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Regularity properties of the Azukawa metric

By Włodzimierz Zwonek

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Abstract. Some regularity properties of the Azukawa pseudometric for a wide class of domains (including bounded hyperconvex domains) are proven. Among others we prove that in this class of domains the Azukawa metric is continuous and the upper limit in its definition may be replaced with the limit. Some properties of the pluricomplex Green function with one (as well as with many) pole are also given.

0. Introduction.

The pluricomplex Green function has been studied by many authors and many regularity properties of this function are known. In many cases properties of the Green function can be transported without large effort to its infinitesimal version, i.e. to the Azukawa pseudometric. Nevertheless, it is not always the case.

In this paper we want to show some new properties of the Azukawa pseudometric, which come from analoguous ones for the Green function. The good class of domains for which the Green function has 'nice' properties are bounded hyperconvex domains. Among others it is known that the Green function of a domain from this class is continuous. Below we consider a class of domains containing bounded hyperconvex domains and using some localization (with the help of sublevel sets) we show that from the point of view of the Azukawa pseudometric this class of domains is the same as bounded hyperconvex domains. We prove in this paper the continuity of the Azukawa pseudometric for this class of domains.

It is interesting to know whether we can get rid of 'limsup' in the definition of the Azukawa pseudometric and to replace it with 'lim'. We prove that the answer is affirmative in the reasonable class of domains. Nevertheless, we find an example showing that this is not true in general. On the other hand in a class of domains (containing bounded hyperconvex domains) we can take the limit over the larger family of points tending along the given vector to the pole. In particular, we need not have in the definition of the Azukawa pseudometric the pole fixed. One of conclusions of our results is that the Green function is 'almost' symmetric for points lying not far from each other (still in the considered class of domains).

We also study the behaviour of the Green function for balanced domains and relate it to the Green function of their holomorphic envelopes.

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Finally, we consider the Green function with many poles and we show some simple properties of this function making use of its alternate definition (to that with the help of plurisubharmonic functions—see [**Lel**]), namely, the one given in [**Lar-Sig**].

1. Notations, definitions and formulation of most important results.

Following [**Kli1**], [**Kli2**] for a given domain $D \subset \boldsymbol{C}^n$ and points $w, z \in D$ we define *the pluricomplex Green function with pole at* w

$$g_D(w,z)$$
$$:= \sup\{u(z) : u \in \mathrm{PSH}(D),\ u < 0 \text{ and } u(\cdot) - \log\|\cdot - w\| \text{ is bounded above near } w\}$$

(we allow plurisubharmonic functions to be identically equal to $-\infty$).

The infinitesimal version of the Green function is *the Azukawa pseudometric* defined as follows (see [**Azu1**], [**Azu2**]):

$$A_D(z;X) := \limsup_{\lambda \to 0}(g_D(z, z + \lambda X) - \log|\lambda|), \quad z \in D, \quad X \in \boldsymbol{C}^n.$$

Let us denote the unit disk in $\boldsymbol{C}$ by E.

Let us recall that a domain D is *hyperconvex* if there is a negative plurisubharmonic continuous function u defined on D such that for any $\varepsilon > 0$ the sublevel set $\{u < -\varepsilon\}$ is relatively compact in D (note that we do not assume that the hyperconvex domain must be bounded).

A plurisubharmonic function $u : D \mapsto \boldsymbol{R}$ is *maximal* if for any open set $U \subset\subset D$ and for any upper semicontinuous function v on $\overline{U}$, which is plurisubharmonic on U, if $v \le u$ on ∂U, then $v \le u$ on U.

Let us recall some well-known properties of the pluricomplex Green function and the Azukawa pseudometric, which will be helpful in our considerations (see [**Azu1**], [**Azu2**], [**Dem**], [**Jar-Pfl1**], [**Jar-Pfl2**], [**Kli1**] and [**Kli2**]), which are also a good starting point for our considerations.

Theorem 1.1. (i) *If* $F : D_1 \mapsto D_2$ (D_1 *and* D_2 *are domains*) *is a holomorphic mapping then*

$$g_{D_2}(F(w), F(z)) \le g_{D_1}(w,z),$$
$$A_{D_2}(F(w); F'(w)X) \le A_{D_1}(w;X), \quad w, z \in D_1, \quad X \in \boldsymbol{C}^n.$$

(ii) *If the mapping above is a biholomorphism then the inequalities above become equalities.*

(iii) $A_D(w; \lambda X) = A_D(w;X) + \log|\lambda|$ *for any* $w \in D$, $X \in \boldsymbol{C}^n$, $\lambda \in \boldsymbol{C}$.

(iv) *For any* $w \in D$ $g_D(w,\cdot) \in \mathrm{PSH}(D, [-\infty, 0))$. *Moreover*, $g_D(w,z) - \log\|w - z\|$ *is bounded above for* z *close to* w.

(v) $A_D(w;\cdot) \in \mathrm{PSH}(\boldsymbol{C}^n)$, *which in connection with* (iii) *gives us that*

$$\mathscr{I}(D,w) := \{X \in \boldsymbol{C}^n : A_D(w;X) < 0\}$$

is a balanced pseudoconvex domain.

(vi) *For any sequence of domains* $\{D_j\}_{j=1}^{\infty}$ *such that* $D_j \subset D_{j+1}$ *and* $\bigcup_{j=1}^{\infty} D_j = D$ *the following convergences hold*:

$$g_{D_j} \to g_D \quad \text{on } D \times D,$$

$$A_{D_j} \to A_D \quad \text{on } D \times \boldsymbol{C}^n.$$

(vii) *The functions* g_D *and* A_D *are upper semicontinuous. If* D *is a bounded hyperconvex domain then* g_D *is continuous and, even more,* $g_D(w,z) \to 0$ *as* $z \to \partial D$.

(viii) *If* $D_h := \{z \in \boldsymbol{C}^n : h(z) < 1\}$, *where* h *is a non-negative homogeneous upper semicontinuous function, then the following formulas hold*:

$$g_{D_h}(0,z) \leq \log h(z),$$

$$A_{D_h}(0;X) \leq \log h(X), \quad z \in D_h, \quad X \in \boldsymbol{C}^n.$$

If h *is plurisubharmonic (or equivalently* D_h *is pseudoconvex) then in the inequalities above we have the equalities.*

(ix) *If* D *is a bounded domain then* $g_D(w,\cdot)$ *is a maximal function on* $D \setminus \{w\}$.

For fixed $w \in D$ we often consider the following number:

$$\varepsilon(w) := \liminf_{z \to \partial D} g_D(w,z).$$

It is easy to see that for any bounded D we have that $\varepsilon(w) > -\infty$ for any $w \in D$. As we shall see latter if $\varepsilon(w) > -\infty$ then $g_D(w,z) > -\infty$ for any $z \in D$, $z \neq w$.

