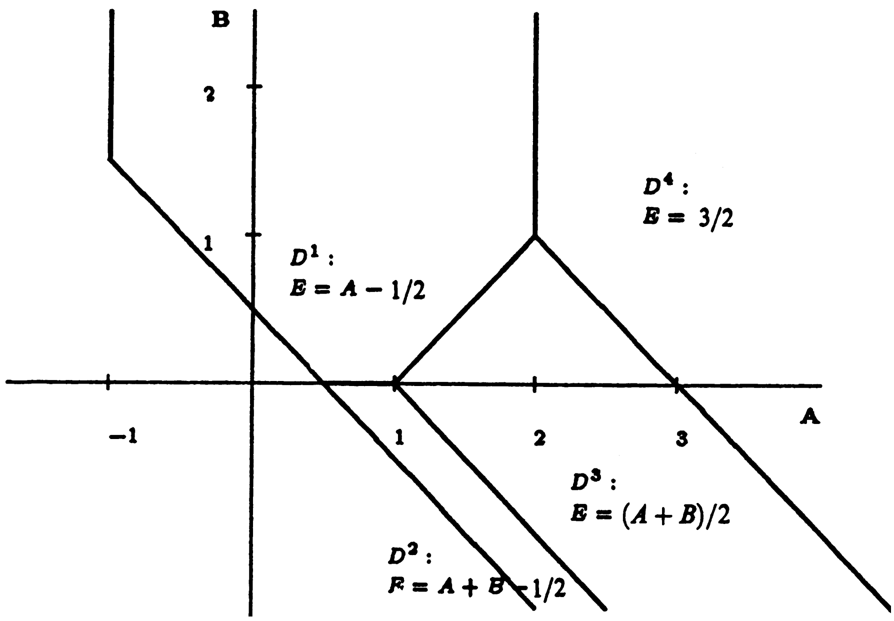

Fig. 2

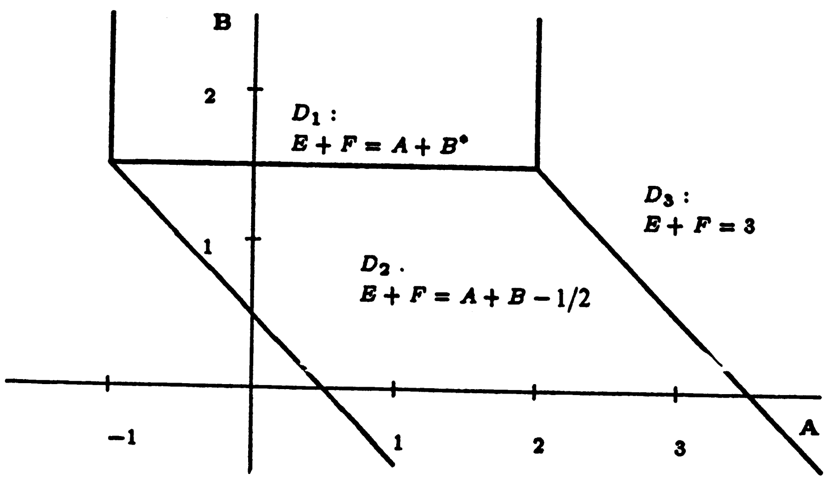

Fig. 3

Let us now regard the logarithmic factors. From Ω_0 we have $\ln_+(\lambda|x|)$ whenever $B = 1$, $A \le 2$ or $A + B^* = 3$ and $e_0 = 3/2 + (1/2)\min(0, A + B^* - 3)$, $e_0 + f_0 = 3 + \min(0, A + B^* - 3)$. Therefore, if $A + B^* = 3$ the factor $\ln_+(\lambda|x|)$ must be taken into account. But for $B = 1$, $-1 < A < 2$ we have $e_0 = (A+1)/2 > A - 1/2$, $e_0 + f_0 = A + B^* > A + B - 1/2$ and therefore, there are in fact no logarithmic factors.

Next in Ω_3 we have $\ln_+(\lambda|x|/(1 + \lambda s(x)))$ for $A = 1 + B$. But if $B \ge 0$ then $e_3 = A - 1/2$ and $e_3 + f_3 = A + B - 1/2$ and then for $0 \le B \le 1$ the factor $\ln_+(\lambda|x|)$ must be taken into account. But for $B < 0$ we have $e_3 > A + B - 1/2$ and for $B > 1$ we have $e_3 > 3/2$. Thus (as if $s(x) \sim |x|$ then $\lambda|x|/(1 + \lambda s(x)) \sim 1$) we can disregard the logarithmic factors in these situations.

Analogously we proceed in other sub-domains and get (iii). The estimate (3.5) for $\lambda = 1$ is therefore shown. In order to show (3.5) for $\lambda \ne 1$, let us recall the homogeneity property of $\mathcal{O}_{ij}(x - y; \lambda)$. Namely, for $N = 3$ we have $\mathcal{O}_{ij}(x - y; \lambda) = \lambda \mathcal{O}_{ij}(\lambda(x - y); 1)$ and therefore

$$\left| \int_{\boldsymbol{R}^3} |\nabla \mathcal{O}(x - y; \lambda)| f(y)\, \mathrm{d}y \right| = \lambda^{-1} \left| \int_{\boldsymbol{R}^3} |\nabla \mathcal{O}(\lambda x - z; 1)| f\left(\frac{z}{\lambda}\right) \mathrm{d}z \right|$$
$$\le \lambda^{-1} \sup_{y \in \boldsymbol{R}^3} |f(y) \eta_B^A(\lambda y; 1)| \eta_{-F}^{-E}(\lambda x; 1) P_1(\ln_+(\lambda|x|)) P_2(\ln_+ s(\lambda x))$$

and so, as $\eta_B^A(\lambda x; 1) = \eta_B^A(x; \lambda)$, we have (3.5).

Let us study the weight $v_B^A(x; \lambda)$. From Lemma 3.2 we have the following conditions $E \ge \max(0, A - 1)$ and $A < 3$ and therefore we get on D^1 that $A \ge 1/2$, on D^2 that $A + B - 1/2 \ge A - 1$, i.e. $B \ge -1/2$, on D^3 that $(A + B)/2 \ge A - 1$, i.e. $A \le B + 2$ and on D^4 that $A \le 5/2$.

Finally, to show (3.6) we proceed as in the case of the estimate (3.5). Evidently, (3.6) holds for $\lambda = 1$. Therefore

$$\left| \int_{\boldsymbol{R}^3} |\nabla \mathcal{O}(x - y; \lambda)| f(y)\, \mathrm{d}y \right|$$
$$\le \lambda^{-1} \sup_{y \in \boldsymbol{R}^3} |f(y) v_B^A(\lambda y; 1)| v_{-F}^{-E}(\lambda x; 1) P_1(\ln_+(\lambda|x|)) P_2(\ln_+ s(\lambda x))$$
$$= \lambda^{-1+A-E} \|f\|_{\infty, (v_B^A(\cdot;\lambda)), \boldsymbol{R}^3} v_{-F}^{-E}(x; \lambda) P_1(\ln_+(\lambda|x|)) P_2(\ln_+ s(\lambda x)). \qquad \square$$

4. L^p-estimates for weakly singular Oseen kernels.

This section is devoted to the L^p-estimates of convolutions with Oseen kernels. Here we shall use the results from the previous section, i.e. the L^∞-theory. The proofs are again similar each to other and we give it at the end of this section only for Theorem 4.3 for $R = |\nabla^2 \mathcal{O} - \nabla^2 \mathcal{S}|$ which is in application the most often used case.

Theorem 4.1. *Let T be an integral operator with the kernel $|\mathcal{O}|$, $T : f \mapsto |\mathcal{O}| * f$ and let $1 < p < \infty$. Then T is a well defined continuous operator*:

a) $$L^p(\boldsymbol{R}^3; \eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \mapsto L^p(\boldsymbol{R}^3; \eta_\beta^{\alpha-p/2-\varepsilon}(\cdot\,;\lambda))$$

for $-\varepsilon(p-1)/p < \beta < p-1+\varepsilon(p-1)/p$, $p/2-1-\varepsilon/p < \alpha+\beta < 5p/2-3+\varepsilon$, $\alpha-\beta < p/2-1+\varepsilon+\varepsilon(p-1)/p$, $-p/2-\varepsilon/p < \alpha < 3p/2-2+\varepsilon$, $0 < \varepsilon \le p$

b) $$L^p(\boldsymbol{R}^3; \nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \mapsto L^p(\boldsymbol{R}^3; \nu_\beta^{\alpha-p/2-\varepsilon}(\cdot\,;\lambda))$$

for $-\varepsilon(p-1)/p < \beta < p-1+\varepsilon(p-1)/p$, $p/2-1-\varepsilon/p < \alpha+\beta < 5p/2-3+\varepsilon$, $\alpha-\beta < p/2-1+\varepsilon+\varepsilon(p-1)/p$, $\max\{-p/2-\varepsilon/p, p/2-3+\varepsilon\} < \alpha < \min\{3p/2-2+\varepsilon, 5p/2-3\}$, $0 < \varepsilon \le p$.

