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Topology of complements of discriminants and resultants

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Abstract. In this paper, we classify the homotopy types of spaces of monic polynomials which have no n-fold real roots or spaces of n-tuples of monic polynomials which have no common real roots, by using the "scanning method" ([**9**]) and Vassiliev's spectral sequence ([**15**], [**16**]). In particular, we show that such spaces are finite dimensional models for the infinite dimensional loop space of spheres.

§1. Introduction.

Let $\mathscr{F}$ be a space of certain functions and $\mathscr{S}$ denote a class of singularities. By $\Sigma(\mathscr{S})$ we denote the space consisting of all functions $f \in \mathscr{F}$ which have a singularity of class $\mathscr{S}$ and call it the *discriminant* of class $\mathscr{S}$. More generally, let $\mathscr{F}$ be a space of n-tuples of certain functions and $\mathscr{S}$ denote a set of (algebraic or analytic) "conditions" on such n-tuples of functions. In this case we also use $\Sigma(\mathscr{S})$ to denote the space consisting of all $f \in \mathscr{F}$ which satisfy the conditions $\mathscr{S}$ and call it the *resultant* of $\mathscr{F}$ of class $\mathscr{S}$. Recently such spaces $\Sigma(\mathscr{S})$ and their complements $\mathscr{M}_{\mathscr{S}} = \mathscr{F} - \Sigma(\mathscr{S})$ have become objects of much interest in a number of areas of mathematics: topology, differential geometry, algebraic geometry, mathematical physics and information science (e.g. [**2**], [**4**], [**5**], [**8**], [**9**], [**11**], [**13**], [**14**], [**15**], [**16**]). In this paper we shall be concerned with the topology of some spaces of this type.

As the first example consider the case when $\mathscr{F}$ is the space $\mathrm{P}^d(\boldsymbol{C})$ consisting of all monic complex polynomials of degree d and $\mathscr{S} = \{$polynomials which have a (complex) n-fold root$\}$. The complement $\mathscr{M}_{\mathscr{S}} = \mathrm{P}^d(\boldsymbol{C}) - \Sigma(\mathscr{S})$ is the space $\mathrm{SP}^d_n(\boldsymbol{C})$ consisting of all monic complex polynomials

$$f(z) = z^d + a_{d-1}z^{d-1} + \cdots + a_1 z + a_0 \in \boldsymbol{C}[z]$$

of degree d which have no n-fold roots. For $n = 2$ this space is homeomorphic to the configuration space $C_d(\boldsymbol{C})$ consisting of all d distinct points of $\boldsymbol{C}$ and it is also homotopy equivalent to the classifying space of Br_d—the braid group on d strings ([**1**], [**15**]). Its topology has been much investigated from the point of view of knot theory and mathematical physics.

Similarly, when $\mathscr{F} = (\mathrm{P}^d(\boldsymbol{C}))^n$ and $\mathscr{S} = \{$polynomials with a common root$\}$, the complement $\mathscr{M}_{\mathscr{S}}$ is the resultant

$$\{(p_1(z), \dots, p_n(z)) \in (\boldsymbol{C}[z])^n : p_j(z) \text{ is a monic polynomial of degree } d,$$
$$p_1(z) = \cdots = p_n(z) = 0 \text{ have no common roots}\}.$$

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Since this space is homeomorphic to the space $\mathrm{Hol}_d^*(S^2, \boldsymbol{C}\mathrm{P}^{n-1})$ consisting of all base-point preserving holomorphic maps $h : S^2 \to \boldsymbol{C}\mathrm{P}^{n-1}$ with degree d, which is important in the study of gauge theory, it has been considered by a number of authors ([**2**], [**5**], [**6**], [**9**]). In particular, in [**6**] we investigated the relationship between the spaces $\mathrm{SP}_n^d(\boldsymbol{C})$ and $\mathrm{Hol}_d^*(S^2, \boldsymbol{C}\mathrm{P}^{n-1})$ and obtained the following result:

Theorem 1.1 ([**6**]). *If $n \geq 3$, there is a homotopy equivalence*

$$(^*) \qquad \mathrm{SP}_n^d(\boldsymbol{C}) \simeq \mathrm{Hol}_{[d/n]}^*(S^2, \boldsymbol{C}\mathrm{P}^{n-1})$$

where $[x]$ denotes the integer part of a real number x. □

Remark. For $n = 2$, the existence of a *stable homotopy* equivalence (*), was proved by F. Cohen, R. Cohen, B. Mann and R. Milgram ([**2**]).

In this paper, we prove a similar result for certain *real* singularity classes (or conditions) $\mathscr{S}_{\boldsymbol{R}}$. For $\boldsymbol{K} = \boldsymbol{R}$ or $\boldsymbol{C}$, let $\mathrm{P}^d(\boldsymbol{K})$ denote the space consisting of all monic polynomials

$$g(z) = z^d + b_{d-1}z^{d-1} + \cdots + b_1 z + b_0 \quad (b_j \in \boldsymbol{K})$$