Our aim is to prove the continuity of the Azukawa pseudometric.

Theorem 4.1. *Let* D *be a domain such that* $\varepsilon(w) > -\infty$ *for any* $w \in D$ *and* g_D *is a continuous function. Then* A_D *is a continuous function (as a function defined on* $D \times \boldsymbol{C}^n$*).*

Note that bounded hyperconvex domains fulfill the assumptions of Theorem 4.1 (as well as the assumptions of all theorems from Section 4)—see Theorem 1.1(vii).

It turns out that in many cases we can replace 'limsup' in the definition of the Azukawa pseudometric with 'lim' as the following result shows:

Theorem 4.2. *Let* $w \in D$, *where* D *is a domain in* $\boldsymbol{C}^n$ *such that* $g_D(w,\cdot)$ *is continuous and* $\varepsilon(w) > -\infty$. *Then*

$$A_D(w;X) = \lim_{\lambda \to 0}(g_D(w, w + \lambda X) - \log|\lambda|), \quad X \in \boldsymbol{C}^n.$$

Let us underline already here that we cannot generalize Theorem 4.2 to all domains—the counterexample is given in Example 4.6, our example is a bounded pseudoconvex domain in $\boldsymbol{C}^2$. However, for many domains some sharper version of Theorem 4.2 remains true, namely:

Corollary 4.4. *Let* D *be a domain such that* g_D *is continuous and* $\varepsilon(w) > -\infty$ *for any* $w \in D$. *Then for any* $w \in D$, $X \in \boldsymbol{C}^n$, $\|X\| = 1$ *the following formula holds*:

$$A_D(w;X) = \lim_{w',w'' \to w,\, w' \neq w'',\, (w'-w'')/\|w'-w''\| \to X}(g_D(w',w'') - \log\|w'-w''\|).$$

As a conclusion of the above results we get some result on symmetry of the Green function. It turns out that the Green function is 'almost' symmetric when both variables are close to each other. More precisely, we have

COROLLARY 4.5. *Let D be as in Theorem* 4.1 *and let $w \in D$ be fixed. Then*

$$\lim_{w', w'' \to w, w' \neq w''} (g_D(w', w'') - g_D(w'', w')) = 0.$$

In Section 3 we prove some result on the Green function for balanced domains, which may be treated as the generalization of Theorem 1.1(viii), which shows that in this case a close relation between the Green function of the domain and that of the envelope of holomorphy exists (at least for points not far from the origin).

Let $D = D_h$ be a balanced domain with the Minkowski function equal to some upper semicontinuous function h. Let $\tilde{h}$ be the largest non-negative homogenuous plurisubharmonic function not larger than h (note that such a function exists). Then we have

THEOREM 3.1. *If D_h is bounded, then*

$$g_{D_h}(0, z) = \log \tilde{h}(z) = g_{D_{\tilde{h}}}(0, z) \quad \text{for } z \in D_h \text{ close enough to } 0.$$

In Section 2 we introduce sublevel sets. We introduce this notion for the Green function with many poles. We give some properties of the sublevel sets, which play later the key role in proofs of the results from Sections 3 and 4.

In Section 5 some properties following from the definition from [**Lar-Sig**] of the pluricomplex Green function with many poles are given.

2. Sublevel sets—definition and basic properties.

Up to now we have dealt with the Green function with one pole. Below we give some properties of the sublevel sets of the Green function. Since the properties of sublevel sets defined below are obtained for the Green function with one pole exactly in the same way as for the Green function with many poles we recall the notion of the Green function with several poles.

Let D be a domain in $\mathbf{C}^n$. Let $\emptyset \neq P \subset D$ be a finite set and let $v : P \mapsto (0, \infty)$. We define *the pluricomplex Green function with poles in P with weights v* as follows (see [**Lel**]):

$$g_D(P; v; z) := \sup\{u(z)\},$$

where the supremum is taken over all $u \in \mathrm{PSH}(D)$, $u < 0$ and such that $u(\cdot) - v(p) \log\|\cdot - p\|$ is bounded from above near p for all $p \in P$.

Note that when $\sharp P = 1$ and $v \equiv 1$ then the above defined function is the pluricomplex Green function defined earlier.

It is well-known that $g_D(P; v; \cdot)$ is plurisubharmonic. Let us recall that if the domain D is bounded, then the Green function $g_D(P; v; \cdot)$ is maximal on $D \setminus P$; if D is a bounded hyperconvex domain, then $g_D(P, v, \cdot)$ is a continuous function, which extends continuously to 0 on the boundary—compare Theorem 1.1(vii) (see [**Dem**] and [**Lel**]).

It turns out that the equivalent definition with the help of analytic disks is possible. Namely, the following equality has been obtained in [**Lar-Sig**] (in case of the Green function with one pole this equality is to be found in [**Edi**] and [**Pol**]):

$$\begin{aligned}(2.1)\qquad & g_D(P;v;z)\\ &= \inf\{g_E(\varphi^{-1}(P)\cap E,\tilde{v},0),\varphi\in\mathcal{O}(\bar{E},D),\varphi(0)=z,E\cap\varphi^{-1}(P)\neq\emptyset\}\\ &= \inf\{g_E(\varphi^{-1}(P)\cap E,\tilde{v},\lambda),\varphi\in\mathcal{O}(\bar{E},D),\varphi(\lambda)=z,E\cap\varphi^{-1}(P)\neq\emptyset\},\end{aligned}$$

where $\tilde{v}(\lambda):=\mathrm{ord}_\lambda(\varphi-\varphi(\lambda))\cdot v(\varphi(\lambda)),\ \lambda\in\varphi^{-1}(P)$.

For $\infty>\varepsilon\geq 0$ let us consider the following *sublevel sets*

$$D(P;v;\varepsilon):=\{z\in D: g_D(P;v;z)<-\varepsilon\}.$$

Note that $D(P;v;\varepsilon)$ is open (the Green function is plurisubharmonic, and therefore upper-semicontinuous). In case of one pole with weight 1 we write $D_\varepsilon(p)$. If it does not lead to misunderstanding, then we omit the set of poles and weight and we simply denote the sublevel set by D_ε.

We start with the problem of the connectivity of sublevel sets.