Moreover we have for α, β specified in a) *and* b), *respectively*

a)

$$\| |\mathcal{O}(\cdot\,;\lambda)| * f\|_{p,(\eta_\beta^{\alpha-p/2-\varepsilon}(\cdot\,;\lambda)),\boldsymbol{R}^3} \le C\lambda^{-2}\|f\|_{p,(\eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\boldsymbol{R}^3}, \tag{4.1}$$

b)

$$\| |\mathcal{O}(\cdot\,;\lambda)| * f\|_{p,(\nu_\beta^{\alpha-p/2-\varepsilon}(\cdot\,;\lambda)),\boldsymbol{R}^3} \le C\lambda^{\varepsilon/p-1}\|f\|_{p,(\nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\boldsymbol{R}^3}. \tag{4.2}$$

Theorem 4.2. *Let T be an integral operator with the kernel $|\nabla\mathcal{O}|$, $T : f \mapsto |\nabla\mathcal{O}| * f$, and let $1 < p < \infty$. Then T is a well defined continuous operator:*

a) $$L^p(\boldsymbol{R}^3; \eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \mapsto L^p(\boldsymbol{R}^3; \eta_\beta^{\alpha}(\cdot\,;\lambda))$$

for $0 < \beta < 3p/2-3/2$, $-1 < \alpha+\beta$, $\alpha < 3p/2-2$, $\alpha-\beta < p/2-1$

b) $$L^p(\boldsymbol{R}^3; \nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \mapsto L^p(\boldsymbol{R}^3; \nu_\beta^{\alpha}(\cdot\,;\lambda))$$

for $0 < \beta < 3p/2-3/2$, $-1 < \alpha+\beta$, $-3 < \alpha < 3p/2-2$, $\alpha-\beta < p/2-1$.

Moreover we have for α, β specified in a) *and* b), *respectively*

a)

$$\| |\nabla\mathcal{O}(\cdot\,;\lambda)| * f\|_{p,(\eta_\beta^{\alpha}(\cdot\,;\lambda)),\boldsymbol{R}^3} \le C\lambda^{-1}\|f\|_{p,(\eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\boldsymbol{R}^3}, \tag{4.3}$$

b)

$$\| |\nabla\mathcal{O}(\cdot\,;\lambda)| * f\|_{p,(\nu_\beta^{\alpha}(\cdot\,;\lambda)),\boldsymbol{R}^3} \le C\lambda^{-1/2}\|f\|_{p,(\nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\boldsymbol{R}^3}. \tag{4.4}$$

Theorem 4.3. *Let $R = |\nabla^2\mathcal{O} - \nabla^2\mathcal{S}|$ or $R = |\partial_1\mathcal{O}|$. Let T be an integral operator with the kernel R, $T : f \mapsto R * f$, and let $1 < p < \infty$. Then T is a well defined continuous operator*:

a) $$L^p(\boldsymbol{R}^3; \eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \mapsto L^p(\boldsymbol{R}^3; \eta_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda))$$

for $-\varepsilon(p-1)/p < \beta < p-1+\varepsilon(p-1)/p$, $-p/2-1-\varepsilon/p < \alpha+\beta < 5p/2-3+\varepsilon$, $\alpha-\beta < p/2-1+\varepsilon+\varepsilon(p-1)/p$, $-3p/2-\varepsilon/p < \alpha < 3p/2-2+\varepsilon$, $0 < \varepsilon \le p$

b) $$L^p(\boldsymbol{R}^3; v_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \mapsto L^p(\boldsymbol{R}^3; v_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda))$$

for $-\varepsilon(p-1)/p < \beta < p-1+\varepsilon(p-1)/p$, $-p/2-1-\varepsilon/p < \alpha+\beta < 5p/2-3+\varepsilon$, $\alpha-\beta < p/2-1+\varepsilon+\varepsilon(p-1)/p$, $\max\{-3p/2-\varepsilon/p, -p/2-3+\varepsilon\} < \alpha < \min\{3p/2-2+\varepsilon, 5p/2-3\}$, $0 < \varepsilon \le p$.

Moreover we have for α, β specified in a) *and* b) *respectively*

a)

$$\|\,|\nabla^2\mathcal{O}(\cdot\,;\lambda) - \nabla^2\mathcal{S}(\cdot)| * f\|_{p,(\eta_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\boldsymbol{R}^3} \le C\|f\|_{p,(\eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\boldsymbol{R}^3}, \tag{4.5}$$

$$\|\,|\partial_1\mathcal{O}(\cdot\,;\lambda)| * f\|_{p,(\eta_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\boldsymbol{R}^3} \le C\lambda^{-1}\|f\|_{p,(\eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\boldsymbol{R}^3}, \tag{4.6}$$

b)

$$\|\,|\nabla^2\mathcal{O}(\cdot\,;\lambda) - \nabla^2\mathcal{S}(\cdot)| * f\|_{p,(v_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\boldsymbol{R}^3} \le C\lambda^{\varepsilon/p}\|f\|_{p,(v_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\boldsymbol{R}^3} \tag{4.7}$$

$$\|\,|\partial_1\mathcal{O}(\cdot\,;\lambda)| * f\|_{p,(v_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\boldsymbol{R}^3} \le C\lambda^{\varepsilon/p-1}\|f\|_{p,(v_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\boldsymbol{R}^3}. \tag{4.8}$$

Theorem 4.4. *Let T be an integral operator with the kernel $|\mathcal{P}|$, $T : f \mapsto |\mathcal{P}| * f$, $i = 1,2,3$, and let $1 < p < \infty$. Then T is a well defined continuous operator:*

a) $$L^p(\boldsymbol{R}^3; \eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \mapsto L^p(\boldsymbol{R}^3; \eta_\beta^{\alpha-p/2}(\cdot\,;\lambda))$$

for $0 < \beta < p-1$, $p/2-3 < \alpha+\beta < 5p/2-3$

b) $$L^p(\boldsymbol{R}^3; v_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \mapsto L^p(\boldsymbol{R}^3; \eta_\beta^{\alpha-p/2}(\cdot\,;\lambda))$$

for $0 < \beta < p-1$, $p/2-3 < \alpha+\beta < 5p/2-3$, $p/2-3 < \alpha < 5p/2-3$.

Moreover we have for α, β specified in a) *and* b), *respectively*

a)

$$\|\,|\mathcal{P}| * f\|_{p,(\eta_\beta^{\alpha-p/2}(\cdot\,;\lambda)),\boldsymbol{R}^3} \le C\lambda^{-1}\|f\|_{p,(\eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\boldsymbol{R}^3}, \tag{4.9}$$

b)

$$\| |\mathscr{P}| * f\|_{p,(v_\beta^{\alpha-p/2}(\cdot;\lambda)),\boldsymbol{R}^3} \le C\|f\|_{p,(v_\beta^{\alpha+p/2}(\cdot;\lambda)),\boldsymbol{R}^3}. \tag{4.10}$$

Next we formulate analogous results also in the two-dimensional case.

THEOREM 4.5. *Let T be an integral operator with the kernel $|\mathscr{O}_{11}| T : f \mapsto |\mathscr{O}_{11}| * f$, and let $1 < p < \infty$. Then T is a well defined continuous operator:*

a) $$L^p(\boldsymbol{R}^2; \eta_\beta^{\alpha+p/2}(\cdot;\lambda)) \mapsto L^p(\boldsymbol{R}^2; \eta_\beta^{\alpha-p/2-\varepsilon}(\cdot;\lambda))$$

for $-\varepsilon(p-1)/p < \beta < p/2 - 1/2 + \varepsilon(p-1)/p$, $p/2 - 1 - \varepsilon/p < \alpha + \beta < 3p/2 - 2 + \varepsilon$, $\alpha - \beta < p/2 - 1 + \varepsilon + \varepsilon(p-1)/p$, $-1/2 - \varepsilon/p < \alpha < p - 3/2 + \varepsilon$, $0 < \varepsilon \le p/2$

b) $$L^p(\boldsymbol{R}^2; v_\beta^{\alpha+p/2}(\cdot;\lambda)) \mapsto L^p(\boldsymbol{R}^2; v_\beta^{\alpha-p/2-\varepsilon}(\cdot;\lambda))$$

for $-\varepsilon(p-1)/p < \beta < p/2 - 1/2 + \varepsilon(p-1)/p$, $p/2 - 1 - \varepsilon/p < \alpha + \beta < 3p/2 - 2 + \varepsilon$, $\alpha - \beta < p/2 - 1 + \varepsilon + \varepsilon(p-1)/p$, $\max\{-1/2 - \varepsilon/p, p/2 - 2 + \varepsilon\} < \alpha < \min\{p - 3/2 + \varepsilon, 3p/2 - 2\}$, $0 < \varepsilon \le p/2$.