of degree d with coefficients in $\boldsymbol{K}$. For $\mathscr{S}_{\boldsymbol{R}} = \{$polynomials with an n fold *real* root$\}$ and $\mathscr{F} = \mathrm{P}^d(\boldsymbol{K})$, we denote by $\mathrm{P}_n^d(\boldsymbol{K})$ the complement $\mathrm{P}^d(\boldsymbol{K}) - \Sigma(\mathscr{S}_{\boldsymbol{R}})$. Thus $\mathrm{P}_n^d(\boldsymbol{K})$ is the space consisting of all monic polynomials $f(z) \in \boldsymbol{K}[z]$ of degree d which have no n fold *real* roots (but may have complex ones of arbitrary multiplicity). Similarly, for $\mathscr{F} = (\mathrm{P}^d(\boldsymbol{K}))^n$ and $\mathscr{S}_{\boldsymbol{R}} = \{$polynomials with a common *real* root$\}$, we denote by $Q_{(n)}^d(\boldsymbol{K})$ the complement $(\mathrm{P}^d(\boldsymbol{K}))^n - \Sigma(\mathscr{S}_{\boldsymbol{R}})$. Thus $Q_{(n)}^d(\boldsymbol{K})$ is the space consisting of all n-tuples $(q_1(z), q_2(z), \ldots, q_n(z)) \in (\boldsymbol{K}[z])^n$ of monic polynomials over $\boldsymbol{K}$ of degree d and such that $q_1(z), q_2(z), \ldots, q_n(z)$ have no *real* common roots (but may have common *complex* roots).

For $\boldsymbol{K} = \boldsymbol{R}$, the topology of the space $\mathrm{P}_n^d(\boldsymbol{K})$ has been investigated by several authors ([**1**], [**5**], [**12**], [**15**], [**16**]). In fact, Vassiliev ([**12**], [**15**], [**16**]) explicitly determined the homotopy type of $\mathrm{P}_n^d(\boldsymbol{R})$:

Theorem 1.2 ([**12**], [**15**]). *If $n \geq 4$, there is a homotopy equivalence*

$$\mathrm{P}_n^d(\boldsymbol{R}) \simeq J_{[d/n]}(\Omega S^{n-1})$$

where $J_m(\Omega S^{k-1}) \simeq S^{k-2} \cup e^{2(k-2)} \cup e^{3(k-2)} \cup \cdots \cup e^{m(k-2)} \subset \Omega S^{k-1}$ denotes the m-th stage of the James filtration of ΩS^{k-1} ([**7**]). □

Complex conjugation induces $\boldsymbol{Z}/2$-actions on the spaces $\mathrm{P}_n^d(\boldsymbol{C})$ and $Q_{(n)}^d(\boldsymbol{C})$. Similarly it also induces a $\boldsymbol{Z}/2$-action on $J_d(\Omega^{2n-1}) \subset \Omega S^{2n-1}$. Remark that their corresponding fixed point sets are

$$\begin{cases} \mathrm{P}_n^d(\boldsymbol{C})^{\boldsymbol{Z}/2} = \mathrm{P}_n^d(\boldsymbol{R}) \\ Q_{(n)}^d(\boldsymbol{C})^{\boldsymbol{Z}/2} = Q_{(n)}^d(\boldsymbol{R}) \end{cases} \quad \text{and} \quad \begin{cases} J_d(\Omega S^{2n-1})^{\boldsymbol{Z}/2} = J_d(\Omega S^{n-1}) \\ (\Omega S^{2n-1})^{\boldsymbol{Z}/2} = \Omega S^{n-1}. \end{cases}$$

The purpose of this paper is to prove the following results:

Theorem 1. *If $n \geq 2$, there exists a homotopy equivalence*

$$f_n^d : \mathrm{P}_n^d(\boldsymbol{C}) \xrightarrow{\simeq} J_{[d/n]}(\Omega S^{2n-1})$$

which is a $\boldsymbol{Z}/2$-equivariant homotopy equivalence when $n \geq 4$, and is a $\boldsymbol{Z}/2$-equivariant homology equivalence when $n = 3$.

Theorem 2. *If $n \geq 2$, there exists a homotopy equivalence*

$$f_{(n)}^d : Q_{(n)}^d(\boldsymbol{C}) \to J_d(\Omega S^{2n-1})$$

which is a $\boldsymbol{Z}/2$-equivariant homotopy equivalence when $n \geq 4$, and is a $\boldsymbol{Z}/2$-equivariant homology equivalence when $n = 3$.

Restricting to the fixed point sets, we obtain:

Corollary 3. *If $n \geq 4$, the maps*

$$(f_n^d)^{\boldsymbol{Z}/2} : P_n^d(\boldsymbol{R}) \xrightarrow{\simeq} J_{[d/n]}(\Omega S^{n-1}) \quad \text{and} \quad (f_{(n)}^d)^{\boldsymbol{Z}/2} : Q_{(n)}^d(\boldsymbol{R}) \xrightarrow{\simeq} J_d(\Omega S^{n-1})$$

are both homotopy equivalences. □

From the theorem of Vassiliev above and theorems 1 and 2, we can deduce:

Corollary 4. (1) *If $n \geq 2$, there exists a homotopy equivalence $\mathrm{P}_n^d(\boldsymbol{C}) \simeq Q_{(n)}^{[d/n]}(\boldsymbol{C})$.*
(2) *If $n \geq 4$, there exists a homotopy equivalence $\mathrm{P}_n^d(\boldsymbol{R}) \simeq Q_{(n)}^{[d/n]}(\boldsymbol{R})$.* □

We can regard both parts of the above corollary as *real* analogues of theorem 1.1 (though their proof is quite different).

We shall call a map $f : X \to Y$ a *homotopy* (respectively *homology*) *equivalence up to dimension k* if the induced homomorphism $f_* : \pi_j(X) \to \pi_j(Y)$ (respectively $f_* : H_j(X, \boldsymbol{Z}) \to H_j(Y, \boldsymbol{Z})$) is bijective when $j < k$ and surjective when $j = k$.