Our aim is the following:

Lemma 2.1. *Let D be a domain. Let P and v be given as above. Then*:

(i) *For any $\varepsilon\geq 0$ any connected component of the set D_ε has non-empty intersection with P*; *in particular, the number of connected components of D_ε is at most $\sharp P$.*

(ii) *There is ε_0 with $\infty\geq\varepsilon_0>0$ such that for any ε with $0<\varepsilon<\varepsilon_0$ the set D_ε is connected and for any ε with $\varepsilon_0<\varepsilon$ the set D_ε is disconnected.*

Proof. Let us put $u(z):=g_D(P;v;z)$, $z\in D$. Suppose that (i) does not hold. Let U be a connected component of D_ε, $U\cap P=\emptyset$. The upper-semicontinuity of the Green function implies that U is open. We know that $u<-\varepsilon$ on U and $u(z)\geq-\varepsilon$ for $z\in\partial U\cap D$. Consequently, the function

$$v(z):=\begin{cases}-\varepsilon, & \text{if } z\in U,\\ u(z), & \text{if } z\in D\setminus U\end{cases}$$

is plurisubharmonic. Moreover, from the definition of the Green function and the fact that $U\cap P=\emptyset$ we have $u\geq v$, which implies that $u(z)\geq-\varepsilon$ for $z\in U$—contradiction.

We are remained with (ii). First we prove the following property:

$$\text{(2.2)}\qquad \begin{array}{l}\text{if for some } \varepsilon>0,\ D_\varepsilon \text{ is connected then } D_{\varepsilon'} \text{ is connected}\\ \text{for any } \varepsilon' \text{ with } 0<\varepsilon'<\varepsilon.\end{array}$$

To prove (2.2) note that there is a compact connected set K with $P\subset K\subset D_\varepsilon$. But $D_\varepsilon\subset D_{\varepsilon'}$, so P is contained in one connected component of $D_{\varepsilon'}$, which in view of (i) finishes the proof of (2.2).

To finish the proof of (ii) it is sufficient to show the existence of $\varepsilon>0$ such that D_ε is connected. Let us take a connected compact set $K\subset D$ such that $P\subset K$. We know that $g_D(P;v;\cdot)_{|K}$ is bounded by some $-\varepsilon_0$. Therefore, $P\subset K\subset D_\varepsilon$ for any ε with

$0 < \varepsilon < \varepsilon_0$, which shows that all poles lie in one connected component of D_ε. This, in connection with (i) finishes the proof. □

Remark 2.2. Note that if D is pseudoconvex then the sets D_ε are pseudoconvex. In general case (D is not pseudoconvex) it need not always be the case. Nevertheless, if D is a bounded domain then for large ε the sublevel sets D_ε are pseudoconvex. It follows from the fact that for large ε the set D_ε satisfies that $D_\varepsilon \subset \bigcup_{p \in P} \boldsymbol{B}(p, r) \subset D$ for some $r > 0$ such that the balls $\bar{\boldsymbol{B}}(p, r)$ are pairwise disjoint for $p \in P$. Even more generally, for any domain D, if $D_\varepsilon \subset U \subset D$, where U is pseudoconvex, then D_ε is pseudoconvex.

Now we come back to the situation of the Green function with one pole. First, note that in view of Lemma 2.1 all sublevel sets in this case are connected.

Below for unbounded domains D we assume that $\infty \in \partial D$.

The lemma below will play a fundamental role in the later considerations:

Lemma 2.3. *Let D be a domain in $\boldsymbol{C}^n$, $p \in D$ and $D_\varepsilon = D_\varepsilon(p)$. Then the following formulas hold*:

$$g_{D_\varepsilon}(p, z) = g_D(p, z) + \varepsilon, \tag{2.3}$$

$$A_{D_\varepsilon}(p; X) = A_D(p; X) + \varepsilon. \tag{2.4}$$

Moreover, for any $\varepsilon_1, \varepsilon_2 \geq 0$, we have

$$D_{\varepsilon_1 + \varepsilon_2} = (D_{\varepsilon_1})_{\varepsilon_2}.$$

Proof. Note that $g_D(p, z) + \varepsilon < 0$ for $z \in D_\varepsilon$. Consequently, we have '$\geq$' in (2.3). Therefore, and additionally because for $z \in \partial D_\varepsilon \cap D$ we have

$$g_D(p, z) \geq -\varepsilon \geq \limsup_{w \to z, w \in D_\varepsilon}(g_{D_\varepsilon}(p, w) - \varepsilon),$$

the function

$$\omega(z) := \begin{cases} g_{D_\varepsilon}(p, z) - \varepsilon, & z \in D_\varepsilon, \\ g_D(p, z), & z \in D \setminus D_\varepsilon \end{cases}$$

is plurisubharmonic. Therefore, $\omega(z) \leq g_D(p, z)$, $z \in D$, which completes the proof of (2.3).

Property (2.4) as well as the last part of the lemma follow from (2.3) and the definition of the Azukawa pseudometric. □

For $w \in D$ let us recall the definition

$$\varepsilon(w) := \liminf_{z \to \partial D} g_D(w, z).$$

We shall be interested in case when $\varepsilon(w) > -\infty$. Note that then $g_D(w, z) > -\infty$ for any $z \in D$, $z \neq w$. In fact, take any $\varepsilon > -\varepsilon(w)$. Then the set $D_\varepsilon(w)$ is bounded; otherwise, in view of Lemma 2.3 there would be a sequence $z^\nu \to \infty$, $z^\nu \in D_\varepsilon(w) \subset D$ such that

$$\limsup_{\nu \to \infty} g_D(w, z^\nu) = \limsup_{\nu \to \infty} g_{D_\varepsilon(w)}(w, z^\nu) - \varepsilon \leq -\varepsilon < \varepsilon(w)$$

—contradiction with the the definition of $\varepsilon(w)$. Take any $z \in D$, $z \neq w$ such that

$g_D(w,z) = -\infty$. Take any $\varepsilon > -\varepsilon(w)$. Then $z \in D_\varepsilon(w)$, the boundedness of $D_\varepsilon(w)$ implies that $g_{D_\varepsilon(w)}(w,z) > -\infty$, which in view of Lemma 2.3 implies that $g_D(w,z) > -\infty$—contradiction.