Moreover we have for α, β specified in a) *and* b), *respectively*

a)

$$\| |\mathscr{O}_{11}(\cdot;\lambda)| * f\|_{p,(\eta_\beta^{\alpha-p/2-\varepsilon}(\cdot;\lambda)),\boldsymbol{R}^2} \le C\lambda^{-2}\|f\|_{p,(\eta_\beta^{\alpha+p/2}(\cdot;\lambda)),\boldsymbol{R}^2}, \tag{4.11}$$

b)

$$\| |\mathscr{O}_{11}(\cdot;\lambda)| * f\|_{p,(v_\beta^{\alpha-p/2-\varepsilon}(\cdot;\lambda)),\boldsymbol{R}^2} \le C\lambda^{\varepsilon/p-1}\|f\|_{p,(v_\beta^{\alpha+p/2}(\cdot;\lambda)),\boldsymbol{R}^2}. \tag{4.12}$$

THEOREM 4.6. *Let T be an integral operator with the kernel $R = |\mathscr{O}_{ij}|$, $i, j = 1, 2$, $i \cdot j \neq 1$ or $R = |\mathscr{P}|$; $T : f \mapsto R * f$, and let $1 < p < \infty$. Then T is a well defined continuous operator*:

a) $$L^p(\boldsymbol{R}^2; \eta_\beta^{\alpha+p/2}(\cdot;\lambda)) \mapsto L^p(\boldsymbol{R}^2; \eta_\beta^{\alpha-p/2}(\cdot;\lambda))$$

for $0 < \beta < p/2 - 1/2$, $p/2 - 2 < \alpha + \beta < 3p/2 - 2$

b) $$L^p(\boldsymbol{R}^2; v_\beta^{\alpha+p/2}(\cdot;\lambda)) \mapsto L^p(\boldsymbol{R}^2; v_\beta^{\alpha-p/2}(\cdot;\lambda))$$

for $0 < \beta < p/2 - 1/2$, $p/2 - 2 < \alpha + \beta < 3p/2 - 2$, $p/2 - 2 < \alpha < 3p/2 - 2$.

Moreover we have for α, β specified in a) *and* b), *respectively*

a)

$$\| |\mathscr{O}_{ij}(\cdot;\lambda)| * f\|_{p,(\eta_\beta^{\alpha-p/2}(\cdot;\lambda)),\boldsymbol{R}^2} \le C\lambda^{-2}\|f\|_{p,(\eta_\beta^{\alpha+p/2}(\cdot;\lambda)),\boldsymbol{R}^2}, \tag{4.13}$$

$$\| |\mathscr{P}| * f\|_{p,(\eta_\beta^{\alpha-p/2}(\cdot;\lambda)),\boldsymbol{R}^2} \le C\lambda^{-1}\|f\|_{p,(\eta_\beta^{\alpha+p/2}(\cdot;\lambda)),\boldsymbol{R}^2}, \tag{4.14}$$

b)

$$\| |\mathscr{O}_{ij}(\cdot\,;\lambda)| * f\|_{p,(v_\beta^{\alpha-p/2}(\cdot\,;\lambda)),\boldsymbol{R}^2} \le C\lambda^{-1}\|f\|_{p,(v_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\boldsymbol{R}^2}, \tag{4.15}$$

$$\| |\mathscr{P}| * f\|_{p,(v_\beta^{\alpha-p/2}(\cdot\,;\lambda)),\boldsymbol{R}^2} \le C\|f\|_{p,(v_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\boldsymbol{R}^2}. \tag{4.16}$$

THEOREM 4.7. *Let T be an integral operator with the kernel $|\partial_2\mathscr{O}_{11}|$, $T : f \mapsto |\partial_2\mathscr{O}_{11}| * f$, and let $1 < p < \infty$. Then T is a well defined continuous operator*:

a) $$L^p(\boldsymbol{R}^2; \eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \mapsto L^p(\boldsymbol{R}^2; \eta_\beta^{\alpha}(\cdot\,;\lambda))$$

for $0 < \beta < p-1$, $-1 < \alpha+\beta$, $\alpha < p-3/2$, $\alpha-\beta < p/2-1$

b) $$L^p(\boldsymbol{R}^2; v_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \mapsto L^p(\boldsymbol{R}^2; v_\beta^{\alpha-p/2}(\cdot\,;\lambda))$$

for $0 < \beta < p-1$, $-1 < \alpha+\beta$, $-2 < \alpha < p-3/2$, $\alpha-\beta < p/2-1$.

Moreover we have for α, β specified in a) *and* b), *respectively*

a)

$$\| |\partial_2\mathscr{O}_{11}(\cdot\,;\lambda)| * f\|_{p,(\eta_\beta^{\alpha}(\cdot\,;\lambda)),\boldsymbol{R}^2} \le C\lambda^{-1}\|f\|_{p,(\eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\boldsymbol{R}^2}, \tag{4.17}$$

b)

$$\| |\partial_2\mathscr{O}_{11}(\cdot\,;\lambda)| * f\|_{p,(v_\beta^{\alpha}(\cdot\,;\lambda)),\boldsymbol{R}^2} \le C\lambda^{-1/2}\|f\|_{p,(v_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\boldsymbol{R}^2}. \tag{4.18}$$

THEOREM 4.8. *Let T be an integral operator with the kernel $R = |\nabla^2\mathscr{O} - \nabla^2\mathscr{S}|$, or $R = |\partial_k\mathscr{O}_{ij}|$, $i, j, k = 1, 2$, except $|\partial_2\mathscr{O}_{11}|$; $T : f \mapsto R * f$, and let $1 < p < \infty$. Then T is a well defined continuous operator*:

a) $$L^p(\boldsymbol{R}^2; \eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \mapsto L^p(\boldsymbol{R}^2; \eta_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda))$$

for $-\varepsilon(p-1)/p < \beta < p/2 - 1/2 + \varepsilon(p-1)/p$, $-p/2 - 1 - \varepsilon/p < \alpha+\beta < 3p/2 - 2 + \varepsilon$, $\alpha - \beta < p/2 - 1 + \varepsilon + \varepsilon(p-1)/p$, $-p - 1/2 - \varepsilon/p < \alpha < p - 3/2 + \varepsilon$, $0 < \varepsilon \le p/2$

b) $$L^p(\boldsymbol{R}^2; v_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \mapsto L^p(\boldsymbol{R}^2; v_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda))$$

for $-\varepsilon(p-1)/p < \beta < p/2 - 1/2 + \varepsilon(p-1)/p$, $-p/2 - 1 - \varepsilon/p < \alpha+\beta < 3p/2 - 2 + \varepsilon$, $\alpha - \beta < p/2 - 1 + \varepsilon + \varepsilon(p-1)/p$, $\max\{-p - 1/2 - \varepsilon/p, -p/2 - 2 + \varepsilon\} < \alpha < \min\{p - 3/2 + \varepsilon, 3p/2 - 2\}$, $0 < \varepsilon \le p/2$.

Moreover we have for α, β specified in a) *and* b), *respectively*

a)

$$\| |\nabla^2\mathscr{O}(\cdot\,;\lambda) - \nabla^2\mathscr{S}(\cdot)| * f\|_{p,(\eta_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\boldsymbol{R}^2} \le C\|f\|_{p,(\eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\boldsymbol{R}^2}, \tag{4.19}$$

$$\| |\partial_k\mathscr{O}_{ij}(\cdot\,;\lambda)| * f\|_{p,(\eta_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\boldsymbol{R}^2} \le C\lambda^{-1}\|f\|_{p,(\eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\boldsymbol{R}^2}, \tag{4.20}$$

b)

$$\| |\nabla^2 \mathcal{O}(\cdot\,;\lambda) - \nabla^2 \mathcal{S}(\cdot)| * f\|_{p,(\nu_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\boldsymbol{R}^2} \leq C\lambda^{\varepsilon/p}\|f\|_{p,(\nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\boldsymbol{R}^2}, \tag{4.21}$$

$$\| |\partial_k \mathcal{O}_{ij}(\cdot\,;\lambda)| * f\|_{p,(\nu_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\boldsymbol{R}^2} \leq C\lambda^{\varepsilon/p-1}\|f\|_{p,(\nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\boldsymbol{R}^2}. \tag{4.22}$$

PROOF OF THEOREM 4.3. We show only the case $R = |\nabla^2\mathcal{O} - \nabla^2\mathcal{S}|$; the other case differs only very slightly. We proceed similarly as in the case of L^∞-weighted estimates. Studying first $\eta_\beta^\alpha(\cdot\,;\lambda)$ weights we show (4.5) for $\lambda = 1$. Applying the homogeneity properties of $\mathcal{O}_{ij}(\cdot\,;\lambda)$ we get (4.5) in the general situation $\lambda \neq 1$. Next, using the results from a) together with Lemma 3.2 we show (4.7).