Corollary 5. *Let $N(d, n) = (d+1)(n-2) - 1$.*
(1) *If $n \geq 2$, there is a map $\mathrm{P}_n^d(\boldsymbol{C}) \to \Omega S^{2n-1}$ which is a homotopy equivalence up to dimension $N([d/n], 2n) = ([d/n]+1)(2n-2) - 1$.*
(2) *If $n \geq 2$, there is a map $Q_{(n)}^d(\boldsymbol{C}) \to \Omega S^{2n-1}$ which is a homotopy equivalence up to dimension $N(d, 2n) = (d+1)(2n-2) - 1$.*
(3) *There is a map $Q_{(n)}^d(\boldsymbol{R}) \to \Omega S^{n-1}$ which is a homotopy equivalence up to dimension $N(d, n) = (d+1)(n-2) - 1$ when $n \geq 4$ and is a homology equivalence up to dimension $N(d, 3) = d$.* □

Hence we may regard the spaces $\mathrm{P}_n^d(\boldsymbol{C})$ and $Q_{(n)}^d(\boldsymbol{C})$ as *finite* dimensional models for the infinite dimensional space ΩS^{2n-1}.

In general, if $\mathscr{F} \cong \boldsymbol{R}^N$ for some N, then $\mathscr{M}_{\mathscr{S}}$ and $\overline{\Sigma(\mathscr{S})}$ are mutually S-dual, where $\overline{\Sigma(\mathscr{S})}$ denotes the one point compactification of $\Sigma(\mathscr{S})$. By Alexander's duality, the computation of $H^*(\mathscr{M}_{\mathscr{S}})$ reduces to that of $H_*(\overline{\Sigma(\mathscr{S})})$. Since $\mathscr{S}$ induces a natural filtration of $\overline{\Sigma(\mathscr{S})}$, we can consider the associated spectral sequence which converges to $H_*(\overline{\Sigma(\mathscr{S})})$. Spectral sequences of this type have been frequently used by Vassiliev ([**13**],

[14], [15]). Our approach is to compute such a spectral sequence and combine it with stable results obtained by means of Segal's "scanning method" ([5], [9]).

The plan of this paper is as follows. In §2 we recall the stability theorems given in [5]. In §3 we consider *geometric resolutions* of discriminants (or resultants) and their induced filtrations, which naturally induce spectral sequences of the Vassiliev type. Finally we prove theorems 1 and 2 using the analysis and the comparison of spectral sequences.

§2. Stable results.

In this section we shall recall several "stability" results.

DEFINITION. Let $Q^d_{(n)}(|z|<d)$ denote the subspace of $Q^d_{(n)}(\boldsymbol{C})$ given by

$$\{(q_1(z),\dots,q_n(z))\in Q^d_{(n)}(\boldsymbol{C}):|\alpha|<d \text{ for any root } \alpha \text{ of } q_j(z)\ (j=1,\dots,d)\}.$$

Since $\boldsymbol{C}\cong\{z\in\boldsymbol{C}:|z|<d\}$, there is a homeomorphism $Q^d_{(n)}(\boldsymbol{C})\cong Q^d_{(n)}(|z|<d)$. Let $\alpha_1,\dots,\alpha_n\in\boldsymbol{R}$ be any mutually distinct real numbers such that $\alpha_j>d$ for each $1\le j\le n$. Define the stabilization map $s_d:Q^d_{(n)}(\boldsymbol{C})\to Q^{d+1}_{(n)}(\boldsymbol{C})$ by

$$\begin{array}{ccccc} Q^d_{(n)}(\boldsymbol{C}) & \overset{\cong}{\to} & Q^d_{(n)}(|z|<d) & \to & Q^{d+1}_{(n)}(\boldsymbol{C}) \\ & & (q_1(z),\dots,q_n(z)) & \to & ((z-\alpha_1)q_1(z),\dots,(z-\alpha_n)q_n(z)). \end{array}$$

Given another such set of real numbers $\alpha'_1,\dots,\alpha'_n\in\boldsymbol{R}$, we can define the map $s'_d: Q^d_{(n)}(\boldsymbol{C})\to Q^{d+1}_{(n)}(\boldsymbol{C})$ in a similar way. We can then choose a path $\theta:[0,1]\to\boldsymbol{R}^n$ such that $\theta(0)=(\alpha_1,\dots,\alpha_n)$, $\theta(1)=(\alpha'_1,\dots,\alpha'_n)$ and that any points of $\theta(t)$ are mutually distinct for any $t\in[0,1]$. This induces a homotopy between s_d and s'_d which shows that the homotopy class of the map s_d is independent of the choices of the numbers $\alpha_1,\dots,\alpha_n$.

Let

$$Q^\infty_{(n)}(\boldsymbol{C})=\lim_{d\to\infty}Q^d_{(n)}(\boldsymbol{C})$$

denote the (homotopy) direct limit of

$$Q^1_{(n)}(\boldsymbol{C})\overset{s_1}{\to}Q^2_{(n)}(\boldsymbol{C})\overset{s_2}{\to}Q^3_{(n)}(\boldsymbol{C})\overset{s_3}{\to}Q^4_{(n)}(\boldsymbol{C})\overset{s_4}{\to}\cdots.$$

In the same way we define the direct limit

$$Q^\infty_{(n)}(\boldsymbol{R})=\lim_{d\to\infty}Q^d_{(n)}(\boldsymbol{R}).$$

Next, we define a map $j^d_n:Q^d_{(n)}(\boldsymbol{C})\to\Omega(\boldsymbol{C}^n-\{0\})/\boldsymbol{R}^*\cong\Omega(\boldsymbol{R}^{2n}-\{0\})/\boldsymbol{R}^*\cong\Omega\boldsymbol{R}\mathrm{P}^{2n-1}$ by

$$j^d_n(q_1(z),q_2(z),\dots,q_n(z))(t)=\begin{cases}[q_1(t):q_2(t):\cdots:q_n(t)] & \text{if } t\in\boldsymbol{R}\\ [1:1:1:\cdots:1] & \text{if } t=\infty\end{cases}$$

for $t\in S^1=\boldsymbol{R}\cup\infty$. If $\Omega_{[d]}\boldsymbol{R}\mathrm{P}^{2n-1}$ denotes the path component corresponding to $[d]\in\pi_1(\boldsymbol{R}\mathrm{P}^{2n-1})=\boldsymbol{Z}/2$, then $j^d_n:Q^d_{(n)}(\boldsymbol{C})\to\Omega_{[d]}\boldsymbol{R}\mathrm{P}^{2n-1}$.