LEMMA 2.4. *Fix $w \in D$. Assume that $g_D(w,\cdot)$ is continuous on D and $\varepsilon(w) > -\infty$. Then for any $\varepsilon \geq -\varepsilon(w)$ the domain $D_\varepsilon(w)$ is hyperconvex. Moreover, if $\varepsilon' > \varepsilon \geq -\varepsilon(w)$ then $D_{\varepsilon'}(w) \subset\subset D_\varepsilon(w)$. Consequently, $D_{\varepsilon'}(w)$ is a bounded hyperconvex domain.*

PROOF. Note that if $z_0 \in \partial D_\varepsilon(w) \cap \partial D$ (z_0 may be equal to ∞) then because of Lemma 2.3

$$0 \geq \limsup_{z \in D_\varepsilon(w), z \to z_0} g_{D_\varepsilon(w)}(w,z) \geq \liminf_{z \in D_\varepsilon(w), z \to z_0} g_{D_\varepsilon(w)}(w,z) =$$

$$\liminf_{z \in D_\varepsilon(w), z \to z_0} g_D(w,z) + \varepsilon \geq \varepsilon(w) + \varepsilon \geq 0.$$

If $z_0 \in \partial D_\varepsilon(w) \cap D$ then because of continuity of $g_D(w,\cdot)$ and Lemma 2.3

$$\lim_{z \in D_\varepsilon(w), z \to z_0} g_{D_\varepsilon(w)}(w,z) = 0.$$

Note that the definition of $\varepsilon(w)$ implies that $D_\varepsilon(w)$ is bounded for any $\varepsilon > -\varepsilon(w)$, which in view of Lemma 2.3 gives us the second statement. □

3. The Green function for balanced domains.

Let $h : \boldsymbol{C}^n \mapsto [0,\infty)$ be a homogeneous function, i.e.

$$h(\lambda z) = |\lambda| h(z), \quad \lambda \in \boldsymbol{C}, \quad z \in \boldsymbol{C}^n,$$

which is upper-semicontinuous.

Then we may define a domain

$$D := D_h := \{z \in \boldsymbol{C}^n : h(z) < 1\}.$$

Let $\tilde{h}$ be the largest plurisubharmonic homogeneous function such that $\tilde{h} \leq h$. It is easy to verify that the function $\tilde{h}$ is well-defined. Note that then $D_{\tilde{h}}$ is the holomorphic envelope of D_h. Certainly, $D_h \subset D_{\tilde{h}}$.

It is trivial that (see Theorem 1.1)

$$g_{D_h}(0,z) \geq \log \tilde{h}(z), \quad z \in D_h. \tag{3.1}$$

Note that for any $z \in D_h$, $\lambda \in E$,

$$g_{D_h}(0,\lambda z) \leq \log|\lambda| + g_{D_h}(0,z). \tag{3.2}$$

In fact, first note that $g_{D_h}(0, e^{i\theta} z) = g_{D_h}(0,z)$ for any $\theta \in \boldsymbol{R}$. For fixed $z \in D_h$ consider the function

$$u : E \ni \lambda \mapsto g_{D_h}(0,\lambda z) - g_{D_h}(0,z).$$

Then u is subharmonic on E, upper-semicontinuous on $\bar{E}$, $u = 0$ on ∂E, $u(\lambda) - \log|\lambda|$ is bounded above near 0. The Riemann removability theorem and maximum principle imply that

$$u(\lambda) \le \log|\lambda|, \quad \lambda \in E,$$

which finishes the proof of (3.2).

Below we shall consider sublevel sets with the pole at 0.

Note that

$$\log h(X) \ge A_{D_h}(0; X) \ge A_{D_{\tilde{h}}}(0; X) = \log \tilde{h}(X), \quad X \in \boldsymbol{C}^n.$$

Therefore, in view of Theorem 1.1(v) (and the definition of $\tilde{h}$) we have that (see [**Azu1**]):

$$A_{D_h}(0; X) = \log \tilde{h}(X)(= A_{D_{\tilde{h}}}(0; X)), \quad X \in \boldsymbol{C}^n. \tag{3.3}$$

Below we shall see that in the case of bounded balanced domains something more is true:

THEOREM 3.1. *Assume that D_h is bounded. Then*

$$g_{D_h}(0, z) = \log \tilde{h}(z) = g_{D_{\tilde{h}}}(0, z), \quad \textit{for } z \in D_h \quad \textit{close enough to } 0. \tag{3.4}$$

PROOF. In view of Remark 2.2 we know that there is $\varepsilon > 0$ such that $D_\varepsilon := (D_h)_\varepsilon$ is pseudoconvex. We claim that D_ε is balanced. To see this take $z \in D_\varepsilon$ and $\lambda \in E$. Then in view of (3.2) we have $g_{D_h}(0, \lambda z) \le \log|\lambda| + g_{D_h}(0, z) < -\varepsilon$. Consequently, $D_\varepsilon = D_{h_1}$ for some plurisubharmonic, homogeneous h_1.

From Theorem 1.1(viii) and Lemma 2.3 we get

$$g_{D_h}(0, z) = g_{D_\varepsilon}(0, z) - \varepsilon = \log h_1(z) - \varepsilon, \quad z \in D_\varepsilon. \tag{3.5}$$

Therefore, to finish the proof it is sufficient to show that $\tilde{h} = \exp(-\varepsilon)h_1$ on $\boldsymbol{C}^n$ (or equivalently on some neighbourhood of 0).

In view of (3.1) and (3.5) we get that $\tilde{h} \le \exp(-\varepsilon)h_1$ on D_ε, which certainly implies that the inequality holds on $\boldsymbol{C}^n$. We are remained with the proof of the inequality $\tilde{h} \ge \exp(-\varepsilon)h_1$, which will be proved if we show that $\exp(-\varepsilon)h_1 \le h$.

To prove the last inequality it is sufficient to show that $\exp(-\varepsilon)h_1(z) < 1$ for any $z \in D_h$. Let us take $z \in D_h$. Then there is $\mu \in E_*$ such that $\mu z \in D_\varepsilon$. So in view of (3.2) and (3.5) we have

$$\log|\mu| + g_{D_h}(0, z) + \varepsilon \ge g_{D_h}(0, \mu z) + \varepsilon = f$$

$$g_{D_\varepsilon}(0, \mu z) = \log h_1(\mu z) = \log|\mu| + \log h_1(z),$$

from which we easily finish the proof. □

As a conclusion of Theorem 3.1 we get a relation between sublevel sets of the balanced domain and that of its holomorphic envelope. By $\hat{D}$ we denote the holomorphic envelope of D.

COROLLARY 3.2. *We have the following*:

$$\widehat{(D_h)_\varepsilon} = (D_{\tilde{h}})_\varepsilon$$

for any $\varepsilon > 0$. In particular, if $(D_{\tilde{h}})_\varepsilon \subset D_h$ then

$$g_{D_h}(0, z) = \log \tilde{h}(z), \quad z \in (D_h)_\varepsilon.$$

Proof. We know that

$$\widehat{(D_h)_\varepsilon} = \{\hat{h} < 1\}$$

for some homogeneous, plurisubharmonic $\hat{h}$. It is sufficient to prove that

$$\exp(-\varepsilon)\hat{h} = \tilde{h}.$$

First let us consider only bounded D_h. Let us take $\delta < \infty$ sufficiently large. Then in view of Lemma 2.3, Theorem 1.1(viii), (3.1) and Theorem 3.1 we have

$$(D_h)_{\varepsilon+\delta} = ((D_h)_\varepsilon)_\delta \subset (\widehat{(D_h)_\varepsilon})_\delta \subset ((D_{\tilde{h}})_\varepsilon)_\delta = (D_{\tilde{h}})_{\varepsilon+\delta} = (D_h)_{\varepsilon+\delta}.$$

From which we conclude that $\hat{h}(z)\exp(\delta) = \tilde{h}(z)\exp(\varepsilon+\delta)$ for small z. This completes the proof in the bounded case.