Let us denote

$$K(x,y) = R(x-y)(\eta_\beta^{\alpha+p/2-\varepsilon}(x))^{1/p}(\eta_\beta^{\alpha+p/2}(y))^{-1/p},$$

$$F(y) = f(y)(\eta_\beta^{\alpha+p/2}(y))^{1/p}.$$

We easily observe that, in order to verify (4.5) with $\lambda = 1$, it is sufficient to show that there exists $C > 0$, independent of f, such that

$$\left\|\int_{\boldsymbol{R}^3} K(\cdot\,,y)F(y)\,\mathrm{d}y\right\|_p \leq C\|F\|_p. \tag{4.23}$$

Let $L(\cdot)$ and $M(\cdot)$ be non-negative functions defined on $\boldsymbol{R}^3$ such that for all $x,\ y \in \boldsymbol{R}^3$

$$J_0(x) = \int_{\boldsymbol{R}^3} |K(x,y)|L(y)^q\,\mathrm{d}y \leq C^q M(x)^q, \tag{4.24}$$

$$J_1(y) := \int_{\boldsymbol{R}^3} |K(x,y)|M(x)^p\,\mathrm{d}x \leq C^p L(y)^p, \tag{4.25}$$

where $C > 0$, $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$. Then relation (4.23) is satisfied. Indeed,

$$\begin{aligned}
&\left\|\int_{\boldsymbol{R}^3} K(\cdot\,,y)F(y)\,\mathrm{d}y\right\|_p^p \\
&\leq \int_{\boldsymbol{R}^3}\left\{\left(\int_{\boldsymbol{R}^3}|K(x,y)|\,|F(y)|^p L(y)^{-p}\,\mathrm{d}y\right)^{1/p} J_0(x)^{1/q}\right\}^p \mathrm{d}x \\
&\leq C^p\int_{\boldsymbol{R}^3} M(x)^p \int_{\boldsymbol{R}^3}|K(x,y)|\,|F(y)|^p L(y)^{-p}\,\mathrm{d}y\mathrm{d}x \\
&= C^p\int_{\boldsymbol{R}^3}|F(y)|^p J_1(y)L(y)^{-p}\,\mathrm{d}y \leq C^{2p}\|F\|_p^p,
\end{aligned}$$

i.e. we get (4.23). We shall suppose the functions $L(\cdot)$, $M(\cdot)$ in the form $L(x) = M(x) = \eta^{-A}_{-\beta/p^2}(x)$, $A \in \boldsymbol{R}^1$. Denoting

$$\begin{aligned} a_0 &= qA + \alpha/p + 1/2 \quad & a_1 &= pA - \alpha/p - 1/2 + \varepsilon/p \\ b_0 &= \beta/(p-1) & b_1 &= 0 \end{aligned} \tag{4.26}$$

we get that in order to verify (4.24) we have to find such a_0 b_0 that

$$\int_{\boldsymbol{R}^3} R(x-y)\eta^{-a_0}_{-b_0}(y)\,\mathrm{d}y \le C\eta^{-a_0+\varepsilon/p}_{-b_0}(x)$$

for all $x \in \boldsymbol{R}^3$ and in order to verify (4.25) we have to find a_1 such that

$$\int_{\boldsymbol{R}^3} R(-x-y)\eta_0^{-a_1}(-x)\,\mathrm{d}x \le C\eta_0^{-a_1+\varepsilon/p}(y)$$

for all $y \in \boldsymbol{R}^3$.

Applying Theorem 3.3 with $f = \eta^{-a_0}_{-b_0}(\cdot)$ we get for the first inequality the following set of conditions

$$\begin{gathered} -\varepsilon/p < b_0 < 1 + \varepsilon/p, \quad a_0 < b_0 + 1 + 2\varepsilon/p, \quad a_0 + b_0^* > 0, \\ a_0 + b_0 < 3 + \varepsilon/p, \quad a_0 < 2 + \varepsilon/p, \end{gathered} \tag{4.27}$$

while for the second inequality we directly apply Tab. 1 and Tab. 2 and get[1]

$$0 < a_1 < 1 + 2\varepsilon/p, \tag{4.28}$$

a_i, b_i defined in (4.26). The conditions on a_i, b_i can be satisfied for some $A \in \boldsymbol{R}^1$ if we have for $0 < \varepsilon \le p$: $-\varepsilon(p-1)/p < \beta < p - 1 + \varepsilon(p-1)/p$, $-p/2 - 1 - \varepsilon/p < \alpha + \beta < 5p/2 - 3 + \varepsilon$, $\alpha - \beta < p/2 - 1 + \varepsilon + \varepsilon(p-1)/p$, $-3p/2 - \varepsilon/p < \alpha < 3p/2 - 2 + \varepsilon$. Thus (4.5) is proved for $\lambda = 1$.

Next let $\lambda \ne 1$. As $|\nabla^2\mathcal{O}(x-y;\lambda) - \nabla^2\mathcal{S}(x)| = \lambda^3|\nabla^2\mathcal{O}(\lambda(x-y);1) - \nabla^2\mathcal{S}(\lambda x)|$ we easily have

$$\begin{aligned} &\int_{\boldsymbol{R}^3}\left|\int_{\boldsymbol{R}^3} R(x-y;\lambda)f(y)\,\mathrm{d}y\right|^p \eta_\beta^{\alpha+p/2-\varepsilon}(x;\lambda)\,\mathrm{d}x \\ &\quad = \int_{\boldsymbol{R}^3}\left|\int_{\boldsymbol{R}^3} R(\lambda x - z;1)f(z/\lambda)\,\mathrm{d}z\right|^p \eta_\beta^{\alpha+p/2-\varepsilon}(\lambda x;1)\,\mathrm{d}x \\ &\quad \le C\lambda^{-3}\int_{\boldsymbol{R}^3}\left|f\left(\frac{z}{\lambda}\right)\right|^p \eta_\beta^{\alpha+p/2}(z;1)\,\mathrm{d}z \\ &\quad = C\lambda^{-2p}\int_{\boldsymbol{R}^3} |f(y)|^p\eta_\beta^{\alpha+p/2}(y;\lambda)\,\mathrm{d}y \end{aligned}$$

and we have (4.5) with $\lambda \ne 1$.

[1] The convolution on the left-hand side of (4.25) is again the same convolution as treated in Section 3, but at the point $-y$.

In order to prove (4.7) we redefine functions $K(\cdot\,,\cdot)$ and $F(\cdot)$:

$$K(x,y) = R(x-y)(v_\beta^{\alpha+p/2-\varepsilon}(x))^{1/p}(v_\beta^{\alpha+p/2}(y))^{-1/p},$$

$$F(y) = f(y)(v_\beta^{\alpha+p/2}(y))^{1/p}.$$

We will now proceed as in the first part of the proof, but now we search the functions $L(\cdot)$, $M(\cdot)$ in the form $L(x) = \mu_{-\beta/p^2}^{-A,-G}(x)$, $M(x) = \mu_{-\beta/p^2}^{-A,-H}(x)$. Denoting

$$\begin{aligned} c_0 &= qG + \alpha/p + 1/2 & c_1 &= pH - \alpha/p - 1/2 + \varepsilon/p \\ d_0 &= qH + \alpha/p + 1/2 - \varepsilon/p & d_1 &= pG - \alpha/p - 1/2 \end{aligned} \tag{4.29}$$

we see that in order to verify (4.24) and (4.25) we have to find such a_i, b_i see (4.26), (4.27) and c_i, d_i such that

$$\int_{\mathbf{R}^3} R(x-y)\mu_{-b_0}^{-a_0,-c_0}(y)\,\mathrm{d}y \le C\mu_{-b_0}^{-a_0+\varepsilon/p,-d_0}(x), \quad x \in \mathbf{R}^3\backslash\{\mathbf{0}\}$$

$$\int_{\mathbf{R}^3} R(-x-y)\mu_0^{-a_1,-c_1}(-x)\,\mathrm{d}x \le C\mu_0^{-a_1+\varepsilon/p,-d_1}(y), \quad y \in \mathbf{R}^3\backslash\{\mathbf{0}\}.$$

Recalling that $|R(x-y)| \le C_0 v_{-1}^{-2}(x-y;1)$ we get from Lemma 3.2 the following two possible sets of conditions for c_i, d_i:

$$\text{(i)} \quad \begin{aligned} &c_i < 3 \\ &c_i + 2 > 3 \\ &d_i \ge c_i - 1 \end{aligned} \qquad \text{(ii)} \quad \begin{aligned} &c_i < 1 \\ &d_i \ge 0, \end{aligned}$$

where in both cases $i = 0,1$. Conditions for a_i, b_i are the same as in the first part of this proof. From the conditions (i) we get the following additional condition

$$-p/2 - 3 + \varepsilon < \alpha < 5p/2 - 3.$$

Case (ii) gives more restrictive conditions on α, no extension of the result. So, (4.7) is proved in the case $\lambda = 1$.