We recall the following result which can be proved by using Segal's "scanning" method ([**5**], [**9**]):

Theorem 2.1 ([**5**]). *If* $n \geq 2$, *the maps* $j_n^d : Q_{(n)}^d(\boldsymbol{C}) \to \Omega_{[d]}\boldsymbol{R}\mathrm{P}^{2n-1}$ *induce a homotopy equivalence*

$$j_{(n)}^{\infty} : \lim_{d\to\infty} Q_{(n)}^d(\boldsymbol{C}) \xrightarrow{\simeq} \lim_{d\to\infty} \Omega_{[d]}\boldsymbol{R}\mathrm{P}^{2n-1} \simeq \Omega_0\boldsymbol{R}\mathrm{P}^{2n-1} \simeq \Omega S^{2n-1}.$$

Moreover, if $n \geq 3$, *the map* $j_{(n)}^{\infty}$ *is a* $\boldsymbol{Z}/2$*-equivariant homotopy equivalence.*

Proof. Let $n \geq 2$. It follows from [**5**] that $j_{(n)}^{\infty}$ is a homotopy equivalence and that it is a $\boldsymbol{Z}/2$-equivariant homotopy equivalence when $n \geq 4$. The case $n = 3$ was not considered in [**5**], but the argument given there works without change in this case also provided one can show that $\pi_1(Q_{(3)}^d(\boldsymbol{R}))$ is abelian. This can be proved by the method used in the appendix of [**4**]. □

Definition. We first treat the case $\boldsymbol{K} = \boldsymbol{C}$. Let $\mathrm{P}_n^d(|z| < d) \subset \mathrm{P}_n^d(\boldsymbol{C})$ be the subspace

$$\mathrm{P}_n^d(|z| < d) = \left\{ f(z) = \prod_{j=1}^{d}(z - \alpha_j) \in \mathrm{P}_n^d(\boldsymbol{C}) : |\alpha_j| < d \ \text{for any} \ j \right\}.$$

We identify $\mathrm{P}_n^d(\boldsymbol{C}) \cong \mathrm{P}_n^d(|z| < d)$ and define a stabilization map $s_d : \mathrm{P}_n^d(\boldsymbol{C}) \to \mathrm{P}_n^{d+1}(\boldsymbol{C})$ by

$$\begin{array}{ccccc} \mathrm{P}_n^d(\boldsymbol{C}) & \xrightarrow{\cong} & \mathrm{P}_n^d(|z| < d) & \to & \mathrm{P}_n^{d+1}(\boldsymbol{C}) \\ & & f(z) & \to & f(z)(z - \alpha) \end{array}$$

where $\alpha \in \boldsymbol{C}$ can be any fixed complex number such that $|\alpha| > d$. Let

$$\lim_{d\to\infty} \mathrm{P}_n^d(\boldsymbol{C})$$

denote the direct limit

$$\mathrm{P}_n^1(\boldsymbol{C}) \xrightarrow{s_1} \mathrm{P}_n^2(\boldsymbol{C}) \xrightarrow{s_2} \mathrm{P}_n^3(\boldsymbol{C}) \xrightarrow{s_3} \mathrm{P}_n^4(\boldsymbol{C}) \xrightarrow{s_4} \cdots\cdots.$$

Finally we define the *jet* map

$$jet_n^d : \mathrm{P}_n^d(\boldsymbol{C}) \to \Omega_{[d]}(\boldsymbol{C}^n - \{0\})/\boldsymbol{R}^* \cong \Omega_{[d]}\boldsymbol{R}\mathrm{P}^{2n-1}$$

by

$$jet_n^d(f)(z) = \begin{cases} [f(z) : f'(z) : \cdots : f^{(n-1)}(z)] & \text{if } z \in \boldsymbol{R} \\ [1 : 1 : \cdots : 1] & \text{if } z = \infty \end{cases}$$

for $f \in \mathrm{P}_n^d(\boldsymbol{C})$ and $z \in S^1 = \boldsymbol{R} \cup \infty$.