The unbounded case follows from the standard approximation process. Put $h^\nu(z) := \max\{h(z), \|z\|/\nu\}$.

Note that $\{h^\nu\}_{\nu=1}^\infty$ tends decreasingly to h. The sequence $\{\tilde{h}^\nu\}_{\nu=1}^\infty$ is also decreasing. Let us denote its limit by $\hat{h}$. Note that $\hat{h}$ is homogenuous and plurisubharmonic on $\boldsymbol{C}^n$. From the definition we have that

$$\tilde{h} \le \tilde{h}^\nu \le h^\nu.$$

Therefore, $\tilde{h} \le \hat{h}$ but $\hat{h}$ is plurisubharmonic and $\hat{h} \le h$, so $\tilde{h} = \hat{h}$. Consequently, $(D_{\tilde{h}^\nu})_\varepsilon \to (D_{\tilde{h}})_\varepsilon$ (see Theorem 1.1(vi)). Similarily, $(D_{h^\nu})_\varepsilon \to (D_h)_\varepsilon$, and the same holds for their holomorphic envelopes (remember that we are still in the class of balanced domains).

The special case $(D_{\tilde{h}})_\varepsilon \subset D_h$ implies, in view of the corollary, (3.1) and Remark 2.2, that $(D_h)_{\varepsilon'} = (D_{\tilde{h}})_{\varepsilon'}$ for any $\varepsilon' > \varepsilon$. Suppose that there is $z \in D_h$ such that for some ε' (see (3.1))

$$-\varepsilon > g_{D_h}(0,z) > -\varepsilon' > \log\tilde{h}(z).$$

Then $z \in (D_{\tilde{h}})_{\varepsilon'} \setminus (D_h)_{\varepsilon'}$—contradiction. □

Example 3.3. If D is a bounded, Reinhardt domain with $0 \in D$ (but not balanced), then Theorem 3.1 does not hold any more. Consider the following example (see Example 4.2.8 from [**Jar-Pfl1**]):

$$D := E^2 \setminus \{z \in E^2 : |z_1| \le \alpha, |z_2| \ge \beta\},$$

where $0 < \beta < \alpha < 1$. Then

$$g_D(0,z) = \max\left\{\log\frac{\alpha}{\beta} + \log|z_2|, \log|z_1|\right\} \ne g_{\hat{D}}(0,z) = g_{E^2}(0,z),$$

for z close to 0. Moreover,

$$A_D(0;X) = \max\left\{\log\frac{\alpha}{\beta} + \log|X_2|, \log|X_1|\right\} \ne A_{\hat{D}}(0;X), \quad X \in \boldsymbol{C}^2.$$

It is worth noting that in the case of balanced domains we do not have in general that (compare Theorem 3.1)

$$g_{D_h}(0,z) = \log \tilde{h}(z) \quad \text{for any } z \in D_h$$

as the following example shows:

Example 3.4. The example is based on the following observation. Assume that for a bounded domain D there is a local strong peak function at $z_0 \in \partial D$, in the following sense:

for any $w \in D$ there are neighbourhoods $U_1 \subset\subset U$ of z_0, $w \notin U$ and a plurisubharmonic function $u : U \cap D \mapsto [-\infty, 0)$ such that $\lim_{\zeta \to z_0, \zeta \in U \cap D} u(\zeta) = 0$ and $\sup_{D \cap (U \setminus U_1)} u < 0$.

Then for any $w \in D$, $\lim_{\zeta \to z_0, \zeta \in D} g_D(w,\zeta) = 0$. In fact, there is $M > 0$ such that $g_D(w,\zeta) \geq Mu(\zeta)$ for $\zeta \in \partial U_1 \cap D$. Define $v(\zeta) := g_D(w,\zeta)$ for $\zeta \in D \setminus U_1$ and $v(\zeta) := \max\{g_D(w,\zeta), Mu(\zeta)\}$ for $\zeta \in D \cap U_1$. Then v is plurisubharmonic and, consequently,

$$\lim_{\zeta \to z_0, \zeta \in D} g_D(w,\zeta) \geq \lim_{\zeta \to z_0, \zeta \in U_1 \cap D} Mu(\zeta) = 0.$$

Let $G \subset E^2$ be any pseudoconvex Reinhardt domain such that $\partial G \cap \partial(E^2) = (\partial E)^2$, $\bar{G}$ contains no points from the axes and such that every point from its boundary is a strong peak point (such domains are easy to construct). Then define

$$D := (rE^2) \cup G,$$

where $r \in (0,1)$ is so large that $(rE^2) \cap G \neq \emptyset$. Then $\hat{D} = E^2$; moreover, we may choose G so that D is a balanced complete Reinhardt domain. In view of the remark above $g_D(0,\zeta) \to 0$ as $\zeta \to \partial G \setminus rE^2$, from which we easily get the desired phenomenon.

4. Regularity properties of the Azukawa pseudometric.

Theorem 4.1. *Let D be a domain such that $\varepsilon(w) > -\infty$ for any $w \in D$ (e.g. D is a bounded domain) and g_D is a continuous function (e.g. D is a bounded hyperconvex domain). Then A_D is a continuous function (as a function defined on $D \times \boldsymbol{C}^n$).*

Proof. Fix $(w;X) \in D \times \boldsymbol{C}^n$. It is sufficient to prove that for any sequence $(w_\nu; X_\nu) \to (w;X)$,

$$\lim_{\nu \to \infty} A_D(w_\nu; X_\nu) = A_D(w;X). \tag{4.1}$$

Since A_D is upper semicontinuous we may assume without loss of generality that $X, X_\nu \neq 0$.