Finally to get (4.7) with $\lambda \ne 1$ we proceed as in the case of the weights $\eta_B^A(\cdot\,;\lambda)$. We have

$$\int_{\mathbf{R}^3}\left|\int_{\mathbf{R}^3} R(x-y;\lambda)f(y)\,\mathrm{d}y\right|^p v_\beta^{\alpha+p/2-\varepsilon}(x;\lambda)\,\mathrm{d}x$$

$$= \lambda^{-\alpha-p/2+\varepsilon}\int_{\mathbf{R}^3}\left|\int_{\mathbf{R}^3} R(\lambda x - z;1)f(z/\lambda)\,\mathrm{d}z\right|^p v_\beta^{\alpha+p/2-\varepsilon}(\lambda x;1)\,\mathrm{d}x$$

$$\leq C\lambda^{-\alpha-p/2+\varepsilon-3}\int_{\boldsymbol{R}^3}|f(z/\lambda)|^p v_\beta^{\alpha+p/2}(z;1)\,\mathrm{d}z$$

$$= C\lambda^{\varepsilon}\int_{\boldsymbol{R}^3}|f(y)|^p v_\beta^{\alpha+p/2}(y;\lambda)\,\mathrm{d}y$$

and we have (4.7) with $\lambda \neq 1$. This completes our proof. $\square$

5. Singular integrals.

The aim of this section is to present some results concerning L^p-estimates of certain singular operators and finally to apply them on the convolutions (defined in the sense of principal value) of the type $\nabla\mathscr{P} * f$, $\nabla^2\mathscr{S} * f$ and $\nabla^2\mathscr{O} * f$.

We shall use the idea of Farwig (see [**2**]). Before formulating the result from [**9**] we need to define some notation.

Definition 5.1. *The weight $w, w \geq 0$ belongs to the Muckenhoupt class A_p, $1 \leq p < +\infty$ if there is a constant C such that*

$$\sup_Q\left[\left((1/|Q|)\int_Q w(x)\,dx\right)\left((1/|Q|)\int_Q w(x)^{-1/(p-1)}\,dx\right)^{p-1}\right] \leq C < +\infty$$
$$\sup_Q\,(1/|Q|)\int_Q w(x)\,dx \leq Cw(x_0), \quad \forall x_0 \in \boldsymbol{R}^N \tag{5.1}$$

for $p \in (1;\infty)$ and $p = 1$, respectively. In the first case, the supremum is taken over all cubes Q in $\boldsymbol{R}^N$, in the second case only over those cubes which contain x_0; $|Q|$ denotes the Lebesgue measure of Q. The constant does not depend on x_0.

Remark 5.1. a) For $p = 1$, the condition $(5.1)_2$ can be replaced by

$$Mw(x) \leq Cw(x) \quad \text{for a.a. } x \in \boldsymbol{R}^N \tag{5.2}$$

where $Mg(x)$ is the Hardy-Littlewood maximal function which is defined by the left hand side in $(5.1)_2$.

b) In (5.1) it is enough to take the supremum over all cubes with edges parallel to an arbitrary chosen Cartesian system X.
To show this, let X be a Cartesian system in $\boldsymbol{R}^N$ and X' another one arising from X by any rotation. Then we have

$$\frac{1}{N^{N/2}}\frac{1}{|Q_1|}\int_{Q_1} w(x)\,\mathrm{d}x \leq \frac{1}{|Q'|}\int_{Q'} w(x)\,\mathrm{d}x \leq \frac{N^{N/2}}{|Q_2|}\int_{Q_2} w(x)\,\mathrm{d}x \tag{5.3}$$

for any locally integrable function $w \geq 0$. In (5.3) Q' is a cube with edges parallel to the axes of X', Q_1 is the greatest cube with edges parallel to the axes of X such that $Q_1 \subset Q'$, and Q_2 is the smallest cube with edges parallel to the axes of X such that $Q' \subset Q_2$.

Next part is devoted to the investigation under what condition the weights defined in Section 1 belong to A_p for some $1 \le p < +\infty$. First we recall several general results:

LEMMA 5.1. a) *If* $w_1, w_2 \in A_1$ *then for any* $1 \le p < +\infty$ *the weight* $w \equiv w_1 w_2^{1-p} \in A_p$.

b) *If* $w_1, w_2 \in A_p$ *for some* $1 \le p < +\infty$ *then for any* $\hbar \in [0;1]$, $w_\hbar \equiv w_1^\hbar w_2^{1-\hbar} \in A_p$.

c) *If* $w \in A_p$ *for some* $1 \le p < +\infty$ *then for any* $\hbar \in [0;1]$, $w^\hbar \in A_p$.

PROOF. It follows directly from the definition of A_p and in case b) from the Hölder inequality. □

DEFINITION 5.2. *Let* μ *be a non-negative Borel measure. We define the maximal function*

$$M\mu(x) = \sup_Q \frac{1}{|Q|} \int_Q d\mu(y) \tag{5.4}$$

where the supremum is taken over all cubes Q *such that* $x \in Q$. *Analogously we define* $Mf(x)$ *for* $f \in L^1_{loc}(\boldsymbol{R}^N)$, *replacing* $d\mu(y)$ *by* $|f|\,dy$. (*See also Remark* 5.1 a) *and* b)).

The proof of the following lemma can be found in [**15**] (Theorems IX.5.5 and IX.3.4).

LEMMA 5.2. a) *If* $M\mu$ *is finite for a.a.* $x \in \boldsymbol{R}^N$ *then, for any* $\hbar \in [0;1)$, $(M\mu)^\hbar \in A_1$.

b) *Let* $w \in A_1$. *Then there exists a function* $f \in L^1_{loc}(\boldsymbol{R}^N)$ *such that* $w \sim (Mf)^\hbar$ *for some* $\hbar \in [0;1)$.

Using Lemma 5.2 we can easily show

LEMMA 5.3. *The weights* $|x|^{-a}$ *and* $(1+|x|)^{-a}$ *satisfy the* A_1*-condition on* $\boldsymbol{R}^N$ *for each* $a \in [0;N)$.

PROOF. We have that for $\mu = \delta_0$ the maximal function $M\mu(x) \sim |x|^{-N}$ and so $|x|^{-N\hbar} \in A_1$, $\forall \hbar \in [0;1)$. Further, if we define $\mu(A) = |A \cap B_1(\mathbf{0})|$ then $M\mu(x) \sim (1+|x|)^{-N}$ and again Lemma 5.2 a) furnishes the result. □

Now we shall concentrate on the two-dimensional case. We have

LEMMA 5.4. *For* $b \in (-1;1]$ *and* $\hbar \in [0;1)$ *the function* $w_0^{(2)}(x) = (|x|^{b-1/2}/(s(x)^{1/2}))^\hbar$, $x \in \boldsymbol{R}^2$ *is a weight of the class* A_1 *in* $\boldsymbol{R}^2$.

PROOF. Let $b \in (-1; 1]$. We define the measure μ by

$$\mu(A) \equiv \int_{A^+} x_1^b \, dx_1$$

where $A^+ = \{x_1 \in \boldsymbol{R}, x_1 > 0 : (x_1, 0) \in A\}$ for $A \subset \boldsymbol{R}^2$, measurable. Evidently μ is a non-negative Borel measure on $\boldsymbol{R}^2$. We shall show that $M\mu \sim r^{b-1/2}/s^{1/2}$; then the assertion of the lemma follows from Lemma 5.2 a).

Let Q_a be a closed square containing x with sides parallel to the axes (see Remark 5.1 b)) with the side length $a > |x_2|$, $q(a) = \mu(Q_a)/|Q_a|$ and $|x|$ denoting the maximum norm of $x = (x_1, x_2) \in \boldsymbol{R}^2$. We have to distinguish several cases:

A) Let $x_1 > 0$, $|x_2| \le x_1$ and $b \in (-1; 0]$. We first consider squares Q_a with $|x_2| \le a \le x_1$ and $Q_a^+ = [x_1 - a; x_1]$. Then we have

$$q(a) = \frac{1}{b+1} \frac{x_1^{b+1} - (x_1 - a)^{b+1}}{a^2} = \frac{1}{b+1} x_1^{b-1} \varphi\left(\frac{a}{x_1}\right)$$

with

$$\varphi(y) = \frac{1 - (1-y)^{b+1}}{y^2} \quad \frac{|x_2|}{x_1} \le y \le 1.$$

Let $b < 0$. Then $\varphi(0^+) = +\infty$, $\varphi(1) = 1$, $\varphi'(1^-) = \infty$. It is possible to show that the function φ has exactly one local minimum on $(0; 1)$ with a value less than 1. Therefore, there exists exactly one point $y_b \in (0, 1)$ such that $\varphi(y_b) = 1$. If $|x_2| \le y_b \cdot x_1$ then φ achieves its maximum on $[|x_2|/x_1, 1)$ at the point $|x_2|/x_1$ and

$$\max\{q(a); |x_2| \le a \le x_1\} = q(|x_2|) \sim \frac{x_1^b}{|x_2|} \sim \frac{r^{b-1/2}}{s^{1/2}}.$$

If $|x_2| > y_b \cdot x_1$ then φ is maximal at the point 1 what yields

$$\max\{q(a); |x_2| \le a \le x_1\} = q(x_1) \sim x_1^{b-1} \sim \frac{r^b}{|x_2|} \quad \text{as } x_1 \sim |x_2|.$$

Now let us consider squares Q_a with $a \ge x_1$ and $Q_a^+ = [0; a)$. Then

$$q(a) = \frac{a^{b-1}}{b+1}$$

is strictly decreasing in $a \ge x_1$. Therefore

$$\max\{q(a); x_1 \le a\} = \frac{x_1^{b-1}}{b+1} \le c \frac{r^b}{|x_2|}.$$

Combining this with the fact that $s(x) \sim |x_2|^2/r$ we get

$$M\mu \sim \frac{r^{b-1/2}}{s^{1/2}}.$$

The case $b = 0$ is trivial.