Next we consider the case $\boldsymbol{K} = \boldsymbol{R}$. We define the space

$$\lim_{d\to\infty} \mathrm{P}_n^d(\boldsymbol{R})$$

and the jet map $jet_n^d : \mathrm{P}_n^d(\boldsymbol{R}) \to \Omega_{[d]}\boldsymbol{R}\mathrm{P}^{n-1}$ exactly as above. Similarly we can prove the following result:

Theorem 2.2 ([**5**]). *If $n \geq 2$, the jet maps $jet_n^d : \mathrm{P}_n^d(\boldsymbol{C}) \to \Omega_{[d]}\boldsymbol{R}\mathrm{P}^{2n-1}$ induce a homotopy equivalence*

$$j_n^\infty : \lim_{d\to\infty} \mathrm{P}_n^d(\boldsymbol{C}) \xrightarrow{\simeq} \lim_{d\to\infty} \Omega_{[d]}\boldsymbol{R}\mathrm{P}^{2n-1} \simeq \Omega_0\boldsymbol{R}\mathrm{P}^{2n-1} \simeq \Omega S^{2n-1}.$$

Moreover, if $n \geq 3$, the map j_n^∞ is a $\boldsymbol{Z}/2$-equivariant homotopy equivalence. □

Observe that it follows from theorems 2.1 and 2.2 that there is a homotopy equivalence

$$\lim_{d\to\infty} \mathrm{P}_n^d(\boldsymbol{K}) \xrightarrow{\simeq} \lim_{d\to\infty} Q_{(n)}^d(\boldsymbol{K}) \tag{2.3}$$

when $\boldsymbol{K} = \boldsymbol{C}$ and $n \geq 2$, or when $\boldsymbol{K} = \boldsymbol{R}$ and $n \geq 3$. In fact, we can describe the homotopy equivalence (2.3) explicitly. For this purpose, we define the *jet* embedding $\widetilde{jet}_n^d : \mathrm{P}_n^d(\boldsymbol{K}) \to Q_{(n)}^d(\boldsymbol{K})$ by

$$f(z) \mapsto (f(z), f(z) + f'(z), f(z) + f''(z), \dots, f(z) + f^{(n-1)}(z)).$$

Then we have:

Proposition 2.4. *Assume that $\boldsymbol{K} = \boldsymbol{C}$ and $n \geq 2$ or that $\boldsymbol{K} = \boldsymbol{R}$ and $n \geq 3$. Then the jet embedding $\widetilde{jet}_n^d : \mathrm{P}_n^d(\boldsymbol{K}) \to Q_{(n)}^d(\boldsymbol{K})$ induces a homotopy equivalence* (2.3) *as $d \to \infty$.*

Proof. Since the proofs are analogous, we only consider the case $\boldsymbol{K} = \boldsymbol{C}$. Note that there is a homotopy commutative diagram

$$\begin{array}{ccc} \mathrm{P}_n^d(\boldsymbol{C}) & \xrightarrow{jet_n^d} & \Omega_{[d]}\boldsymbol{R}\mathrm{P}^{2n-1} \\ {\scriptstyle \widetilde{jet}_n^d}\downarrow & & \downarrow{\scriptstyle =} \\ Q_{(n)}^d(\boldsymbol{C}) & \xrightarrow{j_n^d} & \Omega_{[d]}\boldsymbol{R}\mathrm{P}^{2n-1}. \end{array}$$

Hence if $d \to \infty$, the result follows from theorems 2.1 and 2.2. □

§3. Spectral sequences and unstable results.

In this section, we shall prove theorems 1 and 2. We start with theorem 2. The key ingredients of the proof are the following two theorems:

Theorem 3.1. *Let $n \geq 3$.* (1) *The stabilization map $s_d : Q_{(n)}^d(\boldsymbol{R}) \to Q_{(n)}^{d+1}(\boldsymbol{R})$ is a homotopy equivalence up to dimension $N(d,n) = (d+1)(n-2) - 1$ when $n \geq 4$ and is a homology equivalence up to dimension d when $n = 3$.*

(2) *The cohomology of* $Q^d_{(n)}(\boldsymbol{R})$ *is given by*

$$H^j(Q^d_{(n)}(\boldsymbol{R}), \boldsymbol{Z}) \cong \begin{cases} \boldsymbol{Z} & \text{if } j = k(n-2) \text{ and } 0 \le k \le d \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.2. *Let* $n \ge 2$. (1) *The stabilization map* $s_d : Q^d_{(n)}(\boldsymbol{C}) \to Q^{d+1}_{(n)}(\boldsymbol{C})$ *is a homotopy equivalence up to dimension* $N(d, 2n) = (d+1)(2n-2) - 1$.

(2) *The cohomology of* $Q^d_{(n)}(\boldsymbol{C})$ *is given by*

$$H^j(Q^d_{(n)}(\boldsymbol{C}), \boldsymbol{Z}) \cong \begin{cases} \boldsymbol{Z} & \text{if } j = k(2n-2) \text{ and } 0 \le k \le d \\ 0 & \text{otherwise.} \end{cases}$$

Before proving theorem 3.1 and 3.2, we complete the proof of theorem 2.

Proof of theorem 2. Using the fact that the $\boldsymbol{Z}/2$-action on space $Q^d_{(n)}(\boldsymbol{C})$ is induced by complex conjugation, one can construct an $\boldsymbol{Z}/2$-equivariant simplicial complex structure on $Q^d_{(n)}(\boldsymbol{C})$ whose fixed point set is a simplicial decomposition of $Q^d_{(n)}(\boldsymbol{C})^{\boldsymbol{Z}/2} = Q^d_{(n)}(\boldsymbol{R})$. Since $Q^d_{(n)}(\boldsymbol{C})$ is equivariantly simply connected, by theorem 3.1 and 3.2 we may assume that it has the structure of a $\boldsymbol{Z}/2$-CW complex of dimension $N(d, 2n) = (d+1)(2n-2) - 1$. Note also that $J_d(\Omega S^{2n-1})$ has a natural structure of a $\boldsymbol{Z}/2$-complex with $J_d(\Omega S^{n-1})$ as its $\boldsymbol{Z}/2$-fixed point set. Let us consider the $\boldsymbol{Z}/2$-equivariant map

$$\tilde{f}^d_{(n)} : Q^d_{(n)}(\boldsymbol{C}) \longrightarrow \lim_{k \to \infty} Q^k_{(n)}(\boldsymbol{C}) \xrightarrow[\simeq]{j^{\infty}_{(n)}} \Omega S^{2n-1}.$$

Then by theorem 3.2, $\tilde{f}^d_{(n)}$ is a homology equivalence up to dimension $N(d, 2n)$. By the equivariant cellular approximation theorem there is a $\boldsymbol{Z}/2$-equivariant cellular map

$$f^d_{(n)} : Q^d_{(n)}(\boldsymbol{C}) \to J_d(\Omega S^{2n-1})$$

which is a homology equivalence up to dimension $N(d, 2n)$, and such that that $\tilde{f}^d_n$ and f^d_n are $\boldsymbol{Z}/2$-equivariant homotopic.