Fix $\varepsilon' > \varepsilon > -\varepsilon(w)$. From the assumptions of the theorem, Lemma 2.4 and Theorem 1.1(vi) we can choose a sequence of affine isomorphisms $\Phi_\nu : \boldsymbol{C}^n \mapsto \boldsymbol{C}^n$ such that for large ν:

$$\Phi_\nu(w_\nu) = w, \quad \Phi_\nu'(w_\nu)(X_\nu) = X, \tag{4.2}$$

$$\Phi_\nu(D_{\varepsilon'}(w_\nu)) \subset D_\varepsilon(w), \tag{4.3}$$

$$D_{\varepsilon'}(w) \subset\subset \Phi_\nu(D_\varepsilon(w_\nu)). \tag{4.4}$$

From Lemma 2.3, (4.2) and (4.3) we get

$$\begin{aligned} A_D(w_\nu; X_\nu) + \varepsilon' &= A_{D_{\varepsilon'}(w_\nu)}(w_\nu; X_\nu) \\ &= A_{\Phi_\nu(D_{\varepsilon'}(w_\nu))}(\Phi_\nu(w_\nu); \Phi_\nu'(w_\nu)X_\nu) \\ &= A_{\Phi_\nu(D_{\varepsilon'}(w_\nu))}(w; X) \\ &\geq A_{D_\varepsilon(w)}(w; X) = A_D(w; X) + \varepsilon. \end{aligned}$$

Consequently,

$$A_D(w_\nu; X_\nu) - A_D(w; X) \geq \varepsilon - \varepsilon'.$$

Analoguously, Lemma 2.3, (4.2) and (4.4) give us

$$A_D(w_\nu; X_\nu) - A_D(w; X) \leq \varepsilon' - \varepsilon.$$

Passing with ε' to ε in both inequalities above we get (4.1). □

THEOREM 4.2. *Fix $w \in D$. Assume that D is a domain in $\boldsymbol{C}^n$ such that $g_D(w, \cdot)$ is continuous and $\varepsilon(w) > -\infty$ (e.g. D is a bounded hyperconvex domain). Then*

$$A_D(w; X) = \lim_{\lambda \to 0}(g_D(w, w + \lambda X) - \log|\lambda|), \quad X \in \boldsymbol{C}^n.$$

PROOF. In view of Lemma 2.3 and Lemma 2.4 we lose no generality assuming that D is a bounded hyperconvex domain.

Suppose that the theorem does not hold, so there are a sequence $\{\lambda_k\}_{k=1}^\infty \subset E_*$, $\varepsilon > 0$ and $X \in \boldsymbol{C}^n \setminus \{0\}$ such that $\lambda_k \to 0$ and

$$(4.5) \qquad g_D(w, w + \lambda_k X) - \log|\lambda_k| < A_D(w; X) - 2\varepsilon, \quad k = 1, 2, \ldots.$$

For our convenience we may assume that $w = 0$. We have that $D_\varepsilon \subset\subset D$. Note that there is $\theta_0 \in (0, \pi)$ such that

$$(4.6) \qquad e^{i\theta} D_\varepsilon \subset\subset D \quad \text{for any } |\theta| < \theta_0.$$

Taking if necessary a subsequence we may assume that $\lambda_k X \in D_\varepsilon$, $k = 1, 2, \ldots.$ We know that (see (4.5), (4.6), Theorem 1.1 and Lemma 2.3)

$$\begin{aligned} (4.7) \quad & g_D(0, e^{i\theta}\lambda_k X) - \log|\lambda_k| \\ & \leq g_{e^{i\theta}D_\varepsilon}(0, e^{i\theta}\lambda_k X) - \log|\lambda_k| = g_{D_\varepsilon}(0, \lambda_k X) - \log|\lambda_k| = \\ & g_D(0, \lambda_k X) + \varepsilon - \log|\lambda_k| < A_D(0; X) - \varepsilon, \quad |\theta| < \theta_0, \quad k = 1, 2, \ldots. \end{aligned}$$

Let us define for $\lambda \in U$, where U is a sufficiently small neighbourhood of 0 in $\boldsymbol{C}$, a subharmonic function u as follows:

$$u(\lambda) := g_D(0, \lambda X) - \log|\lambda|, \quad \text{for } \lambda \neq 0$$

$$u(0) := A_D(0; X).$$

Without loss of generality we may assume that $\lambda_k \in U$ for any k. Note that

$\limsup_{\lambda\to 0} u(\lambda) = u(0)$ and u is upper semicontinuous, therefore for k large enough

$$u(e^{i\theta}\lambda_k) < u(0) + \frac{\varepsilon 2\theta_0}{2\pi - 2\theta_0} =: u(0) + \tilde{\varepsilon}, \quad \theta \in [-\pi, \pi]. \tag{4.8}$$

On the other hand we know from (4.7) that

$$u(e^{i\theta}\lambda_k) < u(0) - \varepsilon \quad \text{for any } k \text{ and } |\theta| < \theta_0. \tag{4.9}$$

Subharmonicity of u combined with (4.8) and (4.9) gives us for large k

$$2\pi u(0) \le \int_{-\pi}^{\pi} u(e^{i\theta}\lambda_k)\,d\theta < \int_{|\theta|<\theta_0} (u(0)-\varepsilon)\,d\theta + \int_{\pi\ge|\theta|>\theta_0} u(e^{i\theta}\lambda_k)\,d\theta <$$

$$(u(0)-\varepsilon)2\theta_0 + (2\pi - 2\theta_0)(u(0)+\tilde{\varepsilon}) = 2\pi u(0),$$

which is a clear contradiction. □

Lemma 4.3. *Assume that D is a domain such that $\varepsilon(w) > -\infty$ for any $w \in D$ and g_D is continuous. Fix $w \in D$. Assume that the sequences $\{w_j^\nu\}_{\nu=1}^\infty$, $\{z_j^\nu\}_{\nu=1}^\infty$ of points from D, $j = 1,2$ are such that*

$$w_j^\nu, z_j^\nu \to w, \quad j = 1,2, \quad \frac{w_1^\nu - w_2^\nu}{\|w_1^\nu - w_2^\nu\|} - \frac{z_1^\nu - z_2^\nu}{\|z_1^\nu - z_2^\nu\|} \to 0, \quad \frac{\|w_1^\nu - w_2^\nu\|}{\|z_1^\nu - z_2^\nu\|} \to 1. \tag{4.10}$$

Then $g_D(w_1^\nu, w_2^\nu) - g_D(z_1^\nu, z_2^\nu) \to 0$.

Proof. Fix $\varepsilon' > \varepsilon > -\varepsilon(w)$. From the assumptions of the lemma, Theorem 1.1(vii) and Lemma 2.4 we know that for ν large enough there is $\Phi_\nu : \boldsymbol{C}^n \mapsto \boldsymbol{C}^n$ an affine isomorphism such that

$$\Phi_\nu(w_j^\nu) = z_j^\nu, \quad j = 1,2 \quad \text{and} \quad \Phi_\nu(D_{\varepsilon'}(w_1^\nu)) \subset\subset D_\varepsilon(z_1^\nu).$$

Now we have, in view of Lemma 2.3, for ν large enough:

$$g_D(z_1^\nu, z_2^\nu) + \varepsilon = g_{D_\varepsilon(z_1^\nu)}(z_1^\nu, z_2^\nu) \le g_{\Phi_\nu(D_{\varepsilon'}(w_1^\nu))}(\Phi_\nu(w_1^\nu), \Phi_\nu(w_2^\nu)) =$$

$$g_{D_{\varepsilon'}(w_1^\nu)}(w_1^\nu, w_2^\nu) = g_D(w_1^\nu, w_2^\nu) + \varepsilon'.$$