B) Let $x_1 > 0$, $|x_2| > x_1$, and $b \in (-1; 0]$. It suffices to consider squares Q_a with $a \geq |x'|$ and $Q_a^+ = [0, a]$. But then obviously

$$\max\{q(a); a \geq |x_2|\} \sim q(|x_2|) \sim |x_2|^{b-1} \sim \frac{r^{b-1/2}}{s^{1/2}} \quad \text{as } s(x) \sim |x_2| \sim r.$$

C) Let $x_1 > 0$ and $b \in (0; 1]$. It suffices to consider only squares Q_a such that $a \geq |x_2|$ and $Q_a^+ = [x_1; x_1 + a]$. Therefore

$$q(a) = \frac{1}{b+1}\frac{(x_1 + a)^{b+1} - x_1^{b+1}}{a^2}$$

and since $q(a)$ is evidently decreasing, we get $\max\{q(a); a \geq |x_2|\} = q(|x_2|)$. Now, if $|x_2| < x_1$ then $q(|x_2|) \sim x_1^b/|x_2| \sim r^{b-1/2}/s^{1/2}$. On the other side, if $|x_2| > x_1$ then $q(|x_2|) \sim |x_2|^{b-1} \sim r^{b-1/2}/s^{1/2}$.

D) Let $x_1 < 0$. It suffices to consider only squares Q_a with $a \geq \max(|x_2|, |x_1|)$ and $Q_a^+ = [0; a - |x_1|]$. Then

$$q(a) = \frac{|x_1|^{b-1}}{b+1}\varphi\left(\frac{a}{|x_1|}\right) \quad \text{with } \varphi(y) = \frac{(y-1)^{b+1}}{y^2}.$$

The function $\varphi(y)$ vanishes at the point 1 and, if $b < 1$, at infinity. Thus for $b < 1$ there is a point $y_b > 1$ such that φ is maximal in y_b for $y \geq 1$. If $|x_2| \leq y_b \cdot |x_1|$ then

$$\max\{q(a) : a \geq |x_2|\} = q(y_b \cdot |x_1|) \sim x_1^{b-1} \sim \frac{r^{b-1/2}}{s^{1/2}} \quad \text{as } s \sim r \quad \text{for } x_1 < 0.$$

But if $|x_2| > y_b \cdot |x_1|$ then

$$\max\{q(a); a \geq |x_2|\} \simeq q(|x_2|) \sim |x_2|^{b-1} \sim \frac{r^{b-1/2}}{s^{1/2}}.$$

The case $b = 1$ is trivial since q is maximal for $a \to \infty$ yielding

$$\max\{q(a); a \geq |x_2|\} \sim 1 \sim \frac{r^{1/2}}{s^{1/2}}. \qquad \square$$

Using this result we can now show the following

LEMMA 5.5. *For $b \in (-1; 1/2]$ and $\hbar \in [0; 1)$ the function $w_1^{(2)} = (v_{-1/2}^{b-1/2})^{\hbar}$ is a weight of class A_1 in $\boldsymbol{R}^2$.*

Proof. We have to verify that $(Mw_1^{(2)})(x) \le Cw_1^{(2)}(x)$ a.e. in $\boldsymbol{R}^2$. Let Q_a denotes, similarly as in the previous lemma, a closed square with sides parallel to the axes, and with the side length a; R will be a sufficiently large constant. We again distinguish several cases:

A) $s(x) \le 1,\ r = |x| \ge R$

α) $a \le (1/2)r^{1/2}$

Then for all $y \in Q_a$ we have that $w_1^{(2)}(y) \sim w_1^{(2)}(x)$ and

$$\int_{Q_a} w_1^{(2)}(y)\,\mathrm{d}y \le C|Q_a|w_1^{(2)}(x).$$

β) $a = (1/2)r^{1/2+\sigma},\ \sigma \in (0; 1/2]$

Now $Q_a \subset \{y \in \boldsymbol{R}^2; ||y| - r| \le cr^{1/2+\sigma}; s(y) \le cr^{2\sigma}\}$ and we may use

$$\int_{Q_a} w_1^{(2)}(y)\,\mathrm{d}y \le C\int_{r-cr^{1/2+\sigma}}^{r+cr^{1/2+\sigma}} \mathrm{d}\varrho\varrho^{1/2+(b-1/2)\hbar}\int_0^{Cr^{2\sigma}} \frac{\mathrm{d}s}{s^{1/2}(1+s)^{\hbar}}$$

$$\le C|Q_a|r^{(b-1/2)\hbar-\sigma\hbar} \le C|Q_a|w_1^{(2)}(x)$$

$$\text{as } r = |x| \ge R \gg 1, \quad s(x) \le 1.$$

γ) $a \ge r/2$

In this case Q_a is contained in the ball B_{3a}. We use the following

$$\int_{\partial B_R} v_{-\beta}^{-\alpha}\,\mathrm{d}S \sim R^{1-\alpha}(R+1)^{-\min(\beta,1/2)}\begin{cases}\ln R, & \beta = 1/2\\ 1, & \beta \ne 1/2\end{cases}, \quad R \in (0; +\infty).$$

So we get

$$\int_{S_\tau} w_1^{(2)}\,\mathrm{d}S \sim \tau^{1+(b-1/2)\hbar}(\tau+1)^{-\hbar/2}, \quad \tau \in (0; +\infty). \tag{5.5}$$

This implies for $-1 < b \le 1,\ R \le r \le 2a$

$$\int_{B_{3a}} w_1^{(2)}(y)\,\mathrm{d}S(y) \le C|Q_a|a^{(b-1)\hbar} \le C|Q_a|r^{(b-1)\hbar} \le C|Q_a|r^{(b-1/2)\hbar}.$$

B) $s(x) \ge 1,\ r \ge R$

Because $\hbar \ge 0$ we have

$$w_1^{(2)}(x) = \left(\frac{r^{b-1/2}}{(1+s)^{1/2}}\right)^{\hbar} \le \left(\frac{r^{b-1/2}}{s^{1/2}}\right)^{\hbar} = w_0^{(2)}(x).$$

Since $w_0^{(2)} \in A_1$, we have

$$\int_{Q_a} w_1^{(2)}(y)\,\mathrm{d}y \le C\int_{Q_a} w_0^{(2)}(y)\,\mathrm{d}y \le C|Q_a|w_0^{(2)}(x).$$

As $s(x) \geq 1$, we have $w_0^{(2)}(x) \leq Cw_1^{(2)}(x)$.

C) $r \leq R$

α) If $a/2 \leq R$ then easily

$$\int_{Q_a} w_1^{(2)}(y)\,\mathrm{d}y \leq C|Q_a|w_1^{(2)}(x).$$

Here we use the condition $b \leq 1/2$ so $w_1^{(2)} \geq c' > 0$.

β) If $a/2 > R$ then $Q_a \subset B_{3a}$; using (5.5) we have

$$\begin{aligned}\int_{B_{3a}} w_1^{(2)}(y)\,\mathrm{d}y &\leq C|Q_a|a^{(b-1/2)\hbar}(1+a)^{(-1/2)\hbar}\\ &\leq C_1|Q_a|r^{(b-1/2)\hbar}(1+2R)^{(-1/2)\hbar}\\ &\leq C_1|Q_a|r^{(b-1/2)\hbar}(1+s)^{(-1/2)\hbar}.\end{aligned}$$

In the last two inequalities we used the conditions $b \leq 1/2$, $a/2 > R \geq r$. □

LEMMA 5.6. *For $b \in (-1;1]$ and $\hbar \in [0;1)$ the function $w_2^{(2)} = (\eta_{-1/2}^{b-1/2})^{\hbar}$ is a weight of class A_1 in $\boldsymbol{R}^2$.*

PROOF. We proceed as in Lemma 5.5. The part A) remains the same. In the part B) we use that for some $C > 0$

$$\left(1+\frac{1}{r}\right)^{b-1/2} \leq \left(1+\frac{1}{r}\right)^{1/2} \leq C\left(1+\frac{1}{s}\right)^{1/2}$$

for all $y \in \boldsymbol{R}^2$ such that $s(y) \geq 1$, $r(y) \geq R$, if $b \leq 1$. In the part Cα) the condition $w_2^{(2)} \geq C > 0$ for $|x| \leq R$ is satisfied without the necessity of the additional condition $b \leq 1/2$ unlike in Lemma 5.5. In the part Cβ) we proceed similarly as in Lemma 5.5 but we use

$$\int_{S_\tau} w_2^{(2)}\,\mathrm{d}S \sim \tau(\tau+1)^{1+(b-1)\hbar}, \quad \tau \in (0;+\infty)$$

instead of (5.5). Also here we do not need the condition $b \leq 1/2$. This is why we get the Lemma 5.6 under less restrictive condition on b than in Lemma 5.5. □

Combining Lemmas 5.4, 5.5 and 5.6 with the properties of Muckenhoupt classes A_p we can show the following

THEOREM 5.1. *Let $-1/2 < \beta < (1/2)(p-1)$, $-2 < \alpha + \beta < 2(p-1)$. Then the weights $\eta_\beta^\alpha(x)$ and $\sigma_\beta^\alpha(x)$ are A_p-weights in $\boldsymbol{R}^2$ for $p \in (1;\infty)$.*

Let $-1/2 < \beta < (1/2)(p-1)$, $-2 < \alpha + \beta < 2(p-1)$, $-2 < \alpha < 2(p-1)$. *Then the weight* $v_\beta^\alpha(x)$ *is* A_p*-weight in* $\boldsymbol{R}^2$ *for* $p \in (1;\infty)$.