To show that f^d_n is an equivariant homotopy equivalence we need to show that it induces homotopy equivalences on the fixed point sets under the action of the whole group $\boldsymbol{Z}/2$ and the identity subgroup.

First, consider the latter. Since $N(d, 2n) > d(2n-2)$, the induced homomorphism

$$(f^d_{(n)})_* : H_j(Q^d_{(n)}(\boldsymbol{C}), \boldsymbol{Z}) \xrightarrow{\cong} H_j(J_d(\Omega S^{2n-1}), \boldsymbol{Z})$$

is bijective when $j \le d(2n-2)$. However, since $H_j(Q^d_{(n)}(\boldsymbol{C}), \boldsymbol{Z}) = H_j(J_d(\Omega S^{n-1}), \boldsymbol{Z}) = 0$ for any $j > d(2n-2), (f^d_{(n)})_*$ is bijective for any j. Hence $f^d_{(n)}$ is a homology equivalence. Since $Q^d_{(n)}(\boldsymbol{C})$ and $J_d(\Omega S^{2n-1})$ are simply connected $f^d_{(n)}$ is a homotopy equivalence.

Next consider the induced map on the fixed point sets under the $\boldsymbol{Z}/2$-action,

$$(f^d_{(n)})^{\boldsymbol{Z}/2} : Q^d_{(n)}(\boldsymbol{C})^{\boldsymbol{Z}/2} = Q^d_{(n)}(\boldsymbol{R}) \to J_d(\Omega S^{n-1}) = J_d(\Omega^{2n-1})^{\boldsymbol{Z}/2}$$

It remains to show that $(f^d_{(n)})^{\boldsymbol{Z}/2}$ is a homotopy equivalence when $n \geq 4$ and is a homology equivalence when $n = 3$.

Assume that $n \geq 3$. By theorems 2.1 and 3.1 the restriction $(\tilde{f}^d_{(n)})^{\boldsymbol{Z}/2} : Q^d_{(n)}(\boldsymbol{R}) \to \Omega S^{n-1}$ is a homology equivalence up to dimension $N(d,n)$. Since $\tilde{f}^d_n$ and f^d_n are $\boldsymbol{Z}/2$-equivariant homotopic, the map $(f^d_{(n)})^{\boldsymbol{Z}/2}$ is a homology equivalence up to dimension $N(d,n)$. Because $N(d,n) > d(n-2)$, the induced homomorphism

$$(f^d_{(n)})^{\boldsymbol{Z}/2}_* : H_j(Q^d_{(n)}(\boldsymbol{R}), \boldsymbol{Z}) \xrightarrow{\cong} H_j(J_d(\Omega S^{n-1}), \boldsymbol{Z})$$

is bijective for any $j \leq d(n-2)$. However, since $H_j(Q^d_{(n)}(\boldsymbol{R}), \boldsymbol{Z}) = H_j(J_d(\Omega S^{n-1}), \boldsymbol{Z}) = 0$ for any $j > d(n-2)$, $(f^d_{(n)})^{\boldsymbol{Z}/2}_*$ is bijective for any j. Hence $(f^d_{(n)})^{\boldsymbol{Z}/2}$ is a homology equivalence. However, since both spaces $Q^d_{(n)}(\boldsymbol{R})$ and $J_d(\Omega S^{n-1})$ are simply connected when $n \geq 4$, the restriction $(f^d_{(n)})^{\boldsymbol{Z}/2}$ is, in this case, a homotopy equivalence. □

Proof of theorem 3.1. For a locally connected space X, let $\bar{X}$ denote the one-point compactification of $X, \bar{X} = X \cup \{\infty\}$, and let $\bar{H}_j(X)$ be the Borel-Moore homology group $\bar{H}_j(X) = H_j(\bar{X})$.

We shall first prove assertion (2). Let $\mathrm{P}^d(\boldsymbol{R})$ denote the space consisting of all monic polynomials $f(z) = z^d + a_1 z^{d-1} + \cdots + a_d \in \boldsymbol{R}[z]$ of degree d. (Hence $\mathrm{P}^d(\boldsymbol{R}) \cong \boldsymbol{R}^d$.)

Let $\Sigma^d_n = \Sigma^d_n(\mathscr{S}_{\boldsymbol{R}})$ be discriminant defined by

$$\Sigma^d_n = \{(p_1(z), \ldots, p_n(z)) \in (\mathrm{P}^d(\boldsymbol{R}))^n : p_1(\alpha) = \cdots = p_n(\alpha) = 0 \text{ for some } \alpha \in \boldsymbol{R}\}.$$

Then $Q^d_{(n)} = Q^d_{(n)}(\boldsymbol{R}) = (\mathrm{P}^d(\boldsymbol{R}))^n - \Sigma^d_n$. Since $(\mathrm{P}^d(\boldsymbol{R}))^n \cong \boldsymbol{R}^{dn}$, it follows from the Alexander duality that there is a natural isomorphism

$$(^*) \qquad H^j(Q^d_{(n)}(\boldsymbol{R}), \boldsymbol{Z}) \cong \bar{H}_{dn-1-j}(\Sigma^d_n, \boldsymbol{Z}) \quad \text{for } 1 \leq j \leq dn-2$$

and so we try to compute $\bar{H}_*(\Sigma^d_n)$.