Consequently, for ν large enough,

$$g_D(z_1^\nu, z_2^\nu) \le g_D(w_1^\nu, w_2^\nu) + \varepsilon' - \varepsilon.$$

Similarily, we get for ν large enough:

$$g_D(w_1^\nu, w_2^\nu) \le g_D(z_1^\nu, z_2^\nu) + \varepsilon' - \varepsilon.$$

Passing with ε' to ε we complete the proof of the lemma. □

Corollary 4.4. *Let D be a domain as in Lemma 4.3, $w \in D$ and $X \in \boldsymbol{C}^n$, $\|X\| = 1$. Then*

$$A_D(w;X) = \lim_{w',w''\to w,\, w'\neq w'',\, (w'-w'')/\|w'-w''\|\to X} (g_D(w',w'') - \log\|w'-w''\|).$$

Proof. Take any sequences $\{w_j^\nu\}_{\nu=1}^\infty$ of different points from D such that $w_j^\nu \to w$ $(j = 1, 2)$ and $(w_1^\nu - w_2^\nu)/\|w_1^\nu - w_2^\nu\| \to X$. Define

$$z_1^\nu := w, \quad z_2^\nu := w - \|w_1^\nu - w_2^\nu\| X.$$

Note that $\|z_1^\nu - z_2^\nu\| = \|w_1^\nu - w_2^\nu\|$ and $(z_1^\nu - z_2^\nu)/\|z_1^\nu - z_2^\nu\| \to X$. Therefore, in view of Lemma 4.3

$$\begin{aligned}&\lim_{\nu\to\infty}(g_D(w_1^\nu, w_2^\nu) - \log\|w_1^\nu - w_2^\nu\|)\\ &\quad = \lim_{\nu\to\infty}(g_D(z_1^\nu, z_2^\nu) - \log\|z_1^\nu - z_2^\nu\|)\\ &\quad = \lim_{\nu\to\infty}(g_D(w, w - \|w_1^\nu - w_2^\nu\|X) - \log\|w_1^\nu - w_2^\nu\|),\end{aligned}$$

the last expression is, in view of Theorem 4.2, equal to $A_D(w; X)$. □

Corollary 4.5. *Let D be as Lemma 4.3. Then*

$$\lim_{w', w'' \to w, w' \neq w''}(g_D(w', w'') - g_D(w'', w')) = 0, \quad w \in D.$$

Proof. It is sufficient to consider two sequences $\{w_j^\nu\}_{\nu=1}^\infty$ $(j = 1, 2)$ tending to w such that $(w_1^\nu - w_2^\nu)/\|w_1^\nu - w_2^\nu\| \to X$ for some $X \in \boldsymbol{C}^n$, $\|X\| = 1$. Note that, in view of Corollary 4.4,

$$\begin{aligned}&\lim_{\nu\to\infty}(g_D(w_1^\nu, w_2^\nu) - \log\|w_1^\nu - w_2^\nu\|)\\ &\quad = A_D(w; X) = A_D(w; -X)\\ &\quad = \lim_{\nu\to\infty}(g_D(w_2^\nu, w_1^\nu) - \log\|w_2^\nu - w_1^\nu\|),\end{aligned}$$

from which we get that

$$\lim_{\nu\to\infty}(g_D(w_1^\nu, w_2^\nu) - g_D(w_2^\nu, w_1^\nu)) = 0. \qquad \square$$

Having the result on symmetry of the Green function as in Corollary 4.5 it seems to be probable that the integrated form of the Azukawa pseudometric coincides with some kind of inner pseudodistance related to the Green function (at least for bounded hyperconvex domains), defined similarily as it was done in [**Jar-Pfl1**].

Example 4.6. There is a bounded pseudoconvex domain for which we cannot replace 'limsup' with 'lim' in the definition of the Azukawa pseudometric. Let $D_h = \{z \in \boldsymbol{C}^2 : h(z) < 1\}$ be a bounded pseudoconvex balanced domain, where h is the Minkowski function of D_h, such that $h(1, 1) = 1$ and there are sequences $\{a_k\}_{k=1}^\infty$ and $\{b_k\}_{k=1}^\infty$ of points from E different from 0 such that $a_k \to 0$, $b_k \to 0$ and

$$\lim_{k\to\infty} h(1, \exp(a_k)) \to \delta < 1, \quad \lim_{k\to\infty} h(1, \exp(b_k)) = 1$$

(note that such a function and sequences exist). Define $\Phi : \boldsymbol{C}^2 \ni (z_1, z_2) \mapsto (z_1, z_2 \exp(z_1)) \in \boldsymbol{C}^2$. Note that Φ is a biholomorphism. Put $D := \Phi^{-1}(D_h)$. Remark that D is a bounded pseudoconvex domain. For k large we have the following sequence of equalities (see Theorem 1.1(ii) and (viii)):

$$\begin{aligned}g_D(0, (a_k, a_k)) - \log|a_k| &= g_{D_h}(0, (a_k, \exp(a_k)a_k)) - \log|a_k|\\ &= \log h(a_k, \exp(a_k)a_k) - \log|a_k| = \log h(1, \exp(a_k))\end{aligned}$$

and the last expression tends to $\log\delta < 0$ as k tends to infinity. Similarily we get that

$$g_D(0,(b_k,b_k)) - \log|b_k| \to 0 \quad \text{as } k \text{ tends to infinity},$$

which gives us that there is no limit in the definition of the Azukawa metric of $A_D(0,(1,1))$.

COROLLARY 4.7. *If D is a domain in $\boldsymbol{C}$, then A_D is continuous and for any $w \in D$ $A_D(w;1) = \lim_{\lambda\to 0}(g_D(w,w+\lambda) - \log|\lambda|)$ and if ∂D is not polar then*

$$A_D(w;1) = \lim_{w',w''\to w,\, w'\neq w''}(g_D(w',w'') - \log\|w'-w''\|).$$

PROOF. The result is trivial if ∂D is polar. Therefore, we may assume that ∂D is not polar. We know that g_D is continuous (see [**Ran**]), therefore, in view of Theorems 4.1 and 4.2 and Corollary 4.4, it is sufficient to show that for any $w \in D$, $\varepsilon(w) > -\infty$. Take any point $z_0 \in \partial D$. Without loss of generality we may assume that $z_0 \in \boldsymbol{C}$. There is $r > 0$ such that $\partial D \setminus \bar{\boldsymbol{B}}(z_0,r)$ is not polar. Put $\tilde{D} := D \cup \boldsymbol{B}(z_0,r)$. $\tilde{D}$ is a domain. Then $z_0 \in \tilde{D}$ and $\partial\tilde{D}$ is not polar. Therefore, (see [**Ran**]),

$$\liminf_{z\to z_0,\, z\in D} g_D(w,z) \geq \liminf_{z\to z_0,\, z\in\tilde{D}} g_{\tilde{D}}(w,z) = g_{\tilde{D}}(w,z_0) > -\infty.$$