Now we shall concentrate on the three-dimensional case.

Lemma 5.7. *For* $b \in (-1;2]$ *and* $\hbar \in [0;1)$ *the function* $w_0^{(3)}(x) = (|x|^{b-1}/(s(x)))^{\hbar}$, $x \in \boldsymbol{R}^3$ *is a weight of the class* A_1 *in* $\boldsymbol{R}^3$.

The proof is analogous to the proof of Lemma 5.4. For the proof see also [**2**].

Lemma 5.8. *For* $b \in (-1;1]$ *and* $\hbar \in [0;1)$ *the function* $w_1^{(3)} = (v_{-1}^{b-1})^{\hbar}$ *is a weight of class* A_1 *in* $\boldsymbol{R}^3$.

The proof of the lemma is analogous to the proof of Lemma 5.6. But in the part Aβ) and Cβ) the estimates are slightly less technical than in Lemma 5.6.

Lemma 5.9. *For* $b \in (-1;2]$ *and* $\hbar \in [0;1)$ *the function* $w_2^{(3)} = (\eta_{-1}^{b-1})^{\hbar}$ *is a weight of class* A_1 *in* $\boldsymbol{R}^3$.

Proof. We proceed as in Lemma 5.5. Part A) remains the same. In part B) we use the fact that for some $C > 0$

$$\left(1 + \frac{1}{r}\right)^{b-1} \le \left(1 + \frac{1}{r}\right) \le C\left(1 + \frac{1}{s}\right)$$

for all $y \in \boldsymbol{R}^3$ such that $s(y) \ge 1$, $r(y) \ge R$, if $b \le 2$. In part Cα) the condition $w_2^{(3)} \ge C > 0$ for $|x| \le R$ is satisfied without the necessity of any additional condition on b. In part Cβ) we proceed as in Lemma 5.5 but we use

$$\int_{S_\tau} w_2^{(3)}\,\mathrm{d}S \sim \tau^2(\tau+1)^{1+(b-2)\hbar}. \qquad \square$$

Combining Lemmas 5.7, 5.8 and 5.9 with the properties of Muckenhoupt classes A_p we can show the following

Theorem 5.2. *Let* $-1 < \beta < p-1$, $-3 < \alpha + \beta < 3(p-1)$. *Then the weights* $\eta_\beta^\alpha(x)$ *and* $\sigma_\beta^\alpha(x)$ *are* A_p*-weights in* $\boldsymbol{R}^3$ *for* $p \in (1;\infty)$.

Let $-1 < \beta < p-1$, $-3 < \alpha + \beta < 3(p-1)$, $-3 < \alpha < 3(p-1)$. *Then the weight* $v_\beta^\alpha(x)$ *is* A_p*-weight in* $\boldsymbol{R}^3$ *for* $p \in (1;\infty)$.

In order to formulate the fundamental theorem used in this section we need to define the so called L^∞-Dini condition. We will use the following notation in the definition:

ρ-rotation of $\partial B_1(\mathbf{0})$ with magnitude $|\rho| = \sup_{x \in \partial B_1(\mathbf{0})} |\rho x - x|$.

Definition 5.3. *Let* Ω *be a function defined on* $\partial B_1(\mathbf{0})$, $\Omega \in L^\infty(\partial B_1(\mathbf{0}))$. *We say that function* Ω *satisfies the* L^∞*-Dini condition* ($\Omega \in L^\infty$*-Dini*) *if*

$$\int_0^1 \theta_\infty(\delta)\frac{1}{\delta}\,d\delta < +\infty, \quad \textit{where } \theta_\infty(\delta) := \sup_{|\rho|<\delta} \|\Omega(\rho x) - \Omega(x)\|_{L^\infty(\partial B_1(\mathbf{0}))}.$$

THEOREM 5.3. *Let $N \in \boldsymbol{N}$, $N \geq 2$, $\Omega \in L^\infty(\partial B_1(\mathbf{0}))$, $\Omega \in L^\infty$-Dini, $\int_{\partial B_1(\mathbf{0})} \Omega\, dS = 0$, Ω is a positively homogeneous function of degree zero, $R(x) := \Omega(x')/|x|^N$, $x' = x/|x|$. Let T be an operator with the kernel R, i.e. $Tf(x) = (R * f)(x)$ in the principal-value sense and $w \in A_p$ in $\boldsymbol{R}^N$, $p > 1$. Then T is a continuous operator $L^p(\boldsymbol{R}^N; w) \mapsto L^p(\boldsymbol{R}^N; w)$.*

For a proof of the theorem see [**9**].

REMARK 5.2. The fact that $R(\cdot)$ satisfies the conditions formulated in Theorem 5.3 means that $R(\cdot)$ represents a Calderón-Zygmund singular integral kernel. We will write in this case $R(\cdot) \in CZ$. It is known that $\nabla^2\mathscr{S}$, $\nabla\mathscr{P} \in CZ$ in both the two- and three-dimensional cases. The operator T in Theorem 5.3 is in fact defined on $C_0^\infty(\boldsymbol{R}^N)$; but the closure of $C_0^\infty(\boldsymbol{R}^N)$ in the $L^p(\boldsymbol{R}^N; w)$ norm coincides with $L^p(\boldsymbol{R}^N; w)$ if $w \in A_p$, see e.g. [**16**].

As the corollary of Theorem 5.3 we get

COROLLARY 5.1. *Let $N = 2, 3$, R be either $\partial_i\partial_j\mathscr{S}_{rs}$ or $\partial_i\mathscr{P}_r$, $i, j, r, s = 1, 2, \dots N$ and $f \in L^p(\boldsymbol{R}^N; w)$, where w stands for η^α_β, σ^α_β or ν^α_β with $1 < p < \infty$ and let α, β be such that the corresponding weights are A_p-weights. Then v.p. $(R * f) \in L^p(\boldsymbol{R}^N; w)$ and*

$$\|v.p.(R * f)\|_{p,(w),\boldsymbol{R}^N} \leq C\|f\|_{p,(w),\boldsymbol{R}^N}.$$

We will now formulate two theorems about $\nabla\mathscr{P}$ both of which follow from Theorem 5.3.

THEOREM 5.4. *$\nabla\mathscr{P} : f \mapsto \nabla\mathscr{P} * f$ defines continuous operators*:

a) $$L^p(\boldsymbol{R}^3; \eta_\beta^{\alpha+p/2}) \mapsto L^p(\boldsymbol{R}^3; \eta_\beta^{\alpha+p/2})$$

for $1 < p < \infty$, $-1 < \beta < p - 1$, $-3 - p/2 < \alpha + \beta < 5p/2 - 3$

b) $$L^p(\boldsymbol{R}^3; \nu_\beta^{\alpha+p/2}(\cdot\,; \lambda)) \mapsto L^p(\boldsymbol{R}^3; \nu_\beta^{\alpha+p/2})$$

for $1 < p < \infty$, $-1 < \beta < p - 1$, $-3 - p/2 < \alpha < 5p/2 - 3$, $-3 - p/2 < \alpha + \beta < 5p/2 - 3$.

THEOREM 5.5. *$\nabla\mathscr{P} : f \mapsto \nabla\mathscr{P} * f$ defines continuous operators*:

a)

$$L^p(\boldsymbol{R}^2;\eta_\beta^{\alpha+p/2}) \mapsto L^p(\boldsymbol{R}^2;\eta_\beta^{\alpha+p/2}) \tag{5.6}$$

for $1<p<\infty$, $-1/2<\beta<(1/2)(p-1)$, $-2-p/2<\alpha+\beta<3p/2-2$

b)

$$L^p(\boldsymbol{R}^2;\nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \mapsto L^p(\boldsymbol{R}^2;\nu_\beta^{\alpha+p/2}) \tag{5.7}$$

for $1<p<\infty$, $-1/2<\beta<(1/2)(p-1)$, $-2-p/2<\alpha<3p/2-2$, $-2-p/2<\alpha+\beta<3p/2-2$.