Let $I : \boldsymbol{R} \to \boldsymbol{R}^d$ be the Veronese embedding, $I(t) = (t, t^2, \ldots, t^d)$ for $t \in \boldsymbol{R}$. Let $f = (q_1(z), \ldots, q_n(z)) \in \Sigma^d_n$ and suppose that $q_1(z), q_2(z), \ldots, q_n(z)$ have at least t-distinct common real roots $\{z_1, \ldots, z_t\} \subset \boldsymbol{R}$. We denote by $\Delta(f, \{z_1, \ldots, z_t\}) \subset \boldsymbol{R}^n$ the *open* $(t-1)$-dimensional simplex with vertices $\{I(z_1), \ldots, I(z_t)\}$. Define the *geometric resolution* $G(\Sigma^d_n)$ of Σ^d_n by

$$G(\Sigma^d_n) = \bigcup_{f \in \Sigma^d_n; \{z_1, \ldots, z_t\}} \{f\} \times \Delta(f, \{z_1, \ldots, z_t\}) \subset \Sigma^d_n \times \boldsymbol{R}^n.$$

The first projection defines an open proper map $\pi : G(\Sigma^d_n) \to \Sigma^d_n$, and this induces a map between one point compactification spaces $\bar{\pi} : \overline{G(\Sigma^d_n)} \to \overline{\Sigma^d_n}$. It is known ([**14**]) that the map $\bar{\pi} : \overline{G(\Sigma^d_n)} \xrightarrow{\simeq} \overline{\Sigma^d_n}$ is a homotopy equivalence. Define subspaces $F_p \subset \overline{G(\Sigma^d_n)}$ by

$$F_p = \begin{cases} \{\infty\} \cup \displaystyle\bigcup_{f \in \Sigma^d_n; \{z_1, \ldots, z_t\}, t \leq p} \{f\} \times \Delta(f, \{z_1, \ldots, z_t\}) & \text{if } p \geq 1 \\ \{\infty\} & \text{if } p = 0. \end{cases}$$

There is an increasing filtration

$$F_0 = \{\infty\} \subset F_1 \subset F_2 \subset \cdots \subset F_d = F_{d+1} = \cdots = \overline{G(\Sigma_n^d)} \simeq \overline{\Sigma_n^d}$$

and this induces a spectral sequence

$$\{E^r_{p,q}, d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}\} \Rightarrow \bar{H}_{p+q}(G(\Sigma_n^d)) \cong \bar{H}_{p+q}(\Sigma_n^d)$$

such that

$$E^1_{p,q} = \begin{cases} \bar{H}_{p+q}(F_p - F_{p-1}) & \text{if } 1 \le p \le d \\ 0 & \text{otherwise.} \end{cases}$$

For each $1 \le p \le d$, there is a fibre bundle $F_p - F_{p-1} \to C_p(\boldsymbol{R}) \cong \boldsymbol{R}^p$ with fibre $\boldsymbol{R}^{nd-1-p(n-1)}$. Hence, using Thom isomorphism, for each $1 \le p \le d$,

$$E^1_{p,q} \cong \bar{H}_{p+q-\{nd-1-p(n-1)\}}(\boldsymbol{R}^p) = \begin{cases} \boldsymbol{Z} & \text{if } p+q = dn-1-p(n-2),\ 1 \le p \le d \\ 0 & \text{otherwise.} \end{cases}$$

For dimensional reasons we have

$$E^1_{p,q} = E^2_{p,q} = \cdots = E^\infty_{p,q} = \begin{cases} \boldsymbol{Z} & \text{if } p+q = dn-1-p(n-2),\ 1 \le p \le d \\ 0 & \text{otherwise} \end{cases}$$

Hence from (*),

$$H^j(Q^d_{(n)}(\boldsymbol{R}), \boldsymbol{Z}) \cong \begin{cases} \boldsymbol{Z} & \text{if } j = k(n-2) \text{ and } 0 \le k \le d \\ 0 & \text{otherwise.} \end{cases}$$

and (2) is proved.

Next we shall prove assertion (1). Using the same method as above, there is an increasing filtration

$$'F_0 = \{\infty\} \subset {}'F_1 \subset {}'F_2 \subset \cdots \subset {}'F_d = {}'F_{d+1} = \cdots = \overline{G(\Sigma_n^{d+1})} \simeq \overline{\Sigma_n^{d+1}}$$

and this induces a spectral sequence

$$\{'E^r_{p,q}, 'd^r : {}'E^r_{p,q} \to {}'E^r_{p-r,q+r-1}\} \Rightarrow \bar{H}_{p+q}(G(\Sigma_n^{d+1})) \cong \bar{H}_{p+q}(\Sigma_n^{d+1})$$

such that

$$'E^1_{p,q} = {}'E^2_{p,q} = \cdots = {}'E^\infty_{p,q} = \begin{cases} \boldsymbol{Z} & \text{if } p+q = dn-1-p(n-2),\ 1 \le p \le d+1 \\ 0 & \text{otherwise.} \end{cases}$$