This completes the proof. □

5. Pluricomplex Green function with many poles.

Below we deal with the upper and lower bound from the following formula (see [**Lel**]):

$$\min\{\nu(p)g_D(p,z) : p \in P\} \geq g_D(P;\nu;z) \geq \sum_{p\in P}\nu(p)g_D(p,z), \quad z \in D. \tag{5.1}$$

First we consider the case of the lower bound from the formula (5.1). Let us consider the following sets (see [**Lel**]):

$$\mathscr{E}(D,P,\nu) := \left\{z \in D : g_D(P;\nu;z) = \sum_{p\in P}\nu(p)g_D(p,z)\right\}.$$

Certainly, $P \subset \mathscr{E}(D,P,\nu)$.

We have the following:

LEMMA 5.1. *Let D and P be as above. Then for any $\mu,\nu : P \mapsto (0,\infty)$ the following equality holds*:

$$\mathscr{E}(D,P,\nu) = \mathscr{E}(D,P,\mu).$$

PROOF. Let us take $z \in \mathscr{E}(D,P,\nu)$, $z \notin P$. In view of (5.1) we may assume without loss of generality that $g_D(p,z) > -\infty$ for any $p \in P$. Fix $\varepsilon > 0$ so small that:

$$\min\left\{\sum_{p\in Q}\nu(p)g_D(p,z) : \varnothing \neq Q \subset P, Q \neq P\right\} \tag{5.2}$$

$$> \sum_{p\in P}\nu(p)g_D(p,z) + \min\left\{\frac{\nu(p)}{\mu(p)} : p \in P\right\}\frac{\varepsilon}{\sharp P}.$$

Because of (2.1) there is $\varphi \in \mathcal{O}(\bar{E}, D)$ such that $\varphi(0) = z$, $\varphi^{-1}(P) \cap E \neq \emptyset$ and

$$g_E(\varphi^{-1}(P) \cap E; \tilde{v}; 0) < \sum_{p \in P} v(p) g_D(p, z) + \min\left\{\frac{v(p)}{\mu(p)} : p \in P\right\}\varepsilon \Big/ \sharp P. \tag{5.3}$$

First note that in view of (5.1), (5.2) and (2.1) we get that $\varphi^{-1}(p) \cap E \neq \emptyset$ for any $p \in P$. The left hand-side in the inequality (5.3) equals

$$\begin{aligned}\sum_{\lambda \in E, \varphi(\lambda) \in P} \tilde{v}(\lambda) \log|\lambda| &= \sum_{p \in P} \sum_{\lambda \in E, \varphi(\lambda) = p} \tilde{v}(\lambda) \log|\lambda| \\ &= \sum_{p \in P} g_E(\varphi^{-1}(p) \cap E; \tilde{v}_{|\varphi^{-1}(p) \cap E}; 0).\end{aligned}$$

Each summand in the last expression is at least $v(p) g_D(p, z)$, which gives us, in view of (5.3), that

$$\begin{aligned}&\frac{v(p)}{\mu(p)} g_E(\varphi^{-1}(p) \cap E, \tilde{\mu}_{|\varphi^{-1}(p) \cap E}, 0) \\ &\quad = g_E(\varphi^{-1}(p) \cap E; \tilde{v}_{|\varphi^{-1}(p) \cap E}; 0) \\ &\quad < v(p) g_D(p, z) + \min\left\{\frac{v(p)}{\mu(p)} : p \in P\right\}\varepsilon \Big/ \sharp P, \quad p \in P,\end{aligned}$$

so

$$g_E(\varphi^{-1}(p) \cap E; \tilde{\mu}_{|\varphi^{-1}(p) \cap E}; 0) < \mu(p) g_D(p, z) + \varepsilon / \sharp P.$$

Summing over $p \in P$ we get that (see (5.1))

$$\sum_{p \in P} \mu(p) g_D(p, z) \le g_D(P, \mu, z) < \sum_{p \in P} \mu(p) g_D(p, z) + \varepsilon,$$

and, consequently, $z \in \mathcal{E}(D, P, \mu)$. □

As an immediate corollary we get

Corollary 5.2. *Let* $P \subset \boldsymbol{B}_n$, $\sharp P \ge 2$, $n \ge 2$. *Then* $\mathcal{E}(\boldsymbol{B}_n, P, v) = P \cup (L \cap \boldsymbol{B}_n)$, *where* L *is the complex straight line containing* P ($L = \emptyset$ *if such a line does not exist*).

Proof. Use Lemma 5.1 and Remark 3.4 from [**Edi-Zwo**]. □

We get also the following generalization of the results from [**Com**] (Proposition 4):

Proposition 5.3. *Let* P *and* D *be arbitrary. Assume that* $v, \mu : P \to (0, \infty)$, $\mu(p)/v(p) \le \mu(p_0)/v(p_0)$ *and*

$$g_D(P; v; z) = v(p_0) g_D(p_0, z)$$

for some $z \in D$. *Then*

$$g_D(P; \mu; z) = \mu(p_0) g_D(p_0, z).$$

PROOF. Suppose that the last equality does not hold. So because of (2.1) and (5.1) there is $\varphi \in \mathcal{O}(\bar{E}, D)$ such that $\varphi(0) = z$, $\varphi^{-1}(P) \cap E \neq \emptyset$ and

$$g_E(\varphi^{-1}(P) \cap E; \tilde{\mu}; 0) < \mu(p_0) g_D(p_0, z).$$

The left hand-side in the last inequality equals the left hand-side of the following:

$$\sum_{p \in P, \varphi^{-1}(p) \cap E \neq \emptyset} \frac{\mu(p)}{\nu(p)} g_E(\varphi^{-1}(p) \cap E; \tilde{\nu}_{|\varphi^{-1}(p) \cap E}; 0)$$

$$\geq \frac{\mu(p_0)}{\nu(p_0)} \sum_{p \in P, \varphi^{-1}(p) \cap E \neq \emptyset} g_E(\varphi^{-1}(p) \cap E; \tilde{\nu}_{|\varphi^{-1}(p) \cap E}; 0) \geq \frac{\mu(p_0)}{\nu(p_0)} g_D(P; \nu; z)$$

—contradiction. □

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WŁODZIMIERZ ZWONEK
Instytut Matematyki
Uniwersytet Jagielloński
Reymonta 4
30-059 Kraków, Poland
E-mail: zwonek@im.uj.edu.pl