The next two theorems follow from Theorems 4.3, 4.8, 5.1, 5.2 and Corollary 5.1.

Theorem 5.6. *Let T be an integral operator in the principal-value sense with the kernel $\partial_k\partial_l\mathcal{O}_{ij}(\cdot\,;\lambda)$, $i,j,k,l=1,2,3$, $T: f\mapsto R*f$ and let $1<p<\infty$. Then T is a well defined continuous operator*:

a) $$L^p(\boldsymbol{R}^3;\eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \mapsto L^p(\boldsymbol{R}^3;\eta_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda))$$

for $\max\{-1,-\varepsilon(p-1)/p\}<\beta<p-1$, $-p/2-1-\varepsilon/p<\alpha+\beta<5p/2-3$, $\alpha-\beta<p/2-1+\varepsilon+\varepsilon(p-1)/p$, $-3p/2-\varepsilon/p<\alpha<3p/2-2+\varepsilon$, $0<\varepsilon\le p$

b) $$L^p(\boldsymbol{R}^3;\nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \mapsto L^p(\boldsymbol{R}^3;\mu_\beta^{\alpha+p/2-\varepsilon,\,\alpha+p/2}(\cdot\,;\lambda))$$

$$L^p(\boldsymbol{R}^3\backslash\Omega;\nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \mapsto L^p(\boldsymbol{R}^3;\nu_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda))$$

for $\max\{-1,-\varepsilon(p-1)/p\}<\beta<p-1$, $-p/2-1-\varepsilon/p<\alpha+\beta<5p/2-3$, $\alpha-\beta<p/2-1+\varepsilon+\varepsilon(p-1)/p$, $\max\{-p/2-3,-3p/2-\varepsilon/p\}<\alpha<\min\{5p/2-3,\ 3p/2-2+\varepsilon\}$, $0<\varepsilon\le p$, $\Omega\subset\boldsymbol{R}^3$*—an arbitrary domain,* $\mathbf{0}\in\Omega$.

Moreover, we have for α, β *specified in* a) *and* b), *respectively*

a)

$$\|v.p.(\partial_k\partial_l\mathcal{O}_{ij}(\cdot\,;\lambda)*f)\|_{p,(\eta_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\boldsymbol{R}^3} \le C\|f\|_{p,(\eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\boldsymbol{R}^3}, \tag{5.8}$$

b)

$$\|v.p.(\partial_k\partial_l\mathcal{O}_{ij}(\cdot\,;\lambda)*f)\|_{p,(\nu_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\boldsymbol{R}^3} \le C\|f\|_{p,(\mu_\beta^{\alpha+p/2-\varepsilon,\,\alpha+p/2}(\cdot\,;\lambda)),\boldsymbol{R}^3} \tag{5.9}$$

$$\|v.p.(\partial_k\partial_l\mathcal{O}_{ij}(\cdot\,;\lambda)*f)\|_{p,(\nu_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\boldsymbol{R}^3\backslash\Omega} \le C\lambda^{\varepsilon/p}\|f\|_{p,(\nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\boldsymbol{R}^3}. \tag{5.10}$$

Theorem 5.7. *Let T be an integral operator in the value-principal sense with the kernel $\partial_k\partial_l\mathcal{O}_{ij}(\cdot\,;\lambda)$, $i,j,k,l=1,2$, $T: f\mapsto R*f$, and let $1<p<\infty$. Then T is a well defined continuous operator*:

a) $$L^p(\boldsymbol{R}^2; \eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \mapsto L^p(\boldsymbol{R}^2; \eta_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda))$$

for $\max\{-1/2, -\varepsilon(p-1)/p\} < \beta < p/2 - 1/2$, $-p/2 - 1 - \varepsilon/p < \alpha + \beta < 3p/2 - 2$, $\alpha - \beta < p/2 - 1 + \varepsilon + \varepsilon(p-1)/p$, $-p - 1/2 - \varepsilon/p < \alpha < p - 3/2 + \varepsilon$, $0 < \varepsilon \le p/2$

b) $$L^p(\boldsymbol{R}^2; \nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \mapsto L^p(\boldsymbol{R}^2; \mu_\beta^{\alpha+p/2-\varepsilon,\,\alpha+p/2}(\cdot\,;\lambda))$$

$$L^p(\boldsymbol{R}^2 \backslash \Omega; \nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \mapsto L^p(\boldsymbol{R}^2; \nu_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda))$$

for $\max\{-1/2, -\varepsilon(p-1)/p\} < \beta < p/2 - 1/2$, $-p/2 - 1 - \varepsilon/p < \alpha + \beta < 3p/2 - 2$, $\alpha - \beta < p/2 - 1 + \varepsilon + \varepsilon(p-1)/p$, $\max\{-p/2 - 2, -p - 1/2 - \varepsilon/p\} < \alpha < \min\{3p/2 - 2, p - 3/2 + \varepsilon\}$, $0 < \varepsilon \le p/2$, $\Omega \subset \boldsymbol{R}^2$—*an arbitrary domain*, $\mathbf{0} \in \Omega$.

Moreover, we have for α, β *specified in* a) *and* b), *respectively*

a)

$$\|v.p.(\partial_k \partial_l \mathcal{O}_{ij}(\cdot\,;\lambda) * f)\|_{p,(\eta_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\boldsymbol{R}^2} \le C\|f\|_{p,(\eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\boldsymbol{R}^2}, \tag{5.11}$$

b)

$$\|v.p.(\partial_k \partial_l \mathcal{O}_{ij}(\cdot\,;\lambda) * f)\|_{p,(\nu_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\boldsymbol{R}^2} \le C\|f\|_{p,(\mu_\beta^{\alpha+p/2-\varepsilon,\,\alpha+p/2}(\cdot\,;\lambda)),\boldsymbol{R}^2}, \tag{5.12}$$

$$\|v.p.(\partial_k \partial_l \mathcal{O}_{ij}(\cdot\,;\lambda) * f)\|_{p,(\nu_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\boldsymbol{R}^2\backslash\Omega} \le C\lambda^{\varepsilon/p}\|f\|_{p,(\nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\boldsymbol{R}^2}. \tag{5.13}$$

References

[1] P. Dutto, Solutions physiquement raisonnables des équations de Navier–Stokes compressibles stationnaires dans un domaine extérieur du plan. Ph.D. Thesis, University of Toulon (1998).

[2] R. Farwig, The stationary exterior 3D-problem of Oseen and Navier–Stokes equations in anisotropically weighted Sobolev spaces. Mathematische Zeitschrift, **211** (1992), 409–447.

[3] R. Farwig, Das stationäre Außenraumproblem der Navier-Stokes-Gleichungen bei nicht-verschwindender Anströmgeschwindigkeit in anisotrop gewichteten Sobolev-räumen. SFB 256 preprint no. **110** (Habilitationsschrift), University of Bonn (1990).

[4] R. Finn, Estimates at infinity for stationary solution of Navier–Stokes equations. Bult. Math. de la Soc. Sci. Math. de la R.P.R. Tome 3, **53** 4. (1959), 387–418.

[5] R. Finn, On the Exterior Stationary Problem and Associated Pertubation Problems for the N.S. Eq's. Arch. Rat. Mech. Anal., **19** (1965), 363–406.

[6] G. P. Galdi, An Introduction to the Mathematical Theory of Navier–Stokes Equations I. Springer Verlag (1994).

[7] T. Kobayashi and Y. Shibata, On the Oseen equation in the three-dimensional exterior domains. Math. Ann., **310** (1998), 1–45.

[8] G. A. Korn and T. M. Korn, Mathematical Handbook for Scientists and Engineers. Mc Graw–Hill (1961).

[9] D. S. Kurtz and R. L. Wheeden, Results on Weighted Norm Inequalities for Multipliers. Trans. Amer. Math. Soc., **255** (1979), 343–364.

[10] A. Novotný and M. Padula, Physically reasonable solutions to steady compressible Navier–Stokes equations in $3D$-exterior domains ($v_\infty \neq 0$). Math. Ann., **370** (1997), 439–489.

[11] A. Novotný and M. Pokorný, Three-dimensional steady flow of viscoelastic fluid past an obstacle. (to appear in Journal of Mathematical Fluid Mechanics).
[12] A. Novotný and M. Pokorný, Steady plane flow of viscoelastic fluid past an obstacle. (submitted to Applications of Mathematics).
[13] M. Pokorný, Asymptotic behaviour of some equations describing the flow of fluids in unbounded domains. Ph.D. thesis, Charles University Prague & University of Toulon (1999).
[14] D. Smith, Estimates at Infinity for Stationary Solutions of the N.S. Equations in Two Dimensions. Arch. Rat. Mech. Anal., **20** (1965), 341–372.
[15] A. Torchinski, Real Variable Methods in Harmonic Analysis. Academic Press, Orlando (1986).
[16] B. O. Turesson, Nonlinear Potential Theory and Weighted Sobolev Spaces. Lecture Notes in Mathematics 1736, Springer Verlag (2000).

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