The stabilization map $Q^d_{(n)}(\boldsymbol{R}) \to Q^{d+1}_{(n)}(\boldsymbol{R})$ naturally induces maps between the corresponding filtrations $\{F_p \to {}'F_p\}$ such that for each $p \ge 1$ there is a commutative diagram

$$\begin{array}{ccc} F_p & \longrightarrow & 'F_p \\ \downarrow & & \downarrow \\ C_p(\boldsymbol{R}) & \xrightarrow{=} & C_p(\boldsymbol{R}). \end{array}$$

Hence the maps $F_p \to {}'F_p$ also induce a homomorphism of spectral sequences

$$\{h^r_{p,q} : E^r_{p,q} \to {}'E^r_{p,q}\}$$

such that $h^\infty_{p,q}$ is isomorphic except when $(p,q) \neq (d+1, dn-1-(d+1)(n-1))$. Thus s_d is a homology equivalence up to dimension $N(d,n) = (d+1)(n-2)-1$. However, when $n \geq 4$, since both spaces $Q^d_{(n)}(\boldsymbol{R})$ and $Q^{d+1}_{(n)}(\boldsymbol{R})$ are simply connected, s_d is a homotopy equivalence up to dimension $N(d,n)$. □

PROOF OF THEOREM 3.2. Since the proof is completely analogous to that of theorem 3.1 we omit the details. □

Similar methods also prove the following result whose proof we omit:

THEOREM 3.3. *Let $n \geq 2$. (1) The stabilization map $s_d : \mathrm{P}^d_n(\boldsymbol{C}) \to \mathrm{P}^{d+1}_n(\boldsymbol{C})$ is a homotopy equivalence up to dimension $N([d/n], 2n) = ([d/n]+1)(2n-2)-1$.*

(2) *More precisely, the cohomology of $\mathrm{P}^d_n(\boldsymbol{C})$ is given by*

$$H^j(\mathrm{P}^d_n(\boldsymbol{C}), \boldsymbol{Z}) \cong \begin{cases} \boldsymbol{Z} & \text{if } j = k(2n-2) \text{ and } 0 \leq k \leq [d/n] \\ 0 & \text{otherwise.} \end{cases}$$ □

THEOREM 3.4 ([**15**], [**16**]). *Let $n \geq 3$. (1) The stabilization map $s_d : \mathrm{P}^d_n(\boldsymbol{R}) \to \mathrm{P}^{d+1}_n(\boldsymbol{R})$ is a homotopy equivalence up to dimension $N([d/n], n) = ([d/n]+1)(n-2)-1$ when $n \geq 4$, and is a homology equivalence up to dimension $N([d/3], 3) = [d/3]$ when $n = 3$.*

(2) *More precisely, the cohomology of $\mathrm{P}^d_n(\boldsymbol{R})$ is given by*

$$H^j(\mathrm{P}^d_n(\boldsymbol{R}), \boldsymbol{Z}) \cong \begin{cases} \boldsymbol{Z} & \text{if } j = k(n-2) \text{ and } 0 \leq k \leq [d/n] \\ 0 & \text{otherwise.} \end{cases}$$ □

PROOF OF THEOREM 1. The proof uses theorems 2.2, 3.3, 3.4, and an argument analogous to the one used in proving theorem 2. So we leave the details to the reader. □

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References

[1] V. I. Arnold, Some topological invariants of algebraic functions, Trans. Moscow Math. Soc., **21** (1970), 30–52.

[2] F. R. Cohen, R. L. Cohen, B. M. Mann and R. J. Milgram, The topology of rational functions and divisors of surfaces, Acta Math., **166** (1991), 163–221.

[3] R. L. Cohen, J. D. S. Jones and G. B. Segal, Floer's infinite dimensional Morse theory and homotopy theory, Floer Memorial volume (edited by H. Hofer, C. H. Taubes, A. Weinstein and H. Zehnder), Progress in Math., Birkhäuser, **133** (1995), 297–325.

[4] M. A. Guest, A. Kozlowski, and K. Yamaguchi, The topology of spaces of coprime polynomials, Math. Z., **217** (1994), 435–446.

[5] M. A. Guest, A. Kozlowski, and K. Yamaguchi, Spaces of polynomials with roots of bounded multiplicity, Fundamenta Math., **116** (1999), 93–117.
[6] M. A. Guest, A. Kozlowski, and K. Yamaguchi, Stable splitting of the space of polynomials with roots of bounded multiplicity, J. Math. Kyoto Univ., **38** (1998), 351–366.
[7] I. M. James, Reduced product spaces, Ann. Math., **62** (1953), 170–197.
[8] G. B. Segal, Configuration spaces and iterated loop spaces, Invent. Math., **21** (1973), 213–221.
[9] G. B. Segal, The topology of spaces of rational functions, Acta Math., **143** (1979), 39–72.
[10] G. B. Segal, Some results in equivariant homotopy theory, unpublished manuscript.
[11] S. Smale, On the topology of algorithms. I, J. Complexity, **3** (1987), 81–89.
[12] V. A. Vassiliev, Topology of spaces of functions without complicated singularities, Functional Annal. Appl., **23** (1989), 277–286.
[13] V. A. Vassiliev, Topology of complements of discriminants and loop spaces, Advanced in Soviet Math., **1** (1990), 9–21.
[14] V. A. Vassiliev, Cohomology of knot spaces, Advanced in Soviet Math., **1** (1990), 23–69.
[15] V. A. Vassiliev, Complements of Discriminants of Smooth Maps: Topology and Applications, vol. 98, Translations of Math. Monographs, Amer. Math. Soc., 1992 (revised edition 1994).
[16] V. A. Vassiliev, Topology of discriminants and their complements, Proc. ICM, Zürich (1994), 209–226.

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